A Semi-Lagrangian discretization of non linear Fokker Planck equations

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joint works with
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Mean Field Games and related topics 4,
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Outline

1. A Semi-Lagrangian scheme for non linear Fokker-Planck equation
2. Convergence Analysis
3. Numerical simulations
   - Linear case: Lotka Volterra model
   - Non linear case: two interacting species
   - Non linear implicit case: Mean Field Game model
   - Non linear explicit case: a new Hughes type model

A nonlinear Fokker-Planck equation

The nonlinear FP equation

$$\begin{cases} \partial_t m - \frac{1}{2}\sum_{i,j} \partial_{x_i x_j} (a_{ij}[m]m) + \mathsf{div}(b[m]\, m) = 0 & \mathbb{R}^d \times \mathbb{R}^+ \\ m(\cdot, 0) = m_0(\cdot) & \mathbb{R}^d \end{cases} \tag{1}$$

describes the evolution of the law of the diffusion process $X(t)$

$$\begin{aligned} \mathrm{d}X(s) &= b[m](X(s), s)\mathrm{d}s + \sigma[m](X(s), s)\mathrm{d}W(s), \\ X(0) &= x_0, \end{aligned} \tag{2}$$

where x_0 is a random variable with law m_0 independent from W and

A nonlinear Fokker-Planck equation

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where x_0 is a random variable with law m_0 independent from W and

- $\mathcal{P}_p(\mathbb{R}^d)$ denotes the probability measures $\mathcal{P}(\mathbb{R}^d)$ with bounded p moments;
- $b[m](x,t) : C((0,T) \times \mathcal{P}_1(\mathbb{R}^d)) \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^d$ is a given vector field, depending non locally on m;
- $\sigma[m](x,t) : C((0,T) \times \mathcal{P}_1(\mathbb{R}^d)) \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^{d\times r}$,

  $$(A[m])_{i,j} = a_{ij}[m](x,t) := (\sigma[m]\sigma[m]^\top)_{ij},$$

  is the diffusion matrix (possible degenerate), depending non locally on m;
- the density of the initial law is given by m_0 and it still denoted by m_o

Some applications

- Non local interactions due to collective phenomena (biophysics, social behavior)
- Hughes model: $b[m](x,t) = -f^2(m(x,t))\nabla v[m](x,t)$
  $v[m]$ is the solution of a stationary HJB

$$|\nabla v| = \frac{1}{f(m(x,t))}$$

In this case $b[m](x,t)$ depends on m at time t

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  In this case $b[m](x,t)$ depends on m at time t
- Mean Filed Games : $b[m](x,t) = -\nabla v[m](x,t)$
  in the case $v[m]$ is the solution a backward HJB with $H(x,p) = \frac{1}{2}|p|^2$

  $$v[m](x,t) = \inf_{\alpha} \mathbb{E}[\int_t^T \{\tfrac{1}{2}|\alpha|^2 + F(X^{x,t}(s), m(s))\}\mathrm{d}s + G(X^{x,t}(T)(m(T))]$$

  Then $b[m](x,t)$ depends on m in all time $t \le s \le T$

Some applications

- a new Hughes type model:

$$b[m](x,t) := -\nabla v[m](x,t)$$

$v[m]$ is the solution of a backward HJB

$$\begin{cases} -\partial_t v - \frac{\sigma^2}{2}\Delta v + H(\nabla v) = F(x,s,m(t)) & \mathbb{R}^d \times (t,T) \\ v(x,T) = g(x,m(t)). \end{cases}$$

In this case $b[m](x,t)$ depends on m only at time t.
We propose the following second order *possibly degenerate* system

$$\begin{cases} \partial_t m - \Delta m - \mathrm{div}\big(\nabla H(\nabla v[m])m\big) = 0 & \text{in } \mathbb{R}^d \times (0,T), \\ -\partial_s v(x,s) - \Delta v(x,s) + H(\nabla v[m](x,s)) = F(x,s,m(t)) & \text{in } \mathbb{R}^d \times (t,T), \\ v(x,T) = G(x,m(t)) & \text{for } x \in \mathbb{R}^d, \\ m(\cdot,0) = m_0(\cdot) \in \mathcal{P}_1(\mathbb{R}^d). \end{cases} \tag{3}$$

Numerical approximation of FP
Some references

Linear FP

- Chang and Cooper (1970): finite difference scheme s.t. preserves positivity, equilibrium states and mass of the distribution (explicit version is stable under a parabolic type CFL condition)
- Kushner (1976): finite difference via probabilistic method
- Naess and Johnsen (1993) : Path Integration Method (time integration of the evolution probability density function, where the transition probability density is approximated by a Gaussian)
- Chen, Jakobsen and Naess (2016): Convergence Path Integration Method in L_1
- Jordan, Kinderlehrer,Otto (1998): variational scheme for FP equations for which the drift term is given by the gradient of a potential (JKO) (It preserves positivity, and mass of the distribution)

Numerical approximation of FP
Some references

Non Linear FP

- Drozdov, Morillo (1995): finite difference scheme s.t. preserves equilibrium states and mass of the distribution (high order).
- Achdou,Camilli,Capuzzo Dolcetta (2012): implicit finite difference scheme s.t. preserves positivity, and mass of the distribution.
- Benamou, Carlier, Laborde (2015): semi implicit variant of JKO.

