New estimates on quadratic variational Mean Field Games via techniques from the JKO world

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Mean Field Games and Related Topics
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Outline

Do we really need here a long introduction about MFG?

Mean Field Games (introduced by Lasry and Lions, and at the same time by Huang, Malhamé and Caines) describe the evolution of a population, where each agent has to choose the strategy (i.e., a path) which best fits his preferences, but is affected by the others through a global *mean field* effect.

It is a differential game, with a continuum of players, all indistinguishable and all negligible. It is a typical congestion game (agents try to avoid the regions with high concentrations) and we look for a *Nash equilibrium*, which can be translated into a system of PDEs.

J.-M. Lasry, P.-L. Lions, Jeux à champ moyen. (I & II) *C. R. Math. Acad. Sci. Paris,* 2006 + Mean-Field Games, *Japan. J. Math.* 2007

M. Huang, R.P. Malhamé, P.E. Caines, Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle, *Comm. Info. Syst.* 2006

P.-L. Lions, courses at Collège de France, 2006/12, videos available on the web

P. Cardaliaguet, lecture notes, www.ceremade.dauphine.fr/~cardalia/

MFG with density penalization- 1

In a population of agents everybody chooses its own trajectory, solving

$$\min \int_0^T \left(\frac{|x'(t)|^2}{2} + g(x, \rho_t(x(t)))\right) dt + \Psi(x(T)),$$

with given initial point $x(0)$; here $g(x, \cdot)$ is a given increasing function of the density ρ_t at time t. The agent hence tries to avoid overcrowded regions.

Input: the evolution of the density ρ_t.

A crucial tool is the value function φ for this problem, defined as

$$\varphi(t_0, x_0) := \min\left\{\int_{t_0}^T \left(\frac{|x'(t)|^2}{2} + g(x, \rho_t(x(t)))\right) dt + \Psi(x(T)),\ x(t_0) = x_0\right\}.$$

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MFG with density penalization- 2

Optimal control theory tells us that φ solves

$$(HJ) \qquad -\partial_t\varphi(t,x) + \frac{1}{2}|\nabla\varphi(t,x)|^2 = g(x,\rho_t(x)), \quad \varphi(T,x) = \Psi(x).$$

Moreover, the optimal trajectories $x(t)$ follow $x'(t) = -\nabla\varphi(t,x(t))$.

Hence, given the initial ρ_0, we can find the density at time t by solving

$$(CE) \qquad \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0,$$

which give as **Output:** the evolution of the density ρ_t.
We have an equilibrium if **Input = Output**.
This requires to solve a coupled system (HJ)+(CE):

$$\begin{cases} -\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} = g(x,\rho), \\ \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0, \\ \varphi(T,x) = \Psi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$

Stochastic case : we can also insert random effects $dX = \alpha dt + dB$, obtaining $-\partial_t\varphi - \Delta\varphi + \frac{|\nabla\varphi|^2}{2} - g(x,\rho) = 0$; $\quad \partial_t\rho - \Delta\rho - \nabla\cdot(\rho\nabla\varphi) = 0$.

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Variational principle

It happens that an equilibrium is found by minimizing the (global) energy

$$\mathcal{A}(\rho, v) := \int_0^T \int_\Omega \left(\frac{1}{2}\rho_t |v_t|^2 + G(x, \rho_t) \right) + \int_\Omega \Psi \rho_T$$

among pairs (ρ, v) such that $\partial_t \rho + \nabla \cdot (\rho v) = 0$, with given ρ_0, where $G(x, \cdot)$ is the anti-derivative of $g(x, \cdot)$, i.e. $G(x, \cdot)' = g(x, \cdot)$. This problem is convex in the variables $(\rho, w := \rho v)$ and admits a dual problem:

$$\sup \left\{ -\mathcal{B}(\phi, p) := \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega G^*(x, p) \ : \ \phi_T \leq \Psi, \ -\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 = p \right\},$$

where G^* is the Legendre transform of G (w.r.t. p).

Formally, if (ρ, v) solves the primal problem and (φ, p) the dual, then we have $v = -\nabla \phi$ and $p = g(x, \rho)$, i.e. we solve the MFG system.

