Mean-Field Control Hierarchy

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G. Albi, M. Bongini, Y.-P. Choi, D. Kalise, F. Rossi, F. Solombrino

Mean-field Sparse Optimal Control?

"Ultimately it would be good to have a theory that combined both the collective behaviour of a large number of "ordinary" agents with the decisions of a few key players of unusually large (relative) influence – some complicated combination of PDE and game theory, presumably – but our current mathematical technology is definitely insufficient for even a zeroth approximation to this task".

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`https://terrytao.wordpress.com/2010/01/07/mean-field-equations`

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Next numerics from:

G. Albi, M. Bongini, E. Cristiani, and D. Kalise, *Invisible control of self-organizing agents leaving unknown environments*, SIAM J. Appl. Math.

Evacuating an unknown environment

Simulations I

Simulations II

Technical assumptions

(K) The function $K \in \mathcal{C}^2(\mathbb{R}^d; \mathbb{R}^d)$ is odd and sublinear, i.e., there exists $C_K > 0$ such that for all $x \in \mathbb{R}^d$ it holds

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with $\ell \in \mathcal{C}^2(\mathbb{R}^{dm} \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ and $\Omega \in \mathcal{C}^2(\mathbb{R}^d; \mathbb{R}^d)$.

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(G) The function $g \in \mathcal{C}^2(\mathbb{R}^{dm}; \mathcal{C}^2(\mathbb{R}^d; \mathbb{R}^d))$ satisfies for all $x \in \mathbb{R}^d$ and all $y \in \mathbb{R}^{dm}$

$$g(y)(x) \cdot x \leq G_1\|x\|^2 + G_2 \max_{l=1,\dots,m} \|y_l\|^2 + G_3,$$

where the constants G_1, G_2 and G_3 are independent on x and y.

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(U) The set $\mathcal{U} \subseteq \mathbb{R}^D$ is compact and convex.

(γ) The function $\gamma : \mathcal{U} \to \mathbb{R}$ is strictly convex.

The finite particle sparse optimal control model

For $T > 0$ fixed, find $u^* \in L^1([0,T];\mathcal{U})$ minimizing the cost functional

$$F_N(u) = \int_0^T \left[L(y(t), \mu_N(t)) + \gamma(u(t))\right] dt,$$

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for the given initial datum $(y(0), x(0)) = (y^0, x^0) \in \mathbb{R}^{dm} \times \mathbb{R}^{dN}$,

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is the empirical measure centered on the trajectory $x(\cdot) = (x_1(\cdot), \ldots, x_N(\cdot))$.

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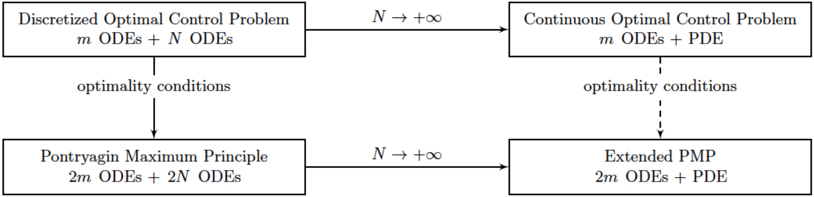
for the given initial datum
$(y(0),\mu(0)) = (y^0,\mu^0) \in \mathbb{R}^{dm} \times \mathcal{P}_c(\mathbb{R}^d)$.

Scheme of results



Mean-field optimal control

Definition

Let $(y^0, \mu^0) \in \mathbb{R}^{dm} \times \mathcal{P}_c(\mathbb{R}^d)$ be given. A optimal control u^* for the ∞-dimensional OC with initial datum (y^0, μ^0) is a *mean-field optimal control* if there exists a sequence $(u_N^*)_{N\in\mathbb{N}} \subset L^1([0, T]; \mathcal{U})$ and a sequence $(\mu_N^0)_{N\in\mathbb{N}} \in \mathcal{P}_c(\mathbb{R}^d)$ such that

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- for every $N \in \mathbb{N}$, u_N^* is a solution of the finite dimensional OC with initial datum (y^0, μ_N^0);
- there exists a subsequence of $(u_N^*)_{N\in\mathbb{N}}$ converging weakly in $L^1([0,T];\mathcal{U})$ to u^*.

