A.E. POINTWISE CONVERGENCE OF THE SOLUTION OF THE SCHRÖDINGER EQUATION

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THE SCHRÖDINGER EQUATION

The solution to the free Schrödinger equation,

$$\partial_t u(x,t) = i\Delta_x u(x,t) \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}$$
$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}^n,$$

is given by

$$e^{it\Delta} u_0 = u(x,t) = \int_{\mathbb{R}^d} e^{2\pi i(x\xi - 2\pi t|\xi|^2)} \widehat{u_0}(\xi)\, d\xi.$$

POINTWISE CONVERGENCE TO THE INITIAL DATA

Question (Carleson) $u(x,t) \longrightarrow_{t\to 0} u_0(x)$ *a.e.*?

L. Carleson 1980: True for $u_0 \in H^{1/4}(\mathbb{R})$ where

$$H^s(\mathbb{R}^n) = \{\, u_0\,;\ \int_{\mathbb{R}^n} |\widehat{u_0}(\xi)|^2 (1+|\xi|)^{2s}\, d\xi < +\infty \,\}.$$

B. Dahlberg and C. Kenig, 1982: False for $s < 1/4$ in all dimensions
P. Sjölin, L. Vega, 1985: True for $s > 1/2$ as a in any dimension.
J. Bourgain, 90's Improved to $u_0 \in H^{1/2-\epsilon}(\mathbb{R}^2)$ in dimension $n = 2$.
Later improved by Moyua–V–Vega, Tao–V–Vega, Tao–V.
Best result known for $n = 2$ is $s > 3/8$ (Sanghyuk Lee 2006).
Bourgain 2012: For $n \geq 3$ a sufficient condition is $s > \frac{1}{2} - \frac{1}{4n}$. For $n \geq 4$, a necessary condition is $s \geq \frac{1}{2} - \frac{1}{2n}$.

Carleson's result

We want to prove

$$\|\sup_t |e^{it\Delta}f(x)|\|_{L^4(\mathbb{R})} \leq C\|f\|_{\dot{H}^{1/4}(\mathbb{R})}.$$

By duality, it will suffice to show that

$$\left|\int_{\mathbb{R}} e^{it(x)\Delta}f(x)w(x)dx\right|^2 \leq \|f\|^2_{\dot{H}^{1/4}(\mathbb{R})}\|w\|^2_{L^{4/3}(\mathbb{R})}$$

for all measurable functions $t:\mathbb{R}\to\mathbb{R}$ and $w\in L^{4/3}(\mathbb{R})$.

$$\left|\int_{\mathbb{R}} e^{it(x)\Delta}f(x)w(x)dx\right|^2 = \left|\int\int \widehat{f}(\xi)e^{2\pi i(x\xi+t(x)\xi^2)}\,d\xi w(x)dx\right|^2.$$

By Fubini's theorem

$$= \left| \int \widehat{f}(\xi) \int e^{2\pi i(x\xi + t(x)\xi^2)} w(x) dx \, d\xi \right|^2$$

and by the Cauchy–Schwarz inequality,

$$\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{1/2} d\xi \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{2\pi i(x\xi + t(x)\xi^2)} w(x) dx \right|^2 \frac{d\xi}{|\xi|^{1/2}}$$

Since

$$\int_{\mathbb{R}^2} |\widehat{f}(\xi)|^2 |\xi|^{1/2} d\xi = \|f\|^2_{\dot{H}^{1/4}(\mathbb{R})},$$

writing the squared integral in as a double integral, it will suffice to show that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\pi i((x-y)\xi + (t(x)-t(y))\xi^2)} w(x) w(y) dx dy \frac{d\xi}{|\xi|^{1/2}} \leq \|w\|^2_{L^{4/3}(\mathbb{R}^2)}.$$

We need the following lemma.

Lemma

Let $a, b \in \mathbb{R}$ and $\alpha \in (0,1)$. Then there is a constant C_α such that

$$\left| \int_{\mathbb{R}} e^{2\pi i(a\xi + b\xi^2)} \frac{d\xi}{|\xi|^\alpha} \right| \leq C_\alpha \left(|b|^{\alpha - 1/2} |a|^{-\alpha} + |a|^{\alpha - 1} \right).$$

We take $\alpha = 1/2$. Then,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\pi i((x-y)\xi + (t(x)-t(y))\xi^2)} w(x) w(y) dxdy \frac{d\xi}{|\xi|^{1/2}}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{w(x) w(y)}{|x-y|^{1/2}} dxdy.$$

