Mean field games with congestion: weak solutions

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Outline

MFG with Congestion

The dynamics of a representative agent is

$$dX_t = \sqrt{2\nu} dW_t + \gamma_t dt$$

where

(W_t) is a d-dimensional Brownian motion

(γ_t) is the control of the agent.

1. **Individual optimal control problem:** the representative agent minimizes

$$\mathbb{E}_{t,x}\left(\int_t^T \mathcal{L}(X_s, \gamma_s; m_s)ds + G(X_T; m_T)\right),$$

where m_s is the distribution of states (a single agent is assumed to have no influence on m_s).
Dynamic programming yields an optimal feedback γ_t^* and an optimal trajectory X_t^*.

2. **Nash equilibria:**

$$m_t = \text{law of } X_t^*.$$

Congestion

- The cost of motion at x depends on $m(x)$ in an increasing manner.
- A typical example was introduced by P-L. Lions (lectures at Collège de France):

$$\mathcal{L}(x,\gamma;m) \sim (\mu+m(x))^{\sigma}|\gamma|^{q'} + F(x,m(x))$$

where $\mu \geq 0$, $\sigma > 0$ and $q' > 1$.

The corresponding Hamiltonian is of the form

$$\mathcal{H}(x,p;m) = \frac{|p|^q}{(\mu+m(x))^{\alpha}} - F(x,m(x)),$$

with $\alpha = \sigma(q-1)$.

Remarks

- Degeneracy of the Hamiltonian H as $m \to +\infty$
- This model is named “Soft Congestion” by Santambrogio and his coauthors. Their “Hard Congestion” models include inequality contraints on m: $m \leq \bar{m}$

The system of PDEs and the main assumptions

$$\begin{cases} -\partial_t u - \nu\Delta u + \frac{1}{q}\frac{|Du|^q}{(m+\mu)^\alpha} = F(m)\,, & (t,x)\in(0,T)\times\Omega \\ \partial_t m - \nu\Delta m - \operatorname{div}\left(m\frac{|Du|^{q-2}Du}{(m+\mu)^\alpha}\right) = 0\,, & (t,x)\in(0,T)\times\Omega \\ m(0,x) = m_0(x)\,,\ u(T,x) = G(x,m(T))\,, & x\in\Omega. \end{cases} \tag{1}$$

The system of PDEs and the main assumptions

$$
\begin{cases}
-\partial_t u - \nu \Delta u + \frac{1}{q} \frac{|Du|^q}{(m+\mu)^\alpha} = F(m)\,, & (t,x) \in (0,T) \times \Omega \\
\partial_t m - \nu \Delta m - \operatorname{div}\left(m \frac{|Du|^{q-2} Du}{(m+\mu)^\alpha} \right) = 0\,, & (t,x) \in (0,T) \times \Omega \\
m(0,x) = m_0(x)\,,\ u(T,x) = G(x, m(T))\,, & x \in \Omega.
\end{cases} \tag{1}
$$

Main assumptions

For simplicity, $\Omega = \mathbb{R}^d/\mathbb{Z}^d$: no difficulty from boundary conditions

The system of PDEs and the main assumptions

$$\begin{cases} -\partial_t u - \nu \Delta u + \frac{1}{q} \frac{|Du|^q}{(m+\mu)^\alpha} = F(m)\,, & (t,x) \in (0,T)\times\Omega \\ \partial_t m - \nu \Delta m - \operatorname{div}\left(m \frac{|Du|^{q-2} Du}{(m+\mu)^\alpha} \right) = 0\,, & (t,x) \in (0,T)\times\Omega \\ m(0,x) = m_0(x)\,,\ u(T,x) = G(x, m(T))\,, & x \in \Omega. \end{cases} \quad (1)$$

Main assumptions

F and G are bounded from below.

$\exists \lambda > 0, \kappa \geq 0$, and a nondecreasing function f such that $s \mapsto f(s)s$ is convex s.t.

$$\lambda\, f(m) - \kappa \leq F(t,x,m) \leq \frac{1}{\lambda} f(m) + \kappa, \qquad \forall m \geq 0.$$

Remark: no restriction on the growth

Same kind of assumption for G.

