Examples of Restoration of Uniqueness in Mean-Field Games

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June 15th 2017

MFG Conference Rome

1. Motivation

Uniqueness in MFG

- MFG of the general form $\rightsquigarrow$ time $[0,T]$, state in $\mathbb{R}^d$
  - freeze a path $(\mu_t)_{t\in[0,T]} \rightsquigarrow$ representative player

$$dX_t = (b(X_t,\mu_t) + \alpha_t)dt + \sigma dW_t$$

$\rightsquigarrow$ with $X_0 \sim \mu_0$ and $\sigma \in \{0,1\}$, T >0

  - cost functional of the form

$$J(\alpha) = \mathbb{E}\Big[g(X_T,\mu_T) + \int_0^T \Big(f(X_t,\mu_t) + \tfrac{1}{2}|\alpha_t|^2\Big)dt\Big]$$

  - find $(\mu_t)_{t\in[0,T]}$ such that $\mu_t = \mathrm{Law}(X_t^{\text{optimal}})$
- Standard example for uniqueness [Lasry-Lions]
  - b independent of μ and f and g monotone in μ

$$\int_{\mathbb{R}^d} (f(x,\mu) - f(x,\mu'))d(\mu-\mu')(x) \geq 0$$

Restoration of uniqueness

• General purpose is to restore uniqueness by forcing the equilibria by a random noise

• Long history for ODEs

∘ ODE driven by bounded non-Lipschitz velocity field

$$\dot{X}_t = b(t, X_t), \quad \text{with prescribed } X_0$$

$\rightsquigarrow$ b continuous $\Rightarrow$ existence but ~~uniqueness~~

∘ well-known: noise may restore ! [Veretennikov, Krylov...]

∘ perturb the dynamics by a Brownian motion $(B_t)_{t \geq 0}$

$$dX_t = b(t, X_t)dt + dB_t$$

∘ based on smoothing properties of the heat kernel $\rightsquigarrow$ use the fact that the PDE

$$\partial_t u(t, x) + \frac{1}{2}\Delta u(t, x) + b(t, x) \cdot D_x u(t, x) = f(t, x)$$

has a strong generalized solution if f is bounded

2. A toy example

Linear quadratic control problem

- Choose $X_0 = 0$, $\sigma = 1$, $d = 1$ and dynamics of the form

$$dX_t = \Big[\big(c_b X_t + b(\mu_t)\big) + \alpha_t\Big]dt + dW_t$$

  - cost functional of the form

$$J(\alpha) = \mathbb{E}\Big[\frac{1}{2}\big(c_g X_T + g(\mu_T)\big)^2 + \int_0^T [\frac{1}{2}\big(c_f X_t + f(\mu_t)\big)^2 + \frac{1}{2}\alpha_t^2]dt\Big]$$

  - coefficients c_b, c_f, c_g may be arbitrarily chosen (say 1)
  - σ may be 0 or 1 $\rightsquigarrow$ does not matter in this toy example $\rightsquigarrow$

analysis relies on the convex structure of the problem

- General form of the optimizer over α when μ is fixed

$$\alpha_t = -\eta_t X_t - h_t$$

  - η and $h \rightsquigarrow$ deterministic and η independent of μ!
  - optimal trajectories

$$dX_t = \Big((1-\eta_t)X_t + b(\mu_t) - h_t\Big)dt + \sigma dW_t$$

Linear quadratic control problem

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- General form of the optimizer over α when μ is fixed

$$\alpha_t = -\eta_t X_t - h_t$$

  - η and h $\rightsquigarrow$ deterministic and η independent of μ!

  - X is an O.-U. process $\rightsquigarrow$ marginal of X is Gaussian with fixed variance $\rightsquigarrow$ fixed point on the mean only!