By Hölder's inequality, this is bounded by

$$\|w\|_{L^{4/3}(\mathbb{R})}\|I_{1/2}w)\|_{L^4(\mathbb{R})},$$

where

$$I_{1/2}f(y) = \int_{\mathbb{R}} \frac{f(y-x)}{|x|^{1/2}}dx_1$$

is a fractionnary integral. Then, the Hardy–Littlewood–Sobolev inequality,

$$\|I_{1/2}w\|_{L^4(\mathbb{R})} \leq \|w\|_{L^{4/3}(\mathbb{R})}.$$

$H^{1/4}$ counterexample (Dahlberg and Kenig's)

For $j \in \mathbb{Z}$, define the functions f_j by

$$\widehat{f_j}(\xi) = \chi_{[2^j, 2^j + \frac{1}{100}2^{j/2}]}(\xi).$$

Then,

$$\|f\|_{H^s} \sim 2^{j/4} 2^{js}.$$

We have

$$e^{it\Delta} f_j(x) = \int_{2^j}^{2^j + \frac{1}{100}2^{j/2}} e^{2\pi i (x\cdot\xi - 2\pi t \xi^2)} d\xi,$$

and by the change of variables $\xi \to \xi + 2^j$, we have

$$|e^{it\Delta} f_j(x)| = \left| \int_0^{\frac{1}{100}2^{j/2}} e^{2\pi i (\xi(x - 2^j 2\pi t) - 2\pi t \xi^2)} d\xi \right|.$$

We consider the sequence of times t_j defined by

$$t_j = 2^{-j}(2\pi)^{-1}x.$$

Then for all $\xi \in [0, \frac{1}{100}2^{j/2}]$ and all $x \in [0,1]$,

$$|\xi(x - 2^j 2\pi t_j) - 2\pi t_j \xi^2| \leq 0 + 2/50.$$

Thus there is a constant C such that,

$$|e^{it_j\Delta} f_j(x)| \geq 2^{j/2}$$

for all $x \in [0,1]$. Hence,

$$\| \sup_t |e^{it\Delta} f_j| \|_{L^p} \geq C2^{j/2}.$$

Hence,

$$\| \sup_t |e^{it\Delta} f_j| \|_{L^p} >> \|f\|_{H^s}$$

for all $s < 1/4$.

To prove divergence for $s < 1/4$, we define, for $s + 1/4 < \alpha < 1/2$,

$$f(x) = \sum_{\ell \geq 2} 2^{-\ell\alpha} f_\ell(x),$$

We note first that $f \in H^s$.

We will show that for $x \in [1/2, 1]$, and $t_j = \frac{x}{2^j} \sim 2^{-j}$,

$$|e^{it_j\Delta} f_\ell(x)| \leq C 2^{-\frac{1}{2}|j-\ell|} 2^{j/2}.$$

With this,

$$|e^{it_j\Delta} f(x)| \geq C 2^{-j\alpha + j/2} \longrightarrow \infty,$$

as $t_j \to 0$.

It remains to show that $|e^{it_j\Delta} f_\ell(x)|$ is small.

As before,

$$|e^{it_j\Delta} f_\ell(x)| = \left| \int_0^{2^{\ell/2}} e^{2\pi i(\xi(x-2^\ell 2\pi t_j)-2\pi t_j\xi^2)} d\xi \right|.$$

For $\ell << j$, $|e^{it_j\Delta} f_\ell(x)| \leq 2^{\ell/2} = 2^{(\ell-j)/2} 2^{j/2} \leq C 2^{-\frac{1}{2}|j-\ell|} 2^{j/2}$.

For $\ell >> j$, use integration by parts, noting that the phase $\phi(\xi) = \xi(x - 2^\ell 2\pi t_j) - 2\pi t_j \xi^2$, satisfies

$$|\phi'(\xi)| = |x - 2\pi t_j(2^\ell + 2\xi)| \geq 2^{\ell-j}.$$

Sjölin and Vega's result

We are going to present a proof of Sjölin and Vega's result:

$$e^{it\Delta}u_0 = u(x,t) = \int_{\mathbb{R}^n} e^{2\pi i(x\xi - 2\pi t|\xi|^2)}\widehat{u_0}(\xi)\,d\xi \to_{t\to 0} u_0(x) \qquad a.e.$$

for $u_0 \in H^s(\mathbb{R}^n)$, $s > 1/2$.