Numerical approximation based on SL for FP

We propose a Semi-Lagrangian (SL) scheme for non linear FP equation s.t.

- it is first order accurate (numerically)
- it allows for large time steps
- it preserves the positivity of the density and conserves its integral equals to 1

Ref. F.Camilli, F.Silva (2013), E.Carlini, F.Silva (2014,2015)

The scheme has been applied to numerically compute the solution of

- Mean Field Games model (with F. Silva)
- a regularized Hughes model for pedestrian flow (with A.Festa, F.Silva, M.T. Wolfram)
- Hughes model for pedestrian flow with different congestion penalty function (with A.Festa, F.Silva)

A Semi-Lagrangian scheme for non linear Fokker-Planck equation

Weak solution

For many problems of interest, (1) has only a formal meaning, henceforth we mean m is a weak solution of (1) if for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\phi(x)\mathrm{d}m(t)(x)\\
&=\int_{\mathbb{R}^d}\left(\tfrac{1}{2}\sum_{i,j}a_{ij}\partial_{x_i x_j}\phi(x)-\sum_i b_i\phi_{x_i}(x)\right)\mathrm{d}m(t)(x) \qquad (4)\\
&=\int_{\mathbb{R}^d}\left(\tfrac{1}{2}\mathrm{tr}(AD^2\phi(x))+b^\top\nabla\phi(x)\right)\mathrm{d}m(t)(x)
\end{aligned}$$

A Semi-discrete in time SL for a nonlinear 1d Fokker-Planck equation

Given Δt, we define $t_k = k\Delta t$, $k = 0, \dots, N$. We integrate (4) in time on $[t_k, t_{k+1}]$:

$$\int_{\mathbb{R}} \phi(x) \mathrm{d}m(t_{k+1}) = \int_{\mathbb{R}} \phi(x) \mathrm{d}m(t_k) \qquad (5)$$

$$+ \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}} \left(\tfrac{1}{2}\sigma^2 D^2\phi(x) + b\nabla\phi(x)\right) \mathrm{d}m(t)\mathrm{d}t.$$

Semi-discrete in time

We first approximate (5) as

$$\int_{\mathbb{R}} \phi(x)\mathrm{d}m(t_{k+1}) =$$
$$\int_{\mathbb{R}} \left(\phi(x) + \Delta t b(x,t_k)\nabla\phi(x) + \frac{\Delta t}{2}\sigma^2(x,t_k)D^2\phi(x) \right) \mathrm{d}m(t_k).$$

Note that the right hand side corresponds to a Taylor expansion. Hence we approximate

$$\int_{\mathbb{R}} \phi(x)\mathrm{d}m(t_{k+1}) \simeq$$
$$\frac{1}{2}\int_{\mathbb{R}} \left(\phi(x + \Delta t b(x,t_k) + \sqrt{\Delta t}\sigma^2(x,t_k) \right) \mathrm{d}m(t_k) +$$
$$\frac{1}{2}\int_{\mathbb{R}} \left(\phi(x + \Delta t b(x,t_k) - \sqrt{\Delta t}\sigma^2(x,t_k) \right) \mathrm{d}m(t_k).$$

Fully-discrete in space

Given $\Delta x > 0$ consider a space grid, for $i = 1, \dots, M$

$$\begin{aligned} E_i &= [x_i - \tfrac{1}{2}\Delta x, x_i + \tfrac{1}{2}\Delta x], \\ m_{i,k} &:= \textstyle\int_{E_i} \mathrm{d}m(t_k). \end{aligned} \tag{6}$$

By the standard rectangular quadrature formula, we get

$$\sum_{j\in\mathbb{Z}} \phi(x_j) m_{j,k+1} = \tag{7}$$

$$\frac{1}{2}\sum_{j\in\mathbb{Z}} \phi(\Phi_j^+) m_{j,k} + \frac{1}{2}\sum_{j\in\mathbb{Z}} \phi(\Phi_j^-) m_{j,k},$$

where, for $\mu \in \mathbb{R}^M$, $j \in \mathbb{Z}$, $k = 0, \dots, N-1$, we have defined

$$\Phi_j^{\pm} := x_j + \Delta t\, b(x_j, t_k) \pm \sqrt{\Delta t}\sigma(x_j, t_k). \tag{8}$$

