Boltzmann-type Optimal Control and Model Predictive Control

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Outline

Constrained self-organized systems

- The mathematical description of emerging collective phenomena and self-organization in systems composed of large numbers of agents has gained increasing interest in various fields in *biology*, *robotics* and *control theory*, as well as *sociology* and *economics*.
- We consider such problems in a constrained setting, where the emergent behaviour (*alignment/consensus*, *patterns*,...) is not spontaneous but enforced by the action of an external policy maker or hierarchical leadership[1].
- *Mean-field type control and game theory* has raised a lot of interest recently[2]. The general setting consists in a control problem involving a very large number of agents where the evolution of the state and the objective functional of each agent may be influenced by the behaviour of other agents.

[1] *M. Caponigro, M. Fornasier, B. Piccoli, E. Trélat '13; G. Albi, L.P. '13; G. Albi, L. P., M. Zanella '14; M. Fornasier, B. Piccoli, F. Rossi '14; S. Wongkaev, A. Borzí '15; B. Düring, P.A. Markowich, J.F. Pietschmann, M.-T. Wolfram '09;*

[2] *J-M. Lasry, P-L. Lions '07; M. Huang, R.P. Malhamé, P.E. Caines '07; A. Bensoussan, J. Frehse, P. Yam '13; P. Cardaliaguet, J-M. Lasry, P-L. Lions, A. Porretta '12; Y. Achdou, F. Camilli, I. Capuzzo-Dolcetta '12; P. Degond, J.-G. Liu, C. Ringhofer '14; M. Fornasier, F. Solombrino '14; D.A. Gomes, J. Saúde '14;...*

Constrained self-organized systems

Classical *examples* in socio-economy, biology and robotics are given by forcing animals/humans/robots to follow a specific path or to reach a desired zone...



... but also influencing consumers towards a given good, persuading voters during political elections, influencing opinions over social networks



Microscopic optimal control problems

Let $v_i(t) \in \Omega \subset \mathbb{R}^d$ evolve

$$\dot{v}_i(t) = F_i(\mathbf{v}(t)) + \mathbf{u}_i(t), \qquad i = 1, \ldots, N$$

where $\mathbf{v} = (v_1, \ldots, v_N)$ and $\mathbf{u} = (u_1, \ldots, u_N)$ is a control term defined as follows

Microscopic optimal control

$$\mathbf{u}^* = \arg\min_{\mathbf{u} \in \mathcal{U}} J^N(u) := \int_0^T \frac{1}{N} \sum_{i=1}^N \left(L_i(\mathbf{v}(t)) + \gamma\psi(u_i(t))\right) dt$$

constrained to the dynamic of $(v_i(t))_{i=1}^N$, $\mathcal{U}$ the admissible space of controls.

- In the above system the control is determined as a minimizer of the common social cost $J^N(u)$.
- For large values of N the computational effort is prohibitive[3] (*curse of dimensionality*).

[3] *R.E. Bellman '57*

Examples

- **Opinion formation**[4]

$$d = 1, \qquad \Omega = [-1, 1], \qquad F_i(\mathbf{v}) = \frac{1}{N}\sum_{j=1}^{N} P(v_i, v_j)(v_j - v_i)$$

$$L_i(\mathbf{v}(t)) = |v_i(t) - \bar{v}|^2, \qquad \psi(u_i(t)) = |u_i(t)|^2,$$

where $\bar{v}$ is a *desired opinion* and P is the compromise function, for example $P(v_i, v_j) = \Psi(|v_i - v_j| \leq \Delta)$ with Δ the bounded confidence interval and $\Psi(\cdot)$ the indicator function.

- **Wealth distribution**[5]

$$d = 1, \qquad \Omega = [0, \infty[, \qquad F_i(\mathbf{v}) = \frac{1}{N}\sum_{j=1}^{N} P(v_i, v_j)(v_j - v_i),$$

$$L_i(\mathbf{v}(t)) = \frac{1}{N}\sum_{j=1}^{N} |v_i(t) - v_j(t)|^2, \quad \psi(u_i(t)) = |u_i(t)|^2,$$

where the control aims at *reducing inequalities* in the wealth distribution.

