Remarks on the construction of invariant measures for KdV

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Itinerant meeting on PDEs Roma, January 10^{th}, 2013

We denote by $d\mu_s$ a probability (Gaussian) measure on the spaces $H^{s-1/2-\epsilon}$ for $\epsilon > 0$

In fact the measure $d\mu_s = e^{-\|u\|_{H^s}^2} du$ can be constructed in a rigorous way by a finite dimensional approximation via a general Kolmogorov result, and it is in principle a measure on $\mathbf{C}^\infty$.

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A more concrete representation of the probability space is given by the randomized vector

$$\varphi_s(x,\omega) = \sum_{n \in \mathbf{Z}\setminus\{0\}} \frac{\varphi_n(\omega)}{|n|^s} e^{\mathbf{i}nx}$$

where $(\varphi_n(\omega))$ is a sequence of centered complex gaussian variables defined on a probability space $(\Omega, \mathcal{A}, p)$

Then for every function $F(u)$ defined on the support of the measure $d\mu_s = e^{-\|u\|_s^2} du$ we have

$$\int F(u) d\mu_s = \int_\Omega F(\varphi_s(\omega)) dp$$

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As a consequence one can deduce that $d\mu_s(H^{s-1/2}) = 0$ and the space $H^{s-1/2-\epsilon}$ is support for $d\mu_s$

We also have $d\mu_1(H^{1/2}) = 0$ and $d\mu_1(\mathcal{F}L^{r,1/2}) = 1$ for $r > 2$ where

$$\|u\|_{\mathcal{F}L^{r,1/2}} = \|\{\hat{u}(n)n^{1/2}\}\|_{l^r}$$

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Let us consider a general Hamiltonian PDE

$$u_t + Au = F(u)$$

$$u(0,x) = u_0(x)$$

$$u(0,t) = u(2\pi,t)$$

with an associate conservation law

$$E_s(u) = \|u\|_{H^s}^2 + R_s(u)$$

For example the KdV, BO, DNLS satisfy this assumption.

Typical probabilistic statements that can be proved for the Cauchy problems above are of the following type.

Theorem

The Cauchy problem is G.W.P. for a.e. u_0 *w.r.t.* $d\mu_s = e^{-\|u\|_{H^s}^2} du$

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The measure $e^{-R_s(u)}e^{-\|u\|_{H^s}^2}du = e^{-R_s(u)}d\mu_s$ *is invariant along the flow of the PDE.*

Consequence: Poincaré Recurrence Theorem Let T_t be a dynamical system on a topological space X endowed with a finite invariant measure $d\mu$, then for a.e. $d\mu$ initial data in the phase space there is a sequence $t_n \to \infty$ such that $T_{t_n}(u_0) \to u_0$

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Φ_N^t is the flow associated with

$$\partial_t u_N + A u_N = \pi_N(F u_N)$$

$$u_N(0) = \pi_N(u_0)$$

where π_N is the projections on the modes with frequency $\leq N$ and u_N are trigonometric polynomials of degree at most N.
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MAIN STEPS

1. There is a suitable Banach space X which is a support of $d\mu_s = e^{-\|u\|^2_{H^s}} du$ where the Cauchy problem is L.W.P. (i.e. there is an unique local solution in the space $\mathcal{C}(0, T(\|u_0\|_X)); X)$
2. Finite dimensional approximation:

$$\|\Phi^t_N(u_0) - \Phi^t(u_0)\|_{L^\infty((0,T);X)} \to 0 \text{ as } N \to \infty$$

3. The energy $E_s(\Phi^t_N u_0)$ is (almost) preserved along the evolution as long as $N \to \infty$ for a.e. $u_0 \in X$.

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Lemma

Assume that

$$\lim_{N\to\infty} \|G_N\|_{L^q(d\mu_s))} = 0$$

where

$$G_N(u_0) = \frac{d}{dt}(E_s(\Phi_N^t(u_0))_{|t=0}$$

Then we have the following:

$$\lim_{N\to\infty} \sup_{\substack{t\in[0,t_0] \\ A\in Borel\subset supp(d\mu_s)}} \left|\frac{d}{dt}\int_{\Phi_N^t(A)} e^{-R_s(\pi_N u)} d\mu_s(u)\right| = 0$$

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Reduction of the analysis at time $t = 0$