Interpretation of the scheme by means of characteristics

Note that $\Phi_j^{\pm}$ defined in (12) (with $\mu_j = m(x_j, t_k)$) can be interpreted as a single Euler step approximation of

$$\begin{aligned} \mathrm{d}X(s) &= b\,(X(s), s)\,\mathrm{d}s + \sigma\,(X(s), s)\,\mathrm{d}W(s), \qquad s \in (t_k, t_{k+1}), \\ X(t_k) &= x_j, \end{aligned} \tag{9}$$

with a random walk discretization of the Brownian motion $W(\cdot)$. Indeed, considering a random value Z in $\mathbb{R}$ such that f

$$\mathcal{P}(Z = 1) = \mathcal{P}(Z = -1) = \frac{1}{2} \tag{10}$$

the function $\Phi_j^{\pm}$ corresponds to one realization of

$$x_j + \Delta t b(x_j, t_k) \pm \sqrt{2\Delta t}\sigma(x_j, t_k)Z.$$

Fully-discrete in space

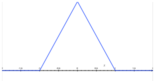

Figure: P1-basis function, β_i

Given $i \in \mathbb{Z}$ and setting in (7) $\phi = \beta_i$ we have for all $i \in \mathbb{Z}$

$$\begin{cases} m_{i,k+1} = & G(m_k, i, k) \\ \quad\quad = & \frac{1}{2}\sum_{j\in\mathbb{Z}}\left(\beta_i(\Phi_j^+) + \beta_i(\Phi_j^-)\right) m_{j,k} \\ m_{i,0} = & \int_{E_i} m_0(x)\mathrm{d}x \end{cases} \quad (11)$$

Fully-discrete Non Linear

In the non linear case, we have

$$\Phi_j^{\pm}[m] := x_j + \Delta t\, b[m](x_j, t_k) \pm \sqrt{\Delta t}\sigma[m](x_j, t_k). \tag{12}$$

Fully-discrete Non Linear

In the non linear case, we have

$$\Phi_j^{\pm}[m] := x_j + \Delta t\, b[m](x_j, t_k) \pm \sqrt{\Delta t}\sigma[m](x_j, t_k). \quad (12)$$

Find $m = \{m_{i,k}\}$ s.t.

$$\left\{ \begin{array}{rl} m_{i,k+1} = & G(m_k, i, k) \\ = & \frac{1}{2}\sum_{j\in\mathbb{Z}} \Big(\beta_i(\Phi_j^+[m]) + \beta_i(\Phi_j^-[m])\Big) m_{j,k} \\ m_{i,0} = & \int_{E_i} m_0(x)\mathrm{d}x \end{array} \right. \quad (13)$$

- Non-negative : $m_{i,k} \geq 0$ for $k = 0, \ldots, N-1, i \in \mathbb{Z}$
- Mass conservative : $\Delta x \sum_i m_{i,k} = 1$ for $k = 0, \ldots, N-1$
- Generalizable to any dimension
- Generalizable to handle Dirichlet and Neumann Boundary conditions
- Generalizable to handle degeneracy of the diffusion matrix

Existence and Uniqueness of the SL scheme

Proposition (Existence)

There exists at least one solution $\{m_{i,k}\}$ *of* (13).

Uniqueness: if the scheme is explicit in the time steps the uniqueness of solution, as well as the existence, of the scheme is straightforward.

$$\begin{cases} m_{i,k+1} = & G(m_k, i, k) \\ \quad\quad\;\;\, = & \frac{1}{2}\sum_{j\in\mathbb{Z}}\Big(\beta_i(\Phi_j^+[m_k]) + \beta_i(\Phi_j^-[m_k])\Big)\, m_{j,k} \\ m_{i,0} = & \int_{E_i} m_0(x)\mathrm{d}x \end{cases}$$

In the general case, the scheme is implicit in the time steps and the uniqueness of solutions is generally not true and its fulfilment depends on the continuous problem.

Dual Problem and Dual Scheme in the linear case

Kolmogorov forward equation (FP)

$$\begin{cases} \partial_t m = \frac{1}{2}\sum_{i,j}\partial_{x_i x_j}\left(a_{ij}(x)m\right) - \mathrm{div}\left(b(x)m\right) & \mathbb{R}^d \times (0,T] \\ m(\cdot,0) = m_0 \end{cases}$$

Kolmogorov backward equation(KB):

$$\begin{cases} -\partial_t u = \frac{1}{2}\sum_{i,j} a_{ij}(x)\partial_{x_i x_j} u + b(x)^\top \nabla u & \mathbb{R}^d \times (0,T] \\ u(\cdot,T) = u_T \end{cases} \tag{14}$$

Dual Problems

Kolmogorov forward equation (FP)

$$\begin{cases} \partial_t m = \boxed{\frac{1}{2}\sum_{i,j}\partial_{x_i x_j}\left(a_{ij}(x)m\right) - \mathrm{div}\left(b(x)m\right) = L^*(m)} & \mathbb{R}^d \times (0,T] \\ m(\cdot,0) = m_0 \end{cases}$$

Kolmogorov backward equation (KB):

