Numerical approaches for MFG

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Workshop "'MEAN FIELD GAMES AND RELATED TOPICS"', Rome May 12-13 2011

Outline

Mean field games

- Mean field games: limits of Nash equilibriums for infinite number of players (P.L.Lions & J.M.Lasry)
- equation for each player $dX_t = \alpha dt + \sigma dW_t$, $\alpha(t,x) = \text{control}$
- $m(t,x)$ = the density of players at time t and position $x \in Q$
- evolution equation

$$\frac{\partial}{\partial t} m(t,x) - \nu \Delta m(t,x) + div(\alpha(t,x) m(t,x)) = 0,$$
$$m(0,x) = m_0(x).$$

- We consider the optimisation setting: $\min_\alpha J(\alpha)$

$J(\alpha) := \Psi(m(\cdot,T)) + \int_0^T \left\{ \Phi(m(t,\cdot)) + \int_Q L(x,\alpha) m(t,x) dx \right\} dt$

- Φ, Ψ can be linear, concave, ... Typical L : $L(x,\alpha) = \frac{\alpha^2}{2}$.

Numerics of MFG : literature overview

• (in)finite horizon: finite-difference discretization: approximation properties, existence and uniqueness, bounds on the solutions. "Mean Field Games: Numerical Methods" Y. Achdou & I. Capuzzo-Dolcetta

• Y. Achdou & I. Capuzzo-Dolcetta: Newton method for the coupled direct-adjoint critical point equations (finite horizon)

• O. Gueant: study of a prototypical case: solution, stability (09), quadratic Hamiltonian (11)

• solution of the MFG equations from an optimization point of view (A. Lachapelle, J. Salomon, G. Turinici, M3AS 2010)

• Lachapelle & Wolfram (2011) (congestion modelling)

Outline

Optimal control of a Fokker-Plank equation (G. Carlier & J. Salomon)

Evolution equation :

$$\partial_t \rho - \epsilon^2 \Delta \rho + div(v\rho) = 0 \quad (1)$$

$$\rho(x, t = 0) = \rho_0(x) \quad (2)$$

- goal: minimize w.r. to v the functional (for some given $V(\cdot)$) :

$$E(v) = \int \int \rho v^2 dxdt + \int \rho(x,1) V(x)$$

Time dependent Schrödinger equation w. BILINEAR interaction (e.g. laser)

$$\begin{cases} i\frac{\partial}{\partial t}\Psi(x,t) = (H_0 - \epsilon(t)^k \mu(x))\Psi(x,t) \\ \Psi(x, t=0) = \Psi_0(x) \end{cases} \tag{3}$$

• vectorial case (rotation control, NMR):
$i\frac{\partial}{\partial t}\Psi(x,t) = [H_0 + (E_1(t)^2 + E_2(t)^2)\mu_1 + E_1(t)^2 \cdot E_2(t)\mu_2]\Psi(x,t)$.
$H_0 = -\Delta + V(x)$, unbounded domain
Evolution on the unit sphere: $\|\Psi(t)\|_{L^2} = 1, \ \forall t \geq 0$.

• evaluation of the quality of a control through a objective functional to minimize

$J(\epsilon) = -2\Re\langle\psi_{target}|\psi(\cdot,T)\rangle + \int_0^T \alpha(t)\epsilon^2(t)dt$

$J(\epsilon) = \|\psi_{target} - \psi(\cdot,T)\|_{L^2}^2 - 2 + \int_0^T \alpha(t)\epsilon^2(t)dt$

$J(\epsilon) = -\langle\Psi(T), O\Psi(T)\rangle + \int_0^T \alpha(t)\epsilon^2(t)dt$

General monotonic algorithms (J. Salomon, G.T.)

state $X \in H$, control $v \in E$, H, E = Banach spaces.

