On a mean field game approach modeling pedestrian motion

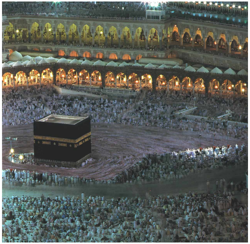
M.-T. Wolfram and A. Lachapelle

Mean Field Games and Related Topics,
Sapienza Università di Roma,
May 2011

universität wien
Faculty of Mathematics

Outline

Crowd motion



(lane formation movie)

Motivations

- Today half of the human population lives in urban areas, in 1950 $\sim$ 30%, prediction for 2050 $\sim$ 70%.
- Fatal accidents in the last decades increased, e.g. Hadj in Mekka, Love Parade in Duisburg, Water Festival in Phnom Penh ....
- Empirical studies of human crowd started about 50 years ago, based on observations, photographs and video data.
- Mathematical modeling and simulations have been used successfully to secure dangerous situations.

Mathematical modeling of human crowds I

- *Microscopic approaches:* Individuals are treated as agents whose motion is determined by the interaction with the surrounding agents and the goal to reach a desired destination.

  Behavioral force models - Helbing and Molnar (1995), Helbing et al. (2002), ...

  Cellular automata models - Burstedde et al. (2001), Kirchner and Schadschneider (2002), Adler and Blue (2000), ...

  Optimal control - Hoogendorn and Bovy (2003)

  Stochastic dynamic games - Huang et al. (2006), ...

- *Mesoscopic approaches:* Mainly kinetic models, ideas from gas kinetic theory are used.

  Henderson (1971), Hoogendorn and Bovy (2000)

Mathematical modeling of human crowds II

- *Macroscopic approaches:* Here the crowd is treated like a density.

  Fluid dynamics - Henderson (1974), Hughes (2002), Colombo and Rossini (2005), Chalons (2007), Venuti et al. (2007), Bellomo and Dogbé (2008), ...

  Optimal transportation - Maury et al. (2010)

  Nonlinear convection diffusion equations - Burger et al. (submitted, 2010) , ...

  Mean field games - Guéant (2009), Guéant et al. (2009), Lachapelle (2010), Dogbé (2010), ...

- *Multiscale approaches:* Coupling of micro- and macroscopic modeling approaches.

  Time evolving measures - Piccoli and Tosin (2009), Cristiani et al. (2010), ...

Mathematical modeling of human crowds II

- *Macroscopic approaches:* Here the crowd is treated like a density.

  Fluid dynamics - Henderson (1974), Hughes (2002), Colombo and Rossini (2005), Chalons (2007), Venuti et al. (2007), Bellomo and Dogbé (2008), ...

  Optimal transportation - Maury et al. (2010)

  Nonlinear convection diffusion equations - Burger et al. (submitted, 2010) , ...

  Mean field games - Guéant (2009), Guéant et al. (2009), Lachapelle (2010), Dogbé (2010), ...

- *Multiscale approaches:* Coupling of micro- and macroscopic modeling approaches.

  Time evolving measures - Piccoli and Tosin (2009), Cristiani et al. (2010), ...

This list is by no means complete !!!!!

Microscopic model

- N-player stochastic differential game

$$\inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^T f(t, X_t^{\alpha,x}, \alpha_t, m_t)dt + g[m_T](X_T^{\alpha,x})\right]$$
$$dX_t^{\alpha,x} = \alpha_t dt + \sigma dW_t\ ,\ X_0^x = x.$$

Macroscopic model

- Limiting equations as $N \to \infty$ one obtains a time dependent mean field game:

$$\frac{\partial u}{\partial t} + \nu\Delta u - H(x, \nabla u) = V[m]$$
$$\frac{\partial m}{\partial t} - \nu\Delta m - \mathrm{div}(\frac{\partial H}{\partial p}(x, \nabla u), m) = 0$$

with the initial and end conditions

$$u(x,T) = V[m(x,T)],\ m(x,0) = m_0(x).$$

Here $\nu \in \mathbb{R}^+$, H is the Legendre transform of the running cost f, i.e.

$$H(x,p) = \sup_{\alpha \in \mathbb{R}^d} (p \cdot \alpha - f(x,\alpha)), \text{ with } \lim_{|\alpha| \to \infty} \inf_x \frac{f(x,\alpha)}{|\alpha|} \to \infty.$$

- Stationary problem: Find (u, m, λ) such that

$$-\nu \Delta u + H(x, \nabla u) + \lambda = V[m]$$
$$-\nu \Delta m - \mathrm{div}(\frac{\partial H}{\partial p}(x, \nabla u) m) = 0$$
$$\int u \; dx = 0, \int m \; dx = 1, m > 0$$