- It is enough to prove, for all $s > 1/2$,

$$\|\sup_{t\in[0,1]} |e^{it\Delta}f|\|_{L^2(B_1)} \leq C_s\|f\|_{H^s}.$$

- By Littlewood–Paley decomposition, it is enough to prove, for all $s > 1/2$ and all f such that $\operatorname{supp}\widehat{f} \subset \{|\xi| \sim R\}$,

$$\|\sup_{t\in[0,1]} |e^{it\Delta}f|\|_{L^2(B_1)} \leq R^s\|f\|_{L^2}.$$

• Due to the finite speed of propagation, it is enough to prove, for all f such that supp $\widehat{f} \subset \{|\xi| \sim R\}$,

$$\| \sup_{t \in [0, \frac{1}{R}]} |e^{it\Delta} f| \|_{L^2(B_1)} \leq R^{1/2} \|f\|_{L^2}.$$

• By scaling, it is enough to prove, for f, supp $\widehat{f} \subset \{|\xi| \sim 1\}$,

$$\| \sup_{t \in [0, R]} |e^{it\Delta} f| \|_{L^2(B_R)} \leq R^{1/2} \|f\|_{L^2}.$$

• By Bernstein's inequality,

$$\| \sup_{t \in [0, R]} |e^{it\Delta} f| \|_{L^2(B_R)} = \| \|e^{it\Delta} f\|_{L^\infty([0,R])} \|_{L^2(B_R[0,R])}$$

$$\leq \|e^{it\Delta} f\|_{L^2(B_R \times [0,R])}.$$

Hence, the problem has been reduced to the Trace lemma:

$$\|e^{it\Delta} f\|_{L^2(\mathbb{R}^n \times [0,R])} \leq C R^{1/2} \|f\|_{L^2}.$$

Bourgain's positive result

Theorem (Bourgain 2012)

For every $n \geq 3$, there is some $\theta_n < 1/2$, such that the a.e. pointwise convergence property holds for all $s > \theta_n$. (Actually, one can take $\theta_n = \frac{1}{2} - \frac{1}{4n}$).

- It is enough to prove, $\| \sup_{t\in[0,1]} |e^{it\Delta} f| \|_{L^2(B_1)} \leq C_s \|f\|_{H^s}$.
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For every $n \geq 3$, there is some $\theta_n < 1/2$, such that for all f such that supp $\widehat{f} \subset \{|\xi| \sim 1\}$,

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We will use an argument of induction on the dimension. For $n = 2$, Sanghyuk Lee's theorem gives us the starting point.

- If $\widehat{f} \subset \{|\xi| \sim 1\}$,

$$\| \sup_{t \in [0,R]} |e^{it\Delta} f| \|_{L^2(B_R)} = \| \, \|e^{it\Delta} f\|_{L^\infty([0,R])} \|_{L^2(B_R[0,R])}$$

can be treated as

$$\leq \|e^{it\Delta} f\|_{L^2(B_R \times [0,R])}.$$

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$$\mathcal{R}^* f(x,t) = \int_{B_1} f(\xi) e^{-2\pi i(\bar{x}\cdot\xi + t|\xi|^2)}\, d\xi = e^{it\Delta}\widehat{f}(\bar{x}).$$

We consider $(x,t) \in [0,K]^{n+1}$ for some big $K >> 1$. We decompose $B(0,1) \subset \mathbb{R}^n$ in "cubes" of sidelength $\frac{1}{K}$, Ω_α, centered at ξ_α. Define

$$\mathcal{R}^* f(x,t) = \sum_\alpha \int_{\Omega_\alpha} f(\xi) e^{-2\pi i(x\cdot\xi + t|\xi|^2)}\, d\xi := \sum_\alpha \mathcal{R}^* f_\alpha(x,t)$$

$$= \sum_\alpha e^{-2\pi i(x\cdot\xi_\alpha + t|\xi_\alpha|^2)} \int_{\Omega_\alpha} f(\xi) e^{-2\pi i(x\cdot(\xi-\xi_\alpha) + t(|\xi|^2 - |\xi_\alpha|^2))}\, d\xi$$

Here, $f_\alpha = f\chi_{\Omega_\alpha}$.

Note that $\mathcal{R}^* f_\alpha \sim a_\alpha e^{-2\pi i(x\cdot\xi_\alpha + t|\xi_\alpha|^2)}$ and $|\mathcal{R}^* f_\alpha|$ is "essentially constant" in $[0,K]^{n+1}$.