Search for equilibria

- Characterization of (η, h) for a given μ
  - equation for $\eta \rightsquigarrow$ Riccati equation (with $c_b = c_f = c_g = 1$)

$$\dot{\eta}_t = \eta_t^2 - 2\eta_t - 1, \quad \eta_T = 1$$

  - equation for $h \rightsquigarrow$ backward linear ODE

$$\dot{h}_t = -\Big((1 - \eta_t)h_t + f(\mu_t) + b(\mu_t)\eta_t\Big), \quad h_T = g(\mu_T)$$

- Equilibrium condition $\rightsquigarrow$ find μ s.t. μ_t is the marginal law of

$$dX_t = \Big((1 - \eta_t)X_t + b(\mu_t) - h_t\Big)dt + dW_t$$

  - key point is $\boxed{\mu_t \sim \mathcal{N}(\bar{\mu}_t, \sigma_t^2)}$ with $\begin{array}{l} \bar{\mu}_t = \int_{\mathbb{R}} x d\mu_t(x) \\ \sigma_t = \sigma_t(\eta_t) \text{ fixed} \end{array}$

$\Rightarrow b(\mu_t) = \bar{b}(t, \bar{\mu}_t)$, same for f and g

- End up with forward backward ODE

$$\dot{\bar{\mu}}_t = \Big((1 - \eta_t)\bar{\mu}_t + \bar{b}(t, \bar{\mu}_t) - h_t\Big)$$

$$\dot{h}_t = -\Big((1 - \eta_t)h_t + \bar{f}(t, \bar{\mu}_t) + \bar{b}(t, \bar{\mu}_t)\eta_t\Big), \quad h_T = \bar{g}(\bar{\mu}_T)$$

Uniqueness to the FB system

- FB system $\rightsquigarrow$ finite-dimensional writing of the MFG system
  - Cauchy-Lipschitz theory in small time only
  - may loose existence / uniqueness on a given time interval
- Characteristics system of finite-dimensional master equation

$$\partial_t v(t,x) + ((1-\bar{\eta}_t)x + \bar{b}(t,x) - v(t,x_t))\partial_x v(t,x)$$
$$+ \Big((1-\eta_t)v(t,x) + \bar{f}(t,x) + \bar{b}(t,x)\eta_t\Big)$$
$$v(T,x) = g(x)$$

  - if smooth solution $\rightsquigarrow$ $\boxed{h_t = v(t,\bar{\mu}_t)}$
- Well-posedness if $\bar{b} \equiv 0, \bar{f}, \bar{g} \nearrow \Rightarrow$! of characteristics
  - if not $\Rightarrow$ shocks may emerge in finite time...
- $\sigma = 1$ does not help but Laplace in master restores uniquess

$\rightsquigarrow$ meaning?

Common noise

- Return to the FB system and add a noise

$$
\begin{aligned}
d\bar{\mu}_t &= \Big((1-\eta_t)\bar{\mu}_t + \bar{b}(t,\bar{\mu}_t) - h_t\Big)dt + \epsilon dB_t \\
dh_t &= -\Big((1-\eta_t)h_t + \bar{f}(t,\bar{\mu}_t) + \bar{b}(t,\bar{\mu}_t)\eta_t\Big)dt \\
h_T &= \bar{g}(\bar{\mu}_T)
\end{aligned}
$$

  - B new Brownian motion $\perp\!\!\!\perp$ of W, $\epsilon > 0$

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- Known fact: If $\bar{b}, \bar{f}$ and $\bar{g}$ are Lipschitz and bounded $\Rightarrow \exists!$
  - roughly speaking, add $\varepsilon^2 \partial^2_{xx}$ in master equation

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- Interpretation of B in the definition of the equilibria?

$$dX_t = \Big(c_b(X_t + b(t,\mu_t)) + \alpha_t\Big)dt + \sigma dW_t + \epsilon dB_t$$

  - fixed point condition $\rightsquigarrow \mu_t = \mathcal{L}(X_t^{\star,\mu}|B)$ and $\bar{\mu}_t = \mathbb{E}[X_t^{\star,\mu}|B]$
  - B is common noise!

