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First-order, stationary MFGs with congestion

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Outline

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Mean-field games

- Mean-field game (MFG) theory is the study of strategic decision making in large populations of interacting agents.
- Here, we are interested in stationary problems where agents face a high cost of moving in high-density regions - the congestion problem.

Congestion

Cost is determined by a Lagrangian

$$L(x, v, m) = m^{\alpha} \frac{|v|^{\gamma'}}{\gamma'} - V(x) + g(m),$$

where

- $1 \leq \alpha \leq \gamma < \infty$, $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$,
- $V \in C^{\infty}(\mathbb{T}^d)$,
- $g : \mathbb{R}^+ \to \mathbb{R}$ with $g(m) = G'(m)$ for some convex function $G : \mathbb{R}_0^+ \to \mathbb{R}$ with $G \in C^{\infty}(\mathbb{R}^+) \cap C(\mathbb{R}_0^+)$

Given the density of agents, m, agents seek to minimize

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T L(\mathbf{x}, \dot{\mathbf{x}}, m(\mathbf{x}(t))).$$

The MFG is determined by the system

$$\begin{cases} \frac{|P+Du|^\gamma}{\gamma m^\alpha} + V(x) = g(m) + \overline{H} \\ -\operatorname{div}(m^{1-\alpha}|P+Du|^{\gamma-2}(P+Du)) = 0, \end{cases} \tag{1}$$

where $u, m : \mathbb{T}^d \to \mathbb{R}$ and $\overline{H} \in \mathbb{R}$, with $m \geq 0$ and $\int_{\mathbb{T}^d} m\, dx = 1$.

Congestion - second-order problems

Theorem - G., Mitake

For $0 < \alpha < 1$, there exists a classical solution (u, m) to the congestion MFG

$$\begin{cases} u + V(x) + \frac{|Du|^2}{2m^\alpha} = \Delta u + \overline{H} \\ m - \operatorname{div}(m^{1-\alpha} Du) = \Delta m + 1. \end{cases}$$

Congestion - second-order problems

Theorem - G., Evangelista

For $0 < \alpha < 2$ and H subquadratic satisfying natural convexity and growth conditions, there exists a classical solution (u, m) to the congestion MFG

$$\begin{cases} u + m^\alpha H(x, \frac{Du}{m^\alpha}) = \Delta u + \overline{H} \\ m - \operatorname{div}(m D_p H) = \Delta m + 1. \end{cases}$$

Monotone operators and MFGs

For suitable $\alpha \geq 0$, $\epsilon \geq 0$, convex H, and increasing g, the operator

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -u - m^{\alpha} H(x, \frac{Du}{m^{\alpha}}) - \epsilon \Delta u - \overline{H} + g(m) \\ m - \operatorname{div}(m D_p H) - \epsilon \Delta m - 1. \end{bmatrix}$$

is a monotone operator in $L^2 \times L^2$.

A weak solution of the MFG is a pair (m, u), $m \geq 0$, such that

$$\left\langle \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix}, A \begin{bmatrix} \eta \\ v \end{bmatrix} \right\rangle_{\mathcal{D}'(\mathbb{T}^d)\times\mathcal{D}'(\mathbb{T}^d), C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)} \geq 0$$

for all $(\eta, v) \in \mathcal{C}^\infty(\mathbb{T}^d; \mathbb{R}^+) \times \mathcal{C}^\infty(\mathbb{T}^d)$.

Existence by Monotone operator methods

Theorem - G., Ferreira

Under suitable but general Assumptions, there exists a weak solution, $(m, u) \in \mathcal{D}'(\mathbb{T}^d) \times \mathcal{D}'(\mathbb{T}^d)$, with $m \geq 0$, to the MFG

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Moreover, $(m, u) \in \mathcal{M}_{ac} \times W^{1,\gamma}$ for some $\gamma > 1$ and $\int_{\mathbb{T}^d} m\, dx = 1$.

