3 TOPICS ON MFG

joint work(s) in progress with Ch. BERTUCCI

I COUPLINGS OF STRATEGIES

II "PLANNING" WITH COMMON NOISE

III FROM MFG TO AGENT BASED MODELS

only samples ... and simple ones!

I COUPLINGS OF STRATEGIES

1) a long time ago, J.-M. Lasry - PL² observed that, in a MFG setting, if the cost and the dynamics for each player depend on the strategies of the other players i.e. for the player i depend upon $\left(\frac{1}{N-1} \sum_{j \neq i} \delta_{\alpha_j^*}\right)$ - a measure on the "space of strategies" (α_j^* optimal strategy for j ...)

THEN, AT LEAST FORMALLY, ONE COULD WRITE MFG (Forward-Backward system if no common noise) or (master equation in the general case) with an Hamiltonian

$H(x, \nabla_x u ; m, \mu)$ and a transport operator

$\operatorname{div}(\alpha^* m)$

where α^* is the optimal control and μ is the image measure of m by α^* : $\mu = \alpha^* \circ m$.

Ex (Fw-Bw case, T-t) $\nu \geq 0$

$$\frac{\partial u}{\partial t} - \nu \Delta u + \sup_\alpha \{ + \alpha \nabla u - L(x, \alpha; m, \mu) \} = 0$$

$$\frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}(\alpha^* m) = 0 \qquad + \text{IC } u + \text{TC } m$$

and $\mu = \alpha^* \circ m$

(α^* max. in α so α^* depend upon ∇u and (m, μ), hence $\mu = \alpha^* \circ m$ is a "third equation" highly nonlinear in μ ...!)

$$\frac{\Sigma}{N-1} \int \varphi(\alpha_j^*) \xrightarrow[N]{} \int \varphi(\alpha^*(x; m, \mu))\, dm(x)$$

See also P. CARDIALAGUET - Ch. A. LEHALLE

2) WITH NATURAL CONDITIONS / RESTRICTIONS EXISTENCE IS OK. WHAT ABOUT SITUATIONS WHICH LEAD TO UNIQUENESS (T small OK, similar to MFG) : "GLOBAL" UNIQUENESS / MON. FOR MFG ??

3) In other contexts (jump diffusion processes for instance), many ways to write "image measure operators" exists and it can make a difference for the math. analysis...

4) Here, a substantial class of problems can be reformulated as

$$\frac{\partial u}{\partial t} - \nu\Delta u + H(x, \nabla u; \nabla u(\cdot), m(\cdot)) = 0$$

$$u|_{t=0} = \varphi[m(0)]$$

$$\frac{\partial m}{\partial t} + \nu\Delta u + \operatorname{div}\left(\frac{\partial H}{\partial p}(-)\, m\right) = 0 \quad , m|_{t=T} = m_1$$

Ex : $H = H(\nabla u) - G[\nabla u, m] \cdot \nabla u$

$$\frac{\partial H}{\partial p} = H'(\nabla u) - G[\nabla u, m]$$

particular case $G[\nabla u, m] = \lambda \int H'(\nabla u)\, dm$ i.e.

dynamics $\alpha - \lambda\bar{\alpha}$ $\quad\bar{\alpha}$ mean flow

obviously $\bar{\alpha} = \int z\, d\mu$!

A GLOBAL UNIQUENESS RESULT

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(\nabla u) - G[\nabla u, m] \cdot \nabla u = f[u,m]$$

$$\frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left((H'(\nabla u) - G)m\right) = g[u,m]$$

$$u|_{t=0} = \varphi[m(0)] \quad , \quad m|_{t=T} = m_1$$

CONDITIONS: H convex ; f, g commute with translations ;

G, φ are invariant by translations ; φ mon ;

$\exists\ \alpha, \beta \in \mathbb{R}$ s.t. $\underline{\text{constants}}$ $(g, f) + (\alpha m, \beta u)$ mon. in (u, m)

(+ some "strict. somewhere" ...)

THM: UNIQUENESS HOLDS

Z No x dep. in $H, f, g \ldots$! Z

PROOF (VERY SIMPLE)

- let $(\tilde{u}, \tilde{m}) = (u, m)\left(x - \int_t^T G\,ds, t\right)$, it solves the MFG with $G \equiv 0$!

