On PDE Models for (some) Socio-Economic Problems: Price Formation, Opinion Formation and Crowd Modeling

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Outline

Price Formation - Introduction

In a series of papers J.-M. Lasry and P.-L. Lions introduced coarse grained models for economical equilibria in markets with a large number of rational players.

Basic setup:

Consider N players, whose investment strategies follow stochastic differential equations with Brownian diffusion and drifts, determined as Nash equilibria of an appropriate cost functional. When the number of players N tends to infinity (in analogy to statistical mechanics), systems of highly non-linear PDEs are obtained, e.g.

- Viscous Hamilton-Jacobi-Bellmann systems
- One-dimensional parabolic equation with a free boundary

J.-M. Lasry and P.-L. Lions
Mean field games.
Jpn. J. Math., 2(1):229-260, 2007.

Given a large group of buyers and vendors, with densities f_B and f_V respectively, the formation of the agreed price $x = p(t) \in \mathbb{R}$ can be described by

$$
\begin{aligned}
&\frac{\partial f_B}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f_B}{\partial x^2} = \lambda(t)\delta\left(x - p(t) + a\right), \quad \text{for } x < p(t)\\
&f_B \geq 0,\ f_B(x,t) = 0 \ \text{ for } x \leq p(t)\\
&\frac{\partial f_V}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f_V}{\partial x^2} = \lambda(t)\delta\left(x - p(t) - a\right), \quad \text{for } x > p(t)\\
&f_V \geq 0,\ f_V(x,t) = 0 \ \text{ for } x \geq p(t),
\end{aligned}
$$

where

$$\lambda(t) = -\frac{\sigma^2}{2}\frac{\partial f_B}{\partial x}\left(p\left(t\right), t\right) = \frac{\sigma^2}{2}\frac{\partial f_V}{\partial x}\left(p\left(t\right), t\right)$$

is the transaction rate, $\sigma > 0$ measures the randomness and $2a$ the bid-ask spread. The initial datum satisfies a compatibility condition

$$
\begin{aligned}
f_B(x, t=0) > 0 \text{ for } x < p_0, f_B(x, t=0) = 0 \text{ for } x \geq p_0\\
f_V(x, t=0) > 0 \text{ for } x > p_0, f_V(x, t=0) = 0 \text{ for } x \leq p_0.
\end{aligned}
$$

Introduce the function

$$f(x,t) = \begin{cases} f_B(x,t) & \text{for } x < p(t) \\ -f_V(x,t) & \text{for } x > p(t). \end{cases}$$

Then the system reduces to

$$\frac{\partial f}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} = \lambda(t)\left(\delta(x - p(t) + a) - \delta(x - p(t) - a)\right)$$
$$f(x,t) > 0 \text{ if } x < p(t),\ \ f(x,t) < 0 \text{ if } x > p(t)$$

with initial conditions

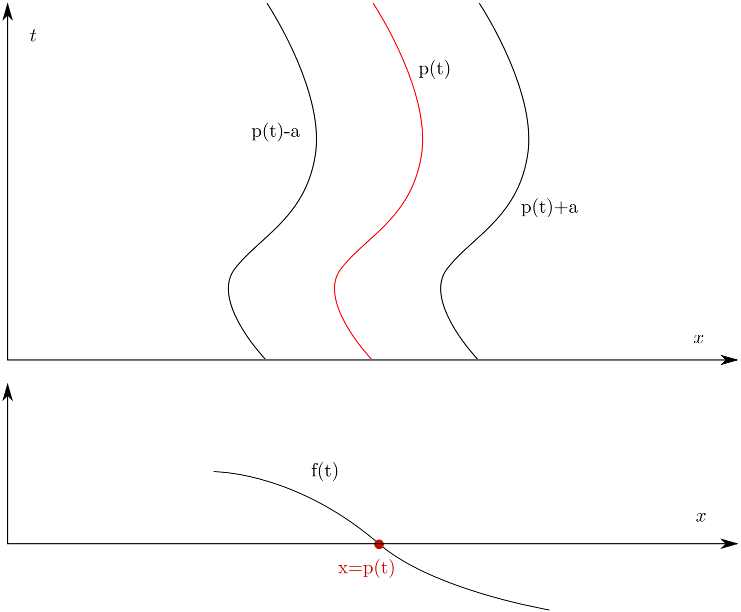
$$f(x,0) = f_I(x),\ p(0) = p_0.$$

Conservation of masses:

$$\frac{d}{dt}\int_{\mathbb{R}} f_B(x,t)dx = \frac{d}{dt}\int_{-\infty}^{p(t)} f(x,t)dx = 0$$
$$\frac{d}{dt}\int_{\mathbb{R}} f_V(x,t)dx = -\frac{d}{dt}\int_{p(t)}^{\infty} f(x,t)dx = 0$$

Min-max principle: $f > 0$ for $x < p(t)$; $f < 0$ for $x > p(t)$ and $\lambda(t) \geq 0$.

Setup



Nonlinear Character of the Problem

- Using the shift $x = p(t) + y$ we obtain

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2} - \frac{\partial g}{\partial y}(0,t)\left[\delta(y+a) - \delta(y-a)\right] + \dot{p}(t) g_y$$
$$\dot{p}(t) = -\frac{g_{yy}(0,t)}{g_y(0,t)},$$

  where we set $g(y,t) = f(y + p(t), t)$.