The system of PDEs and the main assumptions

$$\begin{cases} -\partial_t u - \nu\Delta u + \frac{1}{q}\frac{|Du|^q}{(m+\mu)^\alpha} = F(m)\,, & (t,x)\in(0,T)\times\Omega \\ \partial_t m - \nu\Delta m - \operatorname{div}\left(m\frac{|Du|^{q-2}Du}{(m+\mu)^\alpha}\right) = 0\,, & (t,x)\in(0,T)\times\Omega \\ m(0,x) = m_0(x)\,,\ u(T,x) = G(x,m(T))\,, & x\in\Omega. \end{cases} \quad (1)$$

Main assumptions

$$m_0 \in C(\Omega) \quad \text{and} \quad m_0 \geq 0.$$

The system of PDEs and the main assumptions

$$\begin{cases} -\partial_t u - \nu \Delta u + \frac{1}{q} \frac{|Du|^q}{(m+\mu)^\alpha} = F(m)\,, & (t,x) \in (0,T) \times \Omega \\ \partial_t m - \nu \Delta m - \operatorname{div}\left(m \frac{|Du|^{q-2} Du}{(m+\mu)^\alpha} \right) = 0\,, & (t,x) \in (0,T) \times \Omega \\ m(0,x) = m_0(x)\,,\ u(T,x) = G(x, m(T))\,, & x \in \Omega. \end{cases} \tag{1}$$

Main assumptions

- $$1 < q \leq 2$$
- Either $\mu > 0$ (non singular case) or $\mu = 0$ (singular case)
- $$0 < \alpha \leq 4 \frac{q-1}{q} = \frac{4}{q'}$$

The condition $\alpha \le 4(q-1)/q$

General MFG with local coupling: for systems of the form

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x,p,m) = F(m) & (t,x)\in(0,T)\times\Omega, \\ \partial_t m - \nu\Delta m - \operatorname{div}(mH_p(x,p,m)) = 0\,, & (t,x)\in(0,T)\times\Omega, \\ m(0,x) = m_0(x)\,,\ u(T,x) = G(m(T,x))\,, & x\in\Omega\,, \end{cases}$$

P-L. Lions proved that a sufficient condition for the uniqueness of classical solutions is that F and G be non decreasing and that

$$\begin{pmatrix} -H_m\,(x,p,m) & \frac{1}{2}mH_{m,p}^T(x,p,m) \\ \frac{1}{2}mH_{m,p}(x,p,m) & mH_{p,p}(x,p,m) \end{pmatrix} > 0,$$

for all $x\in\Omega$, $m>0$ and $p\in\mathbb{R}^d$.

In the present congestion model, this condition is equivalent to $\alpha \le 4\frac{q-1}{q}$.

Some references

- P-L. Lions [$\sim$ 2011]: lectures at Collège de France. In particular, the condition for uniqueness of classical solutions.
- Gomes-Mitake [2015]: existence of classical solutions in a specific stationary case: purely quadratic Hamiltonian, i.e. $H(x,p,m) = \frac{|p|^2}{m^\alpha}$, with a very special trick
- Gomes-Voskanyan[2015] and Graber[2015]: short-time existence results of classical solutions for evolutive MFG with congestion

In general, for the existence of classical solutions, restrictive assumptions (e.g. on the growth of F and G) are needed.

In particular, if $H(x,p,m) = \frac{|p|^q}{m^\alpha}$, one needs to prove that m does not vanish.

It seems more feasible to work with weak solutions.

Weak solutions

- Weak solutions of the MFG systems were introduced by Lasry and Lions in 2007
- For Hamiltonians with separate dependencies: $\mathcal{H}(x, p, m) = H(x, p) - F(m)$, Porretta, [ARMA 2015], showed that weak solutions allow to build a very general well-posed setting
- Allow to prove general convergence results for numerical schemes [A.-Porretta 2016]
- If the MFG system of PDEs can be rephrased as the optimality conditions of an optimal control problem driven by some PDE, then
  - weak solutions are the minima of a relaxed functional
  - variational methods can be used

  This occurs often when the Hamiltonian depends separately on p and m.
- Can be used for degenerate diffusion [Cardaliaguet-Graber-Porretta-Tonon 2015]
- The variational approach leads to robust (but often slow) numerical methods [Benamou-Carlier], [A.-Laurière]
- Difficulty with the present congestion model: it is not possible to use a variational approach.

Outline

The main result

Consider the model problem:

$$\begin{cases} -\partial_t u - \nu \Delta u + \frac{1}{q}\frac{|Du|^q}{(m+\mu)^\alpha} = F(m)\,, & (t,x) \in (0,T)\times\Omega \\ \partial_t m - \nu\Delta m - \operatorname{div}\left(m\frac{|Du|^{q-2}Du}{(m+\mu)^\alpha}\right) = 0\,, & (t,x)\in(0,T)\times\Omega \\ m(0,x) = m_0(x)\,,\ u(T,x) = G(x,m(T,x))\,, & x\in\Omega. \end{cases}$$

Definition

A weak solution (u,m) is a distributional solution of the system such that

$$\begin{array}{ll} mF(m) \in L^1, & m_T G(m_T) \in L^1(\Omega), \\ m\frac{|Du|^q}{(\mu+m)^\alpha} \in L^1, & \frac{|Du|^q}{(\mu+m)^\alpha} \in L^1. \end{array}$$

Theorem

Under the previous assumptions and if F and G are non decreasing, then there exists a unique weak solution.

