Ground states of focusing Mean-Field Games on $\mathbb{R}^N$

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Setting

We consider a Mean-Field Game where the dynamics of a typical agent is driven by the controlled SDE

$$dX_s = -v_s ds + \sqrt{2\varepsilon}\, dB_s \qquad \text{on } \mathbb{R}^N,$$

where v_s is the control and B_s is a Brownian motion, and the cost, of long-time average form, is

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T [L(v_s) + V(X_s) - m^\alpha(X_s)]ds.$$

In what follows,

- $\varepsilon, \alpha > 0$,
- $L(q) \sim |q|^{\gamma'}, \quad \gamma' = \frac{\gamma}{\gamma-1} > 1$
- $0 \leq V(x) \to \infty$ as $|x| \to \infty$

Equilibria of this MFG are intimately related to solutions (u, λ, m) of the following system

$$\begin{cases} -\varepsilon \Delta u + H(\nabla u) + \lambda = -m^{\alpha} + V(x) \\ -\varepsilon \Delta m - \operatorname{div}(m \nabla H(\nabla u)) = 0 & \text{on } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} m = 1, \; m > 0 \end{cases} \tag{1}$$

which is a system of stationary coupled viscous HJB and Fokker-Planck equations, where $H(p) = L^*(p) \sim |p|^{\gamma}$.

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We will address the following problems:

1. For fixed $\varepsilon > 0$, prove the existence of a triple $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ to (1),
2. Study the behaviour of solutions as $\varepsilon \to 0$,
3. Understand (1) when $V \equiv 0$.

Key point 1: the "focusing" coupling $-m^\alpha$

A large part of the literature is devoted to systems with competition, namely when the coupling in the cost is monotonically increasing w.r.t *m*. In this case,

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- the coupling term is "regularising"

Few studies consider the opposite case, i.e. when agents are attracted toward congested areas. See, e.g. [Guéant, 09], [Gomes, Nurbekyan, Prazeres, 16], [C., 16], [C., Tonon, 17], ...

In this framework, uniqueness of equilibria has not to be expected, while existence is a delicate issue: if $\alpha >> 0$, non-existence phenomena show up.

Key point 2: $\mathbb{R}^N$ as a state space

Most of the (PDE) literature is restricted to $\mathbb{T}^N$. For a truly non-periodic setting, we mention

- [Arapostathis, Biswas, Carroll, 17]: bounded controls,
- [Bardi, Priuli, 14]: Linear-Quadratic case,
- [Gomes, Pimentel, 16]: local regularity,
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At the PDE level, the main issue of $\mathbb{R}^N$ is its lack of compactness.

From the point of view of the game, a typical player is subject to diffusion, which prevents a stable long-time behaviour.

In other words, an agent playing optimally moves according to

$$dX_s = -\nabla H(\nabla u(X_s))ds + \sqrt{2\varepsilon}\, dB_s,$$

and in a (M-F) equilibrium regime

$$\mathcal{L}(X_s) \to m \quad \text{as } s \to \infty,$$

where m is the overall population density.

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that is itself related to the existence of a Lyapunov function, i.e. $\varphi \in C^2(\mathbb{R}^N)$ such that

$$\varphi \to +\infty \quad \text{and} \quad -\varepsilon\Delta\varphi + \nabla H(\nabla u) \cdot \nabla\varphi \to +\infty \quad \text{as } x \to \infty.$$

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What happens if $V \equiv 0$?

Tools - 1. Variational formulation

We will construct solutions to (1) via minimisers of the non-convex energy

$$\mathcal{E}(m,w) = \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) + V(x)m - \frac{1}{\alpha+1} m^{\alpha+1}\, dx,$$

subject to the constraint

$$(m,w) \in \mathcal{K}_\varepsilon := \left\{ -\varepsilon\Delta m + \mathrm{div}(w) = 0, \quad \int_{\mathbb{R}^N} m\, dx = 1, \quad m > 0 \right\}.$$

(see [Cardaliaguet et al. 13-16]).

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Question: $\qquad e_\varepsilon(M) = \inf_{\mathcal{K}_\varepsilon} \mathcal{E}(m,w) > -\infty\,?$

YES, if

$$0 < \alpha < \frac{\gamma'}{N}$$

(while $\inf \mathcal{E} = -\infty$ if $\alpha > \frac{\gamma'}{N}$).

In particular, in this regime there exist $C, \delta > 0$ s.t.

$$\left(\int_{\mathbb{R}^N} m^{\alpha+1}\right)^{1+\delta} dx \leq C \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx$$

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If, in addition,

$$1 < \gamma < \frac{N}{N-1}, \quad (\text{i.e. } \gamma' > N)$$

then

$$||m||_{C^{0,\beta}(\mathbb{R}^N)} \leq C \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx$$

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$$||m||_{C^{0,\beta}(\mathbb{R}^N)} \leq C \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) dx$$

Proof: Gagliardo-Nirenberg inequality and elliptic regularity.