[4] *M.H. DeGroot '74; R. Hegselmann, U. Krause '02; G. Albi, M. Herty, L. P. '15*
[5] *S. Solomon, M. Levy '96; B. Düring, L.P., G. Toscani '17*

Swarming models

Let $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$, $d = 1, 2, 3$ evolve

$$\dot{x}_i(t) = v_i(t),$$
$$\dot{v}_i(t) = F_i(\mathbf{x}(t), \mathbf{v}(t)) + u_i(t), \qquad i = 1, \dots, N$$

where $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{v} = (v_1, \dots, v_N)$ and $\mathbf{u} = (u_1, \dots, u_N) \in \mathbb{R}^{d \times N}$ is a control term defined as

$$\mathbf{u}^* = \arg\min_{\mathbf{u} \in \mathcal{U}} J^N(u) := \int_0^T \frac{1}{N} \sum_{i=1}^N \left(L_i(\mathbf{x}(t), \mathbf{v}(t)) + \gamma \psi(u_i(t)) \right) dt.$$

For example [6]

$$F_i(\mathbf{x}, \mathbf{v}) = \frac{1}{N} \sum_{j=1}^N H(x_i, x_j)(v_j - v_i), \quad H(x_i, x_j) = \frac{K}{(1 + |x_i - x_j|^2)^\beta}$$

and we can take $L_i(\mathbf{x}, \mathbf{v}) = |v_i - \bar{v}|^2$ to enforce convergence towards a velocity $\bar{v}$.

[6] *F. Cucker, S. Smale '07; M. D'Orsogna, A. Bertozzi et al.'06; S.Motsch, E.Tadmor '11*

Mean-Field Optimal Control

Optimal Control Problem with N agents → (N >> 1) → Mean-Field Optimal Control Problem

Let $f_N(x,v,t) = \frac{1}{N}\sum_{j=1}^N \delta(x - x_i(t))\delta(v - v_i(t))$ then $f_N \to f(x,v,t)$ as $N \to \infty$ which satisfies [7]

Mean-field optimal control

$$\min_{u\in\mathcal{U}} J(f,u) := \int_0^T \int_{\mathbb{R}^{2d}} (L(x,v,t) + \gamma\psi(u))\, f(x,v,t)\; dx\, dv\; dt$$

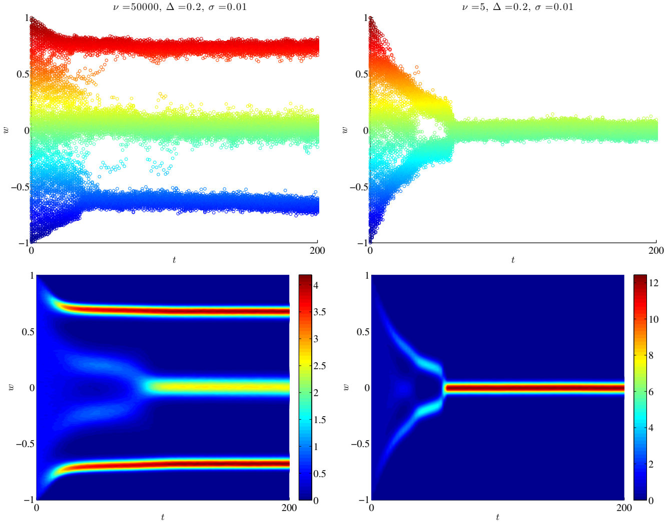
$$s.t. \qquad \partial_t f + v\cdot\nabla_x f = -\nabla_v \cdot ((\mathcal{F}[f] + u)\, f)\,, \qquad f(x,v,0) = f^0(x,v)$$

For example, for the Cucker-Smale model we have

$$\mathcal{F}[f](x,v,t) = \int_{\mathbb{R}^{2d}} H(x,y)(w-v)f(y,w,t)\, dw\, dy.$$

[7] *A. Bensoussan, J. Frehse, P. Yam '13; G.A. Y-P. Choi, M. Fornasier, D. Kalise, F. Solombrino, '13, '16.*

Comparison of microscopic and mean-field solutions



Opinion model with $\Delta = 0.2$. On the left $\gamma = 50000$ on the right $\gamma = 5$.