Extension

Existence and uniqueness of weak solutions holds for

$$\begin{cases} -\partial_t u - \nu\Delta u + H(t,x,p,m) = F(m) & (t,x) \in (0,T)\times\Omega \\ \partial_t m - \nu\Delta m - \mathrm{div}(m H_p(t,x,p,m)) = 0\,, & (t,x) \in (0,T)\times\Omega \\ m(0,x) = m_0(x)\,,\ u(T,x) = G(m(T,x))\,, & x\in\Omega\,, \end{cases}$$

under the structure conditions

$$\begin{aligned} H(t,x,0,m) &\le 0\,, \\ H(t,x,p,m) &\ge c_0\,\frac{|p|^q}{(m+\mu)^\alpha} - c_1\,(1+m^{\frac{\alpha}{q-1}})\,, \\ |H_p(t,x,p,m)| &\le c_2\,(1+\frac{|p|^{q-1}}{(m+\mu)^\alpha})\,, \\ H_p(t,x,p,m)\cdot p &\ge (1+\sigma)\,H(t,x,p,m) - c_3\,(1+m^{\frac{\alpha}{q-1}})\,, \end{aligned}$$

for a.e. $(t,x)\in Q_T$ and every $p\in\mathbb{R}^N$, where $\sigma, c_0,\ldots,c_3$ are positive constants,

and the same assumptions on F, G α and q.

Main arguments in the proof

A regularized problem and energy estimates

$$
\begin{aligned}
-\partial_t u^\epsilon - \nu \Delta u^\epsilon + H(t,x,T_{1/\epsilon} m^\epsilon, Du^\epsilon) &= F^\epsilon(t,x,m^\epsilon)\,, && (t,x)\in(0,T)\times\Omega\\
\partial_t m^\epsilon - \nu\Delta m^\epsilon - \mathrm{div}(m^\epsilon H_p(t,x,T_{1/\epsilon} m^\epsilon, Du^\epsilon)) &= 0\,, && (t,x)\in(0,T)\times\Omega\\
m^\epsilon(0,x) = m_0^\epsilon(x)\,,\ u^\epsilon(T,x) &= G^\epsilon(x,m^\epsilon(T))\,, && x\in\Omega
\end{aligned}
$$

where

$$
\begin{aligned}
T_{1/\epsilon} m &= \min(m, 1/\epsilon),\\
F^\epsilon(t,x,m) &= \rho^\epsilon \star F(t,\cdot,\rho^\epsilon \star m))(x),\\
G^\epsilon(x,m) &= \rho^\epsilon \star G(\cdot,\rho^\epsilon \star m))(x),\\
m_0^\epsilon &= \rho^\epsilon \star m_0,
\end{aligned}
$$

and ρ^ϵ is a standard symmetric mollifier in $\mathbb{R}^d$.

A regularized problem and energy estimates

$$\begin{aligned} -\partial_t u^\epsilon - \nu\Delta u^\epsilon + H(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon) &= F^\epsilon(t,x,m^\epsilon)\,, && (t,x)\in(0,T)\times\Omega \\ \partial_t m^\epsilon - \nu\Delta m^\epsilon - \mathrm{div}(m^\epsilon H_p(t,x,T_{1/\epsilon}m^\epsilon, Du^\epsilon)) &= 0\,, && (t,x)\in(0,T)\times\Omega \\ m^\epsilon(0,x) = m_0^\epsilon(x)\,,\ u^\epsilon(T,x) &= G^\epsilon(x,m^\epsilon(T))\,, && x\in\Omega \end{aligned}$$

where

$$\begin{aligned} T_{1/\epsilon}m &= \min(m,1/\epsilon), \\ F^\epsilon(t,x,m) &= \rho^\epsilon\star F(t,\cdot,\rho^\epsilon\star m))(x), \\ G^\epsilon(x,m) &= \rho^\epsilon\star G(\cdot,\rho^\epsilon\star m))(x), \\ m_0^\epsilon &= \rho^\epsilon\star m_0, \end{aligned}$$

and ρ^ϵ is a standard symmetric mollifier in $\mathbb{R}^d$.