Link to optimal control problems

If the running cost f has the form

$$f(x,t,\alpha,m) = L(x,t,\alpha)m + V[x,m]$$

and V is a Gateaux derivative of the potential Φ, then the MFG can be written as the following optimal control problem:

$$\inf_{\alpha}\left[\int\int L(\alpha,)dxdt + \Phi(m) + \Psi(m(T))\right]$$

under the constraint that

$$\frac{\partial m}{\partial t} - \nu\Delta m - \mathrm{div}(\alpha m) = 0.$$
$$m(x,0) = m_0(x).$$

Optimality conditions:

$$\alpha = \frac{\partial H}{\partial p}(x,\nabla u), \quad \text{and } V = \Phi'.$$

Numerical methods based on the optimal control formulation

One can use different approaches from the theory of parabolic optimal control problems. Simplest scheme is the steepest descent method:

1. Solve the Kolmogorov equation for m^n: $\frac{\partial m}{\partial t} - \nu \Delta m + \operatorname{div}(\alpha^n m) = 0$.
2. Solve the adjoint equation for u^n: $\frac{\partial u}{\partial t} + \nu \Delta u + \alpha^n \nabla u = \Phi'(m^n) + L(x, \alpha^n)$.
3. Update control parameter α: $\alpha^{n+1} = \alpha^n - \tau \frac{d\mathcal{L}}{d\alpha}$, where τ denotes the damping parameter and $\mathcal{L}$ the corresponding Lagrange functional, given by

$$\mathcal{L} = \int_0^T \int_0^1 \left[L(x, \alpha) m \; + u \left(\frac{\partial m}{\partial t} - \nu \Delta m + \operatorname{div}(\alpha m) \right) \right] dxdt + \Phi(m) + \Psi(m(T)).$$

4. Go to 1) until convergence.

Depending on the problem (convexity, ...) different methods like Newton-type methods or monotonic schemes can be used.

References: e.g. Lachapelle et al. (2010), ...

PDE-based methods

If the MFG is not equivalent to an optimal control problem things are not so nice ......

Stationary problems:

- Newton method in space

Time dependent problem:

- Newton scheme in space and time $\Rightarrow$ high computational effort !

See work by Achdou and Capuzzo-Dolcetta (2010)

A mean field games congestion model

We consider the following stochastic mean field game (for a single agent)

$$\mathbb{E}\left(\int_t^T (\frac{|\alpha_t|^q}{q}(m(X_t,s))^a + k(X_t,s)ds + \mu_0(X_t))e^{-rt}\right)$$
$$dX_t = \sigma dW_t + \alpha dt$$

The corresponding mean field game is given by

$$\frac{\partial u}{\partial t} + \nu\Delta u - \frac{1}{p}\frac{|\nabla u|^p}{m^b} - ru = k, \quad u(x,T) = u_T(x)$$
$$\frac{\partial m}{\partial t} - \nu\Delta m - \mathrm{div}(m\frac{(\nabla u)^{p-1}}{m^b}) = 0, \quad m(x,0) = m_0(x),$$

where $b = \frac{a}{q-1}$ and $p = \frac{q}{q-1}$.

Congestion model for two species I

The corresponding two species model looks slightly different. Here we assume that each density would like to avoid congestion within its own group as well as with the other. The corresponding stochastic model is given by

$$\mathbb{E}\left(\int_t^T (\frac{|\alpha_t^i|^q}{q}(m^i(X_t,s))^a(m^j(X_t,s))^{\tilde{a}} + k(X_t,s)ds + \mu_0(X_t))e^{-rt}\right)$$

$$dX_t = \sigma dW_t + \alpha^i dt$$

for i=1,2. The corresponding mean field game for both species reads as

$$-\nu\Delta u_i + \frac{1}{p}\frac{|\nabla u_i|^p}{m_i^b m_j^{\tilde{b}}} - ru_i + \lambda_i = k$$

$$-\nu\Delta m_i - \operatorname{div}(m_i\frac{(\nabla u_i)^{p-1}}{m_i^b m_j^{\tilde{b}}}) = 0$$

$$\int u_i dx = 0, \quad \int m_i \; dx = 1.$$

Congestion model for two species II

To avoid the diffusion by zero we consider a slightly different model, namely

$$
\begin{aligned}
-\nu\Delta u_i + \frac{1}{p}\frac{|\nabla u_i|^p}{(c+m_i)^b(c+m_j)^{\tilde{b}}} - ru_i + \lambda_i &= k \\
-\nu\Delta m_i - \operatorname{div}\Big(m_i\frac{(\nabla u_i)^{p-1}}{(c+m_i)^b(c+m_j)^{\tilde{b}}}\Big) &= 0 \\
\int u_i dx = \ 0, \quad \int m_i \ dx &= 1.
\end{aligned}
$$

for a small positive constant c.