Outline

Lack of classical solutions

Consider the following MFG

$$\begin{cases} \frac{|Du|^2}{2m^\alpha} + V(x) = m + \overline{H} \\ -\operatorname{div}(m^{1-\alpha} Du) = 0, \end{cases} \tag{2}$$

$u, m : \mathbb{T}^d \to \mathbb{R}$, $m \geq 0$, $\int m = 1$.

Multiplying the second equation by u and integrating

$$\int_{\mathbb{T}^d} m^{1-\alpha}|Du|^2\,dx = 0.$$

Hence, u is constant.
Thus

$$m = -\overline{H} + V(x).$$

By adding a constant to V, we assume

$$\int_{\mathbb{T}^d} V dx = 0.$$

Hence, using $\int_{\mathbb{T}^d} m\, dx = 1$

$$m = 1 + V(x).$$

Because $1 + V$ may take negative values we may not have a classical solution with $m > 0$.

$\alpha = 1$ and $\gamma = 2$

The equation

$$-\operatorname{div}(m^{1-\alpha}|P + Du|^{\gamma-2}(P + Du)) = 0$$

becomes $\Delta u = 0$.
Hence, u is constant.

$\alpha = 1$ and $\gamma = 2$

Then,

$$\frac{|P + Du|^{\gamma}}{\gamma m^{\alpha}} + V(x) = g(m) + \overline{H}$$

becomes an algebraic equation for m:

$$\frac{|P|^2}{2m} - g(m) = \overline{H} - V(x).$$

If g is increasing and $P \neq 0$, for each x the preceding equation has at most one solution, $m(x) > 0$. $\overline{H}$ is determined by $\int m = 1$.

Outline

First-order estimates

First-order estimates

Let $\bar{\alpha} := \frac{\alpha}{\gamma-1}$. There exists a constant, $C > 0$, such that, for any smooth solution $(u, m, \overline{H})$ with $m > 0$ and $\int_{\mathbb{T}^d} m\,dx = 1$, we have

$$\int_{\mathbb{T}^d} \left[\left| \frac{P + Du}{m^{\bar{\alpha}}} \right|^{\gamma} (m^{\bar{\alpha}} + m^{\bar{\alpha}+1}) + (m-1)g(m) \right] dx \leq C\left(1 + \int_{\mathbb{T}^d} m^{\bar{\alpha}+1}\,dx\right).$$

Second-order estimates

Second-order estimates

There exists a constant, $C > 0$, such that, for any smooth solution, $(u, m, \overline{H})$ with $m > 0$ and $\int_{\mathbb{T}^d} m\,dx = 1$, we have

$$\int_{\mathbb{T}^d} \operatorname{tr}\left(D^2_{pp}H\left(\frac{P+Du}{m^{\bar{\alpha}}}\right)D^2uD^2u\right)m^{1-\bar{\alpha}}\,dx + \int_{\mathbb{T}^d} g'(m)|Dm|^2\,dx \leq C,$$

where

$$H = \frac{|p|^\gamma}{\gamma}$$

Outline

Separated MFGs without congestion

Solve

$$\begin{cases} H(x, Du) = g(m) + \overline{H} \\ -\operatorname{div}(mD_pH(x, Du)) = 0 \end{cases}$$

for $u, m : \mathbb{T}^d \to \mathbb{R}$, $m \geq 0$, and $\overline{H} \in \mathbb{R}$.
Let $G : \mathbb{R}_0^+ \to \mathbb{R}$, convex, and $g = G'$.

Constrained minimization

Minimize

$$(m, v) \mapsto \int_{\mathbb{T}^d} m(x)L(x, v(x)) + G(m)$$

under the constraint

$$-\operatorname{div}(m(x)v(x)) = 0,$$

$m \geq 0$ and $\int_{\mathbb{T}^d} m dx = 1$.