Standard energy estimates:

$$u^\epsilon(t,x)\geq C, \qquad \|u^\epsilon\|_{L^\infty(0,T;L^1(\Omega))}\leq C,$$

$$\int_\Omega G^\epsilon(x,m^\epsilon(T))m^\epsilon(T)dx + \int_0^T\int_\Omega F^\epsilon(t,x,m^\epsilon)m^\epsilon\,dxdt + \|(T_{1/\epsilon}m^\epsilon)^{\frac{\alpha}{q-1}+1}\|_{\frac{N+2}{N}} \leq C,$$

$$\int_0^T\int_\Omega \frac{|Du^\epsilon|^q}{(T_{1/\epsilon}m^\epsilon+\mu)^\alpha}\,dxdt + \int_0^T\int_\Omega m^\epsilon\,\frac{|Du^\epsilon|^q}{(T_{1/\epsilon}m^\epsilon+\mu)^\alpha}\,dxdt \leq C$$

Properties of Fokker-Planck equations with L^2 drifts. [Porretta, ARMA 2015]

Set $Q_T = (0,T) \times \Omega$. For any $b \in L^2(Q_T;\mathbb{R}^d)$, consider the Fokker-Planck equation

$$\begin{cases} \partial_t m - \nu \Delta m - \mathrm{div}(mb) = 0 & \text{in } (0,T)\times\Omega, \\ m(t=0) = m_0. \end{cases} \tag{2}$$

Properties of Fokker-Planck equations with L^2 drifts. [Porretta, ARMA 2015]

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$$\begin{cases} \partial_t m - \nu\Delta m - \operatorname{div}(mb) = 0 & \text{in } (0,T)\times\Omega, \\ m(t=0) = m_0. \end{cases} \tag{2}$$

- A weak solution is a nonnegative distributional sol. $m \in L^1(Q_T)$ of (2) s.t.

$$m|b|^2 \in L^1(Q_T). \tag{3}$$

- A renormalized solution of (2) is a nonnegative function $m \in L^1(Q_T)$ s.t.
  - for any $k>0$, $T_k(m) \in L^2(0,T,H^1(\Omega))$,
    where $(T_k(m)) = \max(-k, \min(k,m)))$
  - $\lim_{n\to+\infty} \frac{1}{n}\int_{\{n<m(t,x)<2n\}} |Dm|^2 dxdt = 0$
  - for all $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support,

$$\begin{cases} \partial_t S(m) - \nu\Delta S(m) - \operatorname{div}(mbS'(m)) + \nu S''(m)|Dm|^2 + S''(m)mb\cdot Dm = 0, \\ S(m(t=0)) = S(m_0). \end{cases}$$

Properties of Fokker-Planck equations with L^2 drifts. [Porretta, ARMA 2015]

Set $Q_T = (0,T) \times \Omega$. For any $b \in L^2(Q_T;\mathbb{R}^d)$, consider the Fokker-Planck equation

$$\begin{cases} \partial_t m - \nu \Delta m - \operatorname{div}(mb) = 0 & \text{in } (0,T)\times\Omega, \\ m(t=0) = m_0. \end{cases} \tag{2}$$

1. **Uniqueness:** there exists at most one weak solution of (2)
2. **Weak sol. $\Leftrightarrow$ renormalized sol. and $m|b|^2 \in L^1(Q_T)$:** any weak solution m belongs to $C([0,T];L^1(\Omega))$ and is a renormalized solution
3. **Compactness:** if (b, m_0) lies in a bounded subset of $L^2(Q_T) \times L^1(\Omega)$, then m lies in a relatively compact subset of $L^1(Q_T)$
4. **Stability:** consider a sequence m^ϵ of weak solutions of the F.P equation associated to $b^\epsilon \in L^2(Q_T;\mathbb{R}^d)$.
   If $m^\epsilon \to m$ a.e. in Q_T and if $m^\epsilon|b^\epsilon|^2 \to m|b|^2$ in $L^1(Q_T)$,
   then $m^\epsilon \to m$ in $C([0,T];L^1(\Omega))$ and m is a weak solution of the F.P. equation associated to b.

Passage to the limit if $1 < q < 2$: main steps

- Energy estimates $\Rightarrow -\partial_t u^\epsilon - \nu \Delta u^\epsilon$ is bounded in $L^1(Q_T)$: $\Rightarrow$ for subsequences, $u^\epsilon \to u$ and $Du^\epsilon \to Du$ in $L^1(Q_T)$ and a.e.

Passage to the limit if $1 < q < 2$: main steps

- $u^\epsilon \to u$ and $Du^\epsilon \to Du$ in $L^1(Q_T)$ and a.e.
- $\partial_t m^\epsilon - \nu \Delta m^\epsilon - \mathrm{div}(m^\epsilon b^\epsilon) = 0$, with $|b^\epsilon| \approx \frac{|Du^\epsilon|^{q-1}}{(\mu + T_{1/\epsilon} m^\epsilon)^\alpha}$.
  - Energy estimates: $|b^\epsilon|_{L^{q'}(Q_T)} \le C$ with $q' \ge 2$
    $\Rightarrow m^\epsilon$ is compact in $L^1(Q_T) \Rightarrow m^\epsilon \to m$ in $L^1(Q_T)$ and a.e.
  - $b^\epsilon \to b = H_p(x, Du, m)$ a.e.