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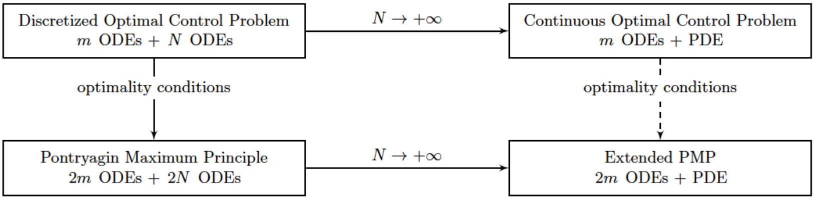

Γ-convergence

Theorem (F., Rossi, Piccoli,'14)

Consider an initial datum $(y^0, \mu^0) \in \mathbb{R}^{dm} \times \mathcal{P}_1(\mathbb{R}^d)$*, and a sequence* $(\mu_N^0)_{N \in \mathbb{N}}$*, where* μ_N^0 *is as in Definition.*

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PMP mean-field convergence

Theorem (Bongini, F., Rossi, Solombrino, '16)

Fix an initial datum $(y^0, \mu^0) \in \mathbb{R}^{dm} \times \mathcal{P}_c(\mathbb{R}^d)$. If u^* is a mean-field optimal control and (y^*, μ^*) is the corresponding trajectory, then (u^*, y^*, μ^*) satisfies the following **extended Pontryagin Maximum Principle***:*

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- *there exists* $R_T > 0$*, depending only on* $y^0, \mathrm{supp}(\mu^0), m, K, g, f_k, B_k, \mathcal{U}$*, and* T*, such that* $\mathrm{supp}(\nu^*(\cdot)) \subseteq B(0, R_T)$ *and it satisfies* $\pi_{1\#}\nu^*(t) = \mu^*(t)$ *for all* $t \in [0, T]$*;*

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- *it holds*

$$
\begin{cases}
\dot{y}_k^* &= \nabla_{q_k} \mathbb{H}_c(y^*, q^*, \nu^*, u^*), \\
\dot{q}_k^* &= -\nabla_{y_k} \mathbb{H}_c(y^*, q^*, \nu^*, u^*), \\
\partial_t \nu^* &= -\nabla_{(x,r)} \cdot \left((J \nabla_\nu \mathbb{H}_c(y^*, q^*, \nu^*, u^*)) \nu^* \right), \\
u^* &= \arg\max_{u \in \mathcal{U}} \mathbb{H}_c(y^*, q^*, \nu^*, u),
\end{cases}
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$$\mathbb{H}_c(y, q, \nu, u) = \begin{cases} \mathbb{H}(y, q, \nu, u) & \text{if } \mathrm{supp}(\nu) \subseteq \overline{B(0, R_T)}, \\ +\infty & \text{elsewhere;} \end{cases}$$

and $\mathbb{H} : \mathbb{R}^{2dm} \times \mathcal{P}_c(\mathbb{R}^{2d}) \times \mathbb{R}^D \to \mathbb{R}$ is defined as

$$\begin{aligned} \mathbb{H}(y, q, \nu, u) =& \frac{1}{2}\int_{\mathbb{R}^{4d}} (r - r')\cdot K(x - x')\, d\nu(x, r)\, d\nu(x', r') \\ &+ \int_{\mathbb{R}^{2d}} r \cdot g(y)(x) d\nu(x, r) + \sum_{k=1}^{m} \int_{\mathbb{R}^{2d}} q_k \cdot K(y_k - x)\, d\nu(x, r) \\ &+ \sum_{k=1}^{m} q_k \cdot (f_k(y) + B_k u) - L(y, \pi_{1\#}\nu) - \gamma(u). \end{aligned}$$

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- $y^*(0) = y^0$ and $\nu^*(0)(E \times \mathbb{R}^d) = \mu^0(E)$ for every Borel set $E \subseteq \mathbb{R}^d$,
- $q^*(T) = 0$ and $\nu^*(T)(\mathbb{R}^d \times E) = \delta_0(E)$ for every Borel set $E \subseteq \mathbb{R}^d$, where δ_0 is the Dirac measure centered in 0.