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L^* is the dual of L with respect to the L_2 inner product:

$$\int L(f) g \mathrm{d}x = \int L^*(g) f \mathrm{d}x$$

Dual Schemes

The SL scheme for FP can be written in vectorial form as

$$\boxed{\mu_{k+1} := B^* \mu_k}$$

where, $\mu_k = (\mu_{j,k})_k$ and $(B^*)_{i,j} = \frac{1}{2}\left(\beta_i\left(\Phi_{j,+}\right) + \beta_i\left(\Phi_{j,-}\right)\right)$.
The SL scheme for KB

$$v_{i,k} = \frac{1}{2}\left(I[v_{k+1}](\Phi_{i,+}) + I[v_{k+1}](\Phi_{i,-})\right) = \frac{1}{2}\sum_{j\in\mathbb{Z}}\left[\beta_j\left(\Phi_{i,+}\right) + \beta_j\left(\Phi_{i,-}\right)\right] v_{j,k+1}$$

can also be written in vectorial form as

$$\boxed{v_k := B v_{k+1}},$$

where, $v_k = (v_{j,k})_k$

Dual Schemes

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can also be written in vectorial form as

$$\boxed{v_k := B v_{k+1}},$$

where, $v_k = (v_{j,k})_k$ and $\boxed{B^\top = B^*}$, i.e.

$$(Bv^{k+1}, \mu^k) = (v^{k+1}, B^* \mu^k)$$

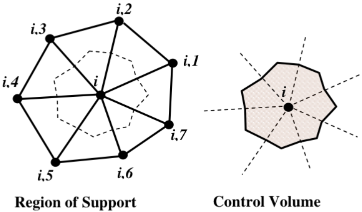
Unstructured grids

Let $\mathcal{T}$ be a triangulation of Ω, with Δx the maximum diameter of the triangles. We call $\{\beta_i \; ; \; i = 1, ..., M\}$ the set of affine shape functions related to the triangular mesh, such that $\beta_i(x_j) = \delta_{i,j}$ and $\sum_i \beta_i(x) = 1$ for each $x \in \overline{\Omega}$.

Median dual control volume

$$E_i := \bigcup_{T \in \mathcal{T} : x_i \in \partial T} E_{i,T},$$

$$E_{i,T} := \{x \in T : \beta_j(x) \leq \beta_i(x) \quad j \neq i\}.$$



We approximate the solution m by a sequence $\{m_k\}_k$, such that

$$m_{i,k} \simeq \int_{E_i} \mathrm{d}m(t_k)$$

Convergence Analysis

Topology

Let us denote

$$m^{\Delta x,\Delta t}(t) := \left(\frac{t-t_k}{\Delta t}\right) m_{k+1} + \left(\frac{t_{k+1}-t}{\Delta t}\right) m_k \text{ si } t \in [t_k, t_{k+1}[.$$

a proper extension in $\mathbb{R}^d \times [0,T]$ of the discrete approximation $m_{i,k}$ and $\mathcal{P}_1$ a propability space, metrized by the Kantorowich-Rubinstein distance:

$$d_1(\mu_1,\mu_2) := \sup\left\{\int_{\mathbb{R}^d} f(x)\mathrm{d}(\mu_1(x)-\mu_2(x))\ ;\ \|\nabla f\|_\infty \leq 1\right\}.$$

Stability

Proposition

Suppose b *and* σ *continuous and s.t. there exists* $C > 0$

$$|b(x,t,\mu)| \le C|x| \quad |\sigma(x,t,\mu)| \le C|x|,$$

for any x, t, μ *and* $\Delta t \ge \bar{C}(\Delta x)^2$, $\bar{C} \ge 0$. *Then,* $\exists C > 0$ *(independent of* $(\Delta x, \Delta t)$*)* *s. t.:*

$$\int_{\mathbb{R}^d} |x|^2 \mathrm{d}m_{\Delta x,\Delta t}(t) \le C \quad \forall\, t \in [0,T]. \tag{15}$$

Proposition

Suppose that $\frac{(\Delta x)^2}{\Delta t} \to 0$. *Then, there exists a constant* $C > 0$ *(independent of* $(\Delta x, \Delta t)$*) such that:*

$$\begin{aligned} d_1(m_{\Delta x,\Delta t}(t), m_{\Delta x,\Delta t}(s)) \quad &\le C|t-s|^{\frac{1}{2}}\ \forall\, t,s \in [0,T] \\ &(\le C|t-s|,\ \sigma = 0). \end{aligned}$$

Stability and Consistency

Proposition (Weak consistency)

Assuming that b *and* σ *are continuous and Lipschitz w.r.t.* x, *for every* $\phi \in C_0^\infty(\mathbb{R})$ *if* $(\Delta x_n, \Delta t_n)$ *is s.t. as* $n \to \infty$ $(\Delta x_n, \Delta t_n) \to 0$ *and* $\frac{(\Delta x_n)^2}{\Delta t_n} \to 0$, *and*

$$m_{\Delta x_n, \Delta t_n} \to m \text{ in } \quad C(0,T;\mathcal{P}_1(\mathbb{R}^d)),$$

then for k^n *such that* $t_{k^n} \to t$

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^d} \phi(x) \left[dm_{\Delta x_n,\Delta t_n}(t_{k^n+1}) - dm_{\Delta x_n,\Delta t_n}(0)\right] \\
&= \int_{\mathbb{R}} \int_0^t \left[\tfrac{1}{2}\mathrm{tr}(A[m](x,s)D^2\phi) + b[m](x,s)^\top \nabla\phi\right] \mathrm{d}m(s).
\end{aligned}$$