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Denote $\nu(\xi) = (-2\xi, 1)$, the normal vector to the surface $\tau = |\xi|^2$, at the point $(\xi, |\xi|^2)$.
Two situations may appear:

Multilinear situation.
There are $n+1$ points $\xi_{\alpha_1}, \xi_{\alpha_2}, \dots, \xi_{\alpha_{n+1}} \in B(0,1)$ such that $\det(\nu(\xi_{\alpha_1}), \nu(\xi_{\alpha_2}), \dots, \nu(\xi_{\alpha_{n+1}})) \geq K^{-1}$ and

$$|\mathcal{R}^* f_{\alpha_1}(x,t)|,\ |\mathcal{R}^* f_{\alpha_2}(x,t)|,\ |\mathcal{R}^* f_{\alpha_{n+1}}(x,t)| \geq K^{-n} \max_{\alpha} |\mathcal{R}^* f_{\alpha}(x,t)|$$

$$\geq K^{-2n} |\mathcal{R}^* f(x,t)|.$$

Then,

$$|\mathcal{R}^* f| \leq K^{2n} \Big(\prod_{k=1}^{n+1} |\mathcal{R}^* f_{\alpha_k}| \Big)^{\frac{1}{n+1}} \leq K^{2n} \Big(\sum_{non\ colinear} \prod_{k=1}^{n+1} |\mathcal{R}^* f_{\alpha_k}| \Big)^{\frac{1}{n+1}}.$$

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$$\| \sup_{0\le t\le R} |\mathcal{R}^* f| \|_{L^2_{\bar{x}}} \le \| |\mathcal{R}^* f| \|_{L^2_{\bar{x}} L^q_t}$$

$$\le K^{2n} \| \Big(\sum_{non\ colinear} \prod_{k=1}^{n+1} |\mathcal{R}^* f_{\alpha_k}| \Big)^{\frac{1}{n+1}} \|_{L^2_{\bar{x}} L^q_t}.$$

By Hölder's inequality

$$\le K^{2n} R^{n(\frac{1}{2}-\frac{1}{q})} \| \Big(\sum_{non\ colinear} \prod_{k=1}^{n+1} |\mathcal{R}^* f_{\alpha_k}| \Big)^{\frac{1}{n+1}} \|_{L^q(B_R\times[0,R])}.$$

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Theorem (Multilinear Strichartz's estimate. Bennett-Carbery-Tao 2006)

Under the above assumption on the normal vectors

$$\|\Pi_{k=1}^{n+1}\mathcal{R}^* f_k\|_{L^{2/n}} \le C_\epsilon R^\epsilon K^C \Pi_{k=1}^{n+1}\|f_k\|_2,$$

for all $\epsilon > 0$.

This gives, for $q = 2\frac{n+1}{n}$,

$$K^{2n}R^{n(\frac{1}{2}-\frac{1}{q})}\|(\sum_{non\ colinear}\prod_{k=1}^{n+1}|\mathcal{R}^* f_{\alpha_k}|)^{\frac{1}{n+1}}\|\|_{L^q(B_R\times[0,R])}$$

$$= K^{2n}R^{\frac{1}{2}-\frac{1}{2(n+1)}}\|\sum_{non\ colinear}\prod_{k=1}^{n+1}|\mathcal{R}^* f_{\alpha_k}|\|\|_{L^{2/n}(B_R\times[0,R])}^{\frac{1}{n+1}}$$

$$\le K^{2n}R^{\frac{1}{2}-\frac{1}{2(n+1)}}C_\epsilon R^\epsilon K^C\left(\sum_{non\ colinear}\prod_{k=1}^{n+1}\|f_k\|_{L^2}\right)^{\frac{1}{n+1}}$$

$$\le K^{2n}R^{\frac{1}{2}-\frac{1}{2(n+1)}}C_\epsilon R^\epsilon K^C\|f\|_{L^2}.$$

Theorem (Multilinear Strichartz's estimate. Bennett-Carbery-Tao 2006)

Under the above assumption on the normal vectors

$$\|\Pi_{k=1}^{n+1}\mathcal{R}^* f_k\|_{L^{2/n}} \leq C_\epsilon R^\epsilon K^C \Pi_{k=1}^{n+1}\|f_k\|_2,$$

for all $\epsilon > 0$.