Boltzmann-type Optimal Control

Let us consider the following Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q_\kappa(f,f)$$

where $Q_\kappa(f,f)(x,v,t)$ is the *constrained Boltzmann-Povzner operator* [8] defined as

$$Q_\kappa(f,f) = \lambda \int_{\mathbb{R}^{2d}} \left(\frac{1}{\mathcal{J}_\kappa} f(x,{}'v) f(y,{}'w) - f(x,v) f(y,w) \right) \, dy \, dw.$$

Here $\lambda > 0$ is the interaction rate, $({}'v,{}'w)$ the *pre-interaction terms* which generate the pair (v,w) and $\mathcal{J}_\kappa$ indicates the jacobian of the binary interaction rule

$$\begin{cases} v' = v + \alpha F(x,y,v,w) + \alpha\kappa(v,w) \\ w' = w + \alpha F(y,x,w,v) + \alpha\kappa(w,v) \end{cases}$$

where (v',w') are now the *post-interaction terms* generated from (v,w).
The factor $\kappa(v,w)$ represents the effect of the *control over the binary dynamic*. If $\kappa(v,w) = u(v)$, $\kappa(w,v) = u(w)$ the *control acts over the single particle*.

[8] *A.Y. Povzner '62; M. Fornasier, J. Haskovec, G. Toscani '10; L.P., G. Toscani '13*

Boltzmann Optimal Control

Boltzmann-type optimal control

$$\min_{\kappa\in\mathcal{K}} J_B(f,\kappa) := \int_0^T\int_{\mathbb{R}^{2d}}\left(L(x,v,t)+\gamma\int_{\mathbb{R}^{2d}}\psi(\kappa)f(y,w,t)dydw\right)f(x,v,t)dxdvdt,$$

$$s.t. \qquad \partial_t f + v\cdot\nabla_x f = Q_\kappa(f,f), \qquad f(x,v,0)=f^0(x,v).$$

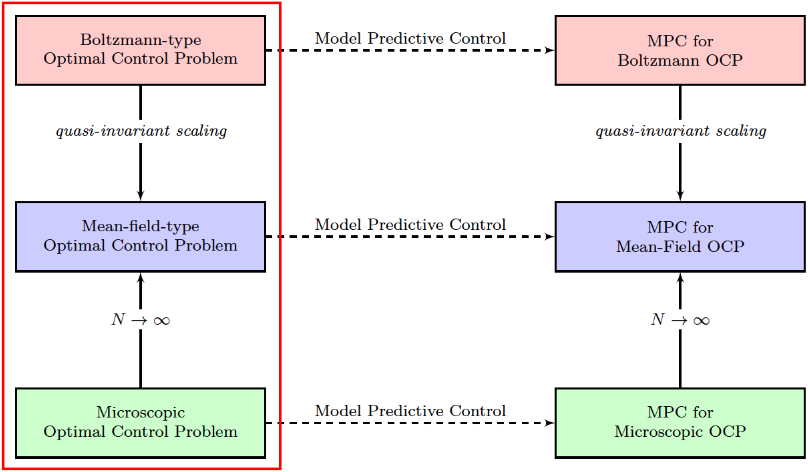
In the case $\kappa(v,w)=u(v)$ if $\int_{\mathbb{R}^{2d}} f(y,w,t)dydw = 1$ we have

$$\int_{\mathbb{R}^{2d}}\psi(u)f(y,w,t)dydw = \psi(u)\int_{\mathbb{R}^{2d}} f(y,w,t)dydw = \psi(u).$$

Questions

- Relations between Boltzmann-type control and mean-field control?
- How can we deal with the curse of dimensionality?