3. A more general case ($\sigma = 0$)

Framework

- Return to the general setting but $\sigma = 0$
  - representative player $\rightsquigarrow dX_t = (b(X_t, \mu_t) + \alpha_t)dt$
  - cost functional

$$J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T \Big(f(X_t, \mu_t) + \tfrac{1}{2}|\alpha_t|^2\Big)dt\Big]$$

- Optimal trajectories when $(\mu_t)_{0 \le t \le T}$ is frozen $\rightsquigarrow$ Pontryagin

$$\begin{aligned} dX_t &= \Big(b(X_t, \mu_t) - Y_t\Big)dt \\ dY_t &= -\Big([D_x b(X_t, \mu_t)]^\top Y_t + D_x f(X_t, \mu_t)\Big)dt \\ Y_T &= D_x g(X_T, \mu_t) \end{aligned}$$

  - $D_x b \equiv 0$, $D_x f$ and $D_x g$ non-decreasing and Lipschitz in $x \Rightarrow \exists!$

$$(D_x f(x, \mu) - D_x f(x', \mu)) \cdot (x - x') \geq 0$$

Framework

- Return to the general setting but $\sigma = 0$
  - representative player $\rightsquigarrow dX_t = (b(X_t, \mu_t) + \alpha_t)dt$
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$$J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T \Big(f(X_t, \mu_t) + \tfrac{1}{2}|\alpha_t|^2\Big)dt\Big]$$

- Optimal trajectories when $(\mu_t)_{0 \le t \le T}$ is frozen $\rightsquigarrow$ Pontryagin

$$\begin{aligned} dX_t &= \Big(b(X_t, \mathrm{Law}(X_t)) - Y_t\Big)dt \\ dY_t &= -\Big([D_x b(X_t, \mathrm{Law}(X_t))]^\top Y_t + D_x f(X_t, \mathrm{Law}(X_t))\Big)dt \\ Y_T &= D_x g(X_T, \mathrm{Law}(X_t)) \end{aligned}$$

  - $D_x b \equiv 0$, $D_x f$ and $D_x g$ non-decreasing and Lipschitz in $x \Rightarrow \exists!$
- Implement the MFG condition
  - solve forward-backward system with $\mu_t = \mathrm{Law}(X_t) \rightsquigarrow$ MKV
  - if monotonicity in $\mu \Rightarrow \exists!$; if no monotonicity in μ?

Randomizing the solution

- From now on $\rightsquigarrow$ b independent of x and $d = 1$
- Force the dynamics to mollify in the direction of the measure

  ◦ pay attention: no reason to have a Gaussian structure $\rightsquigarrow$ forcing must be infinite dimensional

  ◦ somehow must force the law $\rightsquigarrow$ force the random variable itself seen as an element of L^2 space

- Construct the initial condition on $L^2(\mathbb{S}^1)$ with $\mathbb{S}^1$ = circle

  ◦ random variables $X_t, Y_t : \mathbb{S}^1 \to \mathbb{R}$ and $\mathrm{Law}(X_t) = \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}$

- Dynamics rewrite

$$
\begin{aligned}
dX_t(x) &= \Big(b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) - Y_t(x)\Big)dt \\
dY_t(x) &= -\partial_x f(X_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt \\
Y_T(x) &= \partial_x g(X_T(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_T^{-1}), \quad x \in \mathbb{S}^1
\end{aligned}
$$

  ◦ force the dynamics with infinite dimensional white noise!

Infinite dimensional forward-backward

- Look at the system

$$
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\begin{aligned}
dX_t(x) &= \Big(b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) - Y_t(x)\Big)dt \qquad + dB_t(x) \\
dY_t(x) &= -\partial_x f(X_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt \\
Y_T(x) &= \partial_x g(X_T(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_T^{-1}), \quad x \in \mathbb{S}^1
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  - B time space white noise on $\mathbb{S}^1$

$$
B_t(x) = B_t^0(x) + \sum_{n \geq 1} \sqrt{2}\Big(\cos(2\pi n x) B_t^{n,+} + \sin(2\pi n x) B_t^{n,-}\Big)
$$

$\rightsquigarrow (B^{n,\pm})_{n \in \mathbb{N}}$ independent Brownian motions

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dX_t(x) &= \Big(b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) - Y_t(x)\Big)dt \;+\; \partial_x^2 X_t(x)dt + dB_t(x) \\
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∘ B does not belong to $L^2(\mathbb{S}^1) \rightsquigarrow$ need friction term to force X_t to be in $L^2(\mathbb{S}^1) \Rightarrow \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}$ random measure