Convergence

Theorem

Suppose that $\frac{(\Delta x_n)^2}{\Delta t_n} \to 0$, b *and* σ *continuous functions Lipschtiz w.r. to* x *and there exists* $C > 0$ *such that for any* x, t, μ

$$|b(x,t,\mu)| \leq C|x| \quad |\sigma(x,t,\mu)| \leq C|x|,$$

then

$$m_{\Delta x, \Delta t} \to m$$

in $C([0,T], \mathcal{P}_1)$*, where* m *is solution of* (1) *and* $m_{\Delta x, \Delta t}$ *is a sequence of solution of* (13) *(extended to a general dimension* d*).*

Idea of the proof

- By stability properties , if $(\Delta x_n, \Delta t_n) \to (0,0)$ and $\frac{(\Delta x_n)^2}{\Delta t_n} \to 0$, there exists $m \in C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ s.t.

$$m_n := m^{\Delta x_n, \Delta t_n} \to m.$$

- To prove that m is a solution of the Fokker-Planck, given t and $t_{k(n)}$ s.t. $t \in [t_{k(n)}, t_{k(n)+1}[$ we consider

$$\begin{aligned}&\textstyle\int_{\mathbb{R}^d} \phi(x) \mathrm{d} m_n(t_{k(n)})(x) \\ &= \textstyle\int_{\mathbb{R}^d} \phi(x) \mathrm{d} m_0(x) + \sum_{k=0}^{k(n)-1} \int_{\mathbb{R}^d} \phi(x) \mathrm{d}\left(m_n(t_{k+1}) - m_n(t_k)\right).\end{aligned}$$

By consistency

$$= \int_{\mathbb{R}^d} \phi(x) \mathrm{d} m_n(0)(x) + \int_0^{t_{k(n)}} \int_{\mathbb{R}^d} L_\phi^n[m_n] \mathrm{d} m_n(s)(x) \mathrm{d}s + O\left(\frac{(\Delta x_n)^2}{\Delta t_n} + \sqrt{\Delta t_n}\right)$$

where, if $s \in [t_k, t_{k+1}[$,

$$L_\phi^n[m_n](x,s) := \tfrac{1}{2} \sum_{i,j} a_{i,j}[m_n](x,t_k) \partial_{ij}^2 \phi(x) + b[m_n](x,t_k) \cdot \nabla \phi(x).$$

- If $(\Delta x_n)^2/\Delta t_n \to 0$ for all $s \in [0,T]$ we have

$$L_\phi^n[m_n](\cdot,s) \to \frac{1}{2}\sum_{i,j} a_{i,j}[m](\cdot,s)\partial_{ij}^2\phi(x) + b[m](\cdot,s)\cdot\nabla\phi(x)$$

uniformly on compact of $\mathbb{R}^d$, then we can pass to the limit and we get that m solves the Fokker-Planck equation.

Remark:

- if b et σ are not smooth, it is necessary to regularize them, by using mollifiers. This situation arise naturally when b (and or σ) is not explicitly given and we have to approximate it.

  We will apply this technique to approximate the solution of Mean Field Game Problem.

- We can not apply this convergence analysis to local non linearity, since we have only weak convergence of the measure $m_n \rightharpoonup m$

Numerical simulations

Linear case: Lotka Volterra model with seasonality

A two-species prey-predator system

$$\begin{aligned} \mathrm{d}u &= (-u + uv)\mathrm{d}t \\ \mathrm{d}v &= ((1 + \lambda \sin(t))v - uv - \gamma v^2)\mathrm{d}t, \end{aligned}$$

- $u \simeq$ predator which death and growth rate is equal to 1.
- $v \simeq$ prey, which growth has periodic variations $1 + \lambda \sin(t)$ due for instance to seasons, they are also subject to self-limitations, represented by the term v^2 with rate γ.

The sinusoidal term makes the joint probability density function periodically non stationary

Remark: the coefficient of the o.d.e. does not verifies the assumption of the convergence theorem, however the scheme seems working well.

Ref. A. A. King and W. M. Schaffer. The rainbow bridge: Hamiltonian limits and resonance in predator-prey dynamics. J. Math. Biol., 39:439?469, 1996.

Lotka Volterra model with seasonality

The model can be simplified by the logarithmic transformation $x_1 = \ln u, x_2 = \ln v$ into

$$\begin{aligned} \mathrm{d}x_1 &= (-1 + \exp(x_2))\mathrm{d}t \\ \mathrm{d}x_2 &= (1 + \lambda \sin(t)) - \exp(x_1) - \gamma \exp(x_2))\mathrm{d}t. \end{aligned} \tag{16}$$

To allow very large time step and maintain accuracy, we modify our scheme: by defining a second time step δ, such that $\Delta t = P\delta$, with $P \in \mathbb{N}^+$.