Passage to the limit if $1 < q < 2$: main steps

- $u^\epsilon \to u$ and $Du^\epsilon \to Du$ in $L^1(Q_T)$ and a.e.

- $\partial_t m^\epsilon - \nu \Delta m^\epsilon - \mathrm{div}(m^\epsilon b^\epsilon) = 0$, with $|b^\epsilon| \approx \frac{|Du^\epsilon|^{q-1}}{(\mu + T_{1/\epsilon} m^\epsilon)^\alpha}$.
  - $m^\epsilon \to m$ in $L^1(Q_T)$ and a.e.
  - $b^\epsilon \to b = H_p(x, Du, m)$ a.e.
  - Energy estimates and $(1-\alpha)q' \geq -\alpha \Rightarrow \int_{Q_T} m^\epsilon |b^\epsilon|^{q'} \leq C$, for a constant C independent of μ.
  - Since $q' > 2$, $m^\epsilon |b^\epsilon|^2$ is compact in $L^1(Q_T)$.
  - Stability result: $m^\epsilon \to m$ in $C([0, T], L^1(\Omega))$ and m is a weak sol. of the Fokker Planck eq. related to b.

Passage to the limit if $1 < q < 2$: main steps

- $u^\epsilon \to u$ and $Du^\epsilon \to Du$ in $L^1(Q_T)$ and a.e.
- $\partial_t m^\epsilon - \nu \Delta m^\epsilon - \mathrm{div}(m^\epsilon b^\epsilon) = 0$, with $|b^\epsilon| \approx \frac{|Du^\epsilon|^{q-1}}{(\mu + T_{1/\epsilon} m^\epsilon)^\alpha}$.
  - $m^\epsilon \to m$ in $L^1(Q_T)$ and a.e.
  - $b^\epsilon \to b = H_p(x, Du, m)$ a.e.
  - $m^\epsilon \to m$ in $C([0,T], L^1(\Omega))$ and m is a weak sol. of the Fokker Planck eq. related to b.
- $F^\epsilon(m^\epsilon) \to F(m)$ and $G^\epsilon(x, m^\epsilon(T)) \to G(x, m(T))$ in L^1
- Passage to the limit in the Bellman equation: OK from the steps above and from stability results for HJB eq. ([Porretta 99]) because the Hamiltonian has natural growth and the good sign
- The proof is achieved for $q < 2$.

The case $q = 2$

What remains from the steps above?

- $u_\epsilon \to u$, $Du^\epsilon \to Du$, $m_\epsilon \to m$ in $L^1(Q_T)$ and a.e.,
  $b^\epsilon = H_p(t, x, T_{1/\epsilon} m^\epsilon, Du^\epsilon) \to b = H_p(t, x, m, Du)$ a.e.
- u is a subsolution of the Bellman equation (no terminal condition yet) (from Fatou lemma and the equi-integrability of $F^\epsilon(t, x, m^\epsilon)$)
- $$\lim_{\epsilon \to 0} \int_\Omega u^\epsilon|_{t=0}(x) m_0(x) dx \le \langle u|_{t=0}, m_0 \rangle,$$
  where both members of the inequality are well defined

The case $q = 2$

What remains from the steps above?

- $u_\epsilon \to u$, $Du^\epsilon \to Du$, $m_\epsilon \to m$ in $L^1(Q_T)$ and a.e., $b^\epsilon = H_p(t, x, T_{1/\epsilon} m^\epsilon, Du^\epsilon) \to b = H_p(t, x, m, Du)$ a.e.
- u is a subsolution of the Bellman equation (no terminal condition yet) and $\lim_{\epsilon\to 0} \int_\Omega u^\epsilon|_{t=0}(x) m_0(x) dx \le \langle u|_{t=0}, m_0 \rangle$
- $$m^\epsilon b^\epsilon = m^\epsilon H_p(t, x, T_{1/\epsilon} m^\epsilon, Du^\epsilon) \le c_2 m^\epsilon + w^\epsilon + \sqrt{m^\epsilon} z^\epsilon$$
  where w^ϵ and z^ϵ are bounded in $L^2(Q_T)$. Therefore, $m^\epsilon b^\epsilon$ is equi-integrable $\Rightarrow$ m is a distribution sol. of the F.P. related to b

The case $q = 2$

What remains from the steps above?