Boltzmann Control, Mean-field control and MPC



Optimality condition for the Boltzmann control

To simplify notations we restrict to the space homogeneous setting $f = f(v,t)$. Let us introduce the function $p \in C_0^2(\mathbb{R}^d \times [0,T];\mathbb{R})$, and we define the Lagrangian of the Boltzmann-type Optimal Control Problem as follows

$$\mathcal{L}_B(f,u,p) = J_B(f,u) + \int_0^T \int_{\mathbb{R}^d} p\left(\partial_t f - Q_\kappa(f,f)\right) dvdt.$$

For $\kappa(v,w) = u(v)$ computing the variations with respect to f and u we get [9]

Boltzmann optimality system ($\kappa(v,w) = u(v)$)

$$\partial_t f = \lambda \int_{\mathbb{R}^d} \left(\frac{1}{\mathcal{J}_\kappa} f('v) f('w) - f(v) f(w) \right) dw, \qquad f(v,0) = f^0(v)$$

$$\partial_t p = L(v,t) + \gamma\psi(u(v))$$

$$\quad - \lambda \int_{\mathbb{R}^d} \left(p(v') - p(v) + p(w') - p(w)\right) f(w)\, dw, \quad p(v,T) = 0,$$

$$\nabla_u \psi(u) = \frac{\lambda\alpha}{\gamma} \int_{\mathbb{R}^d} \nabla_v p(v') f(w)\, dw$$

[9] *C. Cercignani '88*

After integration by parts we get

$$\mathcal{L}_B(f,u,p) = J_B(f,u) + \int_{\mathbb{R}^d} f(v,T)p(v,T)\,dv - \int_{\mathbb{R}^d} f(v,0)p(v,0)\,dv$$
$$- \int_0^T \int_{\mathbb{R}^d} \partial_t p f\,dv - \lambda \int_0^T \int_{\mathbb{R}^{2d}} (p(v') - p(v)) f(v) f(w)\,dv\,dw\,dt.$$

Then we compute the functional derivatives of the Lagrangian with respect to the state function f and the control u. We have

$$\frac{\delta \mathcal{L}_B(f,u,p)}{\delta f} = L(v) + \gamma\psi(u(v))$$
$$- \partial_t p - \lambda \int_{\mathbb{R}^{2d}} \big(p(v') - p(v) + p(w') - p(w)\big) f(w)\,dw$$
$$\frac{\delta \mathcal{L}_B(f,u,p)}{\delta u} = \gamma \nabla_u \psi(u) - \lambda\alpha \int_{\mathbb{R}^d} \nabla_u p(v') f(w)\,dw.$$

Imposing that the solution to the optimal control problem satisfies

$$\left.\frac{\delta \mathcal{L}_B}{\delta f}\right|_{(f,u,p)} = 0 \quad \text{and} \quad \left.\frac{\delta \mathcal{L}_B}{\delta u}\right|_{(f,u,p)} = 0,$$

yields the optimality system.

The quasi-invariant optimality limit

We can prove [10]

Theorem

Let $T > 0$, $\varepsilon > 0$, and assume that the function $F(\cdot,\cdot) \in L^2_{loc}$ for every $t \geq 0$. We consider a weak solution f of the Boltzmann optimal control with initial datum $f_0(v)$. Introducing the *quasi-invariant scaling*

$$\alpha = \varepsilon, \qquad \lambda = 1/\varepsilon,$$

define by $(f^\varepsilon, k^\varepsilon, p^\varepsilon)$ a solution for the scaled optimality conditions system. Then as $\varepsilon \to 0$, $(f^\varepsilon, u^\varepsilon, p^\varepsilon)$ converges point-wise, up to a subsequence, to (g, u, q) solution of the *mean-field optimality system*

$$\begin{aligned}
\partial_t g &= -\nabla_v \cdot \left(\int_{\mathbb{R}^d} F(v,w) g(w)\, dw + u(v) \right) g(v) \\
\partial_t q &= L(v,t) + \gamma\psi(u) \\
&\quad - \int_{\mathbb{R}^d} \left(\nabla_v q(v) \cdot (F(v,w) + u(v)) + \nabla_v q(w) \cdot (F(w,v) + u(w)) \right) g(w)\ dw \\
\nabla_u \psi(u(v)) &= \frac{1}{\gamma} \nabla_v q(v).
\end{aligned}$$