Tools - 2. Regularity of HJB equations

Suppose that v is a viscosity solution to

$$-\varepsilon\Delta v + H(\nabla v) = F(x) \quad \text{on } \mathbb{R}^N,$$

where

$$C_F^{-1}|x|^b \leq F(x) \leq C_F|x|^b \quad \text{for all } |x| >> 0 \text{ and some } b \geq 0,$$

Theorem [Capuzzo Dolcetta, Leoni, Porretta, 10]

There exists $K > 0$ such that

$$|\nabla v(x)| \leq K(1+|x|)^{b/\gamma} \quad \text{on } \mathbb{R}^N.$$

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Theorem [Barles, Meireles, 16]

There exists $K_1 > 0$ such that

$$v(x) \geq K_1(|x|^{1+\frac{b}{\gamma}} - 1) \quad \text{on } \mathbb{R}^N.$$

Existence of solutions, $\varepsilon > 0$

Theorem 1.

Suppose that for some $b > 0$,

$$\alpha < \frac{\gamma'}{N}, \quad \gamma < \frac{N}{N-1}, \quad C_V^{-1}|x|^b \leq V(x) \leq C_V|x|^b \quad \forall |x| >> 0,$$

Then, for every $\varepsilon > 0$,

i) There exists a minimizer $(m_\varepsilon, w_\varepsilon) \in \mathcal{K}_\varepsilon$ of $\mathcal{E}$, that is

$$\mathcal{E}(m_\varepsilon, w_\varepsilon) = \inf_{(m,w)\in\mathcal{K}_\varepsilon} \mathcal{E}(m, w).$$

ii) For any minimizer $(m_\varepsilon, w_\varepsilon) \in \mathcal{K}_\varepsilon$ of $\mathcal{E}$, there exists $(u_\varepsilon, \lambda_\varepsilon)$ such that $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ is a classical solution to (1).

Note: $u_\varepsilon \to +\infty$ as $|x| \to \infty$.

Existence of solutions, $\varepsilon > 0$

Proof: 1. Consider a minimizing sequence (m_n, w_n). Then, it is bounded in $W^{1,r}(\mathbb{R}^N) \times L^{\gamma'}(\mathbb{R}^N)$. Since $\int_{\mathbb{R}^N} Vm$ is bounded, for all $\eta > 0$ there exists $R >> 0$ s.t.

$$\int_{B_R(0)} m_\varepsilon(x)dx \geq 1 - \eta,$$

that gives convergence of $m_n \to \bar{m}$ in $L^1(\mathbb{R}^N) \cap L^{\alpha+1}(\mathbb{R}^N)$.

2. Full solution of (1) is obtained by considering

$$\widetilde{\mathcal{E}}(m, w) = \int_{\mathbb{R}^N} mL\left(-\frac{w}{m}\right) + (V(x) - \bar{m}^\alpha)m\, dx,$$

and its dual formulation (Fenchel-Rockafellar).

The limit $\varepsilon \to 0$: a rescaling

As $\varepsilon \to 0$ one has to expect just weak-* convergence of m_ε.

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Consider the rescaled functions (x_ε will be chosen later)

$$\tilde{m}_\varepsilon(\cdot) = \varepsilon^{\frac{N\gamma'}{\gamma'-\alpha N}} m(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot + x_\varepsilon), \quad \tilde{u}_\varepsilon(\cdot) = \varepsilon^{\frac{N\alpha(\gamma'-1)-\gamma'}{\gamma'-\alpha N}} \Big(u(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot + x_\varepsilon) - u(x_\varepsilon) \Big).$$

Since $(u_\varepsilon, m_\varepsilon)$ solves

$$\begin{cases} -\varepsilon\Delta u + H(\nabla u) + \lambda = -m^\alpha + V(x) \\ -\varepsilon\Delta m - \operatorname{div}(m\nabla H(\nabla u)) = 0 & \text{on } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} m = 1, \ m > 0 \end{cases}$$

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Then $(\tilde{u}_\varepsilon, \tilde{m}_\varepsilon)$ solves

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where

$$V_\varepsilon(\cdot) = \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} V(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot + x_\varepsilon), \qquad H_\varepsilon(p) \sim |p|^\gamma.$$

The limit $\varepsilon \to 0$: a partial convergence

Choose x_ε to be the global minimum of u_ε. Then,

$$\tilde{u}_\varepsilon(0) = 0, \quad \tilde{u}_\varepsilon \geq 0$$

and

Proposition

Up to subsequences, $(\tilde{u}_\varepsilon, \tilde{m}_\varepsilon) \to (\bar{u}, \bar{m})$ classical solution to

$$\begin{cases} -\Delta \bar{u} + H_0(\nabla \bar{u}) + \bar{\lambda} = -\bar{m}^\alpha + g(x) \\ -\Delta \bar{m} - \mathrm{div}(\bar{m} \nabla H_0(\nabla \bar{u})) = 0 \quad \text{on } \mathbb{R}^N. \end{cases} \tag{2}$$

where g is a bounded function on $\mathbb{R}^N$ and $H_0(p) := \lim_{\varepsilon \to 0} H_\varepsilon(p)$.