Remarks

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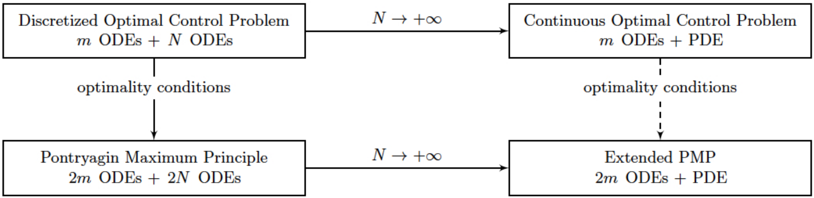
- This fact is remarkably consistent with the state dynamics, since both are flows in a Wasserstein space.
- This formulation of the optimality conditions making use of the formalism of subdifferential calculus in Wasserstein spaces of probability measures constitutes one of the novelties of the work.

Scheme of results



Proof strategy

- The extended PMP is derived after reformulating the finite-dimensional PMP in terms of the empirical measure in the product space of state variables x_i and adjoint variables p_i, defined as

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- Notice that rescaling the adjoint variables p_i by the number N of agents is needed in order to observe a nontrivial dynamics in the limit;

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$$\nu_N(x, r) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i, r - Np_i).$$

- Notice that rescaling the adjoint variables p_i by the number N of agents is needed in order to observe a nontrivial dynamics in the limit;
- A final explicit hard computation is done to verify that the mean-field limit dynamics of the PMP coincides with the symplectic (Wasserstein)-gradient flow of the Hamiltonian.

Mean-field optimal control without isolated leaders

We consider here mainly large particle/agent systems of form:

$$dx_i = \left(\frac{1}{N} \sum_{j=1}^{N} K(x_i, x_j)(x_j - x_i) \right) dt + f_i \, dt + \sqrt{2\sigma} \, dB_i^t, \quad i = 1, \ldots, N,$$

where $K(\cdot, \cdot)$ represents the communication function between agents $x_i \in \mathbb{R}^d$ and B_i^t is a d-dimensional Brownian motion.

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One can use the model for $d = 1$ and $x_i \in I = [-1, 1]$ to formulate opinion models, where x_i represents an opinion in the continuous set between two opposite opinions $\{-1, 1\}$.

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According to the choice of the communication function $K(\cdot, \cdot)$, consensus can emerge or not, and opinion control is of interest.

Mean-field optimal control without isolated leaders

The control

$$f = \arg\min_{g\in\mathcal{U}} J(x,g) := \mathbb{E}\left[\int_0^T \frac{1}{N}\sum_{i=1}^N \left(\frac{1}{2}|x_i - \bar{x}|^2 + \gamma\Psi(g_i)\right)\, dt\right],$$

where $\bar{x}$ represents a target point, γ is the penalization parameter of the control g, which is chosen among the admissible controls in $\mathcal{U}$, and $\Psi : \mathbb{R}^d \to \mathbb{R}_+ \cup \{0\}$ is a convex function.

Mean-field optimal control without isolated leaders

As the number of particles $N \to \infty$, the finite dimensional optimal control problem with SDE constraints converges to the following mean field optimal control problem[1]:

[1] D. Lacker. *Limit theory for controlled McKean-Vlasov dynamics.* SIAM J. Control. Opt. 2016;
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As the number of particles $N \to \infty$, the finite dimensional optimal control problem with SDE constraints converges to the following mean field optimal control problem[1]:

$$\partial_t \mu + \nabla \cdot ((\mathcal{K}[\mu] + f)\, \mu) = \sigma \Delta \mu, \tag{1}$$

where the interaction force $\mathcal{K}$ is given by

$$\mathcal{K}[\mu](x) = \int K(x,y)(y-x)\mu(y,t)\, dy \tag{2}$$

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and the solution μ is controlled by the minimizer of the cost functional

$$J(\mu, f) = \int_0^T \left(\frac{1}{2} \int |x - \bar{x}|^2 \mu(x, t)\, dx + \gamma \int \Psi(f)\mu(x, t)\, dx \right) dt. \tag{3}$$

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Deterministic case: $\sigma = 0$

Definition

For a given T and $q \in [1, \infty)$, we fix a control bound function $\ell \in L^q(0, T)$. Then $f \in \mathcal{F}_\ell([0, T])$ if and only if

(i) $f : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ is a Carathéodory function.

(ii) $f(\cdot, t) \in W^{1,\infty}_{loc}(\mathbb{R}^d)$ for almost every $t \in [0, T]$.

(iii) $|f(0, t)| + \|f(\cdot, t)\|_{\mathrm{Lip}} \leq \ell(t)$ for almost every $t \in [0, T]$.