[10] *G. Albi, L. Pareschi '17*

Let f^ε be a weak solution of the scaled equation

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(v) f^\varepsilon(v)\, dv = \frac{1}{\varepsilon}\int_{\mathbb{R}^{2d}} \left(\varphi(v'_\varepsilon) - \varphi(v)\right) f^\varepsilon(v) f^\varepsilon(w)\, dw\, dv$$

for every test function φ. Now writing $\varphi(v'_\varepsilon) = \varphi(v) + (v'_\varepsilon - v)\cdot\nabla_v\varphi(v) + R(v'_\varepsilon - v;\varphi)$ we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(v) f^\varepsilon(v)\, dv = \int_{\mathbb{R}^{2d}} \left(\nabla_v\varphi(v)\cdot(F(v,w) + u^\varepsilon(v))\right) f^\varepsilon(v) f^\varepsilon(w)\, dw\, dv + \mathcal{R}_1^\varepsilon,$$

where $\mathcal{R}_1^\varepsilon$ is the remainder.
We perform an analogous computation for the adjoint equation

$$\partial_t p^\varepsilon = L(v,t) + \gamma\psi(u(v)) - \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \left(p^\varepsilon(v' - \varepsilon) - p^\varepsilon(v) + p^\varepsilon(w'_\varepsilon) - p^\varepsilon(w)\right) f(w)\, dw,$$

since p^ε is in $C_0^2(\mathbb{R}^d \times [0,T])$ we can write

$$p^\varepsilon(v'_\varepsilon) = p^\varepsilon(v) + (v'_\varepsilon - v)\cdot\nabla_v p^\varepsilon(v) + R(v'_\varepsilon - v; p).$$

Introducing the above expansion yields

$$\begin{aligned}\partial_t p^\varepsilon &= L(v,t) + \gamma\psi(u^\varepsilon(v)) \\ &\quad - \int_{\mathbb{R}^d} \left((F(v,w) + u^\varepsilon(v,w))\cdot\nabla_v p^\varepsilon(v) + (F(w,v) + u^\varepsilon(w,v))\cdot\nabla_v p^\varepsilon(w)\right) f^\varepsilon(w)\, dw + \mathcal{R}_2^\varepsilon.\end{aligned}$$

In the limit $\varepsilon \to 0$ both the remainders vanishes and denoting by (g,u,q) the limit of $(f^\varepsilon, u^\varepsilon, p^\varepsilon)$ we get the desired mean-field optimality system. The same argument permits to show the compatibility condition.

Optimality systems for binary control

In the general case, for a symmetric binary control $\kappa(v,w)=\kappa(w,v)$ we have

Boltzmann optimality system ($\kappa=\kappa(v,w)$)

$$\partial_t f = \lambda \int_{\mathbb{R}^d} \left(\frac{1}{\mathcal{J}_\kappa} f('v) f('w) - f(v) f(w) \right) dw, \qquad f(v,0) = f^0(v)$$

$$\partial_t p = L(v,t) + \gamma \int_{\mathbb{R}^d} \left(\psi(\kappa(v,w)) + \psi(\kappa(w,v))\right) f(w)\, dw$$

$$- \lambda \int_{\mathbb{R}^d} \left(p(v') - p(v) + p(w') - p(w)\right) f(w)\, dw, \quad p(v,T) = 0,$$

$$\nabla_\kappa \psi(\kappa) = \frac{\lambda\alpha}{2\gamma} \left(\nabla_v p(v') + \nabla_v p(w')\right)$$

In the quasi-invariant limit we obtain optimality conditions for a different mean-field problem

Mean-field optimality system ($\kappa=\kappa(v,w)$)