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∘ M $L^2(\mathbb{S}^1)$-valued martingale w.r.t filtration generated by B

∘ the initial condition X_0 is constructed on $\mathbb{S}^1 \rightsquigarrow$ the probability space carrying X_0 also carries the x-position of the white noise

Infinite dimensional forward-backward

- Look at the system

$$dX_t(x) = \Big(b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) - Y_t(x)\Big)dt \;+\; \partial_x^2 X_t(x)dt + dB_t(x)$$
$$dY_t(x) = -\partial_x f(X_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt \;+\; dM_t(x)$$
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- Equivalent to forcing Fourier modes

$$dX_t^{n,\pm} = \Big(b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})\delta_n^0 - Y_t^{n,\pm}\Big)dt - (2\pi n)^2 X_t^{n,\pm} dt + dB_t^{n,\pm}$$
$$dY_t^{n,\pm} = -\Big(\partial_x f(X_t(\cdot), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})\Big)^{n,\pm} dt + dM_t^{n,\pm}$$
$$Y_T^{n,\pm} = \Big(\partial_x g(X_T(\cdot), \mathrm{Leb}_{\mathbb{S}^1} \circ X_T^{-1})\Big)^{n,\pm}$$

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- Formal stochastic Pontryagin for the optimization of

$$\int_{\mathbb{S}^1} g(U_T(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_T^{-1})dx + \int_0^T \int_{\mathbb{S}^1} \Big[f(U_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) + \tfrac{1}{2}|\alpha_t(x)|^2\Big]dxdt$$

  - over $dU_t(x) = b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt + \alpha_t(x)dt + \partial_x^2 X_t(x)dt + dB_t(x)$
  - $(\alpha_t(\cdot))_t$ progressively measurable process with values in $L^2(\mathbb{S}^1)$

$\rightsquigarrow$ rigorously

$$U_t(x) = X_t(x) + \int_0^t (\alpha_s(x) - Y_s(x))ds$$

Solvability results

- Assumptions

  ◦ $\partial_x f$, $\partial_x g$ non-decreasing in x $\rightsquigarrow$ convex optimization

  ◦ b, $\partial_x f$, $\partial_x g$ bounded and Lipschitz $\rightsquigarrow$ use the 2-Wasserstein distance to fit the L^2 framework

- Statement: Existence and uniqueness for any initial condition

$$\boxed{Y_t = \mathcal{V}(t, X_t),\ \mathcal{V} \text{ mild solution of master equation on } L^2(\mathbb{S}^1)}$$

- Form of the master equation

$$\partial_t \mathcal{V}(t,X) + D\mathcal{V}(t,X) \cdot b\Big(\mathrm{Leb}_{\mathbb{S}^1} \circ X^{-1}\Big) - D\mathcal{V}(t,X) \cdot \mathcal{V}(t,X)$$
$$+ \partial_x f\Big(X, \mathrm{Leb}_{\mathbb{S}^1} \circ X^{-1}\Big) + L\mathcal{V}(t,X) = 0$$
$$\mathcal{V}(T,X) = \partial_x g\Big(X, \mathrm{Leb}_{\mathbb{S}^1} \circ X^{-1}\Big)$$

  ◦ where D is Fréchet derivative and L is O.-U. operator on $L^2(\mathbb{S}^1)$

$$LU(t,X) = \frac{1}{2}\mathrm{Trace}(D^2 U(t,X)) + \langle DU(t,X), \partial^2 X\rangle_{L^2(\mathbb{S}^1)}$$

Sketch of proof

- Cauchy Lipschitz theory works in small time

  ◦ small time $\rightsquigarrow$ depends upon Lipschitz constant of terminal condition $\mathcal{V}(T,\cdot)$