This new time step is used to compute the discrete flow on each time interval of size Δt, in the following way:

$$\Phi_{j,k}^P := z_k^P(x_j)$$

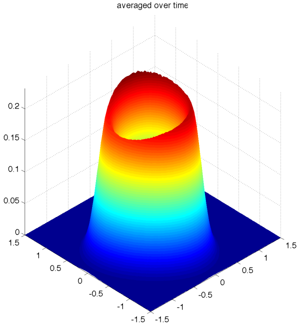
where

$$z_k^{p+1}(x_j) = z_k^p(x_j) + \delta b(x_j, t_k + p\delta), \quad z_k^0 = x_j$$

and $p = 0, \ldots, P-1$.

Lotka Volterra model with seasonality

We show the time averaged of the density on $t \in [100, 150]$ computed with $\Delta x = 0.03, \Delta t = 4\Delta x, P = 16$.



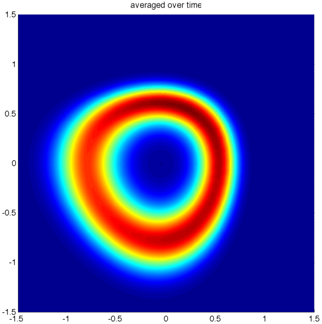


Figure: Time averaged solution of the distribution of (16) with $\Delta x = 0.03, \Delta t = 4\Delta x, P = 16$, using the parameters $\lambda = 0.1$ and $\gamma = 0.05$

Interacting species

Let m_1 and m_2 be the densities of two interacting species

$$\begin{cases} \partial_t m_1 - \mathrm{div}(m_1(\nabla E'(m_1) + \nabla U_1[m_1, m_2])) = 0 \\ \partial_t m_2 - \mathrm{div}(m_2(\nabla E'(m_2) + \nabla U_2[m_1, m_2])) = 0 \\ \text{b.c.} \end{cases}$$

Internal energy

$$E(m) = \frac{1}{2} m^3 \quad \text{and} \quad \mathrm{div}(m(\nabla E'(m)) = \Delta m^3$$

We need to regularize the non linear diffusion term:

$$b_1[m] := \nabla E'(m_1) \star K_\varepsilon = \frac{3}{2} \nabla m^2 \star K_\varepsilon$$

where $K_\varepsilon = C\varepsilon\sqrt{2\pi}\exp(-|x|^2)/(2\varepsilon^2)$ is a regularizing gaussian kernel with $\varepsilon = 0.01$ and $C > 0$ such that $\max K_\varepsilon = 1$

Ref. J.D.Benamou, G. Carlier, M.Laborde

Interacting species

Cross interaction potential

$$U_1[m_1, m_2] = W_{11} \star m_1 + W_{21} \star m_2$$
$$U_2[m_1, m_2] = W_{12} \star m_1 + W_{22} \star m_2$$

where $W_{11} = W_{21} = W_{22} = \frac{|x|^2}{2}$ and $W_{12} = -\frac{|x|^2}{2}$.
Then

- $-\nabla U_1 = -\nabla W_{11} \star m_1 - \nabla W_{21} \star m_2 = -x \star (m_1 + m_2)$
  (self-attraction + attraction toward m_2)
- $-\nabla U_2 = -\nabla W_{12} \star m_1 - \nabla W_{22} \star m_2 = -x \star (-m_1 + m_2)$
  (repulsion from m_1+self-attraction)

Interacting species

Cross interaction potential

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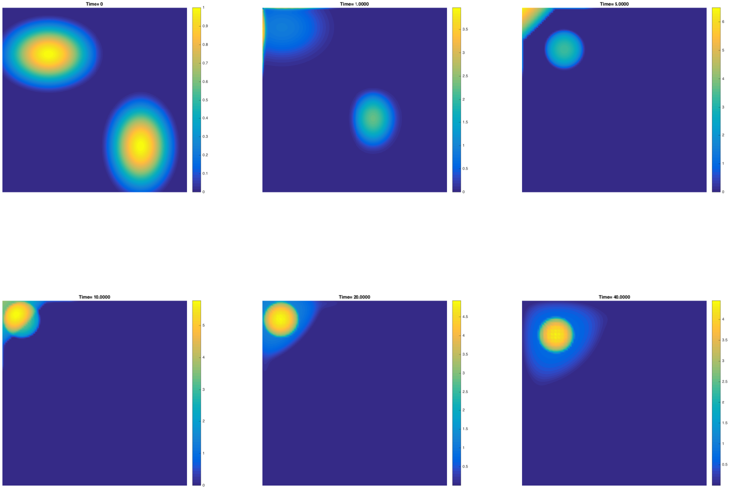
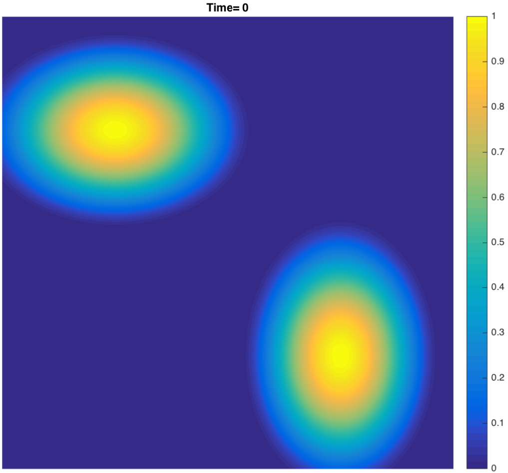
  (repulsion from m_1+self-attraction)