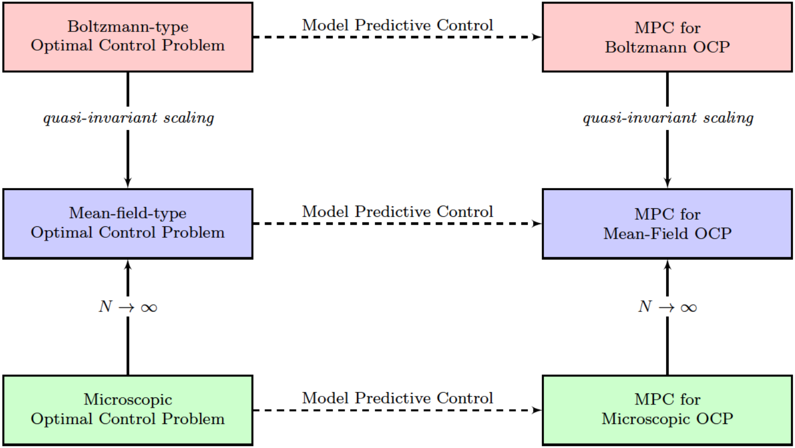
$$\partial_t g = -\nabla_v \cdot \left(\left(\int_{\mathbb{R}^d} (F(v,w) + \kappa)\, g(w)\, dw \right) g(v) \right)$$

$$\partial_t q = L(v,t) + \gamma \int_{\mathbb{R}^d} \left(\psi(\kappa(v,w)) + \psi(\kappa(w,v))\right) g(w)\, dw$$

$$- \int_{\mathbb{R}^d} \left(\nabla_v q(v) \cdot (F(v,w) + \kappa(v,w)) + \nabla_v q(w) \cdot (F(w,v) + \kappa(w,v))\right) g(w)\, dw$$

$$\nabla_\kappa \psi(\kappa) = \frac{1}{2\gamma} \left(\nabla_v q(v) + \nabla_v q(w)\right).$$

Boltzmann Control, Mean-field control and MPC



Model Predictive Control

Model Predictive Control (MPC) MPC compute the next $M-1$, say, optimal moves on a limited time horizon.

- Split the time interval $[0,T]$ in intervals of length Δt, with $t^n = n\Delta t$.
- Compute the value of the control at t^n, solving for the known state $f^n(v)$ a (reduced) time discrete optimization problem on the predictive horizon $[t^n, t^n + (M-1)\Delta t]$.
- Having the control at t^n the new state $f^{n+1}(v)$ is computed.
- This procedure is reiterated until $n\Delta t = T$ is reached.

The method typically yields suboptimal solutions, and is closely related to*Instantaneous Control* (IC) which essentially corresponds to the case $M=2$.

MPC for Boltzmann optimal control

We consider the MPC method to solve the Boltzmann optimal control problem based on a sequence of forward Euler approximations, thus we have [11]

MPC for Boltzmann optimal control

$$\min_u J_M^B(\kappa, f) := \sum_{m=0}^{M-2} \int_{t^m}^{t^{m+1}} \left[\int_{\mathbb{R}^d} \left(L^m(v) + \gamma \int_{\mathbb{R}^d} \psi(\kappa^m) f^m(w)dw \right) f^m(v)dv \right],$$

subject to

$$f^{m+1} = f^m + \Delta t Q_{\kappa^m}(f^m, f^m), \quad m = 0, \dots, M-2,$$

with the binary dynamics

$$v'(\kappa^m) = v + \alpha F(v, w) + \alpha \kappa^m(v, w),$$
$$w'(\kappa^m) = w + \alpha F(w, v) + \alpha \kappa^m(w, v).$$

[11] *G. Albi, L.P. '17*

Boltzmann-MPC feedback control

We obtain the following system of feedback controls for $m = M-2, \dots, 0$

$$\nabla_\kappa \psi(\kappa^{M-1}) = 0,$$

$$\nabla_\kappa \psi(\kappa^m) = -\frac{\alpha\lambda}{2\gamma}\Delta t \left[\nabla_v L^{m+1}(v') + \nabla_v L^{m+1}(w') \right.$$
$$\left. +2\gamma\nabla_v \left(\int_{\mathbb{R}^d} (\psi(\kappa^{m+1}(v',s)) + \psi(\kappa^{m+1}(w',s))) f^m(s) ds \right) \right].$$