- Aim at propagating

  ◦ need a priori bound for Lipschitz constant of $\mathcal{V}(t,\cdot)$

  ◦ given by the smoothing property of O.-U. operator

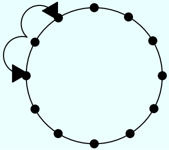
$$\sup_{h \in L^2(\mathbb{S}^1)} \left| D(e^{tL}\varphi)(h) \right| \leq C t^{-1/2} \sup_{h \in L^2(\mathbb{S}^1)} \left| \varphi(h) \right|$$

  ◦ control the Lipschitz constant away from the boundary using mild formulation

$$\begin{aligned} \mathcal{V}^n(t,\cdot) &= e^{(T-t)L}\Big[\Big(\partial_x g(\cdot, \mathrm{Leb}_{\mathbb{S}^1} \circ \cdot^{-1})\Big)^n\Big] \\ &\quad + \int_t^T e^{(s-t)L}\Big[\Big(\partial_x f(\cdot, \mathrm{Leb}_{\mathbb{S}^1} \circ \cdot^{-1})\Big)^n\Big] ds \\ &\quad + \int_t^T e^{(s-t)L}\Big[\Big(\langle b(\mathrm{Leb}_{\mathbb{S}^1} \circ \cdot^{-1}) - \mathcal{V}(s,\cdot), D\mathcal{V}^n(s,\cdot)\rangle\Big)^n\Big] ds \end{aligned}$$

4. Link with MFG

Approximating particle system



- Consider N particles
  - particle k located at $\exp(i2\pi k/N)$ on $\mathbb{S}^1$
  - $\bar{X}_t^k \rightsquigarrow$ state of particle number k

- Discrete version of the stochastic forward-backward system
  - mean field plus local interactions to nearest neighbors

$$d\bar{X}_t^k = \Big(b\big(\underbrace{\bar{\mu}_t^N}_{\frac{1}{N}\sum_{i=1}^{N}\delta_{\bar{X}_t^i}} \big) - \bar{Y}_t^k + \underbrace{N^2(\bar{X}_t^{k+1} + \bar{X}_t^{k-1} - 2\bar{X}_t^k)}_{\text{discrete Laplace}} \Big) dt + \sqrt{N} dB_t^k$$

$$d\bar{Y}_t^k = -\partial_x f(\bar{X}_t^k, \bar{\mu}_t^N) dt + d\text{martingale}_t, \qquad \bar{Y}_T^k = \partial_x g(\bar{X}_T^k, \bar{\mu}_T^N)$$

$\rightsquigarrow B^1, \ldots, B^N$ independent Brownian motions

$$\sqrt{N} dB_t^k = N \int_{k/N}^{(k+1)/N} dB_t(x)$$

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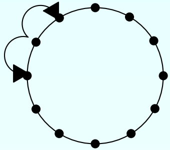

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$\rightsquigarrow B^1, \dots, B^N$ independent Brownian motions

$$\sqrt{N}dB_t^k = N\int_{k/N}^{(k+1)/N} dB_t(x)$$

$\rightsquigarrow$ initial condition $\rightsquigarrow \bar{X}_0^k = N\int_{k/N}^{(k+1)/N} X_0(x)dx \qquad (X_0\ C^0)$

Approximating particle system

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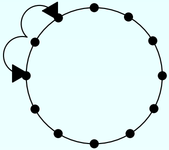
  - particle k located at $\exp(i2\pi k/N)$ on $\mathbb{S}^1$
  - $\bar{X}_t^k \rightsquigarrow$ state of particle number k



- Discrete version of the stochastic forward-backward system
  - mean field plus local interactions to nearest neighbors

$$d\bar{X}_t^k = \Big(b(\bar{\mu}_t^N) - \bar{Y}_t^k + \underbrace{N^2(\bar{X}_t^{k+1} + \bar{X}_t^{k-1} - 2\bar{X}_t^k)}_{\text{discrete Laplace}}\Big)dt + \sqrt{N}dB_t^k$$

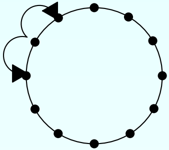
$$d\bar{Y}_t^k = -\partial_x f(\bar{X}_t^k, \bar{\mu}_t^N)dt + d\text{martingale}_t, \qquad \bar{Y}_T^k = \partial_x g(\bar{X}_T^k, \bar{\mu}_T^N)$$