- $u_\epsilon \to u$, $Du^\epsilon \to Du$, $m_\epsilon \to m$ in $L^1(Q_T)$ and a.e., $b^\epsilon = H_p(t, x, T_{1/\epsilon} m^\epsilon, Du^\epsilon) \to b = H_p(t, x, m, Du)$ a.e.
- u is a subsolution of the Bellman equation (no terminal condition yet) and $\lim_{\epsilon\to 0} \int_\Omega u^\epsilon|_{t=0}(x) m_0(x) dx \le \langle u|_{t=0}, m_0 \rangle$
- m is a distribution sol. of the F.P. related to b
- $$\int_0^T \int_\Omega F(t, x, m) m \, dxdt + \int_0^T \int_\Omega \frac{|Du|^2}{(m+\mu)^\alpha} \, dxdt + \int_0^T \int_\Omega m \frac{|Du|^2}{(m+\mu)^\alpha} \, dxdt \le C.$$
- m is a weak solution of the Fokker-Planck equation, so $m \in C([0, T], L^1(\Omega))$

The case $q = 2$

What remains from the steps above?

- $u_\epsilon \to u$, $Du^\epsilon \to Du$, $m_\epsilon \to m$ in $L^1(Q_T)$ and a.e.
- u is a subsolution of the Bellman equation (no terminal condition yet) and $\lim_{\epsilon\to 0} \int_\Omega u^\epsilon|_{t=0}(x) m_0(x) dx \le \langle u|_{t=0}, m_0 \rangle$
- m is a weak solution of the Fokker-Planck eq., and $m \in C([0,T], L^1(\Omega))$
- Finer arguments related to the F.P. equation imply that $m^\epsilon(t) \rightharpoonup m(t)$ in $L^1(\Omega)$ weak for all t, and

$$\int_\Omega G(m(T)) m(T)\, dx \le C.$$

The case $q = 2$

What remains from the steps above?

- $u_\epsilon \to u$, $Du^\epsilon \to Du$, $m_\epsilon \to m$ in $L^1(Q_T)$ and a.e.
- u is a subsolution of the Bellman equation (no terminal condition yet) and $\lim_{\epsilon\to 0}\int_\Omega u^\epsilon|_{t=0}(x)m_0(x)dx \le \langle u|_{t=0}, m_0\rangle$
- m is a weak solution of the Fokker-Planck eq., and $m \in C([0,T], L^1(\Omega))$
- $m^\epsilon(t) \rightharpoonup m(t)$ in $L^1(\Omega)$ weak for all t, and

$$\int_\Omega G(m(T))m(T)\,dx \le C.$$

One needs to work more, because $m^\epsilon|b^\epsilon|^2$ is no longer equi-integrable, so we do not obtain the convergence of m^ϵ to m in $C([0,T]; L^1(\Omega))$, and the convergence of $G^\epsilon(m^\epsilon(T))$ is more difficult to prove.

A crossed energy inequality

Theorem

Consider (u, m) such that

1. $$mF(m) \in L^1, \qquad m_{|t=T}G(m_{|t=T}) \in L^1(\Omega),$$
   $$m\frac{|Du|^q}{(\mu+m)^\alpha} \in L^1, \qquad \frac{|Du|^q}{(\mu+m)^\alpha} \in L^1$$
2. m is a weak sol. of $\Big($ F.P. equation + $m|_{t=0} = m_0$ $\Big)$
3. u is a distrib. subsol. of $\Big($ the Bellman equation + $u|_{t=T} \le G(m_{t=T})$ $\Big)$

For any pair $(\tilde u, \tilde m)$ with the same properties as (u, m), we have the crossed-integrability:

$$\tilde m \frac{|Du|^q}{(\mu+m)^\alpha} \in L^1, \quad m\frac{|D\tilde u|^q}{(\mu+\tilde m)^\alpha} \in L^1, \dots$$

and the energy inequality:

$$\langle \tilde m_0 \, , \, u(0)\rangle \le \int_\Omega G(x, m(T))\, \tilde m(T)\, dx + \int_0^T\!\!\int_\Omega F(t,x,m)\tilde m \, dxdt$$
$$+ \int_0^T\!\!\int_\Omega \left[\tilde m\, H_p(t,x,\tilde m, D\tilde u)\cdot Du - \tilde m\, H(t,x,m,Du)\right] dxdt$$

Passage to the limit using the crossed energy inequality

We start from the energy identity for (u^ϵ, m^ϵ):

$$\int_0^T \int_\Omega m^\epsilon \left(H_p(t,x,T_{1/\epsilon} m^\epsilon, Du^\epsilon) \cdot Du^\epsilon - H(t,x,T_{1/\epsilon} m^\epsilon, Du^\epsilon) \right) dxdt$$

$$+ \int_\Omega m^\epsilon(T) G^\epsilon(x, m^\epsilon(T)) dx + \int_0^T \int_\Omega m^\epsilon F^\epsilon(t,x,m^\epsilon)\, dxdt \;= \int_\Omega u^\epsilon(0) m_0^\epsilon \, dx.$$