- Here the time derivative of the free boundary $\dot{p}(t)$ can be interpreted as the constraint that ensures $g(0,t) = 0$. Note that this formulation, based on mapping the free boundary into the line $y = 0$, shows that the problem under consideration is highly nonlinear.
- *Difficulty*: t-locally uniform estimates.

Literature

Luis A. Caffarelli, P.A. Markowich, J.-F. Pietschmann,
On a price formation free boundary model by Lasry & Lions.
submitted

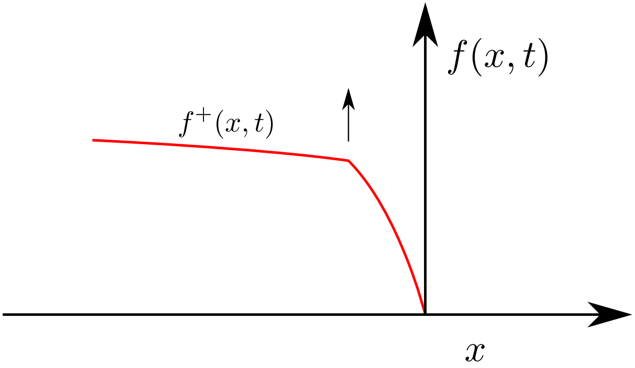
M.d.M Gonzalez, M.P. Gualdani
Asymptotics for a free-boundary problem in price formation.
Nonlinear Analysis 74, 3269-3294, 2011

P.A. Markowich, N. Matevosyan, J.-F. Pietschmann, M.-T. Wolfram
On a parabolic free boundary equation modeling price formation.
Math. Models Methods Appl. Sci., 11(19):1929-1957, 2009

L. Chayes, M.d.M Gonzalez, M.P. Gualdani and I. Kim
Global existence and uniqueness of solutions to a model in price formation.
SIAM J. Math. Anal., 41(5):2107-2135, 2009

M.d.M Gonzalez, M.P. Gualdani
Asymptotics for a symmetric equation in price formation.
Applied Mathematics and Optimization, 59(2):233-246, 2009

FBP → Heat Equation (I)



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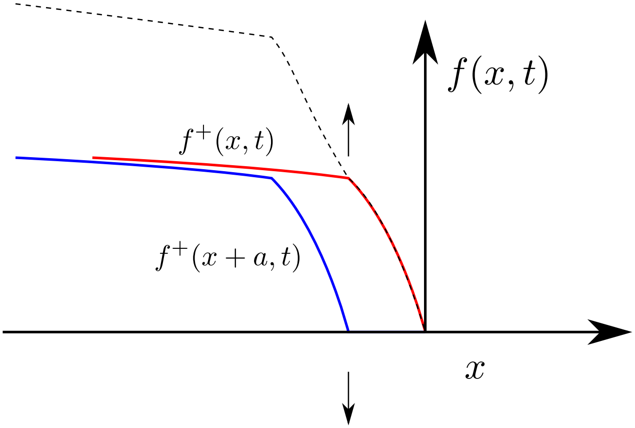

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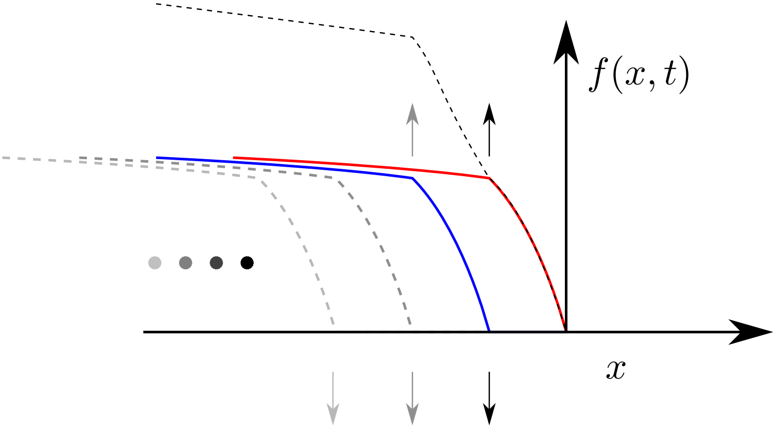

FBP $\rightarrow$ Heat Equation (II)

- Apply construction to positive and negative part, i.e.

$$F(x,t) = \begin{cases} \sum_{n=0}^{\infty} f^{+}(x+na,t), & x < p(t), \\ -\sum_{n=0}^{\infty} f^{-}(x-na,t), & x > p(t). \end{cases}$$

$\Rightarrow$ Then, F fulfils, in the sense of distributions

$$\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0,$$

with initial datum

$$F_I(x) = \begin{cases} \sum_{n=0}^{\infty} f_I^{+}(x+na), & x < p_0, \\ -\sum_{n=0}^{\infty} f_I^{-}(x-na), & x > p_0, \end{cases}$$

The free boundary $p = p(t)$ is the zero-level set of the solution F of the heat equation

Heat Equation $\to$ FBP (I)

- Let F be the solution of the heat equation with intial datum F_I as constructed above.
- Then $f = f(x,t)$ given by

$$f(x,t) = \begin{cases} F^+(x,t) - F^+(x+a), & x < p(t), \\ -F^-(x,t) + F^-(x-a), & x > p(t). \end{cases}$$

is a solution to the FBP.