Warning: the existence of a dual solution (in a suitable weak functional space) is only proven under some growth conditions on G. Also, for non-smooth functions, this is not the same of having optimal trajectories...

P. Cardaliaguet, P.J. Graber. Mean field games systems of first order. *ESAIM: COCV*, 2015.
J.-D. Benamou, G. Carlier Augmented Lagrangian methods for transport optimization, Mean-Field Games and degenerate PDEs, *JOTA*, 2015.
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Measures on the space of trajectories

The same variational problem can also be written in the following way: let $\mathcal{C} = H^1([0,T];\Omega)$ be the space of curves valued in Ω and $e_t : \mathcal{C} \to \Omega$ the evaluation map, $e_t(\gamma) = \gamma(t)$. Solve

$$\min\left\{\int_{\mathcal{C}} K dQ + \int_0^T \mathcal{G}((e_t)_\# Q) + \int_\Omega \Psi d(e_T)_\# Q,\ Q \in \mathcal{P}(\mathcal{C}), (e_0)_\# Q = \rho_0\right\},$$

where $K : \mathcal{C} \to \mathbb{R}$ and $\mathcal{G} : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}$ are given by $K(\gamma) = \frac{1}{2}\int_0^T |\gamma'|^2$ and $\mathcal{G}(\rho) = \int G(x, \rho(x))dx$. ($\#$ denotes image measure, or push-forward).

Existence: by semicontinuity in the space $\mathcal{P}(\mathcal{C})$.

Optimality conditions: take $\overline{Q}$ optimal, $\widetilde{Q}$ another competitor, and $Q_\varepsilon = (1-\varepsilon)\overline{Q} + \varepsilon\widetilde{Q}$. Setting $\rho_t = (e_t)_\# \overline{Q}$ and $p(t,x) = g(x, \rho_t(x))$, differentiating w.r.t. ε gives

$$J_p(\widetilde{Q}) \geq J_p(\overline{Q}),$$

where J_p is the linear functional

$$J_p(Q) = \int K dQ + \int_0^T \int_\Omega p(t,x) d(e_t)_\# Q\, dt + \int_\Omega \Psi d(e_T)_\# Q.$$

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Back to an equilibrium

Look at J_p. It is well-defined for $p \geq 0$ measurable. Yet, if $p \in C^0$ we can also write $\int_0^T \int_\Omega p(t,x) d(e_t)_\# Q\, dt = \int_C dQ \int_0^T p(t,\gamma(t))dt$ (in genera we have problems in the definition a.e.) and hence we get that

$$Q \mapsto \int_C dQ(\gamma) \left(K(\gamma) + \int_0^T p(t,\gamma(t))dt + \Psi(\gamma(T)) \right)$$

is minimal for $Q = \overline{Q}$. Hence $\overline{Q}$ is concentrated on curves minimizing $\mathcal{L}_{p,\Psi}(\gamma) := K(\gamma) + \int_0^T p(t,\gamma(t))dt + \Psi(\gamma(T))$. This means **Input=Output**.

A rigorous proof can also be done even for $p \notin C^0$ but one has to choose a precise representative. Techniques from incompressible fluid mechanics **(incompressible Euler** *à la* **Brenier)** allow to handle some interesting cases using $\hat{p}(x) := \limsup_{r \to 0} \fint_{B(x,r)} p(t,y)dy$ (maximal function Mp needed to justify some convergences. . .).

L. Ambrosio, A. Figalli, On the regularity of the pressure field of Brenier's weak solutions to incompressible Euler equations, *Calc. Var. PDE*, 2008.

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Precise equilibrium statements and need for summability

An adaptation of Ambrosio-Figalli's statement is

Theorem

If $\overline{Q}$ is optimal, then $\overline{Q}$-a.e. curve γ is an optimal trajectory in the following sense:

$$\mathcal{L}_{\hat{p},\Psi}(\gamma) \leq \mathcal{L}_{\hat{p},\Psi}(\tilde{\gamma})$$

on every interval $[t_0, T]$ and for every curve $\tilde{\gamma}$ such that $\int_{t_0}^T M(p_+)(\tilde{\gamma}) < +\infty$.