$\rightsquigarrow B^1, \ldots, B^N$ independent Brownian motions

$$\sqrt{N}dB_t^k = N\int_{k/N}^{(k+1)/N} dB_t(x)$$

$\rightsquigarrow$ ansatz $\rightsquigarrow$ $\underbrace{\bar{X}_t^k}_{\text{discrete state}} \approx N\int_{k/N}^{(k+1)/N} \underbrace{X_t(x)}_{\text{limiting state}} dx$

Approximating particle system



- Consider N particles
  - particle k located at $\exp(i2\pi k/N)$ on $\mathbb{S}^1$
  - $\bar{X}_t^k \rightsquigarrow$ state of particle number k

- Discrete version of the stochastic forward-backward system
  - mean field plus local interactions to nearest neighbors

$$d\bar{X}_t^k = \Big(b(\bar{\mu}_t^N) - \bar{Y}_t^k + \underbrace{N^2(\bar{X}_t^{k+1} + \bar{X}_t^{k-1} - 2\bar{X}_t^k)}_{\text{discrete Laplace}}\Big)dt + \sqrt{N}dB_t^k$$

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$\rightsquigarrow B^1, \dots, B^N$ independent Brownian motions

$$\sqrt{N}dB_t^k = N\int_{k/N}^{(k+1)/N} dB_t(x)$$

$$\rightsquigarrow \quad \sum_{k=0}^{N-1} \bar{X}_t^k \mathbf{1}_{[k/N,(k+1)/N)} \approx X_t$$

Connection with a game

- Connect the previous particle system with a game?
  - natural candidate $\rightsquigarrow$ replace $-\bar{Y}^k$ by a general control $\bar{\alpha}^k$
  - difficulty $\rightsquigarrow$ local interaction too sensitive to variations of $\bar{\alpha}^k$
- Strategy $\rightsquigarrow$ consider N particles per site instead of 1 $\Rightarrow N^2$ particles
  - $X_t^{k,j} \rightsquigarrow$ state of jth particle at site k
- Consider controlled dynamics

$$dX_t^{k,j} = \Big(b(\mu_t^N) + \alpha_t^{k,j} + N^2 \frac{1}{N} \sum_{j=1}^{N} (X_t^{k+1,j} + X_t^{k-1,j} - 2X_t^{k,j})\Big)dt + \sqrt{N} dB_t^k$$

  - empirical measure $\mu_t^N = N^{-2} \sum_{k=0}^{N-1} \sum_{j=1}^{N} \delta_{X_t^{k,j}}$
  - cost to k

$$\mathbb{E}\Big[g(X_T^{k,j}, \bar{\mu}_T^N) + \int_0^T \Big(f(X_s^{k,j}, \bar{\mu}_s^N) + \frac{1}{2}|\alpha_s^{k,j}|^2\Big)ds\Big]$$

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- Use limiting system as nearly Nash for the game?
  - open-loop version $\alpha_t^{k,j} = N\int_{k/N}^{(k+1)/N} Y_t(x)dx$

Connection with a game

- Connect the previous particle system with a game?
  - natural candidate $\rightsquigarrow$ replace $-\bar{Y}^k$ by a general control $\bar{\alpha}^k$
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$$dX_t^{k,j} = \Big(b(\mu_t^N) + \alpha_t^{k,j} + N^2 \frac{1}{N} \sum_{j=1}^{N} (X_t^{k+1,j} + X_t^{k-1,j} - 2X_t^{k,j})\Big)dt + \sqrt{N} dB_t^k$$

- Use limiting system as nearly Nash for the game?
  - closed-loop

$$\alpha_t^{k,j} = N \int_{k/N}^{(k+1)/N} \underbrace{\mathcal{V}\Big(\frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=1}^{N} X_t^{k,j} \mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\Big)}_{\in L^2(\mathbb{S}^1)}(x) dx$$