This gives, for $q = 2\frac{n+1}{n}$,

$$K^{2n}R^{n(\frac{1}{2}-\frac{1}{q})}\|\Big(\sum_{\text{non colinear}}\prod_{k=1}^{n+1}|\mathcal{R}^* f_{\alpha_k}|\Big)^{\frac{1}{n+1}}\|_{L^q(B_R\times[0,R])}$$

$$= K^{2n}R^{\frac{1}{2}-\frac{1}{2(n+1)}}\|\sum_{\text{non colinear}}\prod_{k=1}^{n+1}|\mathcal{R}^* f_{\alpha_k}|\|_{L^{2/n}(B_R\times[0,R])}^{\frac{1}{n+1}}$$

$$\leq K^{2n}R^{\frac{1}{2}-\frac{1}{2(n+1)}}C_\epsilon R^\epsilon K^C\Big(\sum_{\text{non colinear}}\prod_{k=1}^{n+1}\|f_k\|_{L^2}\Big)^{\frac{1}{n+1}}$$

$$\leq K^{2n}R^{\frac{1}{2}-\frac{1}{2(n+1)}}C_\epsilon R^\epsilon K^C\|f\|_{L^2}.$$

Remark. We can compare the multilinear theorem with Strichartz's estimate,

$$\|\mathcal{R}^* f\|_{L^{\frac{2(n+2)}{n}}} \leq C\|f\|_2.$$

Using Hölder's inequality, we obtain, for any functions f_k,

$$\|\Pi_{k=1}^{n+1}\mathcal{R}^* f_k\|_{L^{\frac{2(n+2)}{n(n+1)}}} \leq \Pi_{k=1}^{n+1}\|\mathcal{R}^* f_k\|_{L^{\frac{2(n+2)}{n}}} \leq C\Pi_{k=1}^{n+1}\|f_k\|_2.$$

Concentration near a hyperplane.

There is a $n-1$–dimensional hyperplane, $\mathcal{L}$, such that, $|\mathcal{R}^* f_\beta| \le K^n \max_\alpha |\mathcal{R}^* f_\alpha(x,t)|$ whenever $\mathrm{dist}(\Omega_\beta, \mathcal{L}) \ge K^{-1}$.

Denote $\widetilde{\mathcal{L}}$ a $\frac{1}{K}$–neighborhood of $\mathcal{L}$.

$$|\mathcal{R}^* f| \le \Big|\sum_{\Omega_\beta \subset \mathcal{L}} \mathcal{R}^* f_\beta\Big| + \max_\alpha |\mathcal{R}^* f_\alpha|.$$

To deal with the first term, remember that $\mathcal{R}^* f_\alpha \sim a_\alpha e^{-2\pi i(x\cdot\xi_\alpha + t|\xi_\alpha|^2)}$, on K-cubes in space-time.
Assume that $\mathcal{L} = \{\xi_n = c\}$, and denote $x' = (x_1, x_2, \ldots, x_{n-1})$.
Then, we can choose ξ_α so that $\xi_{\alpha,n} = c$, so that $\mathcal{R}^* f_\alpha \sim \tilde{a}_\alpha e^{-2\pi i(x'\cdot\xi'_\alpha + t|\xi'_\alpha|^2)}$ on K-cubes in space-time.
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Decompose $[-R,R]^{n+1} = \cup B_{\gamma,\ell}$, where $B_{\gamma,\ell} = B_\gamma \times I_\ell$, are K-cubes.

$$\|\max_{|t|\le R}|\sum_{\Omega_\beta\subset\mathcal{L}}\mathcal{R}^* f_\beta|\|_{L^2(|x|\le R)}$$

$$\le\Big[\sum_{\gamma,\ell}\|\max_{t\in I_\ell}|\sum_{\Omega_\beta\subset\mathcal{L}}\mathcal{R}^* f_\beta|\|_{L^2(B_\gamma)}\Big]^{1/2}$$

Denote by Ω'_α a $\frac{1}{K}$–neighborhood of ξ'_α in $\mathbb{R}^{n-1}$ and $g(\xi')=\sum\tilde{a}_\alpha|\Omega'_\alpha|^{-1}\chi_{\Omega'_\alpha}$. Then $\|g\|_{L^2}\sim K^{\frac{n-1}{2}}\big(\sum|\tilde{a}_\alpha|^2\big)^{1/2}$ and

$$\sum\mathcal{R}^* f_\alpha\sim\sum\tilde{a}_\alpha e^{i(x'\cdot\xi'_\alpha+t|\xi'_\alpha|^2)}\sim\int g(\xi')e^{i(\bar{x}'\cdot\xi'+t|\xi'|^2)}\,d\xi'=R^*g(x'),$$

on $(x',t)\in B_\gamma\times I_\ell$.

$$\|\max_{|t|\le R}|\sum_{\Omega_\beta\subset\mathcal{L}}\mathcal{R}^*f_\beta|\|_{L^2(|x|\le R)}$$

$$\le\Big[\sum_{\gamma,\ell}\|\max_{t\in I_\ell}|\sum_{\Omega_\beta\subset\mathcal{L}}\mathcal{R}^*f_\beta|\|_{L^2(B_\gamma)}\Big]^{1/2}$$

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on $(x',t)\in B_\gamma\times I_\ell$.