Dual minimization

Minimize

$$u \mapsto \int_{\mathbb{T}^d} G^*(H(x, Du))dx,$$

Joint minimization

Minimize (or look for critical points)

$$(m,u) \mapsto \int_{\mathbb{T}^d} -m(x)H(x,D_xu) + G(m(x))dx,$$

$m \geq 0$ and $\int_{\mathbb{T}^d} m dx = 1$.

A functional for congestion problems

Let

$$J[u,m]=\int_{\mathbb{T}^d}\left(\frac{|P+Du|^\gamma}{\gamma(\alpha-1)m^{\alpha-1}}-Vm+G(m)\right)dx.$$

Then, if (u,m), with $u,m:\mathbb{T}^d\to\mathbb{R}$ and $m>0$, is a smooth enough minimizer of J under the constraint

$$\int_{\mathbb{T}^d} m\,dx=1,$$

then (u,m) solves (1).

Set

$$\bar{J}[u,m] = \int_{\mathbb{T}^d} \left[\bar{f}(\nabla u, m) - Vm + G(m)\right] dx, \tag{3}$$

where, for $(p,m) \in \mathbb{R}^d \times \mathbb{R}_0^+$,

$$\bar{f}(p,m) = \begin{cases} \frac{|P+p|^\gamma}{\gamma(\alpha-1)m^{\alpha-1}} & \text{if } m \neq 0, \\ +\infty & \text{if } m = 0 \text{ and } p \neq -P, \\ 0 & \text{if } m = 0 \text{ and } p = -P. \end{cases} \tag{4}$$

We aim at proving the existence and uniqueness of solutions to the variational problem

$$\min_{(u,m)\in\mathcal{A}_{q,r}} \bar{J}[u,m], \tag{5}$$

where $\bar{J}$ is given by (3) and $\mathcal{A}_{q,r}$ is the set

$$\mathcal{A}_{q,r} = \left\{(u,m) \in W^{1,q} \times L^r : \int u\,dx = 0, \int m\,dx = 1, m \geq 0\right\},$$

with $q \geq 1$ and $r \geq 1$ to be chosen later.

Outline

Existence of minimizers

Assume that $V \in L^\infty(\mathbb{T}^d)$ and that there exit positive constants, θ and C, such that

$$G(z) \geq \frac{1}{C} z^{\theta+1} - C \text{ for all } z > 0,$$

then (5) has a solution $(u, m) \in \mathcal{A}_{\gamma(1+\theta)/(\alpha+\theta),1+\theta}$ for all $1 < \alpha \leq \gamma$.

Proof: Convexity and Coercivity.

Proof - Convexity

Suppose that $1 < \alpha \leq \gamma$. Then, the function $\bar{f}$ is convex and lower semi-continuous in $\mathbb{R}^d \times \mathbb{R}_0^+$.

Proof - Coercivity

Suppose

$$\sup_{n\in\mathbb{N}} \left|\bar{J}[u_n, m_n]\right| \leq C.$$

Let

$$U_n = \{x \in \mathbb{T}^d \colon \nabla u_n \neq -P\}$$

Then

$$\int_{\mathbb{T}^d} \bar{f}(\nabla u_n, m_n)\, dx = \int_{U_n} \frac{|P + \nabla u_n|^\gamma}{\gamma(\alpha - 1) m_n^{\alpha-1}}\, dx \leq C.$$

Proof - Coercivity

Recalling that $q = \frac{\gamma}{\alpha}$ and using the preceding estimate and Young's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{T}^d} |P + \nabla u_n|^q \, dx &= \int_{U_n} |P + \nabla u_n|^{\frac{\gamma}{\alpha}} \, dx = \int_{U_n} \frac{|P + \nabla u_n|^{\frac{\gamma}{\alpha}}}{m_n^{\frac{\alpha-1}{\alpha}}} m_n^{\frac{\alpha-1}{\alpha}} \, dx \\ &\leq \frac{1}{\alpha} \int_{U_n} \frac{|P + \nabla u_n|^{\gamma}}{m_n^{\alpha-1}} \, dx + \frac{\alpha-1}{\alpha} \int_{\mathbb{T}^d} m_n \, dx \\ &\leq C.\end{aligned}$$