Connection with a game

- Connect the previous particle system with a game?
  - natural candidate $\rightsquigarrow$ replace $-\bar{Y}^k$ by a general control $\bar{\alpha}^k$
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$$dX_t^{k,j} = \Big(b(\mu_t^N) + \alpha_t^{k,j} + N^2 \frac{1}{N} \sum_{j=1}^{N} (X_t^{k+1,j} + X_t^{k-1,j} - 2X_t^{k,j})\Big)dt + \sqrt{N} dB_t^k$$

- Use limiting system as nearly Nash for the game?
  - open/closed-loop
- Statement: form approximate Nash equilibrium
  - Sketch of proof [Gyongy, Nualart...] $\rightsquigarrow$ use discrete semi-group and L^∞ stability of solutions w.r.t. L^2 norms of the controls

5. Zero noise limit

Small noise system

- Consider small viscosity $\varepsilon > 0$

$$dX_t(x) = \Big(b(\mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1}) - Y_t(x)\Big)dt + \varepsilon^2 \partial_x^2 X_t(x)dt + \varepsilon dB_t(x)$$

$$dY_t(x) = -\partial_x f(X_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_t^{-1})dt + d\mathit{martingale}_t$$

$$Y_T(x) = \partial_x g(X_T(x), \mathrm{Leb}_{\mathbb{S}^1} \circ X_T^{-1}), \quad x \in \mathbb{S}^1$$

  - $(X_t, Y_t)_{0 \le t \le T} \rightsquigarrow (X_t^{\varepsilon}, Y_t^{\varepsilon})_{0 \le t \le T}$

- Limits as $\varepsilon \searrow 0$? (initial law of X_0 being fixed)
  - $\big((\mu_t^{\varepsilon} = \mathrm{Leb}_{\mathbb{S}^1} \circ (X_t^{\varepsilon})^{-1})_{0 \le t \le T}\big)_{\varepsilon \in (0,1)}$ tight on $\mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}))$

- Claim: Weak limits $(\mu_t)_{0 \le t \le T}$ are random equilibria of original MFG
  - $(\mu_t)_{0 \le t \le T}$ random process $\perp\!\!\!\perp X_0 \sim \mu_0$, $\mathbb{F} \rightsquigarrow$ canonical filtration

$$dX_t = \Big(b(X_t, \mu_t) + \alpha_t\Big)dt, \quad X_0 \sim \mu_0$$

  - with cost $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T \Big(f(X_t, \mu_t) + \frac{1}{2}|\alpha_t|^2\Big)dt\Big]$

$$\mu_t = \mathcal{L}(X_t^{\star,\mu} | (\mu_s)_{0 \le s \le t}), \quad t \in [0,T]$$

6. Selection of equilibria: An example

Selection of equilibria

- Use vanishing viscosity to select equilibria
  - focus on simpler (but typical of LQ models) case ($X_0 = 0$)

$$dX_t = \alpha_t dt + dW_t, \quad J(\alpha) = \mathbb{E}\Big[X_T g(\mu_T) + c_g g(\mu_T)^2 + \frac{1}{2}\int_0^T \alpha_t^2 dt\Big]$$

- Same analysis as before $\rightsquigarrow$ ODE system

$$\dot{\bar{\mu}}_t = -h_t, \quad \dot{h}_t = 0, \quad h_T = \bar{g}(\bar{\mu}_T) \quad \big(\bar{\mu}_0 = 0\big)$$

  - choose $\bar{g}(x) = \begin{cases} -x & x \in [-1, 1] \\ -\text{sign}(x) & |x| \geq 1 \end{cases}$

- Equilibria parametrized by $A = h_T \Leftrightarrow A = \bar{g}(-TA)$
  - $T > 1$ (1 = time to observe a shock) $\Rightarrow A \in \{-1, 0, 1\}$

$$A = 0 \Rightarrow J^{opt} = 0, \quad A = \pm 1 \Rightarrow J^{opt} = -TA^2 + c_g A^2 + \frac{1}{2} TA^2$$

  - if c_g large then equilibrium of lower cost is $A = 0$!