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- *Finite dimensional optimal control problem:*

$$\min_{f \in \mathcal{F}_\ell} J(x, f) := \min_{f \in \mathcal{F}_\ell} \int_0^T \frac{1}{N} \sum_{i=1}^N \left(\frac{1}{2} |x_i - \bar{x}|^2 + \gamma \Psi(f(x_i, t)) \right) dt, \qquad (4)$$

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where $\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d))$ is the unique weak solution of

$$\begin{aligned}\partial_t\mu_t &= \nabla\cdot\left(\left(\mathcal{K}[\mu_t]+f\right)\mu_t\right), \quad (x,t)\in\mathbb{R}^d\times[0,T],\\ \mathcal{K}[\mu_t](x) &= \int_{\mathbb{R}^d} K(x,y)(y-x)\mu_t(dy).\end{aligned} \qquad (7)$$

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Theorem (F. and Solombrino, '14)

Let $T > 0$*. Suppose that* $K \in W^{1,\infty}_{loc}(\mathbb{R}^{2d})$ *and* Ψ *si such that for* $1 \le q < \infty$

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The proof follows by a combination of a Γ-limit and mean-field limit.

Stochastic case: $\sigma > 0$

Let Ω denote an open, bounded, smooth subset of $\mathbb{R}^d$. We introduce:

$$V := L^2(0,T;H^1(\Omega)) \cap \dot{H}^1(0,T;H_*^{-1}(\Omega)), \quad \text{and} \quad H_*^{-1}(\Omega) = H^1(\Omega)',$$

and the set of admissible controls

$$Q_M := \left\{ \|f\|_{L^2(0,T;L^\infty(\Omega))} \leq M \,:\, f \in L^2(0,T;L^\infty(\Omega)) \right\},$$

for a given $M > 0$.

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$$\min_{f\in Q_M} J(\mu, f) := \min_{f\in Q_M} \int_0^T \left(\frac{1}{2} \int_\Omega |x - \bar{x}|^2 \mu(x,t)\, dx + \gamma \int_\Omega \Psi(f)\mu(x,t)\, dx \right) dt, \tag{8}$$

where μ is a weak solution to the following parabolic equation:

$$\partial_t \mu + \nabla \cdot (\mathcal{K}[\mu]\mu + f\mu) = \sigma \Delta \mu, \quad (x,t) \in \Omega_T := \Omega \times [0,T], \tag{9}$$

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$$\mu(\cdot, 0) = \mu_0(x) \quad x \in \Omega,$$

and the zero-flux boundary condition

$$\langle \sigma \nabla \mu - (\mathcal{K}[\mu] + f)\mu, n(x) \rangle = 0, \quad (x,t) \in \partial\Omega \times [0,T],$$

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Theorem (mathematical folklore)

For a given $T, M > 0$, let us assume $\mu_0 \in L^2(\Omega)$. Furthermore, we assume that $K \in L^\infty(\Omega^2)$ and Ψ satisfies that for all $R > 0$

$$W^{1,\infty}(\Psi, B(0,R)) \leq CR,$$

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One rigorous and simple proof based on the direct method is reviewed in the survey:

G. Albi, Y.-P. Choi, M. Fornasier and D. Kalise. *Mean field control hierarchy*, to appear in Applied Mathematics and Optimization (special issue on Mean-Field Games)

Rigorous derivation of first order optimality

Let X and Y be Banach spaces, and let a functional $J : U(x^*) \subseteq X \to \mathbb{R}$ and a mapping $G : U(x^*) \subseteq X \to Y$ be continuously differentiable on an open neighbourhood of x^*.

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$$J(x) \to \inf, \quad G(x) = 0. \tag{10}$$

Theorem (Lagrange multiplier theorem in Banach spaces)

Let x^* be a solution and let the range of the operator $G'(x^*) : X \to Y$ be closed. Then there exists a nonzero pair $(\lambda, p) \in \mathbb{R} \times Y'$ such that

$$\mathcal{L}'_x(x^*, \lambda, p)(x) = 0 \quad \text{for all } x \in X,$$

where

$$\mathcal{L}(x, \lambda, p) = \lambda J(x) + G(x)(p).$$

Moreover, if Im $G'(x^*) = Y$, then $\lambda \neq 0$ in the above, thus we can assume that $\lambda = 1$.