By Fatou lemma:

$$\limsup_{\epsilon \to 0} \int_\Omega m^\epsilon(T) G^\epsilon(x, m^\epsilon(T)) dx \le \langle u(0), m_0 \rangle - \int_0^T \int_\Omega m F(t,x,m)\, dxdt$$
$$- \int_0^T \int_\Omega m \left(H_p(t,x,m,Du) \cdot Du - H(t,x,m,Du) \right) dxdt$$

But thanks to the crossed energy inequality applied to $(\tilde u, \tilde m) = (u, m)$,

$$\langle u(0), m_0 \rangle - \int_0^T \int_\Omega m F(t,x,m)\, dxdt$$
$$- \int_0^T \int_\Omega m \left(H_p(t,x,m,Du) \cdot Du - H(t,x,m,Du) \right) dxdt \le \int_\Omega m(T) G(x, m(T)) dx$$

Using the monotonicity of G, we then get $G^\epsilon(x, m^\epsilon(T)) \to G(x, m(T))$ in $L^1(\Omega)$.

Conclusion as in the case $q < 2$.

To be more correct ...

I cheated a bit in the previous slides, because I did not prove that u is a subsolution of the boundary value problem with the terminal condition $u(T) \leq G(x, m(T))$.

The rigorous argument goes through the parametrized Young measure generated by the sequence $\rho^\epsilon \star m^\epsilon(T)$:

$$f(x, \rho^\epsilon \star m^\epsilon(T)) \rightharpoonup \int_{\mathbb{R}} f(x, \lambda) d\nu_x(\lambda) \qquad \text{weakly in } L^1(\Omega)$$

for every Carathéodory function $f(x, s)$ such that $f(x, \rho^\epsilon \star m^\epsilon(T))$ is equi-integrable.

We get, with the previous argument,

$$\lim_{\epsilon \to 0} \int_\Omega m^\epsilon(T) G^\epsilon(x, m^\epsilon(T)) dx = \int_\Omega \int_{\mathbb{R}} m(T) G(x, \lambda) d\nu_x(\lambda) dx,$$

which allows to conclude that $G^\epsilon(x, m^\epsilon(T)) \to G(x, m(T))$.

Uniqueness

Needs the following lemma:

Lemma

A weak solution (u, m) satisfies the energy identity:

$$\begin{aligned}\langle m_0\,,\,u(0)\rangle = &\int_\Omega G(x, m(T))\, m(T)\, dx + \int_0^T \int_\Omega F(t,x,m) m\, dxdt \\ &+ \int_0^T \int_\Omega [m\, H_p(t,x,m,Du) \cdot Du - m\, H(t,x,m,Du)]\, dxdt\end{aligned}$$

Then, take two weak solutions (u, m) and $(\tilde{u}, \tilde{m})$. Use

1. the 2 energy identities for (u, m) and $(\tilde{u}, \tilde{m})$
2. the 2 crossed energy inequalities for
    1. $(\tilde{u}, \tilde{m})$ and (u, m)
    2. (u, m) and $(\tilde{u}, \tilde{m})$

Adding all the identities/inequalities, we conclude as P-L. Lions for classical solutions, using the monotonicity of F and G and $\alpha \le 4/q'$.

Outline

The case $\mu = 0$: weak solutions (1)

$$\begin{cases} -\partial_t u - \nu \Delta u + \frac{1}{q} \frac{|Du|^q}{m^\alpha} = F(m)\,, & (t,x) \in (0,T) \times \Omega \\ \partial_t m - \nu \Delta m - \operatorname{div}(m \frac{|Du|^{q-2} Du}{m^\alpha}) = 0\,, & (t,x) \in (0,T) \times \Omega \\ m(0,x) = m_0(x)\,,\ u(T,x) = G(x, m(T))\,, & x \in \Omega. \end{cases}$$

The case $\mu = 0$: weak solutions (2)

Definition The pair (u, m) is a weak solution if

1. $$mF(m) \in L^1, \qquad m_T G(m_T) \in L^1(\Omega),$$
$$m \mathbb{1}_{\{m>0\}} \frac{|Du|^q}{m^\alpha} \in L^1, \qquad \mathbb{1}_{\{m>0\}} \frac{|Du|^q}{m^\alpha} \in L^1.$$

2. u is a subsolution of the Bellman equation: for any $0 \le \varphi \in C_c^\infty((0,T] \times \Omega)$,
$$\int_0^T \int_\Omega u\, \varphi_t \, dxdt - \nu \int_0^T \int_\Omega u\, \Delta\varphi \, dxdt + \int_0^T \int_\Omega H(t,x,m,Du) \mathbb{1}_{\{m>0\}} \varphi \, dxdt$$
$$\le \int_0^T \int_\Omega F(t,x,m)\varphi \, dxdt + \int_\Omega G(x, m(T))\varphi(T)\, dx$$