Theorem (Global Existence)

There exists a unique smooth solution $f = f(x,t)$ of the FBP for $t \in [0,\infty)$. Furthermore, $p \in \mathcal{C}([0,\infty))$.

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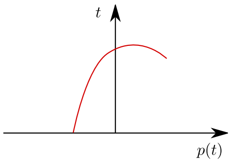
Proof.



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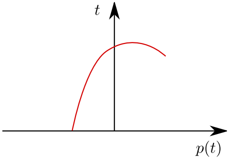

$f_x(p(t),t) < 0$ for all $t > 0$ (by the Hopf Lemma) and the min-max principle implies that $p = p(t)$ is graph of a function.

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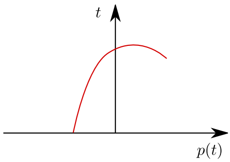

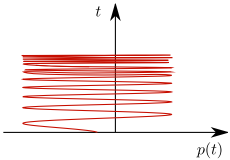

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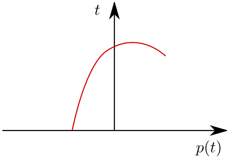

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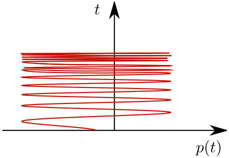
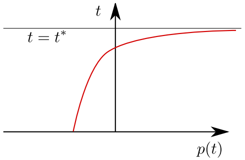
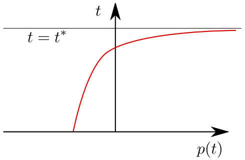


Contradicts x-analyticity of $F = F(x,t)$ (solution of the heat equation).

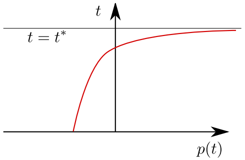


Global Existence Theorem (II)



Global Existence Theorem (II)

Need to exclude the existence of t^* s.t. $|p(t)|$ becomes unbounded as $t \to t^*$



Global Existence Theorem (II)

Need to exclude the existence of t^* s.t. $|p(t)|$ becomes unbounded as $t \to t^*$

We have

$$F(x,t) = \int_{-\infty}^{x-p_0} G(t,z)F_I^-(x-z)\,dz - \int_{x-p_0}^{\infty} G(t,z)F_I^+(x-z)\,dz$$

Bounded from below

$$\int_{p_0}^{p_0+a} |F_I(x+z)|\,dz$$
$$= \sum_{n=0}^{\infty} \int_{x+p_0-na}^{x+p_0-(n-1)a} f_I^-(y)\,dy$$
$$\geq \text{const} > 0.$$

tends to zero as $x \to +\infty$
(as F grows at most linearly)

$\Rightarrow$ $\exists$ unique x with $-\infty < x < \infty$ such that $F(x,t) = 0$.

Theorem

Let $f = f(x,t)$ be a solution of the FBP. If

$$M^+ := \int_{-\infty}^{0} f^+(z)\, dx \neq \int_0^\infty f^-(z)\, dz =: M^-,$$

then $p(t) \sim \sqrt{t} q_\infty$ with $\operatorname{erf}(q_\infty) = M^- / M^+$ as $t \to \infty$.

If $M^- = M^+$, i.e. the total mass of f is zero, then

$$p(t) = \frac{\int_{-\infty}^{\infty} z|f(z)|\, dz}{M^+ + M^-} + O\left(\frac{1}{\sqrt{t}}\right).$$

where

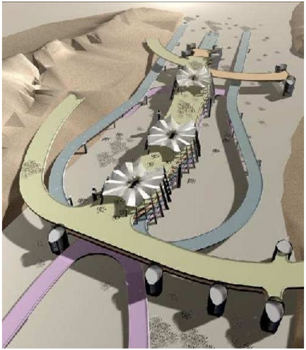
$$\operatorname{erf}(u) := \frac{1}{\sqrt{4\pi}} \int_u^\infty e^{-\frac{x^2}{4}}\, dx$$

Human Crowds - Introduction

- Why model the behaviour of human crowds ?
  - Today, half of the human population lives in urban areas (1950: only 30%)
  - Prediction: In 2050, expected 70 %
  - Increasing need for tools to understand motion of large crowds
- Two distinct approaches: particle and density models
- 1st approach: Discrete model
  - Particle models: each person treated individually.
  - Realistic, easy to simulate (for small crowds)
- Continuum model
  - Continuum models: treat crowd as a density $\rho(x, t)$
  - Suitable for large crowd (or probabilistic interpretation)
- Here: Continuum model introduced by Hughes 2002

Roger L. Hughes
A continuum theory for the flow of pedestrians.
Transportation Research Part B: Methodological, 36(6), p. 507-535, 2002.

