Global existence for the Euler-Maxwell equation

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Outline

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Plasma Physics

A plasma can be modeled by a superposition of several **charged** compressible gases (one for each species), interacting through their electromagnetic fields.

We will consider the case of **two** species:

- a gas of **ions** of density $n_+ \in \mathbb{R}_+$, velocity $v_+ \in \mathbb{R}^3$, mass $m_+ = M_i$, charge Ze and pressure law $p_i = T_+ n_+^2/2$.
- a gas of **electrons** of density n_-, velocity v_-, mass $m_- = m_e$, charge $-e$ and pressure law $p_e = T_- n_-^2/2$.

We need to also consider the **electromagnetic field** (E, B).

Remark: we could consider any a priori pressure law.

This is one of the three main models for plasma physics (with Vlasov-Maxwell and MHD).

The System

The Euler-Maxwell system is simply the law of conservation of each species, the balances of momentum and the Maxwell equations, namely

$$\begin{aligned}
\partial_t n_\pm + \nabla \cdot (n_\pm v_\pm) &= 0 \\
n_\pm m_\pm (\partial_t v_\pm + v_\pm \cdot \nabla v_\pm) + T_\pm \nabla n_\pm &= \pm e n_\pm (E + v_\pm \times B) \\
\partial_t B + c\nabla \times E &= 0 \\
\partial_t E - c\nabla \times B &= -4\pi e(n_+ u_+ - n_- u_-) \\
\mathrm{div}(E) &= 4\pi e(n_+ - n_-) \\
\mathrm{div}(B) &= 0
\end{aligned}$$

when $Z = 1$. These are 14 dynamical equations and 2 elliptic constraints. It corresponds to 12 scalar equations.

Non-dimensionalization

After rescaling, we are reduced to the following system

$$
\begin{aligned}
&\partial_t n + \mathrm{div}((n+1)v) = 0,\\
&\varepsilon\left(\partial_t v + v\cdot\nabla v\right) + T\nabla n + \tilde{E} + v\times\tilde{B} = 0,\\
&\partial_t \rho + \mathrm{div}((\rho+1)u) = 0,\\
&\left(\partial_t u + u\cdot\nabla u\right) + \nabla\rho - \tilde{E} - u\times\tilde{B} = 0,\\
&\partial_t \tilde{B} + \nabla\times\tilde{E} = 0,\\
&\partial_t \tilde{E} - \frac{C_b}{\varepsilon}\nabla\times\tilde{B} = \left[(n+1)v - (\rho+1)u\right],\\
&\mathrm{div}(\widetilde{B}) = 0, \quad \mathrm{div}(\widetilde{E}) = \rho - n,
\end{aligned}
$$

où $\varepsilon = m_e/m_i < 1/2000 \ll 1$, $T = T_e/T_i \geq 1$ et $C_b = c^2/(V_e V_i) \gg 1$.

Different dynamics

The parameters involved have very different sizes and lead to several natural limits:

- $\varepsilon = m_e/M_i \to 0$, this is the **1 fluid** model, where we follow only one dynamics ($M_i = 1$, ion dynamics; $m_e = 1$, electron dynamics).
- $C_b \to +\infty$, this is the electrostatic model (the Maxwell system then reduces to the Poisson equation).

This gives different intermediate models (EP/e), (EP/i), (EM/e), (EP) which are important to understand each partial dynamics. Letting $T \to +\infty$ in (EP/i) gives the cold ion model.

Some of these limits have been studied (Grenier-Cordier, Gérard-Varet-Han-Kwan-Rousset)

Yet other dynamics

But beyond these examples, one can also look at particular regimes (e.g. "long-wave" regimes, "high frequency"...) and derives many important equations.

- One can derive the Zakharov equation (Texier)
- One can derive the KdV, KP and ZK equations (Guo-Pu, Lannes-Linares-Saut).

See also studies by Peng-Wang

Outline

Instability for a compressible neutral gas

One of the main motivation from this program comes from a results of Sideris:

Shocks for Compressible Euler [Sideris, 85]

Many nice (small, smooth, localized, irrotational) initial perturbations of a flat equilibrium $(n, v) = (1, 0)$ lead to solutions which **do not remain** C^2 for all time.

The formation of shocks has been studied in details by Alinhac and recently by Christodoulou-Miao.

Stability for a compressible charged gas

However, Y. Guo realized that the introduction of a self-consistent electromagnetic field could stabilize the system

Stability of a neutral equilibrium for the Euler-Poisson/e [Guo 98]

Any small smooth irrotational perturbation of a constant background leads to solutions which remain smooth for all time and decay.

This was extended to $2D$ by Ionescu-P. (see also Jang-Li-Zhang, Li-Wu). The *neutrality* condition was recently removed in $3D$ by Germain-Masmoudi-P.