- $\partial_t X + A(t, v(t))X = B(t, v(t))$
- $\min_v J(v)$, $J(v) := \int_0^T F\big(t, v(t), X(t)\big)\, dt + G\big(X(T)\big)$.
- F, G: C^1 + concavity with respect to X (not v!)

$$\forall X, X' \in H,\ G(X') - G(X) \leq \langle \nabla_X G(X), X' - X \rangle$$

$\forall t \in \mathbb{R}, \forall v \in E, \forall X, X' \in H$:

$$F(t, v, X') - F(t, v, X) \leq \langle \nabla_X F(t, v, X), X' - X \rangle.$$

Direct-adjoint equations and first lemma

$$\partial_t X + A(t, v(t))X = B(t, v(t))$$
$$X(0) = X_0$$

$$\partial_t Y_v - A^*\big(t, v(t)\big) Y_v + \nabla_X F\big(t, v(t), X_v(t)\big) = 0$$
$$Y_v(T) = \nabla_X G\big(X_v(T)\big).$$

Lemma

Suppose that A, B, F are differentiable everywhere in $v \in E$, then there exists $\Delta(\cdot, \cdot; t, X, Y) \in C^0(E^2, E)$ such that, for all $v, v' \in E$

$$J(v') - J(v) \le \int_0^T \Delta(v', v; t, X_{v'}, Y_v) \cdot_E \Big(v' - v\Big) dt \qquad (4)$$

Well-posedness

$$J(v') - J(v) \leq \int_0^T \Delta(v', v; t, X_{v'}, Y_v) \cdot_E \Big(v' - v \Big) dt \quad (5)$$

Remark: useful factorisation because can test at each step if J goes the right way; also can choose $v'(t^*) = v(t^*)$ if pb.
Remark: $\Delta(v', v; t, X, Y)$ has an explicit formula once the problem is given; also note the dependence on Y_v any not $Y_{v'}$.

Lemma

Under hypothesis on $A, B, F, G, \theta > 0$

$$\Delta(v', v; t, X, Y) = -\theta(v' - v) \quad (6)$$

has an unique solution $v' = \mathcal{V}_\theta(t, v, X, Y) \in E$.

Well-posedness

Theorem (J. Salomon, G.T. Int J Contr, 84(3), 551, 2011)

Under hypothesis ...

- *the following eq. has a solution:*

$$\partial_t X_{v'}(t) + A(t, v')X_{v'}(t) = B(t, v') \quad (7)$$

$$v'(t) = \mathcal{V}_\theta(t, v(t), X_{v'}(t), Y_v(t)) \quad (8)$$

$$X_{v'}(0) = X_0 \quad (9)$$

- $\exists\ (\theta_k)_{k\in\mathbb{N}}$ *such that* $v^{k+1}(t) = \mathcal{V}_{\theta_k}(t, v^k(t), X_{v^{k+1}}(t), Y_{v^k}(t))$
- $J(v^{k+1}) - J(v^k) \leq -\theta_k \|v^{k+1} - v^k\|^2_{L^2([0,T])}$;
- *if* $v^{k+1}(t) = v^k(t)$: $\nabla_v J(v^k) = 0$.

Outline

The Model : framework

- large economy: continuum of consumer agents
- time period: $[0, T]$
- any household owns exactly one house and cannot move to another one until T

The Model : the agents

- arbitrage between insulation and heating. A generic player (agent) has an insulation level $x \in [0,1]$ ($x = 0$: no insulation, $x = 1$: maximal insulation)

- controlled process of the agent: $dX_t = \sigma dW_t + v_t dt + dN_t(X_t)$ where v is the control parameter (insulation effort), the noise level σ is given.

- note that X_t is a diffusion process with reflexion, in the above equality, $dN_t(X_t)$ has the form $\chi_{\{0,1\}}(X_t)\vec{n}d\xi_t$ (ξ = local time at the boundary $\{0,1\} = \partial[0,1]$ cf. Freïdlin)

- initial density: $X_0 \sim m_0(dx)$

The Model : the costs

An agent of the economy solves a minimization problem composed of several terms:

- *Insulation acquisition cost:* $h(v) := \frac{v^2}{2}$
- *Insulation maintenance cost:* $g(t,x,m) := \frac{c_0 x}{c_1 + c_2 m(t,x)}$ increasing in x decreasing in m : economy of scale, positive externality. The agents should do the same choice, stay together. The higher is the number of players having chosen an insulation level, the lower are the related costs.
- *Heating cost:* $f(t,x) := p(t)(1 - 0,8x)$ where $p(t)$ is the unit heating cost (unit price of energy, say)

The model - The minimization problem and MFG (1)

- Define the aggregate state cost:

$$\Phi(m) := \int_0^1 \left(p(t)(1-0,8x) + \frac{c_0 x}{c_1 + c_2 m(t,x)} \right) m(t,x) dx$$

and $V = \Phi'$.