How many curves do satisfy $\int_{t_0}^T M(p_+)(\tilde{\gamma}) < +\infty$? If $G(x,\rho) \approx \rho^q$, $M|p| \in L^{q'}$ then for every $\tilde{Q}$ with finite cost, $\tilde{Q}$-a.e. curves do it, since

$$\int \int_{t_0}^T M|p|(\tilde{\gamma}(t))dt\, d\tilde{Q}(\tilde{\gamma}) = \int_{t_0}^T \int_\Omega M|p|\, d\rho_t\, dt.$$

Need for estimates: we should prove $\rho \in L^q$ and $M|p| \in L^{q'}$ (for $q' > 1$, equivalent to $p \in L^{q'}$): this is easy for G growing as ρ^q, difficult for more exotic G ($G(\rho) = \exp(\rho), (1-\rho)^{-1}) \dots$).
Should we prove $\rho \in L^\infty$ and in particular $p_+ = (g(x,\rho)_+) \in L^\infty$, then the optimality would be among all curves, and we should not care about Mp.

Trajectories on the space of measures, time-discretization

The very same variational problem can also be written in a third way. Use the space of probabilities $\mathbb{W}_2(\Omega)$ endowed with the Wasserstein distance W_2 (enduced by optimal transport) and look for a curve $(\rho(t))_{t\in[0,T]}$ solving

$$\min\left\{\int_0^T\left(\frac{1}{2}|\rho'|(t)^2+\mathcal{G}(\rho(t))\right)dt+\int_\Omega \Psi d\rho_T\ :\ \rho(0)=\rho_0\right\},$$

(here $|\rho'|(t):=\lim_{s\to t}\frac{W_2(\rho(s),\rho(t))}{|s-t|}$ is the metric derivative of the curve ρ).

Existence is also easy by semicontinuity and by Ascoli-Arzelà applied in the space of curves from $[0,T]$ to the compact metric space $\mathbb{W}_2(\Omega)$.

A useful approximation can be obtained via time-discretization: fix $\tau = T/N$ and look for a sequence $\rho_0,\rho_1,\dots,\rho_N$ solving

$$\min\left\{\sum_{k=0}^{N-1}\left(\frac{W_2^2(\rho_k,\rho_{k+1})}{2\tau}+\tau\mathcal{G}(\rho_k)\right)+\int_\Omega \Psi d\rho_N\right\}.$$

Optimality conditions in a JKO-like scheme

If $\rho_0, \rho_1, \ldots, \rho_N$ solves

$$\min \left\{ \sum_{k=0}^{N-1} \left(\frac{W_2^2(\rho_k, \rho_{k+1})}{2\tau} + \tau \mathcal{G}(\rho_k) \right) + \int_\Omega \Psi d\rho_N \right\}$$

then, for each $0 < k < N$, the measure ρ_k solves

$$\min \left\{ \frac{W_2^2(\rho, \rho_{k-1})}{2\tau} + \frac{W_2^2(\rho, \rho_{k+1})}{2\tau} + \tau \mathcal{G}(\rho) \right\},$$

i.e. it solves a minimization problem similar to what we see in the JKO scheme for gradient flows:

$$\min \left\{ \frac{W_2^2(\rho, \rho_{k-1})}{2\tau} + \mathcal{G}(\rho) \right\}.$$

For $k = N$, we have a true JKO-style problem with one only Wasserstein distance.

R. Jordan, D. Kinderlehrer, F. Otto. The variational formulation of the Fokker-Planck equation. *SIAM J. Math. An.*, 1998.

The flow-interchange estimates

Let ρ_s be the gradient flow of a functional $\mathcal{F}(\rho) := \int F(\rho(x))dx$, i.e. a solution of $\partial_s\rho - \nabla\cdot(\rho\nabla(F'(\rho))) = 0$, with initial datum at $s = 0$ equal to the optimal ρ at step k. We suppose Ω to be convex and we choose F so that $\mathcal{F}$ is geodesically convex functional on $\mathbb{W}_2(\Omega)$. This provides

$$\frac{d}{ds}\frac{W_2^2(\rho_s,\nu)}{2} \leq \mathcal{F}(\nu) - \mathcal{F}(\rho_s)$$