- *Neumann boundary conditions:*

  In- and outflow or people m_i, i.e.

$$\frac{\partial m_i}{\partial n} = j_i^{in} \quad \text{for all } x \in \Gamma_i^{in} \quad \text{and} \quad \frac{\partial m_i}{\partial n} = j_i^{out} \quad \text{for all } x \in \Gamma_i^{out}$$

$$\text{with} \int_{\Gamma_i^{out}} j_i^{out} \cdot n \, ds = \int_{\Gamma_i^{in}} j_i^{in} \cdot n \, ds.$$

  $\Rightarrow$ homogenous Neumann boundary conditions for u_i, i.e. $\frac{\partial u_i}{\partial n} = 0$, for all $x \in \Gamma$.

  If $k = const$ a trivial solution is $u_i = 0$, $\lambda_i = k$ and m_i is the solution of

$$-\nu \Delta m_i = 0, \quad \int m_i dx = 1$$

  subject to the boundary conditions stated above.

- *Dirichlet boundary conditions:*

  Homogeneous Dirichlet conditions for m_i at the exit (people leave the room, hence the density has to be zero) and a homogenous Neumann boundary conditions on the rest of the boundary, i.e.

$$m_i = 0 \quad \text{for all } x \in \Gamma_i^{out} \text{ and } \frac{\partial m_i}{\partial n} = 0 \quad \text{on the rest of the boundary.}$$

  $\Rightarrow$ same boundary conditions for u_i, no additional variable λ_i and the integral condition for u_i necessary.

  Integral condition for m_i is replaced by a source term in the Kolmogorov equation, i.e.

$$-\nu \Delta m_i - \operatorname{div}(m_i \frac{(\nabla u_i)^{p-1}}{(c + m_j)^b}) = f(x).$$

  This source term can be interpreted as an exit of an underground or supermarket.

Hybrid discontinuous Galerkin methods for elliptic problems

Let's consider the "mother problem" on the domain Ω

$$-\Delta u = 0.$$

Notation: $\mathcal{T}_h$ denote the triangulation of Ω into triangles T, $\mathcal{F}_h$ the set of facets F.

Basic idea: Choose discontinuous Ansatzfunctions on the triangle and enforce continuity via Lagrange functions that life on the element interface (representing the trace of the continuous function u). We choose the following spaces

$$V_h := \{(u, u_F) : u \in P^k(T)\ \forall T \in \mathcal{T}_h, u_h \in L^2(F)\ \forall F \in \mathcal{F}_h\}$$

where P^k denotes the space of polynomials of degree less or equal to k.

Then the hybrid discontinuous Galerkin (HDG) method reads as:

$$\sum_{T\in\mathcal{T}_h}\Big[\int_T \nabla u \nabla v dx - \int_{\partial T}\frac{\partial u}{\partial n}(v-v_F)ds - \overbrace{\int_{\partial T}(u-u_F)\frac{\partial v}{\partial n}ds}^{\text{symmetry}}$$

$$+\underbrace{\frac{\alpha}{h}\int_{\partial T}(u-u_F)(v-v_F)ds}_{\textit{stability}}\Big] = 0$$

where α denotes the stability parameter and h the maximum mesh size.

HDG methods for hyperbolic problems

We consider

$$\mathrm{div}(bu) = 0$$

where the normal component of the vector field b is continuous across element interfaces.
The HDG formulation of the problem reads as

$$\sum_{T\in\mathcal{T}_h} \int_T \mathrm{div}(bu)vdx = \sum_{T\in\mathcal{T}_h} [-\int_T ub\cdot\nabla vdx + \int_{\partial T} u^{up} b_n vds]$$

where b_n denotes the normal component of the vector field b and u^{up} is the upwind value define by

$$u^{up} = \begin{cases} u & \text{if} b_n > 0 \\ u_F & \text{if } b_n < 0. \end{cases}$$

Problem: element only couple on the downwind element, to obtain a coupling with the upwind element we add the term

$$\int_{T^{out}} b_n(u_F - u)v_F ds \quad \text{where } T^{out} = \{x \in \partial T : b_n > 0\}.$$