- Prominent Example: Jamarat Bridge near Mekka
- In the past: frequent accidents
    - 1994: 270 killed
    - 1998: 118 killed, 180 injured
    - 2004: 251 killed
    - etc.
- 2007: Completion of new bridge, designed also using simulations
- 2007: No fatal accident



Modelling: Basic Setup

- Crowd density $\rho = \rho(x, t)$
- Valid for high densities or for probabilistic interpretation
- Continuity Equation (people neither created nor destroyed)

$$\rho_t + \mathrm{div}(\rho v) = 0$$

- Main question: reasonable model for velocity field v

Modelling: Velocity Field

Based on three hypotheses

1. The speed of the pedestrians is determined by the density of the surrounding pedestrian flow and the behavioral characteristics of the pedestrians only.

$$v = f(\rho)u, \quad |u| = 1$$

2. Pedestrians have a common sense of the task (called potential ϕ) they face to reach their common destination.

$$u = -\frac{\nabla\phi}{|\nabla\phi|}$$

3. Pedestrians seek to minimize their (accurately) estimated travel time, but temper this behavior to avoid high densities.

$$|\nabla\phi| = \frac{1}{g(\rho)f(\rho)}$$

- Combining hypotheses leads to

$$\begin{cases} \rho_t - \operatorname{div}(\rho g(\rho) f^2(\rho) \nabla \phi) = 0, \\ \\ |\nabla \phi| = \frac{1}{f(\rho) g(\rho)} \end{cases}$$

- Analytical Issues
  - **nonlinear** hyperbolic conversation law
  - **density dependent** stationary Hamilton-Jacobi equation (eikonal type)

  $\rightarrow$ $\phi \in \mathcal{C}^{0,1}$ only
  - fully coupled system
- Possible models for f: $f(\rho) = \rho_{max} - \rho$ or $f(\rho) = (\rho_{max} - \rho)^2$.
- In the following: $f(\rho) = \rho_{max} - \rho$ and $g(\rho) = 1$.

Regularization

- Regularize system

$$\begin{cases} \rho_t^\epsilon - \mathrm{div}(\rho^\epsilon g(\rho^\epsilon) f^2(\rho^\epsilon)\nabla\phi^\epsilon) & = \underbrace{\epsilon\Delta\rho^\epsilon}_{\text{diffusion of pedestrians}} \quad , \\ \underbrace{-\delta_1\Delta\phi}_{\text{diffusion of the pedestrian path}} \quad +|\nabla\phi^\epsilon| & = \frac{1}{f(\rho^\epsilon)+\delta_2} \end{cases}$$

→ Existence can be obtained easily

- Realistic B.C.: $\rho^\epsilon g(\rho^\epsilon) f^2(\rho^\epsilon)\nabla\phi^\epsilon \cdot n = \rho^\epsilon$, $\phi^\epsilon = 0$

→ People leave domain with speed one and stop

- Questions
  - Limit $\epsilon \to 0$ (Existence, Uniqueness, BC)
  - Limit $\delta_1,\ \delta_2 \to 0$

M. Di Francesco, P.A. Markowich, J.-F. Pietschmann, M.T. Wolfram
On the Hughes' model of pedestrian flow: The one-dimensional case.
J. Diff. Eq., Volume 250, Issue 3, 1 pp. 1334-1362, 2011.

A Priori Estimates, $\epsilon \to 0$ & Uniqueness

- Suppose $\rho_0 \in W^{1,1}((-1,1))$. Then

$$\|\rho_x(t)\|_{L^1} \leq (\|(\rho_0)_x\|_{L^1} + C)e^{Ct}$$

  for all $t \geq 0$, $C \neq C(\epsilon)$

- Assuming $\rho_0 \in W^{2,1}(\Omega)$ and $\epsilon \in \mathbb{R}_+$. Then

$$\|\rho_t(t)\|_{L^1} \leq Ce^{Ct}$$

  for all $t \geq 0$, C independent of ϵ

- There exists a $\rho \in BV(x,t)$ and $\{\epsilon_k\}$ s.t.

$$\|\rho^{\epsilon_k} - \rho\|_{L^1((0,T);L^1)} \to 0 \text{ for } \epsilon_k \to 0.$$

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- Existence of weak solution, uniqueness?
- Suitable tool: Entropy solutions (formally derived by multiplying the equation with $\mathrm{sgn}(\rho - k)\psi$, $k \in \mathbb{R}$, see Karlsen, Risebro 2003)

Theorem

As $\epsilon \to 0$, the solution ρ^ϵ converges to an entropy solution in the sense of the above Definition.

Boundary Conditions

- For parabolic approximation $\rho^\epsilon = 0$ on $\partial\Omega$
- Hyperbolic limit: BC only on outflow regions

Theorem

Let ρ, $\bar{\rho}$ be the two entropy solutions of

$$\rho_t - \operatorname{div}(h(\rho)\phi_x) = 0,$$
$$\bar{\rho}_t - \operatorname{div}(h(\bar{\rho})\bar{\phi}_x) = 0$$

with initial data ρ_0, $\bar{\rho}_0 \in L^1(\Omega) \cap L^\infty(\Omega) \cap BV(\Omega)$. Then for almost all $t \in (0, T)$,

$$\begin{aligned}\|\rho(\cdot,t) - \bar{\rho}(\cdot,t)\|_{L^1} \le \|\rho_0 - \bar{\rho}_0\|_{L^1(\Omega)} &+ t\|h\|_{L^\infty}\|\phi_{xx} - \bar{\phi}_{\bar{x}\bar{x}}\|_{L^\infty((0,T);L^1(\Omega))} \\ &+ t\|h\|_{Lip(\Omega)}\|\rho_x(\cdot,t)\|_{L^1(\Omega)}\|\phi_x - \bar{\phi}_{\bar{x}}\|_{L^\infty((0,T);L^\infty)}\end{aligned}$$

holds.