We also have

$$\frac{d}{ds}\mathcal{G}(\rho_s) = \int \nabla(g(x,\rho_s))\cdot\nabla(F'(\rho_s))\,d\rho_s.$$

the optimality of ρ_k hence gives

$$\int \nabla(g(x,\rho_k))\cdot\nabla(F'(\rho_k))\,d\rho_k \leq \frac{\mathcal{F}(\rho_{k+1}) - 2\mathcal{F}(\rho_k) + \mathcal{F}(\rho_{k-1})}{\tau^2}.$$

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Let ρ_s be the gradient flow of a functional $\mathcal{F}_m(\rho) := \int F_m(\rho(x))dx$, i.e. a solution of $\partial_s\rho - \nabla\cdot(\rho\nabla(F_m'(\rho))) = 0$, with initial datum at $s = 0$ equal to the optimal ρ at step k. We suppose Ω to be convex and use $F_m(\rho) = \rho^m$, so that we have a geodesically convex functional on $\mathbb{W}_2(\Omega)$. This provides

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We also have

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L^m and L^∞ estimates

Suppose $g(x,\rho) = V(x) + g(\rho)$. We start from $V = 0$:

$$0 \leq \int g'(\rho_k) F_m''(\rho_k)\rho_k |\nabla \rho_k|^2 \quad \leq \quad \frac{\mathcal{F}_m(\rho_{k+1}) - 2\mathcal{F}_m(\rho_k) + \mathcal{F}_m(\rho_{k-1})}{\tau^2}$$

$k \mapsto \mathcal{F}_m(\rho_k)$ is discretely convex. If $\rho_0 \in L^m$, and we suppose $\rho_T \in L^m$, so is ρ_t, uniformly in t.

With a final penalization Ψ, if $\Psi \in C^{1,1}$, then we also obtain

$$\mathcal{F}_m(\rho_N) \leq (1 + C\tau m)\mathcal{F}_m(\rho_{N-1}),$$

hence, not only $k \mapsto \mathcal{F}_m(\rho_k)$ is convex, but we control its final derivative, which also implies boundedness of $\mathcal{F}_m$.

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$k \mapsto \mathcal{F}_m(\rho_k)$ is discretely convex. If $\rho_0 \in L^m$, and we suppose $\rho_T \in L^m$, so is ρ_t, uniformly in t. This also works for $m = \infty$.
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$k \mapsto \mathcal{F}_m(\rho_k)$ is discretely convex. Don't suppose anything on ρ_0, ρ_T and/or Ψ: we can obtain local estimates. Suppose $g'(s) \ge s^{-\alpha}$. We use

$$\int g'(\rho_k)F''_m(\rho_k)\rho_k|\nabla\rho_k|^2 \ge c\int \rho_k^{m-1-\alpha}|\nabla\rho_k|^2 \quad = \quad c||\nabla(\rho_k^{(m+1-\alpha)/2})||^2_{L^2}$$

$$\ge \quad c||(\rho_k^{(m+1-\alpha)/2})||^2_{L^\beta},$$

for $\beta \in (2, 2^*) > 2$ and we use **Moser's iteration** on exponents $m_j \approx (\beta/2)^j$.

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L^m and L^∞ estimates

Suppose $g(x,\rho) = V(x) + g(\rho)$. Do not suppose anymore $V = 0$:

$$? \leq \int g'(\rho_k) F''_m(\rho_k)\rho_k |\nabla \rho_k|^2 \leq \frac{\mathcal{F}_m(\rho_{k+1}) - 2\mathcal{F}_m(\rho_k) + \mathcal{F}_m(\rho_{k-1})}{\tau^2} - \int (\nabla V \cdot \nabla \rho_k) F''_m(\rho_k)\rho_k.$$

The new term needs to be estimated in terms of V and $\mathcal{F}_m$. $k \mapsto \mathcal{F}_m(\rho_k)$ is no more convex (but rather it satisfies $u'' + C(m)u \geq 0$). We can go on...

$$\int g'(\rho_k) F''_m(\rho_k)\rho_k |\nabla \rho_k|^2 \geq c \int \rho_k^{m-1-\alpha} |\nabla \rho_k|^2 = c||\nabla(\rho_k^{(m+1-\alpha)/2})||^2_{L^2} \geq c||(\rho_k^{(m+1-\alpha)/2})||^2_{L^\beta},$$

for $\beta \in (2, 2^*) > 2$ and we use **Moser's iteration** on exponents $m_j \approx (\beta/2)^j$.