3. m is a weak solution of
$$\begin{aligned} \partial_t m - \nu \Delta m - \operatorname{div}\left(m \mathbb{1}_{\{m>0\}} \frac{|Du|}{m^\alpha} \right) &= 0 \\ m(t=0) &= m_0 \end{aligned}$$

4. The energy identity holds:
$$\int_\Omega m_0\, u(0)\, dx = \int_\Omega G(x, m(T))\, m(T)\, dx + \int_0^T \int_\Omega F(t,x,m) m \, dxdt$$
$$+ \int_0^T \int_\Omega m \left[H_p(t,x,m,Du) \cdot Du - H(t,x,m,Du) \right] \mathbb{1}_{\{m>0\}} dxdt$$

where the first term is understood as the trace of $\int_\Omega u(t)\, m_0\, dx$ in $BV(0,T)$.

Existence and uniqueness

Theorem

If either $\left(q < 2 \text{ and } \alpha \le \frac{4}{q'} \right)$ *or* $\left(q = 2 \text{ and } \alpha < 4/q' = 2 \right)$, *and if* F *and* G *are nondecreasing, then there exists a unique weak solution.*

Remark We miss the limit case $q = 2$ and $\alpha = 2$.

Proof

- Consists of passing to the limit as $\mu \to 0^+$ in the non singular case discussed previously.
- It goes through a careful adaption of the previous arguments with a special attention to the regions where $m = 0$.

Outline

MFG vs. Mean field type control (MFTC) (1)

- MFG: look for Nash equilibria with N identical agents, then let $N \to \infty$
- Carmona and Delarue / Bensoussan et al have studied the control of McKean-Vlasov dynamics:
  - Assume that the all N agents use the same feedback law γ
  - The perturbations of γ impact the empirical distribution
  - Pass to the limit as $N \to \infty$ first, then minimize the asymptotic cost

MFG vs. Mean field type control (MFTC) (1)

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  - The perturbations of γ impact the empirical distribution
  - Pass to the limit as $N \to \infty$ first, then minimize the asymptotic cost
- MFTC models consists of an optimal control problem driven by a Fokker-Planck equation:

  Find a feedback $\gamma_s = \gamma(s, X_s; m_s)$ which minimizes

$$J(t) = \mathbb{E}\left\{\int_t^T \mathcal{L}\left(X_s, \gamma_s; m_s\right) ds + \mathcal{G}\left(X_T; m_T\right)\right\}$$

subject to

$$\begin{aligned} &dX_t = \sqrt{2\nu} dW_t + \gamma_t dt \\ &m_t \text{ is the law of } X_t, \end{aligned}$$

therefore

$$\begin{aligned} &\frac{\partial m}{\partial t} - \nu \Delta m + \operatorname{div}\left(m\gamma\right) = 0 \\ &m|_{t=0} = m_0. \end{aligned}$$

MFG vs. MFTC (2)

Assume local dependency: $\mathcal{L}(x,\gamma;m) = L(x,\gamma,m(x))$ and $\mathcal{G}(x;m) = G(x,m(x))$: the cost can be expressed as

$$J(t) = \int_t^T \int_\Omega L\left(x,\gamma(s,x,m_s)m(s,x)\right) m(s,x) ds dx + \int_\Omega G\left(x; m(T,x)\right)) m(T,x) dx$$

and the optimality conditions read

$$\begin{aligned}
\frac{\partial u}{\partial t} + \nu\Delta u - H(x,\nabla u, m(t,x)) - m(t,x)\frac{\partial H}{\partial m}(x,\nabla u(x,t), m(t,x)) &= 0 \\
\frac{\partial m}{\partial t} - \nu\Delta m - \mathrm{div}\left(m\frac{\partial H}{\partial p}(x,\nabla u; m)\right) &= 0
\end{aligned}$$

with the terminal and initial conditions

$$\begin{aligned}
u(t=T,x) &= G(x,m(T,x)) + m(T,x)\frac{\partial G}{\partial m}(x,m(T,x)) \\
m(0,x) &= m_0(x)
\end{aligned}$$

MFG vs. MFTC (3)

- The latter PDE system enjoys uniqueness if

$$\begin{pmatrix} -(mH)_{m,m} & 0 \\ 0 & mH_{p,p} \end{pmatrix} > 0$$

  for all $x \in \Omega$, $m > 0$ and $p \in \mathbb{R}^d$.

  If $H(x, p, m) = \frac{|p|^q}{(\mu+m)^\alpha}$, then the latter condition holds if $\alpha \leq 1$, (while the condition is $\alpha \leq \frac{4q}{q-1}$ for the MFG system with the same Hamiltonian)

- Existence and uniqueness for weak solutions of the mean field type control problem with congestion and possibly degenerate diffusion were proved in [A., Laurière 2015], using a variational approach.

- Numerical methods using the variational approach were studied in [A., Laurière 2016].