- Together with stability estimates $\to$ uniqueness

Numerics

Regularized problem

$$\rho_t - \mathrm{div}(\rho g(\rho) f^2(\rho) \nabla\phi) = \epsilon \rho_{xx} \tag{2a}$$

$$|\nabla\phi| = \frac{1}{f(\rho)g(\rho)} \tag{2b}$$

Iterative solving strategy

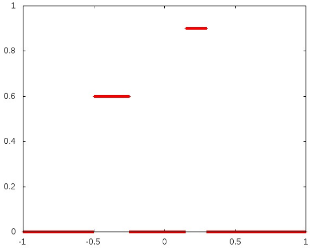
- Given ρ solve Eikonal equation (2b) with fast sweeping or fast marching method
- Solve nonlinear conservation law (2a) for a given ϕ solve (2a) using an ENO scheme.

Initial data:

$$\rho(x,0) = \begin{cases} 0.6 & \text{if } -0.5 \le x \le -0.25 \\ 0.9 & \text{if } 0.15 \le x \le 0.3 \\ 0 & \text{else.} \end{cases}$$

Parameters:

$$\epsilon = 10^{-4},\ \Delta x = 2 \times 10^{-3},\ \Delta t = 10^{-4}$$



Opinion Formation - Classical Boltzmann equation

B. Düring, D. Matthes, and G. Toscani
A Boltzmann-type approach to the formation of wealth distribution curves.
Riv. Mat. Univ. Parma (8)1, 199-261, 2009

B. Düring and G. Toscani
International and domestic trading and wealth distribution.
Comm. Math. Sci. 6(4), 1043-1058, 2008

First formulated by Ludwig Boltzmann in 1872 to describe the dynamics of a dilute gas:

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F - E_{eff} \cdot \nabla_v F = \left(\frac{dF}{dt}\right)_{coll}$$

- $F(x, v, t)$ is the number of particles per unit volume in an infinitesimal neighborhood of (x, v)
- E_{eff} denotes the effective external field
- $\left(\frac{dF}{dt}\right)_{coll}$ denotes the collision term, given by

$$\left(\frac{dF}{dt}\right)_{coll}(x, v, t) = \int \left[P(x, v' \to v, t) - P(x, v \to v', t)\right] dv'$$

where $P(x, v' \to v, t)$ is the rate of a particle with position x at time t to change its velocity vector v' to v (due to scattering).

Toscani's model for opinion formation in a society of commoners

Basic idea: Interactions between humans are modeled by collisions of particles.

Let v and w in $\mathcal{I} = [-1, 1]$ denote the pre-interaction opinions of two individuals, their post-trade opinions v^* and w^* are given by the collision rules

$$
\begin{aligned}
v^* &= v - \overbrace{\gamma P(|v-w|)(v-w)}^{\text{negotiation}} + \overbrace{\eta_1 D(v)}^{\text{self-thinking}}\,, \\
w^* &= w - \gamma P(|w-v|)(w-v) + \eta_2 D(w).
\end{aligned}
$$

- The constant γ denotes the compromise parameter, the parameters $\eta_i \in \mathcal{N}(0, \sigma^2)$ for $i = 1, 2$ model self-thinking via a random diffusion (e.g. through press, television or Internet).
- The functions $P(\cdot)$ and $D(\cdot)$ model the local relevance of compromise and self-thinking.

Tosani's model for opinion formation in a society of commoners

The distribution of opinion f satisfies a Boltzmann type equation

$$\frac{\partial f}{\partial t} = \mathcal{Q}(f, f),$$

where $\mathcal{Q}$ denotes the collision operator. The symmetric weak form of the Boltzmann type equation is

$$\frac{d}{dt}\int_{\mathcal{I}} \phi(w) f(w,t) dw = \left(\mathcal{Q}\left(f,f\right),\phi\right) = \\ \left\langle \int_{\mathcal{I}} \left(\beta_{(v,w)\to(v^*,w^*)} f(v) f(w) \left(\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)\right)\right) dv dw \right\rangle.$$

The transition rate β takes the form

$$\beta_{(v,w)\to(v^*,w^*)} = \chi(|v^*| \leq 1)\chi(|w^*| \leq 1)$$

where $\chi(A)$ is the indicator function on the set A.

Opinion formation with strong leaders

B. Düring, P.A. Markowich, J.-F. Pietschmann, M.-T. Wolfram
Boltzmann and Fokker-Planck Equations modelling Opinion Formation in the Presence of Strong Leaders.
Proc. Royal Soc. A., 465(2112), p. 3687-3708, 2009.

Model Assumptions:

- Two groups of people - commoners and strong leaders
- Commoners change their opinion based on Toscani's model
- Strong leaders do not change their opinion when "colliding" with commoners, but influence each other




Opinion formation with strong leaders (II)

Let $v, w \in \mathcal{I}$ denote the pre-interaction opinions, v^*, w^* the opinions after the collision. If two individuals from the same group meet, the interactions are given by (for $i = 1, 2$)

$$\begin{aligned} v^* &= v - \gamma_i P_i(|v - w|)(v - w) + \eta_{i1} D_i(v), \\ w^* &= w - \gamma_i P_i(|w - v|)(w - v) + \eta_{i2} D_i(w). \end{aligned}$$

If one individual from the group of ordinary people with opinion v meets a strong opinion leader with opinion w their post-interaction opinions are given by

$$\begin{aligned} v^* &= v - \gamma_3 P_3(|v - w|)(v - w) + \eta_{11} D_1(v), \\ w^* &= w. \end{aligned}$$