Hence

$$\| \max_{|t|\le K} | \sum_{\Omega_\beta \subset \mathcal{L}} \mathcal{R}^* f_\beta ||\|_{L^2(|x|\le K)}$$

$$\le K^{1/2} \| \max_{|t|\le K} |\mathcal{R}^* g||\|_{L^2(|x'|\le K)}.$$

By induction on the dimension

$$\le K^{1/2} K^{\theta_{n-1}} \|g\|_{L^2} \le K^{1/2} K^{\theta_{n-1}} K^{\frac{n-1}{2}} \Big(\sum |a_\alpha|^2\Big)^{1/2}$$

$$\le K^{\theta_{n-1}} \left(\frac{1}{K} \sum \|R^* f_\alpha\|^2_{L^2(B_\gamma \times I_\ell)} \right)^{1/2}.$$

$$\le K^{\theta_{n-1}} \left(\frac{1}{K} \sum \|R^* f_\alpha\|_{L^2(B_R \times [0,R])} \right)^{1/2}.$$

Hence

$$\|\max_{|t|\le K}|\sum_{\Omega_\beta\subset\mathcal{L}}\mathcal{R}^*f_\beta|\|\|_{L^2(|x|\le K)}$$

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$$\le K^{1/2}K^{\theta_{n-1}}\|g\|_{L^2}\le K^{1/2}K^{\theta_{n-1}}K^{\frac{n-1}{2}}\big(\sum|a_\alpha|^2\big)^{1/2}$$

$$\le K^{\theta_{n-1}}\left(\frac{1}{K}\sum\|R^*f_\alpha\|^2_{L^2(B_\gamma\times I_\ell)}\right)^{1/2}.$$

$$\le K^{\theta_{n-1}}\left(\frac{1}{K}\sum\|R^*f_\alpha\|_{L^2(B_R\times[0,R])}\right)^{1/2}.$$

By Plancherel,

$$\leq K^{\theta_{n-1}} \left(\frac{R}{K}\right)^{1/2} \left(\sum_{\beta} \|f_\beta\|_{L^2}^2\right)^{1/2} = K^{\theta_{n-1}} \left(\frac{R}{K}\right)^{1/2} \|f\|_{L^2}.$$

With this, and the calculation for the multilinear case, we estimate

$$\| \sup_{|t| \leq R} |\mathcal{R}^* f| \|_{L^2_x} \leq \left[R^{\frac{1}{2} - \frac{1}{2(n+1)}} K^C + R^{1/2} K^{\theta_{n-1} - \frac{1}{2}} \right] \|f\|_{L^2}.$$

We choose $K = R^{\varepsilon_n}$, where $\varepsilon_n = \frac{1}{2(n+1)[C + \frac{1}{2} - \theta_{n-1}]}$, and obtain

$$\leq R^{\frac{1}{2} - \varepsilon_n (\frac{1}{2} - \theta_{n-1})} \|f\|_{L^2}.$$

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$$\leq R^{\frac{1}{2}-\varepsilon_n(\frac{1}{2}-\theta_{n-1})} \|f\|_{L^2}.$$

Concentration near a hyperplane.

There is a $n-1$–dimensional hyperplane, $\mathcal{L}$, such that, $|\mathcal{R}^* f_\beta| \le K^n \max_\alpha |\mathcal{R}^* f_\alpha(x,t)|$ whenever $\mathrm{dist}(\Omega_\beta, \mathcal{L}) \ge K^{-1}$.

Denote $\widetilde{\mathcal{L}}$ a $\frac{2}{K}$–neighborhood of $\mathcal{L}$.

$$|\mathcal{R}^* f| \le \Big|\sum_{\Omega_\beta \subset \mathcal{L}} \mathcal{R}^* f_\beta\Big| + \max_\alpha |\mathcal{R}^* f_\alpha|.$$

To deal with the first term, define a function ϕ by

$$\Big|\sum_{\Omega_\beta \subset \mathcal{L}} \mathcal{R}^* f_\beta\Big| =: \phi \Big(\sum_{\Omega_\beta \subset \mathcal{L}} |\mathcal{R}^* f_\beta|^2 \Big)^{1/2}.$$