- In the model, the agents have rational expectations, i.e they see m as given; we can write the individual agent's problem:

$$\begin{cases} \inf_{v \text{ adm}} \mathbb{E}\left[\int_0^T h(v(t,X_t^x)) + V[m](X_t^x) dt \right] \\ dX_t = v_t dt + \sigma dW_t + dN_t(X_t), X_0 = x \end{cases}$$

The model - The minimization problem and MFG (2)

• We already know that it is linked with the optimal control problem:

$$\begin{cases} \inf_{v \text{ adm}} \int_0^T \int_0^1 h(v(t,x)) + \Phi(m_t)(t) dt \\ \partial_t m - \frac{\sigma^2}{2}\Delta m + \mathrm{div}(vm) = 0 \ , \ m|_{t=0} = m_0(.) \ , \\ m'(.,0) = m'(.,1) = 0 \end{cases}$$

• Finally, if $\nu := \frac{\sigma^2}{2}$, a Mean field equilibrium (Nash equilibrium with an infinite number of players) corresponds to a solution of the following system:

$$\begin{cases} \partial_t m - \nu\Delta m + \mathrm{div}(vm) = 0 \ , \ m|_{t=0} = m_0 \\ \nabla u = v \\ \partial_t u + \nu\Delta u + v \cdot \nabla u - \frac{u^2}{2} = \Phi'(m) \ , \ v|_{t=T} = 0 \end{cases} \tag{10}$$

The model - externality & scale effect

The MFG framework is interesting to describe a situation which lives between two economical ideas: positive externality and economy of scale

• positive externality: positive impact on any agent utility NOT INVOLVED in a choice of an insulation level by a player

• economy of scale: economies of scale are the cost advantages that a firm obtains due to expansion (unit costs decrease)

Criticism of the model:

- stylised from the "industrial" point of view
- not realistic (heating price, maintenance...)
- transition effect (continuous time, continuous space)
- atomised agent (her/his action has no influence on the global density, micro-macro approach)
- non-cooperative equilibrium with rational expectations

Numerical simulations

- Optimization method: Monotonic algorithm

$$\begin{cases} \partial_t m^{k+1} - \nu \Delta m^{k+1} + \mathrm{div}(v^{k+1} m^{k+1}) = 0 \ , \ m^{k+1}(x,0) = m_0 \\ v^{k+1} = \frac{(\theta - 1/2) v^k - \nabla u^k}{(\theta + 1/2)} \\ \partial_t u^{k+1} + \nu \Delta u^{k+1} + v^{k+1} \cdot \nabla u^{k+1} - \frac{(u^{k+1})^2}{2} = \Phi'(m^{k+1}), \ \ v^{k+1}(T) = 0 \end{cases} \tag{11}$$

- Discretization of the PDEs: Godunov scheme (to preserve the positivity of the density m)
- The costs:

heating: $f(t,x) = p(t)(1 - 0,8x)$
insulation: $g(t,x,m) = \frac{x}{0.1 + m(t,x)}$

- *1st example*: $p(t)$ constant / same choices
- *2d example*: $p(t)$ reaching a peak (non constant) / multiplicity of equilibria

Numerical results - First case

- the initial density of the householders is a gaussian centered in $\frac{1}{2}$
- the time period and the noise are respectively $T = 1$ and $\nu = 0.07$
- the energy price is constant ($p(t) \equiv 0$, 3.2 and 10)