- $P_i(\cdot)$ are non-increasing functions - the higher the difference of opinions the lower the possibility to find a compromise
- $D(\cdot)$ is a decreasing function of $|w|$ and $D = 0$ for $|w| = \pm 1$ - extreme opinions do not change that much

Then the distribution function of each group (f_1 corresponds to commoners, f_2 to strong leaders) satisfies a Boltzmann equation given by

$$\frac{\partial}{\partial t} f_1(w,t) = \frac{1}{\tau_{11}} \mathcal{Q}_{11}(f_1, f_1)(w) + \frac{1}{\tau_{12}} \mathcal{Q}_{12}(f_1, f_2)(w),$$
$$\frac{\partial}{\partial t} f_2(w,t) = \frac{1}{\tau_{22}} \mathcal{Q}_{22}(f_2, f_2)(w).$$

where τ_{ij} denote suitable relaxation times that allow to control the interaction frequencies of opinion leaders and commoners. The variational formulations of the collision operators are given by

$$\int_{\mathcal{I}} \mathcal{Q}_{ij}(f_i, f_j)(w)\phi(w)\, dw$$
$$= \frac{1}{2} \left\langle \int_{\mathcal{I}^2} \beta\big(\phi(w^*) + \phi(v^*) - \phi(w) - \phi(v)\big) f_i(v) f_j(w)\, dv\, dw \right\rangle .$$

for all smooth functions $\phi(w)$,

Derivation of the limiting Fokker-Planck system

To study the situation for large times, i.e. close to the steady state, we introduce for $\gamma_i = \gamma \ll 1$ and transform

$$\tau = \gamma t, \quad g_i(w,\tau) = f_i(w,t), \quad i = 1, 2.$$

This implies $f_{i,0} = g_{i,0}$ and the evolution of the scaled densities $g_i(w,\tau)$ satisfies

$$\begin{aligned}
\frac{d}{d\tau}\int_{\mathcal{I}} g_1(w,\tau)\phi(w)\,dw =& \frac{1}{\gamma}\int_{\mathcal{I}} \frac{1}{\tau_{11}}\mathcal{Q}_{11}(f_1,f_1)(w)\phi(w)\,dw \\
&+ \frac{1}{\gamma}\int_{\mathcal{I}} \frac{1}{\tau_{12}}\mathcal{Q}_{12}(f_1,f_2)(w)\phi(w)\,dw, \\
\frac{d}{d\tau}\int_{\mathcal{I}} g_2(w,\tau)\phi(w)\,dw =& \frac{1}{\gamma}\int_{\mathcal{I}} \frac{1}{\tau_{22}}\mathcal{Q}_{22}(f_2,f_2)(w)\phi(w)\,dw.
\end{aligned}$$

Quasi-invariant opinion limit: Taylor expansion of ϕ up to the second order term, then consider the limit $\gamma \to 0$ and $\sigma_{ij} \to 0$ such that $\frac{\sigma_{ij}^2}{\gamma} = \lambda_{ij}$.

Limiting Fokker-Planck system I

The limiting equations are given by

$$
\begin{aligned}
\frac{\partial}{\partial \tau} g_1(w,\tau) =& \frac{\partial}{\partial w}\left(\left(\frac{1}{\tau_{11}}\mathcal{K}_1(w,\tau) + \frac{1}{2\tau_{12}}\mathcal{K}_3(w,\tau)\right) g_1(w,\tau)\right) \\
&+ \left(\frac{\lambda_{11} M_1}{2\tau_{11}} + \frac{\lambda_{12} M_2}{4\tau_{12}}\right)\frac{\partial^2}{\partial w^2}\left(D_1^2(w) g_1(w,\tau)\right), \\
\frac{\partial}{\partial \tau} g_2(w,\tau) =& \frac{\partial}{\partial w}\left(\frac{1}{\tau_{22}}\mathcal{K}_2(w,\tau) g_2(w,\tau)\right) + \frac{\lambda_{22} M_2}{2\tau_{22}}\frac{\partial^2}{\partial w^2}\left(D_2^2(w) g_2(w,\tau)\right).
\end{aligned}
$$

Here $M_i = \int g_i dv$. The operators $\mathcal{K}_i$ for $i = 1, 2$ read

$$
\mathcal{K}_i(w,\tau) = \int_{\mathcal{I}} P_i(|w-v|)(w-v) g_i(v,\tau) dv
$$

and $\mathcal{K}_3$:

$$
\mathcal{K}_3(w,\tau) = \int_{\mathcal{I}} P_3(|w-v|)(w-v) g_2(v,\tau) dv
$$

Limiting Fokker-Planck System II

We supplement this system with no flux boundary conditions

$$\begin{aligned}&\left(\frac{1}{\tau_{11}}\mathcal{K}_1(w,\tau)+\frac{1}{2\tau_{12}}\mathcal{K}_3(w,\tau)\right)g_1(w,\tau)\\&+\left(\frac{\lambda_{11}M_1}{2\tau_{11}}+\frac{\lambda_{12}M_2}{4\tau_{12}}\right)\frac{\partial}{\partial w}\left(D_1^2(w)g_1(w,\tau)\right)=0\end{aligned}\qquad \text{on } w=\pm 1$$

and

$$\frac{1}{\tau_{22}}\mathcal{K}_2(w,\tau)g_2(w,\tau)+\frac{\lambda_{22}M_2}{2\tau_{22}}\frac{\partial}{\partial w}\left(D_2^2(w)g_2(w,\tau)\right)=0 \qquad \text{on } w=\pm 1.$$

Toscani proposed the following functions for the diffusion of opinion

$$D(w):=\left(1-w^2\right)^{\alpha},$$

with $\alpha \geq \frac{1}{2}$. The compromise propensities $P_i(\cdot)$ $(i=1,2,3)$ are given by

$$P_i(|v-w|)=\mathbf{1}_{\{|v-w|\leq r_i\}}.$$

for positive constants r_i.