H. Lavenant, F. Santambrogio Optimal density evolution with congestion: L^∞ bounds via flow interchange techniques and applications to variational Mean Field Games, preprint.

L^m and L^∞ estimates - summary

Theorem

Suppose $g(x,\rho) = V(x) + g(\rho)$. Suppose $g'(s) \geq s^{-\alpha}$ for $s \geq s_0$.

- *If V is Lipschitz, $\alpha \geq -1$, and $s_0 = 0$ then $\rho \in L^\infty_{loc}((0,T) \times \overline{\Omega})$.*
- *Same result if $s_0 > 0$ but $V \in C^{1,1}$ and $\partial V/\partial n \geq 0$.*
- *These results extend to $(0,T]$ is $\Psi \in C^{1,1}$ and $\partial \Psi/\partial n \geq 0$.*
- *If $\alpha < -1$, then the same results, for $V, \Psi \in C^{1,1}$, $\partial V/\partial n \geq 0$ and $\partial \Psi/\partial n \geq 0$, are true if we already know $\rho \in L^{m_0}((0,T) \times \overline{\Omega})$ for $m_0 > d|\alpha + 1|/2$. This is true in particular if $\rho_0 \in L^{m_0}$ and T is small enough.*

If g is a convex function finite on $\mathbb{R}_+$, then we also obtain upper bounds on $p = V(x) + g(\rho(x))$.

Generalizations: If the Hamiltonian is not quadratic (agents minimize $\int H(x') + g(x,\rho)$) we can replace W_2^2 with the transport cost $H(x-y)$; if we have x-dependance in the Hamiltonian then geodesic convexity in the Wasserstein space on a manifold is involved (Ricci bounds...).

The case of density constraints - 1

Consider $g(x,\rho) = V(x) + \rho(x)^m$, and the limit $m \to \infty$. In the variational problem this gives $G = \frac{1}{m+1} F_{m+1}$ and, at the limit, the density constraint $\rho \leq 1$. In this case p is a priori only a measure satisfying

$$p = \tilde{p} + V, \quad \tilde{p} \geq 0, \ \tilde{p}(1 - \rho) = 0.$$

The term $\tilde{p}$ is just the limit of the terms ρ^m, and its regularity is crucial to prove $M|p| \in L^1$.

Results inspired by Incompressible fluid mechanics and based on convex duality gave

$$V \in C^{1,1} \Rightarrow p \in L^2_{loc}((0,T); BV_{loc}(\Omega)).$$

P. Cardaliaguet, A. Mészáros, F. Santambrogio, First order Mean Field Games with density constraints: Pressure equals Price, *SIAM J. Contr. Opt.*, 2016
Y. Brenier, Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations, *Comm. Pure Appl. Math.*, 1999.

The case of density constraints - 2

Applying the flow interchange technique to the case $G = \frac{1}{m+1} F_{m+1}$ we get

$$\begin{aligned} \int \frac{F''_{m+1}(\rho_k)}{m+1} F''_m(\rho_k)\rho_k |\nabla \rho_k|^2 \quad &\le \quad \frac{\mathcal{F}_m(\rho_{k+1}) - 2\mathcal{F}_m(\rho_k) + \mathcal{F}_m(\rho_{k-1})}{\tau^2} \\ &- \quad \int (\nabla V \cdot \nabla \rho_k) F''_m(\rho_k)\rho_k . \end{aligned}$$

This means, for $\tilde{p} = \rho^m$,

$$\int |\nabla \tilde{p}|^2 \le \frac{\mathcal{F}_m(\rho_{k+1}) - 2\mathcal{F}_m(\rho_k) + \mathcal{F}_m(\rho_{k-1})}{(m-1)\tau^2} + \int |\nabla V||\nabla \tilde{p}|.$$

Using $\mathcal{F}_m(\rho) \le \mathcal{F}_{m+1}(\rho)$ and the bound on $\int \frac{1}{m+1}\mathcal{F}_{m+1}(\rho)$ coming from optimality, this allows to conclude a bound on $\tilde{p}$ in $L^2_{loc}((0,T); H^1(\Omega))$ under the only assumption $V \in H^1$.

The End

Thanks for your attention