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where g is a bounded function on $\mathbb{R}^N$ and $H_0(p) := \lim_{\varepsilon \to 0} H_\varepsilon(p)$. Moreover, there exists $a \in (0, 1]$ such that

$$\int_{\mathbb{R}^n} \bar{m}\, dx = a.$$

How to prove that $a = 1$?

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Using $e^{\bar{u}}$ as a Lyapunov function, exponential decay of $\bar{m}$ follows.

The limit $\varepsilon \to 0$: concentration-compactness

Note that $(\tilde{m}_\varepsilon, \tilde{w}_\varepsilon)$ are minimizers of a rescaled energy $\mathcal{E}_\varepsilon$, that is

$$\mathcal{E}_\varepsilon(\tilde{m}_\varepsilon, \tilde{w}_\varepsilon) = \inf_{(m,w)\in\mathcal{K}} \mathcal{E}_\varepsilon(m, w)$$

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Moreover, $\mathcal{E}_\varepsilon$ is sub-additive, that is, if $a < 1$,

$$\mathcal{E}_\varepsilon(\tilde{m}_\varepsilon, \tilde{w}_\varepsilon) = \inf_{\int m=1} \mathcal{E}_\varepsilon < \inf_{\int m=a} \mathcal{E}_\varepsilon + \inf_{\int m=1-a} \mathcal{E}_\varepsilon.$$

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The concentration-compactness Lemma [Lions, 84] states that for some $R = R_\varepsilon \to \infty$, $\int_{B_R(0)} \tilde{m}_\varepsilon \simeq a$ and $\int_{\mathbb{R}^N \setminus B_{2R}(0)} \tilde{m}_\varepsilon \simeq 1 - a$, so

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If we were able to prove

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we would contradict sub-additivity, so $\tilde m_\varepsilon \to \bar m$ in $L^1(\mathbb{R}^N)$.

Instead of splitting $\tilde{m}_\varepsilon$ via cut-offs, we write

$$\tilde{m}_\varepsilon = \bar{m} + (m_\varepsilon - \bar{m})$$
$$\tilde{w}_\varepsilon = \bar{w} + (w_\varepsilon - \bar{w})$$

Instead of splitting $\tilde{m}_\varepsilon$ via cut-offs, we write

$$\begin{aligned} \tilde{m}_\varepsilon &\simeq \bar{m} + (m_\varepsilon - \bar{m} + \nu_\varepsilon(x)) \\ \tilde{w}_\varepsilon &\simeq \bar{w} + (w_\varepsilon - \bar{w} + \nabla \nu_\varepsilon(x)), \end{aligned}$$

where $\nu(x)$ decays exponentially and vanishes as $\varepsilon \to 0$.

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where $\nu(x)$ decays exponentially and vanishes as $\varepsilon \to 0$.

$(\tilde{m}_\varepsilon, \tilde{w}_\varepsilon)$ then splits into two (couples of) admissible competitors, and with some technical work,

$$\begin{aligned}\mathcal{E}_\varepsilon(\tilde{m}_\varepsilon, \tilde{w}_\varepsilon) &\simeq \mathcal{E}_\varepsilon(\bar{m}, \bar{w}) + \mathcal{E}_\varepsilon(m_\varepsilon - \bar{m} + \nu_\varepsilon, w_\varepsilon - \bar{w} + \nabla\nu_\varepsilon) \\ &\gtrsim \inf_{\int m = a} \mathcal{E}_\varepsilon + \inf_{\int m = 1-a} \mathcal{E}_\varepsilon,\end{aligned}$$

contradicting sub-additivity of $\mathcal{E}_\varepsilon$.

tracking the concentration point x_ε

By the previous compactness argument, for all $\eta > 0$ there exists $R >> 0$ s.t.

$$\int_{B_R(0)} \tilde{m}_\varepsilon dx \geq 1 - \eta,$$

namely,

$$\int_{|x - x_\varepsilon| \leq R\varepsilon^{\frac{\gamma'}{\gamma' - \alpha N}}} m_\varepsilon \, dx \geq 1 - \eta,$$

that is: most of the mass of m_ε is located around a small ball centered at x_ε.