Stationary solutions I

Consider the special case $P_i(|w-v|)=1$ and $D(w)=(1-w^2)^\alpha$. Then the equilibrium solution satisfies

$$
\begin{aligned}
0 =& \left(\frac{wM_1-m_1}{\tau_{11}}+\frac{wM_2-m_2}{2\tau_{12}}\right) g_{1,\infty}(w) \\
&+ \left(\frac{\lambda_{11}M_1}{2\tau_{11}}+\frac{\lambda_{12}M_2}{4\tau_{12}}\right)\left(D^2(w)g_{2,\infty}(w)\right)_w \\
0 =& \frac{wM_2-m_2}{\tau_{22}} g_{2,\infty} + \frac{\lambda_{22}M_2}{2\tau_{22}}\left(D^2(w)g_{2,\infty}\right)_w .
\end{aligned}
$$

We denoted the masses of the opinion leaders and followers by $M_i=\int g_{i,\infty}\,dv$ with $i=1,2$ and their first order moments by $m_i=\int v g_{i,\infty}\,dv$, $i=1,2$. The second equation can be solved explicitely

$$
g_{2,\infty} = \frac{c_2}{(1-w^2)^{2\alpha}} e^{-\frac{2}{\lambda_{22}M_2}\int_0^w \frac{vM_2-m_2}{(1-v^2)^{2\alpha}}\,dv}.
$$

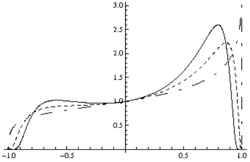
Stationary solutions II

The solution of the first equation can be calculated using the same arguments

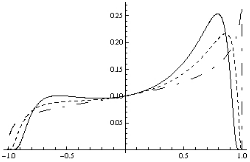
$$g_{1,\infty} = \frac{c_1}{(1-w^2)^{2\alpha}} e^{-k(\frac{M_1}{\tau_{11}}+\frac{M_2}{2\tau_{12}}) \int_0^w \frac{v}{(1-v^2)^{2\alpha}}\, dv} e^{km_2\left(\frac{1}{2\tau_{12}}+\frac{M_1}{\tau_{11}M_2}\right) \int_0^w \frac{1}{(1-v^2)^{2\alpha}}\, dv}.$$

We conclude that if $\alpha > \frac{1}{2}$ then c_1 and c_2 can be determined uniquely such that the mass of $g_{1,\infty}$ is M_1 and the mass of $g_{2,\infty}$ is M_2, if

$$M_2 m_1 - M_1 m_2 = 0.$$



(g) Stationary solution $g_{1,\infty}$



(h) Stationary solution $g_{2,\infty}$

Figure: Solid line $\alpha = 1$, dashed $\alpha = 0.75$, dashed dotted $\alpha = 0.5025$ with $M_1 = 1,\ M_2 = 0.05,\ m_2 = 0.01,\ m_1 = \frac{m_2 M_1}{M_2} = 0.2,\ \tau_{11} = 1,\ \tau_{12} = 10,\ \lambda_{ii} = 1$ for all i

Emergence and decline of opinion leaders

Assumptions

(A1) The overall mass of opinion leaders and followers is constant in time, i.e.

$$\frac{d}{d\tau}\int \left(g_1(w,\tau)+g_2(w,\tau)\right)\,dw = 0.$$

(A2) The society has a certain characteristic percentage of strong opinion leaders in the long-run average, e.g. 5% of the whole population may typically be opinion leaders.

(A3) The exchange of information between followers causes the formation of 'groups' sharing a similar opinion, even if no strong leaders are present. If such a 'group' is sufficiently large, it is likely for somebody to step up and take the lead. Hence, if the density of followers sharing a similar opinion exceeds a certain threshold $\underline{c}$ and the overall number of leaders is less than the typical 5%, then a leader promoting this opinion emerges.

(A4) If leaders promoting a certain opinion have not enough followers, i.e. less than a particular threshold $\bar{c}$, and if there are more than the typical 5% of leaders present in the whole society, then the leaders promoting this opinion decline.

Emergence and decline of leaders

Then the extended model is given by

$$
\begin{aligned}
\frac{\partial}{\partial \tau} g_1(w,\tau) =& \frac{\partial}{\partial w}\left(\left(\frac{1}{\tau_{11}}\mathcal{K}_1(w,\tau) + \frac{1}{2\tau_{12}}\mathcal{K}_3(w,\tau)\right) g_1(w,\tau)\right) \\
&+ \left(\frac{\lambda_{11} M_1}{2\tau_{11}} + \frac{\lambda_{12} M_2}{4\tau_{12}}\right) \frac{\partial^2}{\partial w^2}\left(D^2(w) g_1(w,\tau)\right) - a(g_1)g_1 + b(g_1)g_2, \\
\frac{\partial}{\partial \tau} g_2(w,\tau) =& \frac{\partial}{\partial w}\left(\frac{1}{\tau_{22}}\mathcal{K}_2(w,\tau) g_2(w,\tau)\right) \\
&+ \frac{\lambda_{22} M_2}{2\tau_{22}} \frac{\partial^2}{\partial w^2}\left(D^2(w) g_2(w,\tau)\right) + a(g_1)g_1 - b(g_1)g_2.
\end{aligned}
$$