We solve the system on $\Omega = [-1, 1] \times [-1, 1]$ and $[0, T] = [0, 40]$, we set $h = 0.02$ and $\Delta t = h/6$, Homogenous Neumann Condition

Interacting species



Interacting species



A degenerate second order Mean Field Game system

Model introduced independently by Huang-Malhamé-Caines and, independently, by Lasry-Lions in 2006.

$$\begin{cases} -\partial_t v - \frac{1}{2}\sigma^2(t)\Delta v + \frac{1}{2}|\nabla v|^2 = F(x, m(t)), \text{ in } \mathbb{R}\times(0,T), \\ \partial_t m - \frac{1}{2}\sigma^2(t)\Delta m - \mathrm{div}\big(\nabla v m\big) = 0, \quad \text{in } \mathbb{R}\times(0,T), \\ v(x,T) = G(x, m(T)) \quad \text{for } x\in\mathbb{R}, \;\; m(0) = m_0. \end{cases} \tag{17}$$

In this case the velocity field in the FP is

$$b[m](x,t) = \nabla v[m](x,t)$$

where $v[m]$ is the solution of the first equation in the system.
Since $b[m](x,t)$ depends on m in all time $t \leq s \leq T$ we gets an Implicit scheme: Fixed Point Iterations needed to solve it.

Convergence result

We approximate $\nabla v[m](x, t_k)$ by $\nabla v^{\varepsilon}_{\Delta x, \Delta t}[m](x, t_k)$
where $v^{\varepsilon}_{\Delta x, \Delta t}(\cdot, t_k)$ is a regularization of the linear interpolation $v_{\cdot,k}$, which solve a SL scheme for HJB, then

$$b[m](x_i, t_k) \sim -\nabla v^{\varepsilon}_{\Delta x, \Delta t}[m](x_i, t_k).$$

- If $\sigma = 0$ then $\nabla v^{\varepsilon_n}_{\Delta x_n, \Delta t_n}[m_n] \to \nabla v[m]$ a.e.
- If $\sigma \neq 0$ then $\nabla v^{\varepsilon_n}_{\Delta x_n, \Delta t_n}[m_n] \to \nabla v[m]$ uniformly

- The first property has been used in Carlini-Silva'14 (resp. Carlini-Silva'15) to prove the convergence when $\sigma = 0$ and $d = 1$ (resp. $\sigma = \sigma(t)$ and $d = 1$).

- The second property allow us to prove convergence in general dimension for the case $\sigma \neq 0$.

Numerical test First order MFG

Domain $\Omega \times (0,T) = (-3,3) \times (0,5)$.
Running cost

$$F(x,t,m(t)) = d(x,\mathcal{D})^2 V_\delta(x,m(t)),$$

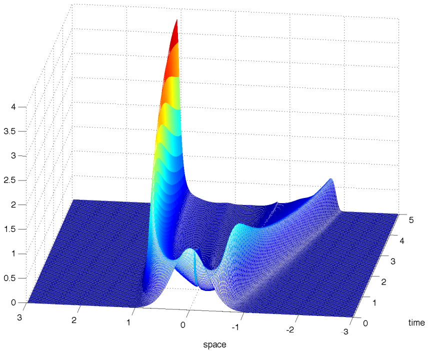
$V_\delta(x,m) = (\phi_\delta \star (\phi_\delta \star m))(x)$, $\phi_\delta(x) := \frac{1}{\delta\sqrt{2\pi}} \exp\left(-x^2/(2\delta^2)\right)$.
$d(x,\mathcal{D})$ is the distance function from the set $\mathcal{D} := [-2,-2.5] \cup [1,1.5]$.
Final cost: $G(x,T,m(T)) = F(x,T,m(T))$
Initial mass distribution:

$$m_0(x) = \frac{\nu(x)}{\int_\Omega \nu(x)dx} \quad \text{with} \quad \nu(x) = e^{-x^2/0.2}$$

Regularizing kernel $\phi_\varepsilon(x)$, with $\varepsilon = 0.15$.
Diffusion term $\sigma = 0$**,** first order MFG system
Discretization step $\Delta x = \Delta t = 0.02$ and $\delta = 0.01$.