Rigorous derivation of first order optimality

In order to apply the above theorem, we set

$$X = V \times L^2(\Omega_T), \quad Y = L^2(0, T; H^{-1}(\Omega)),$$

$$J(\mu, f) = \int_0^T \left(\frac{1}{2} \int_\Omega |x - \bar{x}|^2 \mu(x,t)\, dx + \gamma \int_\Omega \Psi(f) \mu(x,t)\, dx \right) dt,$$

and

$$\begin{aligned} G(\mu, f)(\psi) = &- \int_0^T \int_\Omega \partial_t \psi \, \mu \, dxdt + \int_0^T \int_\Omega \nabla \psi \cdot (\mathcal{K}[\mu]\mu)\, dxdt \\ &+ \int_0^T \int_\Omega \nabla \psi \cdot (f\mu)\, dxdt - \sigma \int_0^T \int_\Omega \nabla \mu \cdot \nabla \psi \, dxdt, \end{aligned}$$

for $\psi \in Y' = L^2(0, T; H_0^1(\Omega))$.

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$$\begin{aligned} G'_\mu(\mu^*, f^*)(\nu, \psi^*) &= J'_\mu(\mu^*, f^*)(\nu), \quad \textit{for all} \quad \nu \in V, \\ G'_f(\mu^*, f^*)(g, \psi^*) &= J'_f(\mu^*, f^*)(g), \quad \textit{for all} \quad g \in L^2(\Omega_T). \end{aligned} \tag{11}$$

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Let us comment the positivity principle on the existence of $\mu_\ell > 0$ such that $\mu^* \geq \mu_\ell$ for all $(x, t) \in \Omega_T$.

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If we assumed that $\mu_0, f, P \in \mathcal{C}^2$ and μ_0 is bounded from below by a positive constant, then by Feynman-Kac formula, we can show that μ is bounded from below by some positive constant until the fixed time T.

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We have only numerical evidence of the persistent positivity more in general.

Rigorous derivation of first order optimality

Formally, forward-backward system in strong form:

$$\begin{cases} \partial_t \psi^* + \frac{1}{2}|x-\bar{x}|^2 + \gamma\left(\Psi(f^*) - \nabla\Psi(f^*)\cdot f^*\right) + \sigma\Delta\psi^* \\ \qquad\qquad + \int_\Omega \left(K(x,y)\nabla\psi^*(x,t) - K(y,x)\nabla\psi^*(y,t)\right)\cdot(y-x)\mu^*(y,t)\,dy = 0, \\ \partial_t\mu^* + \nabla\cdot\left((\mathcal{K}[\mu^*] + f^*)\mu^*\right) = \sigma\Delta\mu^* \\ \nabla\Psi(f^*) = -\frac{1}{\gamma}\nabla\psi^*. \end{cases}$$

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Uniqueness and guaranteed numerical solutions are still open problems.

Numerical approaches: mean-field control hierarchy?

The numerical solution of the mean-field optimal control system can be approached by sweeping, grad. desc., or aug. Lagrangian alg.:

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[2] G. Albi, M. Herty, L. Pareschi, Kinetic description of optimal control problems and applications to opinion consensus , Comm. Math. Scien., 2015
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Two agents have positions $x, y \in \Omega$ and modify them according to

$$\begin{aligned} x^* &= x + \alpha K(x,y)(y-x) + \alpha U_\alpha(x,y,t) + \sqrt{2\alpha}\xi^x, \\ y^* &= y + \alpha K(y,x)(x-y) + \alpha U_\alpha(y,x,t) + \sqrt{2\alpha}\xi^y\,, \end{aligned}$$

[2] G. Albi, M. Herty, L. Pareschi, Kinetic description of optimal control problems and applications to opinion consensus , Comm. Math. Scien., 2015
G. Albi, L. Pareschi, M. Zanella, Boltzmann type control of opinion consensus through leaders. Proc. of the Roy. Soc. A., 2014.

Numerical approaches: mean-field control hierarchy?

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where (x^*, y^*) are the post-interaction positions, α measures the influence strength, (ξ^x, ξ^y) is a vector of i.i.d. random variables with zero mean and variance σ, and $U_\alpha(x,y,t)$ indicates a feedback control.