Vanishing viscosity

- Restore uniqueness by adding a common noise

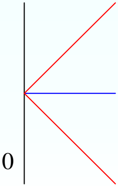
$$d\bar{\mu}_t^\epsilon = -h_t^\epsilon dt + \epsilon dB_t,$$
$$dh_t^\epsilon = dM_t^\epsilon, \quad h_T^\epsilon = \bar{g}(\bar{\mu}_T^\epsilon)$$

- PDE interpretation $\rightsquigarrow h_t^\epsilon = v^\epsilon(t, \bar{\mu}_t^\epsilon)$
  - v^ϵ solves viscous Burgers equation

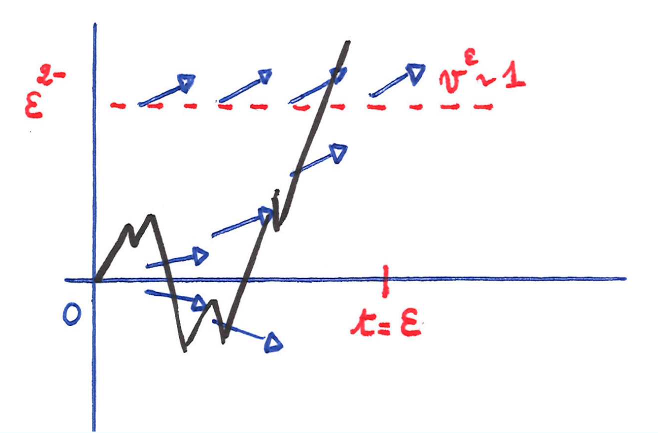
$$\partial_t v^\epsilon - v^\epsilon \partial_x v^\epsilon + \frac{\epsilon^2}{2} v^\epsilon = 0, \quad v^\epsilon(T, \cdot) = \bar{g}$$

  - known fact: $v^\epsilon(t,x) \to -\text{sign}(x)$ as $\epsilon \searrow 0$ for $t < T-1$

- Statement: As $\epsilon \searrow 0$ $(\bar{\mu}_t^\epsilon)_t$ converges (in law) to $\frac{1}{2}\delta_{(t)_t} + \frac{1}{2}\delta_{(-t)_t}$
  - do not see $A = 0$!



Sketch of proof



- In time ϵ, the particle should go beyond ϵ^{2-} with high probability
  - then, the drift is very close to ± 1 $\rightsquigarrow$ the particle follows the drift with very high probability

7. If $\sigma = 1$...

With independent Brownian motions

- In the previous example $\rightsquigarrow$ no Brownian motion in the dynamics

  ◦ difficulty $\rightsquigarrow$ would require to define $W(x)$, for $x \in \mathbb{S}^1$, but hardly compatible with adaptedness constraints

- Strategy is to disentangle the dynamics of the representative player and the dynamics of the environment

  ◦ dynamics of the representative player

$$dX_t = b(\mathrm{Leb}_{\mathbb{S}^1} \circ \chi_t{}^{-1})dt + \alpha_t dt + dW_t$$

$\rightsquigarrow$ typical form of $\alpha_t = \alpha(t, X_t, \chi_t(\cdot))$ with $\chi_t(\cdot)$ in $L^2(\mathbb{S}^1)$

$\rightsquigarrow$ cost of $f(X_t, \chi_t(\cdot))$ and $g(X_T, \chi_T(\cdot))$

  ◦ dynamics of the environment on $L^2(\mathbb{S}^1)$

$$\begin{aligned} d\chi_t(x) &= b(\mathrm{Leb}_{\mathbb{S}^1} \circ \chi_t{}^{-1})dt + \psi(t, \chi_t(x), \chi_t(\cdot))dt \\ &\quad + \Delta\chi_t(x)dt + dw_t(x) + dB_t(x) \end{aligned}$$

$\rightsquigarrow$ w Brownian constructed on $(\mathbb{S}^1, \mathrm{Leb})$ and B white noise on $\mathbb{S}^1$

  ◦ fix ψ and find α^{optimal} $\Rightarrow$ fixed point $\psi = \alpha^{\text{optimal}}$