Fix point: computed by a learning procedure as proposed by Cardaliaguet and Hadikhanloo.

Numerical test First order MFG



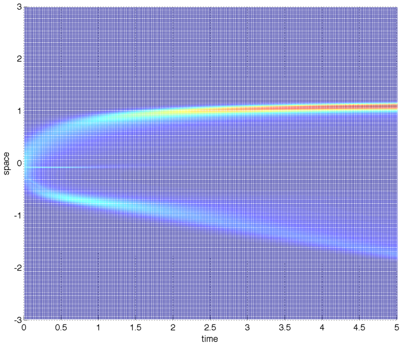


Figure: **Density evolution 3d and 2d view in the (x, t) domain**

Numerical test First order MFG

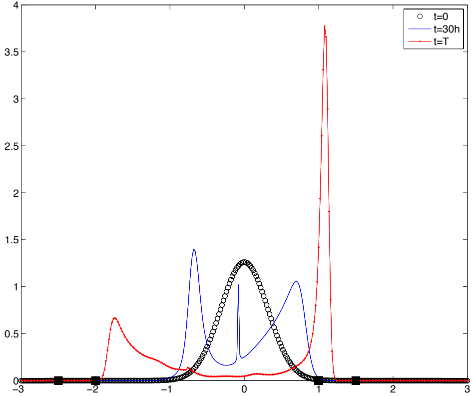


Figure: **Density at time** $t = 0, 0.6, T$ (black squares on the x axis represents the 'meeting areas')

A new Hughes type model

In this model the velocity field in the FP is

$$b[m](x,t) = \nabla v[m](x,t)$$

where $v[m]$ is the solution of the first equation in the system.

$$\begin{cases} -\partial_s v(x,s) + \frac{1}{2}|\nabla v(x,s)|^2 = F(x,s,m(t))) & \text{in } \mathbb{R} \times (t,T), \\ \partial_t m - \text{div}(\nabla v m) = 0 & \text{in } \mathbb{R} \times (0,T), \\ v(x,T) = G(x,m(t)) & \text{for } x \in \mathbb{R}, \\ m(\cdot,0) = m_0(\cdot) \end{cases} \tag{18}$$

where

$$F(x,s,m(t)) = d(x,\mathcal{P})^2 V_\delta(x,m(t)).$$

Since $b[m](x,t)$ depends on $m(s)$ only in time past $0 \le s \le t$ we get an Explicit scheme

Numerical test new Hughes type model

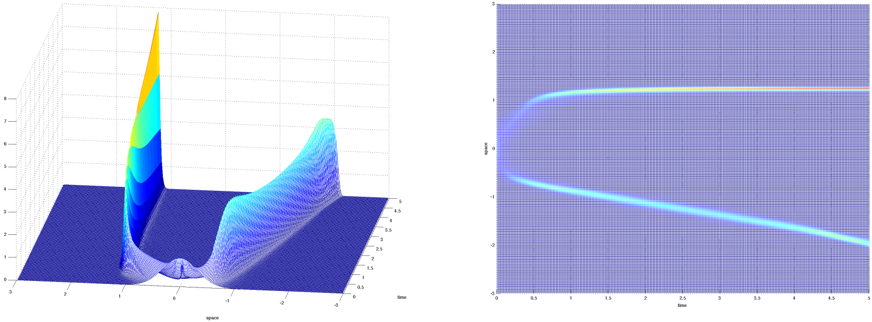


Figure: **Density evolution 3d and 2d view in the (x, t) domain**

Numerical test new Hughes type mode

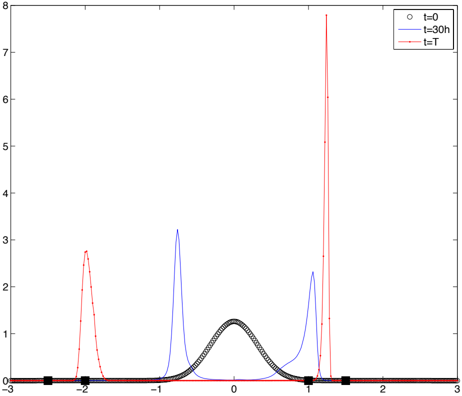


Figure: **Density at time** $t = 0, 30h, T$ (black squares on the x axis represents the 'meeting areas')

Numerical test new Hughes type mode

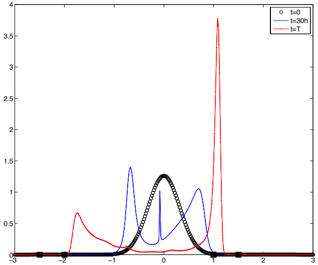


Figure: **MFG(left) vs Hughes type model (right)**

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- E.Carlini, A. Festa, F.J. Silva, M.T. Wolfram, *A Semi-Lagrangian scheme for a modified version of the Hughes model for pedestrian flow*, Dynamic Games and Applications, Springer, 2016.

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THANK YOU