Decompose $[-R,R]^{n+1} = \cup B_{\gamma,\ell}$, where $B_{\gamma,\ell} = B_\gamma \times I_\ell$, are K-cubes.

$$\| \max_{|t|\le R} \phi\Big(\sum |\mathcal{R}^* f_\beta|^2\Big)^{1/2} \|_{L^2(|\bar{x}|\le R)}$$

$$\le \Big[\sum_{\gamma,\ell} \sum_{\Omega_\beta \subset \mathcal{L}} |\mathcal{R}^* f_\beta|^2_{|B_{\gamma,\ell}} \int_{B_\gamma} \max_{t\in I_\ell} |\phi_{B_{\gamma,\ell}}|^2\, dx\Big]^{1/2}$$

Claim:

$$\int_{B_\gamma} \max_{t\in I_\ell} |\phi_{B_{\gamma,\ell}}|^2\, dx \le K^{2\theta_{n-1}} K^n.$$

$$\le K^{\theta_{n-1}} \Big[\frac{1}{K} \sum_{\gamma,\ell} \int_{B_{\gamma,\ell}} \sum_{\Omega_\beta \subset \mathcal{L}} |\mathcal{R}^* f_\beta|^2_{|B_{\gamma,\ell}}\, dx\Big]^{1/2}$$

$$= K^{\theta_{n-1}} \Big[\frac{1}{K} \int_{B_R} \sum_{\Omega_\beta \subset \mathcal{L}} |\mathcal{R}^* f_\beta|^2\, dx\Big]^{1/2}.$$

By the trace lemma,

$$\leq K^{\theta_{n-1}}\left(\frac{R}{K}\right)^{1/2}\left(\sum_{\beta}\|f_\beta\|_{L^2}^2\right)^{1/2} = K^{\theta_{n-1}}\left(\frac{R}{K}\right)^{1/2}\|f\|_{L^2}.$$

With this, and the calculation for the multilinear case, we estimate

$$\|\sup_{|t|\leq R}|\mathcal{R}^* f|\|_{L^2_{\bar{x}}} \leq \left[R^{\frac{1}{2}-\frac{1}{2(n+1)}}K^C + R^{1/2}K^{\theta_{n-1}-\frac{1}{2}}\right]\|f\|_{L^2}.$$

We choose $K = R^{\varepsilon_n}$, where $\varepsilon_n = \frac{1}{2(n+1)[C+\frac{1}{2}-\theta_{n-1}]}$, and obtain

$$\leq R^{\frac{1}{2}-\varepsilon_n(\frac{1}{2}-\theta_{n-1})}\|f\|_{L^2}.$$

About the claim:

Since $R^* f_\beta \sim a_\beta e^{-2\pi i(x\cdot\xi_\alpha + t|\xi_\alpha|^2)}$ in B_K, we seek for an estimate

$$\| \max_{|t|\le K} | \sum_{\xi_\alpha \in \widetilde{\mathcal{L}}} a_\alpha e^{-2\pi i(x\cdot\xi_\alpha + t|\xi_\alpha|^2)}| \|_{L^2(|\bar{x}|\le K)} \le K^{\theta_{n-1}} K^{n/2} \big(\sum |a_k|^2 \big)^{1/2}.$$

Assume that $\mathcal{L} = \{\xi_n = c\}$.

$$\| \max_{|t|\le K} | \sum_{\xi_\alpha \in \widetilde{\mathcal{L}}} a_\alpha e^{-2\pi i(x\cdot\xi_\alpha + t|\xi_\alpha|^2)}| \|_{L^2(|\bar{x}|\le K)}$$

$$\le K^{1/2} \| \max_{|t|\le K} | \sum_{\xi_\alpha \in \widetilde{\mathcal{L}}} a_\alpha e^{-2\pi i(x'\cdot\xi'_\alpha + t|\xi'_\alpha|^2)}| \|_{L^2(|\bar{x}'|\le K)}.$$

Notation: $x' = (x_1, x_2, \ldots, x_{n-1})$.