HDG for the stationary congestion model

- Stationary problem is a coupled system of four nonlinear partial differential equations $\Rightarrow$ Newton's method.
- Two nonlinear convection-diffusion equations for m_i

$$-\nu\Delta m_i - \mathrm{div}\Big(\frac{m_i}{(c+m_i)^b(c+m_j)^{\tilde{b}}}\nabla u_i\Big) = f_i(x)$$

  $\Rightarrow$ HDG for diffusion and convection part (with upwind).
- Two nonlinear Hamilton Jacobi equations for u_i:

$$-\nu\Delta u_i + \frac{1}{2}\frac{|\nabla u_i|^2}{(c+m_i)^b(c+m_j)^{\tilde{b}}} - ru_i = 0$$

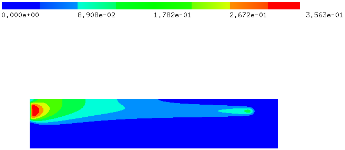
  $\Rightarrow$ HDG for diffusion and Hamiltonian (no stabilization).

Example I - validation of the model

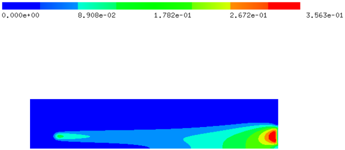
- Computational domain $\Omega = [-1, 1] \times [-0.2, 0.2]$
- Single source of people for every species, i.e. $f(x) = 50 \times \exp(-\frac{(x \pm 0.8)^2 + y^2}{10^{-3}})$
- The parameters are

$$a = 0.5, \ \tilde{a} = 0.5, \ q = 2, \ , \nu = 0.05, \ k = 1, \ r = 1.$$

- The maximum mesh size is $h = 0.03$ and we choose $c = 0.01$.



(a) Population m_1 (top view)

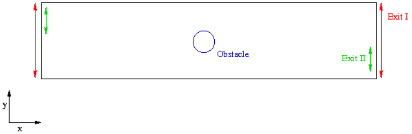


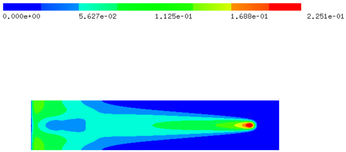
(b) Population m_2 (top view)

Figure: Formation of predefined lanes

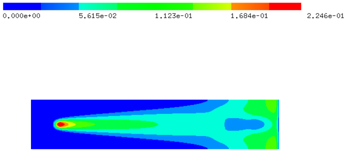
Example II - avoidance behavior

Same parameters as in the previous examples but the exits are different





(a) Population m_1

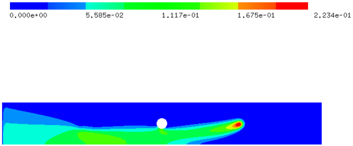


(b) Population m_2

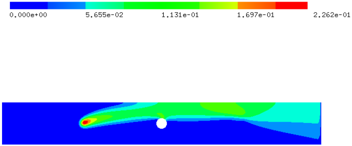
Example III - lane formation

- Computational domain $\Omega = [-1.5, 1.5] \times [-0.2, 0.2]$
- Single source of people for every species, i.e. $f(x) = 50 \times \exp(-\frac{(x \pm 0.75)^2 + y^2}{10^{-3}})$
- The parameters are

$$a = 0.25,\ \tilde{a} = 0.75,\ q = 2,\ ,\nu = 0.05,\ k = 1,\ r = 1.$$



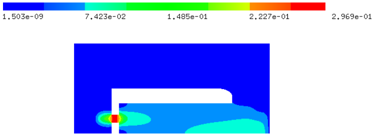
(c) Population m_1



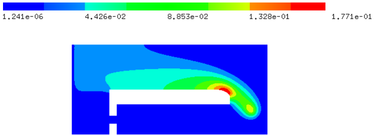
(d) Population m_2

- Rectangular domain with two corridors and a small door (bottleneck).
- Two sources placed in the lower left and lower right corner
- The parameters are

$$a = 0.25,\;\; \tilde{a} = 2,\;\; q = 2,\;\; ,\nu = 0.1,\;\; k = 1,\;\; r = 1.$$



(e) Population m_1



(f) Population m_2

Open problems

- Time dependent congestion simulations - efficient numerical methods.
- Stable HDG method for Hamilton Jacobi equations.
- Boundary conditions for the congestion model ....
- Analytical results for 2 species model - existence, uniqueness ....

Open problems

- Time dependent congestion simulations - efficient numerical methods.
- Stable HDG method for Hamilton Jacobi equations.
- Boundary conditions for the congestion model ....
- Analytical results for 2 species model - existence, uniqueness ....

Thank you very much for your attention !