The functions a (emergence) and b (decline) are given by

$$
a(g_1) = \underbrace{\mathbf{1}_{\{g_1(w) \geq \underline{c}\}}}_{\substack{\text{Commoners above} \\ \text{the threshold}}} \underbrace{e^{-\frac{M_2^2}{\sqrt{2\pi}\sigma_1}}}_{\substack{\text{Leaders emerge if they} \\ \text{make up less than 5\%} \\ \text{of the whole pop.}}}, \quad b(g_1) = \underbrace{\mathbf{1}_{\{g_1(w) \leq \bar{c}\}}}_{\substack{\text{Commoner below} \\ \text{the threshold}}} \underbrace{e^{-\frac{M_1^2}{\sqrt{2\pi}\sigma_2}}}_{\substack{\text{Leaders decline if they} \\ \text{make up more than 5\%} \\ \text{of the whole pop.}}},
$$

where $\mathbf{1}_A$ is the indicator function of the set A.

Solutions of the Fokker-Planck system with $r = 0.5$

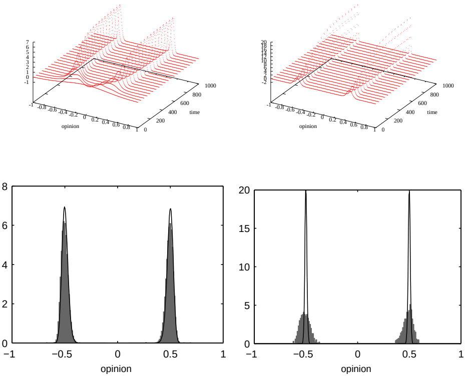


Figure: Evolution of commoners (upper left) and strong leaders (upper right) in time; MC-Fokker-Planck comparison of the equilibrium solutions for commoners (lower left) and strong leaders (lower right).

Results of the state elections in Carinthia

	Grüne	SPÖ	ÖVP	FPÖ	BZÖ
2004	6.7%	38.4 %	11.6 %	42.5 %	—
2009	5.2%	28.8 %	16.8 %	3.8 %	44.9 %

We assume that the extreme opinions $v = \pm 1$ correspond to the right and left wing of the political spectrum. We place the parties according to their political views (i.e. FPÖ at 0.8) and choose the weights such that the masses of commoners correspond to the results of the 2004 elections.

$$g_1(w,0) = \frac{0.07}{\sigma_1\sqrt{2\pi}} e^{-\frac{(w+0.75)^2}{2\sigma_1^2}} + \frac{0.385}{\sigma_1\sqrt{2\pi}} e^{-\frac{(w+0.25)^2}{2\sigma_1^2}} + \frac{0.115}{\sigma_1\sqrt{2\pi}} e^{-\frac{(w-0.25)^2}{(2\sigma_1^2)}} + \frac{0.43}{\sigma_1\sqrt{2\pi}} e^{-\frac{(w-0.8)^2}{2\sigma_1^2}},$$

$$g_2(w,0) = \frac{0.1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(w+0.75)^2}{2\sigma_2^2}} + \frac{0.15}{\sigma_2\sqrt{2\pi}} e^{-\frac{(w+0.2)^2}{2\sigma_2^2}} + \frac{0.3}{\sigma_2\sqrt{2\pi}} e^{-\frac{(w-0.25)^2}{2\sigma_2^2}} + \frac{0.45}{\sigma_2\sqrt{2\pi}} e^{-\frac{(w-0.8)^2}{2\sigma_2^2}}.$$

We choose the following parameters

$$\overbrace{\alpha = 1.5}^{\text{Low diffusion at extreme opinions}}, \lambda = 3\times 10^{-3}, \overbrace{r_1 = r_2 = 0.2}^{\text{small interaction radius between commoners}}, r_3 = 0.45, \tau_{11} = \tau_{12} = 1, \overbrace{\tau_{22} = 10}^{\text{Few interactions between leaders}}.$$

Understanding Carinthia

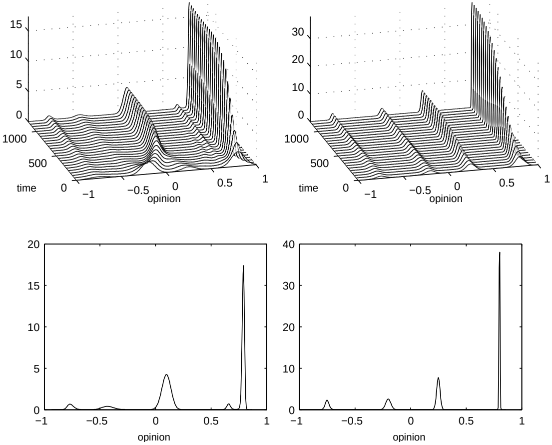


Figure: Evolution of commoners (upper left) and strong leaders (upper right) in time; equilibrium solutions for commoners (lower left) and strong leaders (lower right). Note that the commoners assemble at $v = 0.7$ without a strong leader representing this opinion - formation of a new party.