- This reduced system has a unique solution

$$\frac{d}{dt} \int (U-V)(m-n) = \underbrace{(\quad\quad)}_{\geq 0} - (\alpha+\beta) \int (U-V)(m-n) \;!$$

- $G[\nabla \tilde{u}, \tilde{m}] = G[\nabla u, m]$! ∎

Ex. $\frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 - \nu \Delta u - \lambda \left(\int \nabla u\, m\right) \cdot \nabla u = f(m)$

$$u|_{t=0} = u_0 \in \mathbb{R} \quad , \; m|_{t=T} = m_1$$

$$\frac{\partial m}{\partial t} + \operatorname{div}\left(\left\{\nabla u - \lambda \left(\int \nabla u\, m\right)\right\} m\right) = 0$$

has a unique solution for all $\lambda \in \mathbb{R}$ if $f \nearrow$

RK: nonuniqueness if G is not invariant by translations

what about $u_0(x)$ in the above example?

A SIMPLE AND INSTRUCTIVE CASE

$$\frac{\partial u}{\partial t} - \nu \Delta u + \frac{1}{2}|\nabla u|^2 - \lambda\left(\int \nabla u\, m\right) \cdot \nabla u = 0$$

$$\frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(\{\nabla u - \lambda \int \nabla u\, m\}\, m\right) = 0$$

$$u|_{t=0} = u_0(x), \quad m|_{t=T} = m_1(x) \quad \left(\geq 0, \int m_1 = 1\right)$$

$\lambda = 0$ $\exists !$ sol. ...

Facts: i) If ∇u_0 bded, $\exists$ sol., ii) $\frac{d}{dt}\int \nabla u\, m = 0$,

iii) $\frac{d}{dt}\int u\, m = \frac{1}{2}\int m|\nabla u|^2$, iv) $\frac{d}{dt}\int m x = (1-\lambda)\int \nabla u\, m$

and $\exists$ implicit formula for $A = \int \nabla u\, m$.

Let $u_0(x,t)$ be the solution of $\frac{\partial u}{\partial t} - \nu\Delta u + \frac{1}{2}|\nabla u|^2 = 0$, $u|_{t=0} = u_0(x)$

then $u(x,t) = u_0(x + \lambda A t, t)$ and thus

$$A = \int m_1 \nabla u(x,T) = \int m_1(x) \nabla u_0(x + \lambda A T, T)\, dx\,!$$

- leads to nonuniqueness results
- if $u_0'' \leq C_0$ ($C_0 \geq 0$, $u(T)'' \leq \frac{C_0}{1 + C_0 T}$ and if $\lambda > 0$

uniqueness holds if $\lambda \frac{C_0 T}{1 + C_0 T} < 1$

- $m_1 = \delta_0$, $u_0(0) = c_0 \frac{|x - x_0|^2}{2}$

then $u = \frac{c_0}{1 + c_0 T} \frac{|x - x_0|^2}{2}$ and $\nabla u(x + \lambda A T, T) = \frac{c_0}{1 + c_0 T}(x + \lambda A T - x_0)$

$$\longrightarrow A \overset{?}{=} \lambda \frac{c_0 T}{1 + c_0 T} A - \frac{c_0 x_0}{1 + c_0 T}$$

$\lambda < \frac{1 + c_0 T}{c_0 T}$ OK ($>$ also but changes of sign ...)

$\lambda = \frac{1 + c_0 T}{c_0 T}$ non existence

Finally, the same formula holds for

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}|\nabla u|^2 - \lambda\left(\int \nabla u \, m\right) \cdot \nabla u - \nu \Delta u = f(m(x,t)) \\ \frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(\left(\nabla u - \lambda \int \nabla u \, m\right) m\right) = 0 \end{cases}$$

$u|_{t=0} = u_0$, $m|_{t=T} = m_1$

...

II "PLANNING" WITH COMMON NOISE

Prescribing m at $t=0$ <u>and</u> at $t=T$

no common noise: J.-M. Lasry-PL², A. Porretta ...

common noise - master equation? cond. on U?

To simplify, here only the finite state space

<u>Ex</u>. $(U^i)_{1\le i\le K}$, $\forall i$ $U^i : \mathbb{R}^d \to \mathbb{R}^d$

$$\begin{cases} \dfrac{\partial U^i}{\partial t} + \left(F^i(U^i).\nabla\right)U^i + \sum_{j\neq i} \pi^{ij}(U^j - U^i) = f^i(X) \\ \forall 1\le i\le d \end{cases}$$

($\pi^{ij} \ge 0$ csts)

F^i strongly monotone $\forall i$

f^i monotone $\forall i$

$\left(\begin{matrix} F^i(X,U) \\ f^i(X,U) \end{matrix} \text{ OK} \right)$

?, IC on U^i for planning pb?