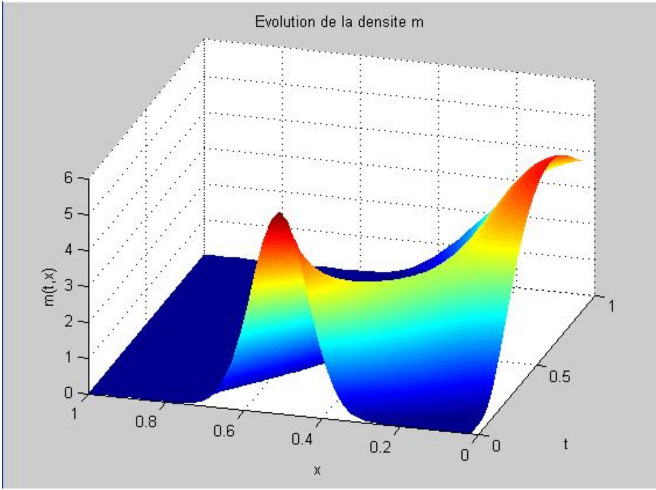

Figure: Numerical results : $p(t) \equiv 0$. Since the cost of energy is null all agents choose to heat their house, move to this choice together.

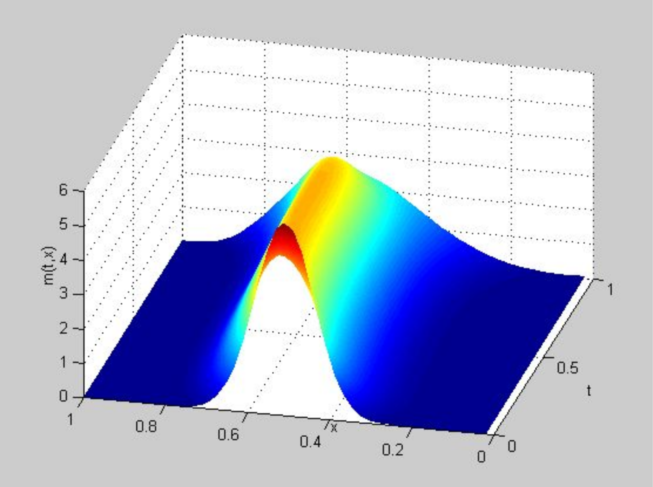

Figure: Numerical results : $p(t) \equiv 3.2$. Cost of energy is intermediary, agents keep their status.

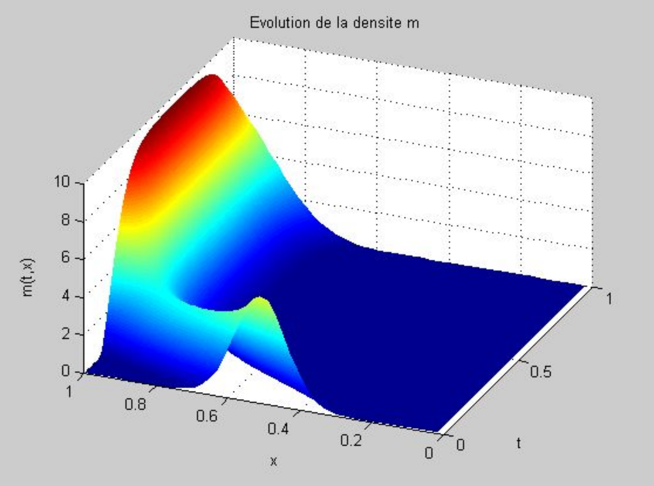

Figure: Numerical results : $p(t) \equiv 10$. Cost of energy is high, agents choose to better insulate, all have the same behavior.

Numerical results - Second case

- the initial density of the agents is an approximation of a Dirac in 0.1 (*i.e* agents are not equipped in insulation material)
- the energy price is not a constant parameter, we look at the following case: the price first reaches a peak and then decreases to its initial level.