After this, the challenge was to understand the following:
Question: how and how much does an electromagnetic field stabilize a compressible gas.

Compressible Euler and Euler-Poisson

The compressible Euler equations read

$$\begin{aligned}\partial_t \rho + \mathrm{div}((1+\rho)u) &= 0\\ \partial_t u + u\cdot\nabla u + \nabla\rho + \rho\nabla\rho &= 0\end{aligned}$$

and the Euler-Poisson equation reads

$$\begin{aligned}\partial_t \rho + \mathrm{div}((1+\rho)u) &= 0\\ \partial_t u + u\cdot\nabla u + \nabla\rho + \rho\nabla\rho &= \nabla\phi\\ \Delta\phi &= \rho.\end{aligned}$$

Wave and Klein-Gordon

The difference in the results for CE and EP can be understood by looking at the linearization of these equations: For CE, we have

$$(\partial_{tt} - \Delta)\,\rho = \mathcal{O}((\partial\rho)^2 + |\partial v|^2),$$

This is a quasilinear wave equation without null forms. In $3D$, this is expected to blow-up in finite time.
For EP, we have instead

$$(\partial_{tt} - \Delta + 1)\,\rho = \mathcal{O}((\partial\rho)^2 + |\partial v|^2)$$

And for Klein-Gordon equations in $3D$, one can use the normal form technique of Shatah to get global existence.

The EP/i and EM/e

Recently two other intermediate models were also settled, the ion case

Stability for EP/i [Guo-P. 10]

Small smooth, irrotational initial perturbations of a flat background lead to global solutions for EP/i.

The difficulty was to understand the ion dispersion and the difficulty related to it.

and a breakthrough by Germain-Masmoudi

Stability for EM/e [Germain-Masmoudi 11]

Small smooth, irrotational, neutral and localized initial perturbations of a flat background lead to global solutions for EM/e.

This was subject to a genericity condition that was later removed by Ionescu-P.

The difficulty here was to deal with systems of quasilinear Klein-Gordon

The EP/i

The Euler-equation for ions reads as follows:

$$\begin{aligned}\partial_t \rho + \mathrm{div}((1+\rho)u) &= 0\\ \partial_t u + u\cdot\nabla u + \nabla\rho + \rho\nabla\rho &= -\nabla\phi\\ -\Delta\phi &= 1+\rho - e^{\phi}.\end{aligned}$$

And the linearization is

$$\left(\partial_{tt} - \Delta - \Delta(1-\Delta)^{-1}\right) u = 0.$$

This supports only solutions which decay slowly $t^{-4/3}$ at best. In addition, the dispersion relation

$$\omega(\xi) = |\xi|\sqrt{\frac{2+|\xi|^2}{1+|\xi|^2}} \simeq_{\xi\to 0} \sqrt{2}|\xi| + O(|\xi|^3)$$

is close to the *wave* dispersion and introduces degenerate bilinear operator upon using normal forms.

Systems of Klein-Gordon equations

The small data theory for *systems of Klein-Gordon equations*

$$\left(\partial_{tt} - \Delta^2 + m_1\right) u = Q_1(u, v, \nabla u, \nabla v, \nabla^2 u, \nabla^2 v)$$
$$\left(\partial_{tt} - C^2\Delta^2 + m_2\right) u = Q_2(u, v, \nabla u, \nabla v, \nabla^2 u, \nabla^2 v)$$

is very challenging and is only understood in the non degenerate case.

Global existence for KG systems [Germain, Ionescu-P.]

Assume that

$$2m_i \neq m_j \quad \text{and} \quad (m_1 - m_2)(m_1 - m_2 C^4) \geq 0$$

Then, small smooth compactly supported data lead to global solutions decaying like $t^{-1-\varepsilon}$.

Outline

Main result

We prove that the stability result of Guo remains true even in the whole system

Stability for the Euler-Maxwell system

Small smooth neutral and irrotational initial perturbations of a constant equilibrium lead to solutions which are global, remain smooth and behave asymptotically as linear solutions in the case of the full 2-fluid Euler-Maxwell system.

Remarks

- The irrotational assumptions

$$B = \varepsilon\nabla \times v = -\nabla \times u$$

  are *restrictive*. They remove a mode which does not decay at the linear level.

- For smooth data, the neutral assumptions $\int(n-\rho) = 0 = \int B$ are forced by Maxwell's compatibility conditions and the irrotational assumption

$$\mathrm{div}(E) = \rho - n, \quad B = -\nabla \times u.$$

- For large data, some solutions **DO** blow-up (Guo-Tahvildar-Zadeh).
- The smoothness condition is a bit restrictive, but the only conserved energy is at the L^2-level.