For $\psi(\cdot) = |\cdot|^2$ and a target cost $L(v) = |v - \bar{v}|^2$ we have the feedback control

$$\kappa^{M-1} = 0,$$

$$\kappa^m = -\frac{\alpha\lambda}{\gamma}\Delta t \left[\bar{v} - \frac{1}{2}(v'+w') + \gamma\nabla_v \left(\int_{\mathbb{R}^d} (|\kappa^{m+1}(v',s)|^2 + |\kappa^{m+1}(w',s)|^2) f^m(s) ds \right) \right].$$

In the *instantaneous control* case $M = 2$ we have

$$\hat{\kappa}(v,w) = \frac{\alpha\lambda\Delta t}{\gamma + \alpha^2\lambda\Delta t}\left[\bar{v} - \frac{1}{2}(v+w) - \frac{\alpha}{2}(F(v,w) + F(w,v)) \right].$$

Mean-field-MPC feedback control

Similarly, we can consider a MPC approach based on forward Euler steps to solve the mean field optimal control, thus we have [12]

MPC for mean-field optimal control

$$\min_u J_M(u,f) := \sum_{m=0}^{M-2} \int_{t^m}^{t^{m+1}} \left[\int_{\mathbb{R}^d} \left(L(v) + \gamma\psi(u^m)\right) f^m(v)dv \right],$$

subject to

$$f^{m+1} = f^m - \Delta t\, \nabla_v \cdot \left(\left(\mathcal{F}[f^m] + u^m \right) f^m \right) \qquad m = 0, \ldots, M-2,$$

In this case the system of feedback controls for $m = M-2, \ldots, 0$ reads

$$\nabla_u \psi(u^{M-1}) = 0,$$

$$\nabla_u \psi(u^m) = -\frac{\Delta t}{\gamma} \left[\nabla_v L(v) + 2\gamma \nabla_v \left(\psi(u^{m+1}) \right) \right], \quad m = 0, \ldots, M-2$$

[12] *M.Herty, M.Zanella '16; G. Albi, L.P. '17*

Mean-field-MPC feedback control

For a quadratic penalization $\psi(\cdot) = |\cdot|^2$ and a target cost $L(v) = |\bar{v} - v|^2$ we have an explicit formulation of the feedback control as follows

$$u^{M-1} = 0,$$
$$u^m = \frac{\Delta t}{\gamma}\left[(\bar{v} - v) - \gamma \nabla_v \left(|u^{m+1}|^2\right)\right], \quad m = M-2, \dots, 0$$

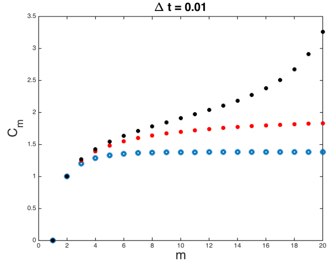
From which we can compute the explicit formula for the control as follows

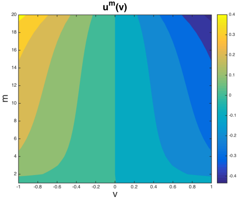
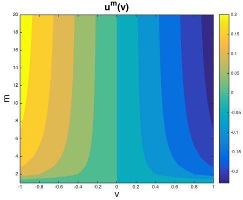
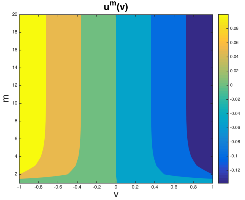
$$u^{M-1} = 0,$$
$$u^m = \frac{\Delta t}{\gamma} C_m (\bar{v} - v)$$

where for $m = M-2, \dots, 0$, $C_m := 1 + 2\Delta t^2/\gamma C_{m+1}^2$, and $C_{M-1} = 0$.
For control horizon such that $M = 2$, we obtain the *instantaneous control*

$$\hat{u}(v) = \frac{\Delta t}{\gamma}(\bar{v} - v).$$

Explicit MPC mean-field feedback control



$M = 20$	$\gamma = 7$	$\gamma = 8$	$\gamma = 10$
$\bar{v} = 0$			

Monte Carlo methods

- High computational cost of solving the Boltzmann and mean-field operators in dimension d, for a product-type quadrature formula based on N parameters is $O(N^d)$.
- Structural properties (positivity, conservation of mass, momentum, ...) are difficult to preserve at the discrete level.