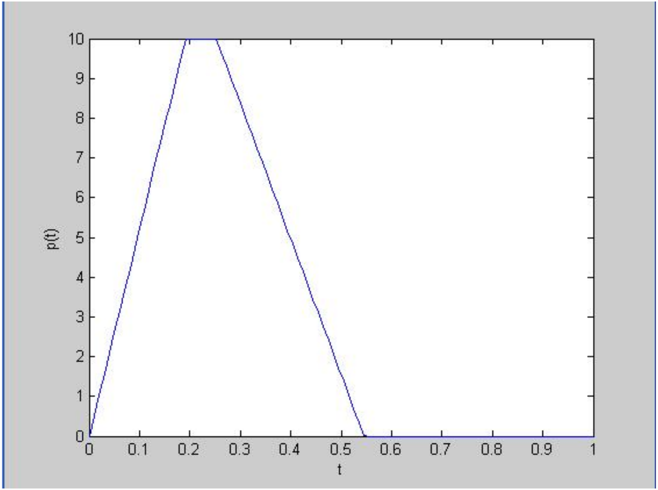

Figure: Numerical results - $p(t)$. Question: In such a case, can we find two Mean Field equilibria, the first related to the expectation of a higher insulation level, the second to the expectation of heating ?

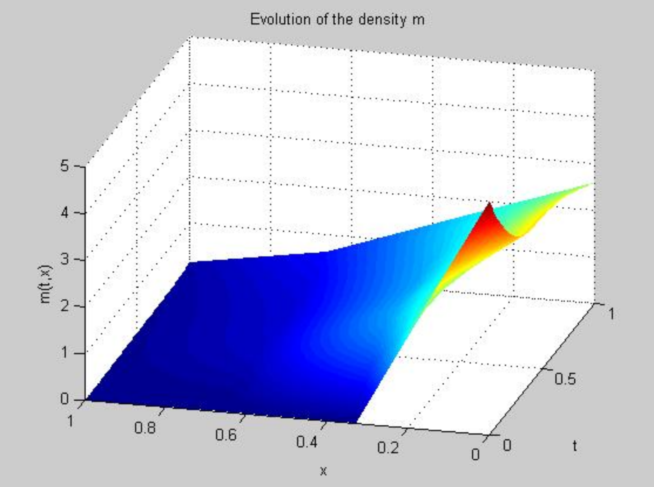

Figure: Numerical results - One of the two equilibria: the energy consumption equilibrium. Agents expect that everybody will keep a low insulation level so there are no gains in insulating.

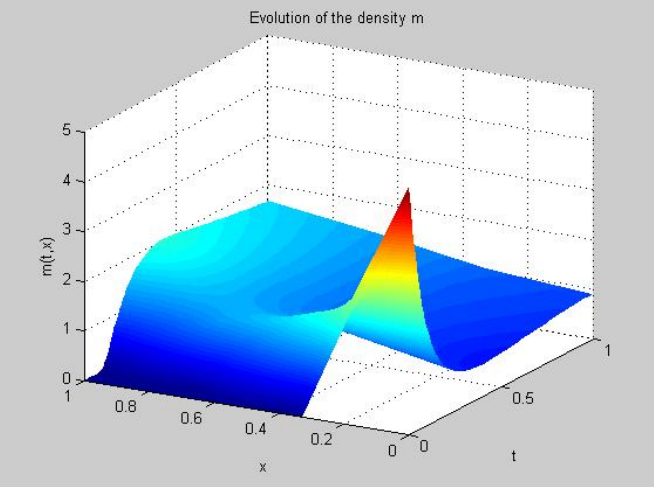

Figure: Numerical results - One of the two equilibria: the insulation equilibrium. Agents expect that everybody will better insulate, which makes insulating attractive.

Multiplicity of equilibria - Incentive policy

- we found an insulation-equilibrium and an energy consumption-equilibrium

- from the ecological point of view: the best is the insulation-equilibrium

- incentive public policies could steer towards the "best" equilibrium (from a certain point of view) when the solution is not unique.

Outline

Liquidity from heterogenous beliefs and analysis costs (joint work with Min Shen, Université Paris Dauphine)

- Why do agents trade ? Here: heterogenous beliefs and expectations
- Liquidity : many definitions (bid/ask spread, rapidity to recover price after shock, max volume traded at same price etc). Here: trading volume.
- Several approaches: limit order book modeling and optimal order submission (Avellaneda et al. 2008) Heterogenous beliefs: asset pricing (working paper by Emilio Osambela), short sale constraints (Gallmeyer and Hollifield 2008) etc.,
- Specific investigation of this work: question on analysis time/cost