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First of all a Boltzmann model

We consider now a kinetic model ruled by the following Boltzmann-type equation

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$$Q_\alpha(\mu,\mu)(x,t) = \mathbb{E}\left[\int_\Omega \left(\mathcal{B}_* \frac{1}{\mathcal{J}_\alpha}\mu(x_*,t)\mu(y_*,t) - \mathcal{B}\mu(x,t)\mu(y,t)\right)\, dy\right],$$

where (x_*, y_*) are the pre-interaction positions that generate arrivals (x, y).

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First of all a Boltzmann model

The collisional operator $Q_\alpha(\cdot,\cdot)$ includes the expected value with respect to ξ^x and ξ^y, while $\mathcal{J}_\alpha$ represents the Jacobian of the transformation $(x,y) \to (x^*,y^*)$. Here $\mathcal{B}_* = \mathcal{B}_{(x_*,y_*)\to(x,y)}$ and $\mathcal{B} = \mathcal{B}_{(x,y)\to(x^*,y^*)}$ are the transition rate functions. More into the details we take into account

$$\mathcal{B}_{(x,y)\to(x^*,y^*)} = \eta\chi_\Omega(x^*)\chi_\Omega(y^*),$$

as the functions with an interaction rate $\eta > 0$, and where χ_Ω is the characteristic function of the domain Ω.

From Boltzmann to mean-field equations

Theorem (grazing collision limit)

Fix some control $U_\alpha(x,y,t)$*. Introducing*

$$\alpha = \varepsilon, \qquad \eta = 1/\varepsilon,$$

for the binary interaction and defining by $\mu^\varepsilon(x,t)$ *the corresponding solution,* $\mu^\varepsilon(x,t)$ *converges for* $\varepsilon \to 0$ *to* $\mu(x,t)$ *where* μ *satisfies the following Fokker-Planck-type equation,*

$$\partial_t \mu + \nabla \cdot ((\mathcal{K}[\mu] + f)\mu) = \sigma \Delta \mu,$$

where the control

$$f(x,t) = \int_{\mathbb{R}^d} U(x,y,t)\mu(y,t)\, dy.$$

with $U(x,y,t) = \lim_{\alpha\to 0} U_\alpha(x,y,t)$*.*[3]

[3] G. Albi, Y.-P. Choi, M. Fornasier and D. Kalise. *Mean field control hierarchy*, to appear in AMO (special issue MFG)

Mean-field control hierarchy

Inspired by the BBGKY herarchy in kinetic theory: we choose the control $f(x, t)$, in three ways:

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- Instantaneous control: in this case $U_\alpha(x, y, t)$ is computed in such a way that the post-collisional positions minimize the cost function:

$$\frac{1}{2}(|x^* - \bar{x}|^2 + |y^* - \bar{x}|^2) + \gamma(\Psi(U_\alpha(x, y, t)) + \Psi(U_\alpha(y, x, t))),$$

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  then

  $$f(x,t) = \int_{\mathbb{R}^d} U(x,y,t)\mu(y,t)\,dy,$$

  e.g., for $\Psi(\cdot) := |\cdot|^2/2$

  $$\begin{aligned} U(x,y,t) &= \lim_{\alpha\to 0} U_\alpha(x,y,t) \\ &= \lim_{\alpha\to 0} \frac{1}{\gamma+\alpha}\left((\bar{x}-x) + \alpha K(x,y)(y-x)\right) \\ &= \frac{(\bar{x}-x)}{\gamma}, \end{aligned}$$

Mean-field control hierarchy

- Binary optimal control: $U(x,y,t)$ is the true solution of (just!) the $N=2$ particle finite time optimal control problem (computed by solving the Hamilton-Jacobi-Bellman equation for 2 particles only); again

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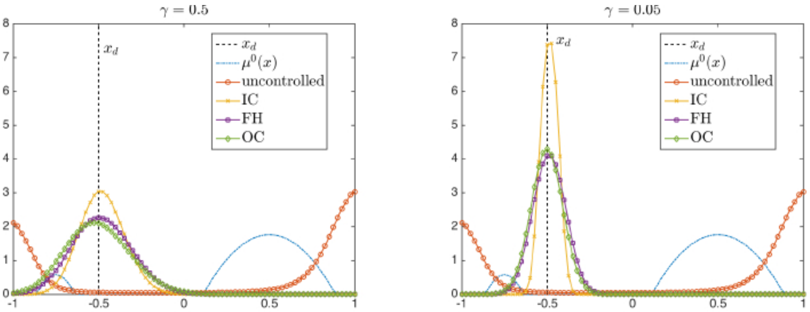
  This control is unfortunately NOT mean-field optimal as it does not fully solve the forward-backward system.
- Mean-field optimal control: $f(x, t)$ is the "true" mean-field optimal control, solving the forward-backward system