Outline

Current transformation - 2d

In 2 dimensions

$$-\operatorname{div}(m^{1-\alpha}|P+Du|^{\gamma-2}(P+Du))=0$$

implies there exists $Q\in\mathbb{R}^2$, and a scalar function, ψ, such that

$$m^{1-\alpha}|P+Du|^{\gamma-2}(P+Du)=Q^{\perp}+D\psi^{\perp}.$$

Current transformation - 2d

Consequently,

$$\frac{|P+Du|^{\gamma}}{m^{\alpha}} = \frac{|Q+D\psi|^{\gamma'}}{m^{\alpha-(\alpha-1)\gamma'}}.$$

Therefore,

$$\frac{|P+Du|^{\gamma}}{m^{\alpha}} + V(x) - g(m) - \overline{H} = \frac{|Q+D\psi|^{\gamma'}}{m^{\tilde{\alpha}}} + V(x) - g(m) - \overline{H},$$

with

$$\tilde{\alpha} = \alpha - (\alpha-1)\gamma'.$$

Current transformation - 2d

Moreover,

$$P^{\perp} + Du^{\perp} = m^{1-\tilde{\alpha}}|Q + D\psi|^{\gamma'-2}(Q + D\psi).$$

Accordingly,

$$\mathrm{div}(m^{1-\tilde{\alpha}}|Q + D\psi|^{\gamma'-2}(Q + D\psi)) = 0.$$

Current transformation - 2d

Thus, (1) can be rewritten as

$$\begin{cases} \frac{|Q+D\psi|^{\gamma'}}{m^{\tilde{\alpha}}} + V(x) - g(m) = \overline{H} \\ \operatorname{div}(m^{1-\tilde{\alpha}}|Q + D\psi|^{\gamma'-2}(Q + D\psi)) = 0. \end{cases}$$

If $0 < \alpha < 1$ and $\gamma > 1$, we have $1 < \tilde{\alpha} < \gamma'$. Thus, $\tilde{\alpha}$ and γ' belong to the range where our prior results apply.

$\alpha = 1$ and $\gamma = 2$

Consider

$$\begin{cases} -\Delta u + \frac{|P+Du|^2}{2m} + V(x) = g(m) + \overline{H} \\ -\Delta m - \Delta u = 0. \end{cases}$$

From the second equation, we get

$$u = \mu - m$$

for $\mu \in \mathbb{R}$.

$\alpha = 1$ and $\gamma = 2$

Replacing u in the first equation

$$\Delta m + \frac{|P - Dm|^2}{2m} + V(x) = g(m) + \overline{H}.$$

$\alpha = 1$ and $\gamma = 2$

We take $P = 0$ and multiply the equation by $m^{1/2}$. Then,

$$m^{1/2}\Delta m + \frac{|Dm|^2}{2m^{1/2}} + V(x)m^{1/2} = m^{1/2}g(m) + \overline{H}m^{1/2}.$$

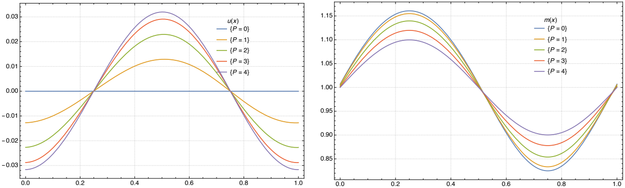
Set $\psi = m^{3/2}$ and conclude that

$$\frac{2}{3}\Delta\psi + V(x)\psi^{1/3} = g(\psi^{3/2})\psi^{1/3} + \overline{H}\psi^{1/3}.$$