We have

$$\frac{d}{dt}\Big(\int_{\phi_N^t(A)} e^{-R_s(\pi_N u)}\,d\mu\Big)_{t=\bar{t}}$$

$$= \lim_{h\to 0} h^{-1}\Big(\int_{\Phi_N^{\bar{t}+h}(A)} e^{-R_s(\pi_N u)}\,d\mu - \int_{\phi_N^{\bar{t}}(A)} e^{-R_s(\pi_N u)}\,d\mu\Big)$$

$$= \lim_{h\to 0} h^{-1}\Big(\int_{\Phi_N^h\circ\Phi_N^{\bar{t}}(A)} e^{-R_s(\pi_N u)}\,d\mu - \int_{\phi_N^{\bar{t}}(A)} e^{-R_s(\pi_N u)}\,d\mu\Big)$$

and hence

$$\frac{d}{dt}\Big(\int_{\phi_N^t(A)} e^{-R_s(\pi_N u)}\,d\mu\Big)_{t=\bar{t}} = \frac{d}{dt}\Big(\int_{\phi_N^t(\tilde{A})} e^{-R_s(\pi_N u)}\,d\mu\Big)_{t=0}$$

where $\tilde{A} = \phi_N^{\bar{t}}(A)$. Hence we are reduced to estimate the time derivative derivative at time $t = 0$.

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Analysis for KdV We compute concretely $\frac{d}{dt}E_m(\Phi_N^t(u))_{|t=0}$ where

$$E_m(u) = \int (\partial_x^m u)^2 \, dx + \int u(\partial_x^{m-1} u)^2 \, dx + ...$$

Roughly speaking, in general, it is sufficient to compute (by Leibnitz rule) the time derivative of the density of E and to replace u_t by $\pi_{>N}(F(u^N(t)))$. More precisely to compute $\frac{d}{dt}E_m(u^N(t)))_{|t=0}$ (for KdV) we get:

$$\frac{d}{dt}E_m(u^N(t)) =$$

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$$+2\int u^N(\partial_x^{m-1}u^N)\partial_x^{m-1}u_t^N \, dx + ...{}_{|u_t^N = u_{xx}^N + u^N\partial_x u^N - \pi_{>N}(u^N\partial_x u^N)} =$$

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To compute $\frac{d}{dt}E_m(u^N(t))_{|t=0}$ it is sufficient to replace above u^N by $u^N(0) = \pi_N \varphi_s(\omega)$.

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To compute $\frac{d}{dt}E_m(u^N(t))_{|t=0}$ it is sufficient to replace above u^N by $u^N(0) = \pi_N \varphi_s(\omega)$.

By the computation above the worst term to be estimated is

$$\int u^N(\partial_x^{m-1}u^N)\partial_x^{m-1}\pi_{>N}(u^N\partial_x u^N)\,dx_{|u^N=\sum_{n\in\mathbf{Z}\setminus\{0\},|n|<N}\frac{\varphi_n(\omega)}{|n|^m}e^{\mathbf{i}nx}}$$

then we get

$$\int \sum_{\substack{|j_1|,|j_2|,|j_3|,|j_4|<N\\ |j_3+j_4|>N}} \frac{\varphi_{j_1}(\omega)}{|j_1|^m}\frac{\varphi_{j_2}(\omega)}{|j_2|}\frac{\varphi_{j_3}(\omega)}{|j_3|^m}\varphi_{j_4}(\omega)e^{i(j_1+j_2+j_3+j_4)x}\,dx$$

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and hence

by integration dx we get

$$\ldots = \sum_{\substack{|j_1|,|j_2|,|j_3|,|j_4|<N \\ |j_3+j_4|>N, j_1+j_2+j_3+j_4=0}} \frac{\varphi_{j_1}(\omega)}{|j_1|^m} \frac{\varphi_{j_2}(\omega)}{|j_2|} \frac{\varphi_{j_3}(\omega)}{|j_3|^m} \varphi_{j_4}(\omega)$$

By Minkowski

$$\|G_N(u)\|_{L^q(d\mu_m)} = \|G_N(\varphi(\omega))\|_{L^q(dp_\omega)}$$

$$\leq \sum_{\substack{|j_1|,|j_2|,|j_3|,|j_4|<N \\ |j_1+j_2|>N}} \frac{1}{|j_1|^m} \frac{1}{|j_2|} \frac{1}{|j_3|^m} \to 0 \text{ as } N \to \infty$$

Thank you!