Fact 1 : $\pi^{ij} = 0$ Fw.-Bw pb $\Longleftrightarrow$ method of characteristics

$$\dot{X}^i = -F(U) \qquad \text{or } F(X,U)$$
$$\dot{U} = f^i(X) \qquad \text{or } f(X,U)$$

and $X(0) = X_0$, $X(T) = X_1$

(f, F) "strictly mon. in (X,U)" $\Rightarrow \exists !$ sol.

$$\frac{d}{dt} F^{-1}(-\dot{X}) + f(X) = 0 \ , \ X(0) = X_0, \ X(T) = X_1$$

elliptic system!

Fact 2 π^{ij} $F = H'$, $f^i = g'$, look for $U = \Phi'$

~~char~~ characteristics : geodesics CV

and HJ eq

$$\frac{\partial \Phi}{\partial t} + H(\nabla \Phi) = g(X)$$

with IC $= \mathbb{1}_{\{X_1\}}(X)$ $\left(= +\infty \text{ if } X \neq X_1, = 0 \text{ if } X = X_1 \right)$

hence $U|_{t=0} = \begin{cases} \mathbb{R}^d & \text{if } X = X_1 \\ \emptyset & \text{if } X \neq X_1 \end{cases}$ $\quad (\mathcal{D}(U) = \{X_1\})$

(max. mon. op. on $\mathbb{R}^d$)

denote by A_{X_1} this op.

THM $$\frac{\partial U^i}{\partial t} + (F^i(U).\nabla)U^i + \sum_{j\neq i} \pi_{ij}(U^j - U^i) = f^i(X)$$

with " $U|_{t=0} = A_{x_1}$ "

has a unique solution which is monotone ($\forall i$) in X

for all $t>0$, U^i Lip in X for $t>0$...

Extension to ∞D possible:

sample (potential/optimal control case): $H = L^2$ (R. Var.)

common Brownian noise

solve uniquely ($\nu \geq 0$)

$$\frac{\partial \Phi}{\partial t} + H(X, \nabla\Phi) - \nu \Delta_d \Phi(X) = 0$$

$$\Phi_d(X)\Big|_{t=0} = \begin{pmatrix} 0 & \text{if } \mathcal{L}(X) = m_1 \\ +\infty & \text{if } \mathcal{L}(X) \neq m_1 \end{pmatrix}$$

...

III FROM MFG TO AGENT BASED MODELS

intertemporal preference rate $\to +\infty$ (noise(s) or not)

Ex. $$\begin{cases} \frac{\partial u}{\partial t} + H(x, \nabla u; m) - \nu \Delta u + \lambda u = 0 \\ \frac{\partial m}{\partial t} + \operatorname{div}\left(\frac{\partial H}{\partial p}(\ldots) m\right) + \nu \Delta m = 0 \end{cases}$$

$$m(T) = m_1 \quad , u(0) = u_0$$

Formally λu bded $\underset{\text{estimates}}{\Longrightarrow}$ $u \to 0$ as $\lambda \to +\infty$

$\underset{\text{estimates}}{\Longrightarrow}$ $\nabla u \to 0$ (initial layer at $t=0$)

and we deduce an evolution equation for m

$$\begin{cases} \frac{\partial m}{\partial t} + \operatorname{div}\left(\frac{\partial H}{\partial p}(x, 0; m) m\right) + \nu \Delta m = 0 \\ m|_T = m_1 \end{cases}$$

(agent based models, effective models for the dynamics of the densities, OK with "source" terms - "death terms" in the m eq $= f(x, u; m) \to f(x, 0; m)$

CAN BE JUSTIFIED MATH (COND. on H, ν ...)

Conversely any "agent based" model can be written as a degenerate MFG or a limit MFG

$$\frac{\partial m}{\partial t} + \operatorname{div}\left(B(x;m)\, m\right) + \nu \Delta m = 0$$

- set $H_0 = B \cdot \nabla u$ linear! MFG!

- set $H_\varepsilon = B \nabla u + \varepsilon |\nabla u|^2 \ldots$

ex. $B(x;m) = m * x$

$\left(= x - \underbrace{\left(\int y m\right)}_{\bar{x}} \text{ if } m \text{ proba} \right)$

cost $\frac{1}{2\varepsilon} |\alpha \pm (x - \bar{x})|^2$

...