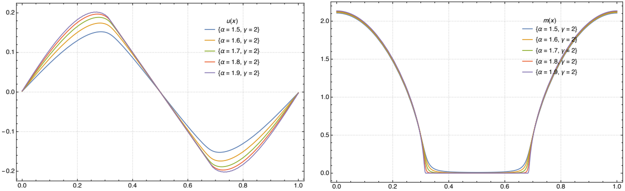
The foregoing equation is the Euler–Lagrange equation of the functional

$$\hat{J}(\psi) = \int_{\mathbb{T}^d} \frac{|D\psi|^2}{3} - \frac{3}{4}V(x)\psi^{4/3} + \hat{G}(\psi) + \frac{4}{3}\overline{H}\psi^{4/3}.$$

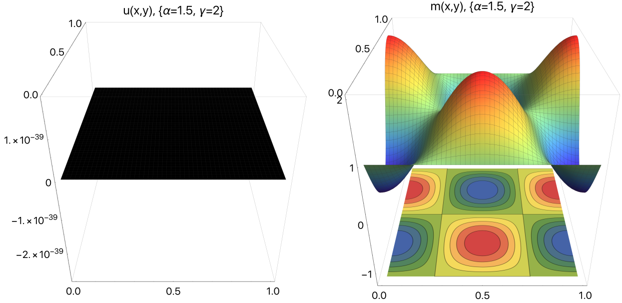
Outline



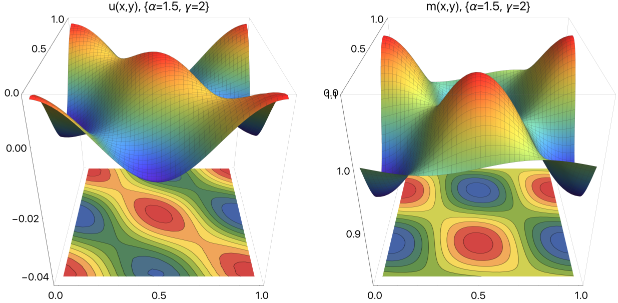
Numerical solution (1d) for $G(m) = \frac{m^2}{2}$, $P = 2$, $V(x) = \cos\left(2\pi\left(x - \frac{1}{4}\right)\right)$, $\alpha = 1.5$ and $\gamma = 2$.



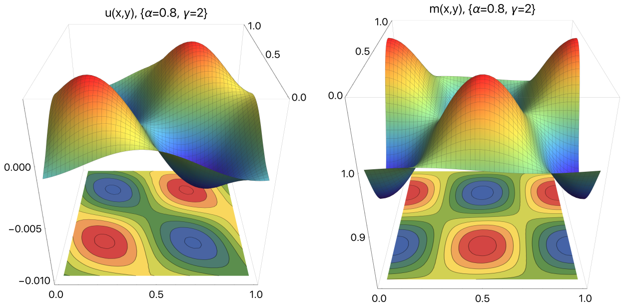
Numerical solution (1d) for $G(m) = m^3$, $P = 1$, $V(x) = 10\sin\left(2\pi\left(x - \frac{1}{4}\right)\right)$, and $\gamma = 2$.



Numerical solution (2d) for $G(m) = m^2/2$, $P = (0, 0)$, $V(x, y) = \sin\left(2\pi\left(x + \frac{1}{4}\right)\right)\cos\left(2\pi\left(y + \frac{1}{5}\right)\right)$, $\alpha = 1.5$ and $\gamma = 2$.



Numerical solution (2d) for $G(m) = m^3$, $P = (3, 1)$,
$V(x, y) = \sin\left(2\pi\left(x + \frac{1}{4}\right)\right)\cos\left(2\pi\left(y + \frac{1}{4}\right)\right)$, $\alpha = 1.5$ and $\gamma = 2$.



Numerical solution for $G(m) = m^3$, $P = (1, 3)$,
$V(x, y) = \sin\left(2\pi\left(x + \frac{1}{4}\right)\right)\cos\left(2\pi\left(y + \frac{1}{4}\right)\right)$, $\alpha = 0.8$ and $\gamma = 2$.