Heterogenous beliefs and liquidity: the model

- One security of "true" value V.
- each agent has each own estimation $V \cdot A$ with A random variable with density $\rho(A)$, mean=1.
- The estimation is uncertain with precision $B(A)$ (B can be e.g. $V^2/\sigma^2(A)$ ($\sigma^2(A)$ = variance of the estimate).
- Precision can be improved paying $f(B)$ and/or waiting for the estimation to converge or new data to be revealed.
- Each agent has an utility function $U(\mathrm{mean}(\theta, B), \mathrm{variance}(\theta, B))$ (equivalent: expected utility framework for normal variable). Linear situation $U(x, y) = x - \lambda y$. Here $V \cdot \theta(A)$ = size of the position of agent at A
- Price Vp^A maximizes liquidity and equals offer and demand.

Note: p^A is not necessarily equal to 1 even if the mean $\mathbb{E}(A) = 1$.

Heterogenous beliefs and liquidity: theoretical results

Technical framework: Mean Field Games by Lasry & Lions; Nash equilibrium

$\mathrm{mean}(\theta, B) = V\theta(A - p^A) - f(B)$; $\mathrm{variance}(\theta, B) = \theta^2 V^2 / B$.

Theorem (M Shen, G.T. 2011)

Under assumptions on functions f and U the equilibrium exists. Offer and demand functions are monotone with respect to p^A.

Theorem (M Shen, G.T. 2011)

Under assumptions on functions f and U if ρ is symetric around p^1 then (liquidity is maximal for $p = p^1$ i.e.) $p^A = p^1$.

Heterogenous beliefs and liquidity: linear case $U = x - \lambda y$

Theorem (M Shen, G.T. 2011)

For the linear case the equlibrium relative price is:

$$P^A = \frac{\int_0^\infty AB(A)\rho(A)dA}{\int_0^\infty B(A)\rho(A)dA}. \tag{12}$$

The relative accuracy $B(A)$ cost is

$$B = (f')^{-1}\left(\frac{(A - P^A)^2}{2\lambda}\right). \tag{13}$$

Heterogenous beliefs and liquidity: linear case $U = x - \lambda y$

The relative market price P^A is solution to the equation:

$$\frac{1}{2V\lambda}\int_0^\infty (A-P^A)(f')^{-1}\left(\frac{(A-P^A)^2}{2\lambda}\right)\rho(A)dA = 0 \qquad (14)$$

The trading volume TV_f is

$$TV_f = \frac{P^A}{2\lambda}\int_0^\infty (A-P^A)_+(f')^{-1}\left(\frac{(A-P^A)^2}{2\lambda}\right)\rho(A)dA \qquad (15)$$

Theorem (anti-monotony of trading volume)

Let f, g be two information cost functions such that $g'(b) \geq f'(b)$ for any $b \in \mathbb{R}_+$. Then the trading volume satisfies $TV_f > TV_g$.

Application: for constant total cost $\int f(B)\rho(A)$ which is the greatest volume : is volume brought by best paid analysts ?

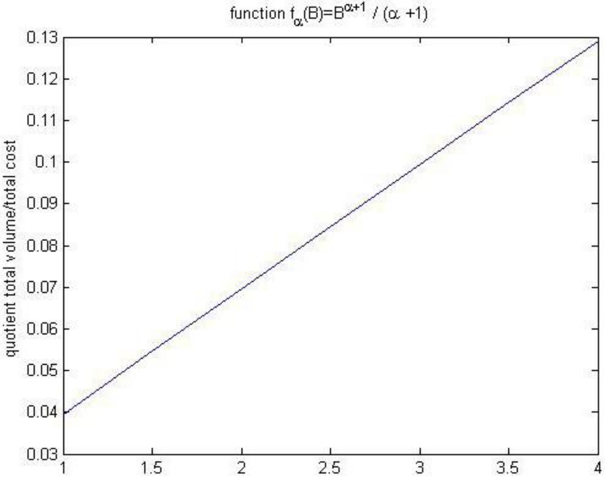

Figure: Quotient of the total volume over total cost for functions $f(B) = \frac{B^{\alpha+1}}{\alpha+1}$