Denote by Ω'_α a $\frac{1}{K}$–neighborhood of ξ'_α in $\mathbb{R}^{n-1}$ and $g(\xi') = \sum a_\alpha |\Omega'_\alpha|^{-1} \chi_{\Omega'_\alpha}$. Then $\|g\|_{L^2} \sim K^{\frac{n-1}{2}} \big(\sum |a_\alpha|^2\big)^{1/2}$ and

$$\sum a_\alpha e^{i(x'\cdot\xi'_\alpha + t|\xi'_\alpha|^2)} \sim \int g(\xi') e^{i(\bar{x}'\cdot\xi' + t|\xi'|^2)}\, d\xi',$$

on $|\bar{x}'| \le K$. Hence

$$K^{1/2} \| \max_{|t|\le K} | \sum_{\xi_\alpha \in \widetilde{\mathcal{L}}} a_\alpha e^{i(x'\cdot\xi'_\alpha + t|\xi'_\alpha|^2)} | \|_{L^2(|x'|\le K)}$$

$$\le K^{1/2} \| \max_{|t|\le K} | \int g(\xi') e^{i(x'\cdot\xi' + t|\xi'|^2)}\, d\xi' | \|_{L^2(|x'|\le K)}.$$

By induction on the dimension

$$\le K^{1/2} K^{\theta_{n-1}} \|g\|_{L^2} \le K^{1/2} K^{\theta_{n-1}} K^{\frac{n-1}{2}} \big(\sum |a_\alpha|^2\big)^{1/2}.$$

The counterexample (Bourgain's)

Theorem (Bourgain 2012)

The estimate fails for $\theta_n < \frac{1}{2} - \frac{1}{n}$*).*

We need two results from number theory

Lemma

Given $\delta > 0$*, there is* $\theta \in \mathcal{S}^{n-1}$ *and* $C = C(\delta, n)$ *such that* $\mathbb{R}^n \subset R^{1/n}\mathbb{Z}^n + 0(\delta) + \theta[-CR, CR]$.

Remark: the proof in $\mathbb{R}^2$ is very easy.

Lemma

We can find R *as big as we want so that* $\sharp(\mathbb{Z}^n \cap R\mathcal{S}^{n-1}) \geq R^{n-2}$.

Denote

$$E = \left[R^{-1/n}\mathbb{Z}^n \cap \mathcal{S}^{n-1}\right] + \theta + B(0, \epsilon R^{-1}).$$

$$H = \{(\xi, \tau) : \ (\xi, \tau) \cdot (2\theta, 1) = 0\}.$$

Note that,

$$H \cap \{(\xi, -|\xi|^2) : \ \xi \in \mathbb{R}^n\} = \{(\xi, -|\xi|^2) : \ -2\theta\xi + |\xi|^2 = 0\}$$

$$= \{(\xi, -|\xi|^2) : \ |\xi - \theta| = 1\}.$$

Hence,

$$E \subset \mathcal{S}^{n-1} + \theta + 0(\epsilon R^{-1}) \subset \Pi(H \cap \{(\xi, |\xi|^2) : \ \xi \in \mathbb{R}^n\}) + 0(R^{-1}),$$

where Π denotes the orthogonal projection onto the $\mathbb{R}^n$.

Take $\widehat{f} = \chi_E$.

$$u(t,x) = \int_E e^{2\pi i (x\cdot\xi - t|\xi|^2)}\, d\xi.$$

For any $x \in B_R \cap \left[R^{1/n}\mathbb{Z}^n + 0(\delta)\right]$, and any $\xi \in E \cap B_4$, $x \cdot \xi \in \mathbb{Z} + 0(\epsilon + \delta) + x \cdot \theta$.
We estimate,

$$|u(0,x)| = |\int_E e^{2\pi i x\cdot\xi}\, d\xi| \sim |E|.$$

Moreover, for $|t| \leq CR$,

$$|u(t, x + 2t\theta)| = |\int_E e^{2\pi i (x\cdot\xi + 2t\theta\xi - t|\xi|^2)}\, d\xi|$$

$$= |\int_E e^{2\pi i (x\cdot\xi + 0 + 0(C\epsilon))}\, d\xi| \sim |E|.$$

Hence,

$$\sup_{|t|\leq CR} |u(t,y)| \geq c|E|,$$

for all $y \in B_R$.

Assume that

$$\| \sup_{t\in[0,R]} |e^{it\Delta} f| \|_{L^2(B_R)} \leq CR^{\theta} \|f\|_{L^2}.$$

Then, $|E|R^{n/2} \leq |E|^{1/2} R^{\theta}$ and thus, $|E| \leq R^{2\theta - n}$. On the other hand, we can find R as big as we want such that $\sharp \mathbb{Z} \cap R^{\frac{1}{n}} \mathcal{S}^{n-1} \geq R^{\frac{n-2}{n}}$. This gives $|E| \geq R^{\frac{n-2}{n}} \epsilon^n R^{-n}$. Hence, $\theta \geq \frac{n-2}{2n}$.