Applications to opinion control



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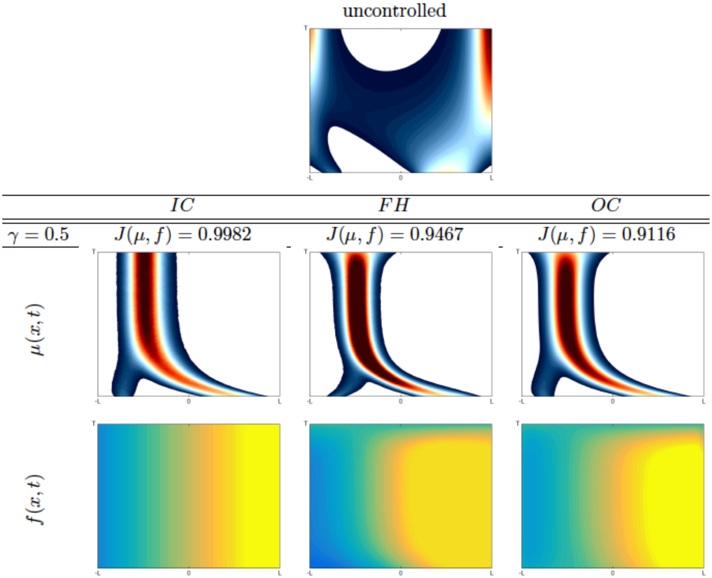

Sznajd model, $K(x, y) = \beta(1 - x^2)$.

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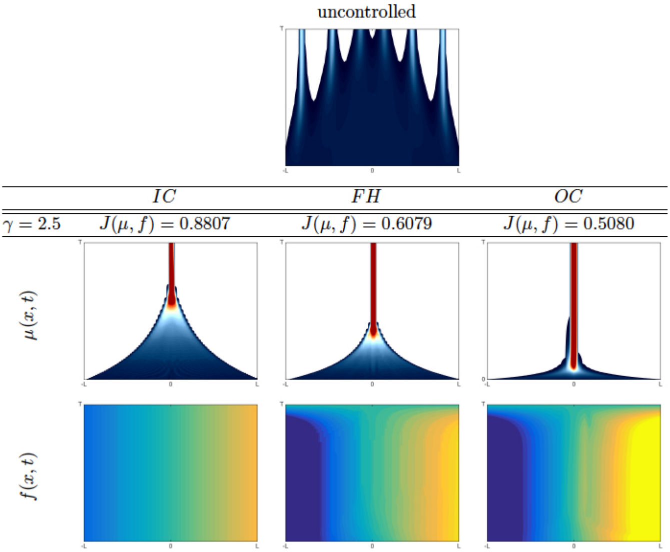

Hegselmann-Krause model, $K(x,y) = \chi_{\{|x-y|\leq\kappa\}}(y)$.

Remarks and open issues

- From the numerical experiments, we observe that the numerical realization of the mean field optimality system yields the best controller in terms of the cost functional value.

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- Feedback controllers obtained for the binary system perform reasonably well, and provide a much simpler control synthesis.
- A proof of a convergence of a hierarchy is open: is there a form of BBGKY hierarchy for controls?

A few info

- **WWW:** http://www-m15.ma.tum.de/
- **References**:

  - G. Albi, Y.-P. Choi, M. Fornasier and D. Kalise. *Mean field control hierarchy*, to appear in Applied Mathematics and Optimization (special issue on Mean-Field Games)
  - G. Albi, M. Bongini, E. Cristiani, and D. Kalise, *Invisible control of self-organizing agents leaving unknown environments*, SIAM J. Appl. Math.
  - M. Bongini, M. Fornasier, F. Rossi, and F. Solombrino, *Mean-field Pontryagin maximum principle*, submitted
  - M. Fornasier, F. Rossi, and B. Piccoli, *Mean field sparse optimal control*, Phil. Trans. Royal Soc. A, 2014
  - M. Fornasier and F. Solombrino, *Mean field optimal control*, ESAIM: Control, Optimization, and Calculus of Variations, 2014