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To track x_ε, we exploit that $\tilde{m}_\varepsilon$ is a minimizer of $\mathcal{E}_\varepsilon$.

Heuristically, x_ε should approach $\mathrm{argmin}_{\mathbb{R}^N} V(x)$.

To complete this program, we will assume also that V has a finite number of minima, that is, for some $\hat{b} > 0$, $x_j \in \mathbb{R}^N$, $j = 1, \ldots, n$,

$$V(x) = h(x) \prod_{j=1}^{n} |x - x_j|^{\hat{b}}, \qquad C_V^{-1} \leq h(x) \leq C_V \text{ on } \mathbb{R}^N.$$

Note that

$$\min_{\mathbb{R}^N} V = 0.$$

Suppose also

$$C_H(|p|^\gamma - 1) \leq H(p) \leq C_H|p|^\gamma,$$

and

$$\text{either} \quad H(p) = C_H|p|^\gamma, \quad \text{or} \quad n\hat{b} < \frac{\gamma}{2}.$$

On one hand, if one chooses suitable competitors (m, w),

$$\mathcal{E}(m_\varepsilon, w_\varepsilon) \le \mathcal{E}(m, w) \quad \rightsquigarrow \quad \int_{\mathbb{R}^N} V m_\varepsilon dx \to 0$$

On the other hand, for some $\delta \to 0$,

$$\frac{1}{2} \inf_{B_\delta(x_\varepsilon)} V(x) \le \int_{B_\delta(x_\varepsilon)} V m_\varepsilon,$$

therefore

$$V(x_\varepsilon) \to 0.$$

as $\varepsilon \to 0$.

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as $\varepsilon \to 0$.

To summarize...

$\varepsilon \to 0$: the convergence result

Theorem 2. Under the standing assumptions, there exist sequences $\varepsilon \to 0$ and x_ε, such that for all $\eta > 0$ there exists R s.t.,

$$\int_{|x-x_\varepsilon|\le R\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}}} m_\varepsilon \, dx \ge 1-\eta,$$

and for some $J = 1, \dots, n$, $C > 0$,

$$|x_\varepsilon - x_J| \le C\varepsilon^{\frac{\gamma'}{n(\gamma'-N\alpha)}}.$$

Moreover, $(\varepsilon^{\frac{N\alpha(\gamma'-1)-\gamma'}{\gamma'-\alpha N}} u_\varepsilon(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot + x_\varepsilon), \varepsilon^{\frac{N\alpha\gamma'}{\gamma'-\alpha N}} m_\varepsilon(\varepsilon^{\frac{\gamma'}{\gamma'-\alpha N}} \cdot + x_\varepsilon))$ converges to a classical solution of

$$\begin{cases} -\Delta u + H_0(\nabla u) + \bar{\lambda} = -m^\alpha \\ -\Delta m - \mathrm{div}(m \nabla H_0(\nabla u)) = 0 \\ \int_{\mathbb{R}^N} m = 1, \quad m > 0. \end{cases}$$

By-products

- If one introduces the "V-free energy"

$$\overline{\mathcal{E}}(m,w) = \int_{\mathbb{R}^N} m L_0\left(-\frac{w}{m}\right) - \frac{1}{\alpha+1} m^{\alpha+1} dx$$

then

$$\mathcal{E}_\varepsilon(\tilde{m}_\varepsilon, \tilde{w}_\varepsilon) \to \overline{\mathcal{E}}(\bar{m},\bar{w}) = \min_{(m,w)\in\mathcal{K},\ m(1+|x|^b)\in L^1(\mathbb{R}^N)} \overline{\mathcal{E}}(m,w)$$

- There exists a classical solution, or ground state, to

$$\begin{cases} -\Delta u + H_0(\nabla u) + \bar{\lambda} = -m^\alpha \\ -\Delta m - \operatorname{div}(m\nabla H_0(\nabla u)) = 0 \\ \int_{\mathbb{R}^N} m = 1, \quad m > 0. \end{cases}$$

A comment on Lyapunov functions and ergodicity

Solutions of V-free MFG cannot be unique, by translation invariance.

More subtle is uniqueness of m for fixed u to

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This holds true because $\varphi(x) = u(x)$ (and $\varphi(x) = e^{u(x)}$) satisfies

$$\varphi \to +\infty \quad \text{and} \quad -\varepsilon\Delta\varphi + \nabla H(\nabla u) \cdot \nabla\varphi \to +\infty \quad \text{as } x \to \infty,$$

that implies also ergodicity of the optimal trajectories.

Final remarks

Symmetry of solutions to MFG on $\mathbb{R}^N$ is a completely open problem, but break of symmetry may show up if ε is small.

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behaviour of time-dependent systems? (joint work w. D. Tonon)

Thank you for your attention.