Main difficulties

- The system is coupled even at the linear level. This introduces many terms and produces multiple interactions and makes the dispersion relations less explicit.
- We need to face all the previous difficulties (slow decay, singular bilinear multipliers, multiple resonances) at the same time and some combine to create new difficulties (Cases B and D later).

Main method

We use the framework of the "space-time resonances" introduced by Germain-Masmoudi-Shatah and Gustafson-Nakanishi-Tsai. We want to combine two type of estimates:

A) An energy estimate to get a rough control on very high-regularity norms. This gives uniform bounds in H^N. It depends on the delicate nonlinear structure of the equation and demands an integrable decay of the solutions. This gives the regularity we need to overcome the loss of derivative from the nonlinearity.

B) A semi linear dispersive analysis of the system. This yields the wanted integrable decay. It gives a very fine description of the solution in low-regularity norms. It depends essentially on the linearized system, but very little on the nonlinear structure (except for possible null forms).

Cartoon

From now on, we only focus on B). We want to describe the solution as a free evolution of a slowly-varying profile

$$U(t) = S(t)f(t)$$

where $S(t)$ is the linearized flow and $f(t)$ remains in a compact set.

To illustrate this, consider the quadratic Schrödinger equation

$$(i\partial_t + \Delta)\, u = N(u) = |u|^2.$$

Then, we want to describe the function under the form

$$u(t) = e^{it\Delta} f(t)$$

Then we can use the Duhamel formula to get that

$$\hat{f}(\xi, t) = \hat{f}(\xi, 0) + \int_0^t \int_{\mathbb{R}^3} e^{is\left[|\xi|^2 + |\xi-\eta|^2 - |\eta|^2\right]} \overline{\hat{f}(\xi-\eta, s)} \hat{f}(\eta, s) d\eta ds$$

Resonances

In general, we are lead to study operators

$$T[f,g] = \mathcal{F} \int_{\mathbb{R}} \int_{\mathbb{R}^3} m(\xi,\eta,s) e^{is\Phi(\xi,\eta)} \hat{f}(\xi-\eta,s) \hat{g}(\eta,s) d\eta ds$$

and to find norms B so that $T : B \times B \to B$.

In general the key information about the interaction $\mu \times \nu \to \sigma$ is contained into the nonlinear phase

$$\Phi(\xi,\eta) = \Lambda_\sigma(\xi) - \Lambda_\mu(\xi-\eta) - \Lambda_\nu(\eta).$$

and particularly in its critical points (resonances) $\nabla_{(s,\eta)} s\Phi(\xi,\eta) = 0$.

The linearized system is

$$\partial_t n + \mathrm{div}(v) = 0, \quad \partial_t v + \frac{T}{\varepsilon}\nabla n + E = 0,$$
$$\partial_t \rho + \mathrm{div}(u) = 0, \quad \partial_t u + \nabla \rho - E = 0,$$
$$\partial_t B + \nabla \times E = 0, \quad \partial_t E - \frac{C_b}{\varepsilon}\nabla \times B = v - u,$$

It can be diagonalized into

$$\partial_t U + M \cdot U = 0$$

where $U = (U_i, U_e, U_b)$ and $M = \mathrm{diag}(\Lambda_i, \Lambda_e, \Lambda_b)$ with eigenvalues

$$\Lambda_i = \varepsilon^{-1/2}\sqrt{\frac{(1+\varepsilon) - (T+\varepsilon)\Delta - \sqrt{((1-\varepsilon) - (T-\varepsilon)\Delta)^2 + 4\varepsilon}}{2}},$$
$$\Lambda_e = \varepsilon^{-1/2}\sqrt{\frac{(1+\varepsilon) - (T+\varepsilon)\Delta + \sqrt{((1-\varepsilon) - (T-\varepsilon)\Delta)^2 + 4\varepsilon}}{2}},$$
$$\Lambda_b = \varepsilon^{-1/2}\sqrt{1 + \varepsilon - C_b\Delta}.$$

Norms

We now need to find appropriate norms such that

- The relevant bilinear operators are bounded $T : B \times B \to B$
- Boundedness of the linear profile in B yields decay of the free solutions:

$$\|e^{it\Lambda} f\|_{L^\infty} \lesssim (1+t)^{-1-\varepsilon} \|f\|_B.$$

We want to control two aspects: *position* and *momentum*. We localize the inputs

$$f = \sum_{N \cdot X \geq 1} Q_X P_N f = \sum_{N \cdot X \geq 1} f_{X,N},$$

$$Q_X f(x) = \mathbb{1}_{\{x \sim X\}} f(x), \qquad \mathcal{F} P_N f(\xi) = \mathbb{1}_{\{\xi \sim N\}} \mathcal{F} f(\xi)$$

and we define the norm on these atoms

$$\|f\|_B = \sup_{N \cdot X \geq 1} \|f_{X,N}\|_{\tilde{B}}$$

The simplest norm that provides the decay is

$$\|f_{X,N}\|_{B^1} = X^{1+\varepsilon}\|f_{X,N}\|_{L^2}.$$

Unfortunately, the presence of strong resonances does not allow us to get boundedness of the corresponding operator and we need to allow another kind of "weak" inputs:

$$\|f_{X,N}\|_{B^2} = X^{1-\varepsilon}\|f_{X,N}\|_{L^2} + \sup_{\xi,R} R^{-2}X^{\frac{3}{2}-\varepsilon}\|\hat{f}\|_{L^1(B(\xi,R))}.$$