- Staring point is a standard *splitting method* between transport and interaction in the scaled (quasi-invariant) Boltzmann equation

$$\partial_t f^\varepsilon = -v \cdot \nabla_x f^\varepsilon, \qquad \partial_t f^\varepsilon = \frac{1}{\varepsilon} Q_\kappa^\varepsilon(f^\varepsilon, f^\varepsilon).$$

- Transport step is solved by shift of the statistical samples (free transport).
- Interaction step can be rewritten as

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon}\left[Q_\kappa^{\varepsilon,+}(f^\varepsilon, f^\varepsilon) - f^\varepsilon\right], \quad \int_{\mathbb{R}^{2d}} f^\varepsilon \, dx \, dv = 1,$$

where $Q_\kappa^{\varepsilon,+} \geq 0$ is the *gain part* of the interaction operator.

MPC Monte Carlo methods

We rewrite the forward Euler scheme in the MPC approximation as

$$f^{\varepsilon,m+1} = \left(1 - \frac{\Delta t}{\varepsilon}\right) f^{\varepsilon,m} + \frac{\Delta t}{\varepsilon} Q^{\varepsilon,+}_{\kappa^m}(f^{\varepsilon,m}, f^{\varepsilon,m}), \quad m = 0, \dots, M-1$$

If we assume $f^{\varepsilon,m}$ is a probability density also $Q^{\varepsilon,+}_{\kappa^m}(f^{\varepsilon,m}, f^{\varepsilon,m})$ is a probability density. Under the restriction $\Delta t \leq \varepsilon$ then $f^{\varepsilon,m+1}$ is a *convex combination* of probability densities and we can construct a Monte Carlo simulation process [13].

- In order to sample from $Q^{\varepsilon,+}_{\kappa^m}(f^{\varepsilon,m}, f^{\varepsilon,m})$ we need to evaluate the *feedback control* κ^m, except for instantaneous control this may require a suitable numerical method.
- The computational cost to advance one time step is *linear*, $O(N_s)$, where N_s is the number of statistical samples from $f^{\varepsilon,m}$.
- Taking $\varepsilon = \Delta t$, for $\Delta t \ll 1$ we approximate the mean-field model through the *asymptotic Monte Carlo algorithm* [14]

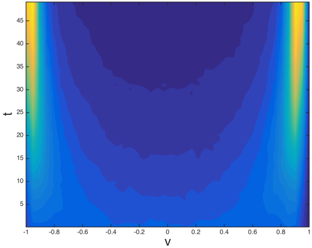
$$f^{\Delta t,m+1} = Q^{\Delta t,+}_{\kappa^m}(f^{\Delta t,m}, f^{\Delta t,m}).$$

[13] *K. Nanbu ('78); G. Bird ('95)*

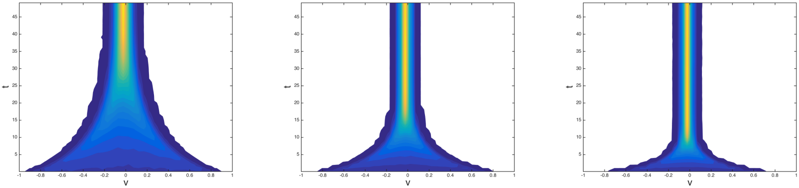
[14] *A.V. Bobylev, K. Nanbu '00; R.E. Caflisch, L.P., G. Dimarco '10; G. Albi, L.P. '13*

MPC of consensus: opinion model

$f(v,t)$



$\gamma = 8$	$M = 2\ (IC)$	$M = 5$	$M = 15$



$\bar{v} = 0$

Conclusions

- We introduced *Boltzmann-type optimal control* problems and studied their relations with classical mean-field optimal control.
- Derivation of the corresponding *MPC* approximations in combination with Monte Carlo methods permits to construct effective stochastic numerical schemes which defeat the curse of dimensionality.

Future directions

- rigorous analysis
- effect of uncertainties in the interaction parameters
- non cooperative case