Our space is then $B = B^1 + B^2$. In this discussion, we have neglected the dependence in N which is less important.

Analysis of the operators

We now need to estimate the relevant bilinear operators

$$f^{\alpha}_{X,N}(t) = \sum_{N_1\cdot X_1\geq 1, N_2\cdot X_2\geq 1, T\leq t, \beta\gamma} T_{[T,2T]}[f^{\beta}_{X_1,N_1}, f^{\gamma}_{X_2,N_2}].$$

This is a huge sum, but now all the relevant information is quantified.

Using Energy estimates, we may assume that $N, N_1, N_2 \lesssim 1$.

Using the finite speed of propagation, we may assume that $X, X_1, X_2 \leq T$.

This way, we get rid of most terms in the sum above and can treat them one by one.

Nonstationary phase analysis

This is the natural framework of the space-time resonance method. Using integration by parts, we can reduce to the study of the following operators

$$R_T^{\alpha}[f,g] = \int_T^{2T}\int_{\mathbb{R}^3} e^{is\Phi^{\alpha\beta\gamma}(\xi,\eta)} m(\xi,\eta)\varphi(\delta_X^{-1}\nabla_\eta\Phi)\varphi(\delta_t^{-1}\Phi) \hat{f}^{\beta}_{X_1,N_1}(\xi-\eta)\hat{f}^{\gamma}_{X_2,N_2}(\eta)d\eta ds$$

where $\delta_t \ll 1$ and

$$\delta_X = \begin{cases} T^{-1/2} & \text{if } X_1+X_2 \leq T^{\frac{1}{2}} \\ (X_1+X_2)T^{-1} & \text{if } X_1+X_2 \geq T^{\frac{1}{2}}. \end{cases}$$

In some cases, the integration domain is then empty.

The Stationary phase analysis

It remains to consider the sets

$$\mathcal{S}^{\alpha\beta\gamma}=\{(\xi,\eta):\ \Phi^{\alpha\beta\gamma}=0,\ \nabla_\eta\Phi^{\alpha\beta\gamma}=0\}.$$

Out of the 63 possible interactions, only about 15 have non empty space-time resonant sets. These cases can be classified into 4 different sets (A,B,C,D).

- Case A: $\mathcal{S}=\{(R\omega,r\omega),\ \omega\in\mathbb{S}^2\}$ for some $R,r>0$. These sets appear for systems of Klein-Gordon equations (see Germain) and have already been studied in (Germain, Ionescu-P.). These interactions create the second kind of input.
- Case B: $\mathcal{S}=\{(R\omega,0),\ \omega\in\mathbb{S}^2\}$.
- Case C: $\mathcal{S}=\{(0,0),\ \omega\in\mathbb{S}^2\}$. This is the case that appear for the ion equation.
- Case D: $\mathcal{S}=\{(0,r\omega),\ \omega\in\mathbb{S}^2\}$.

Zakharov

Note that from the fact that the original system was rotationally symmetric, $\mathcal{S}$ is always invariant under $(\xi, \eta) \to (O\xi, O\eta)$ for any rotation O.

The Zakharov system seems to be made of interactions from Cases B and D. Indeed, it corresponds to a "blow-up" of the closed interactions $i \pm \times e \to e$; $(0, \xi_2) \to \xi_2$ and $e + \times e- \to i$; $(\xi_1, -\xi_1) \to 0$.

Stationary phase analysis

We need to distinguish two situations:

- $X_1 + X_2 \leq T^{\frac{1}{2}}$. In this case, we can obtain no information from the precise nature of the norms. We are on our own. On the other hand, we may assume that the inputs obey symbol bounds and therefore perform an efficient stationary phase analysis. This allows to describe very precisely the output and thus gives the nature of the "weak norm".
- $X_1 + X_2 \geq T^{\frac{1}{2}}$. As $X_1 \to T$, the input become less and less smooth and the stationary analysis becomes less and less efficient, so we have a poorer and poorer description of the output. But this is compensated for by the gains from the norms (e.g. the way it penalizes X). Here a precise choice of the norms is crucial so that the two phenomenon balance each other.