Long time behavior of Mean Field Games

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Outline

The Mean Field Game system

We investigate the long-time average of the solution (u^T, m^T) to

$$(MFG) \quad \left\{ \begin{array}{cl} (i) & -\partial_t u^T - \Delta u^T + \frac{1}{2}|Du^T|^2 = F(x, m^T) \\ (ii) & \partial_t m^T - \Delta m^T - \mathrm{div}(m^T Du^T) = 0 \\ (iii) & m^T(0) = m_0, \ u^T(x,T) = G(x, m^T(T)) \end{array} \right.$$

(model introduced in Lasry-Lions '06. See also Huang, M., Caines, P.E., Malhame, '06)

Heuristic interpretation

An average agent controls the stochastic differential equation

$$dX_t = \alpha_t dt + \sqrt{2} B_t$$

where (B_t) is a standard B.M. He aims at minimizing

$$\mathbf{E}\left[\int_0^T \frac{1}{2}|\alpha_s|^2 + F(X_s, m_s^T)\, ds + G(X_T, m^T(T))\right] .$$

- His value function u^T then satisfies

$$\begin{cases} (i) & -\partial_t u^T - \Delta u^T + \frac{1}{2}|Du^T|^2 = F(x, m^T) \qquad \text{in } \mathbb{R}^N \times (0, T) \\ (iii) & u^T(x, T) = G(x, m^T(T)) \qquad \text{in } \mathbb{R}^N \end{cases}$$

- His optimal control is given by $\alpha^*(x, t) = -Du(x, t)$.

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Heuristic interpretation (continued)

If all agents argue in this way, their density $m^T(x,t)$ evolves according to the Kolmogorov equation

$$\left\{\begin{array}{lll} (ii) & \partial_t m^T - \Delta m^T - \mathrm{div}(m^T Du^T) = 0 & \text{in } \mathbb{R}^N \times (0,T) \\ (iii) & m^T(0) = m_0 \quad \text{in } \mathbb{R}^N & \end{array}\right.$$

By definition, $m^T(\cdot,t)$ is a probability measure.

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Main questions

What is the behavior of the agents when the horizon T is large ?

In other words, we look for the asymptotics of (u^T, m^T) as $T \to +\infty$?

To fix the ideas, we work in the periodic setting :

$$x \to (F(x,m), G(x,m)) \text{ are } \mathbb{Z}^N\text{−periodic.}$$

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Analogy with the asymptotic behavior for HJ

For a single HJ equation

$$(HJ) \quad \left\{ \begin{array}{lll} (i) & -\partial_t u^T - \Delta u^T + \frac{1}{2}|Du^T|^2 = F(x) & \text{in } \mathbb{R}^N \times (0,T) \\ (iii) & u^T(x,T) = G(x) & \text{in } \mathbb{R}^N \end{array} \right.$$

- $\dfrac{u^T(\cdot,0)}{T}$ converges uniformly to a constant $\bar{\lambda}$.
- $u^T(\cdot,0) - \bar{\lambda}T$ converges uniformly to a solution $\bar{u}$ of

$$\bar{\lambda} - \Delta\bar{u} + \frac{1}{2}|D\bar{u}|^2 = F(x) \qquad \text{in } \mathbb{R}^N$$

References : Lions-Papanicolau-Varadhan, Evans, Fathi, Namah-Roquejoffre, Barles-Souganidis,...

The ergodic system

In the MFG framework, the limit system “should" be

$$(MFG-ergo)\quad \left\{\begin{array}{lll} (i) & \bar\lambda-\Delta\bar u+\frac12|D\bar u|^2=F(x,\bar m) & \text{in } \mathbb{R}^N \\ (ii) & -\Delta\bar m-\mathrm{div}(\bar m D\bar u)=0 & \text{in } \mathbb{R}^N \end{array}\right.$$

(introduced in Lasry-Lions 06)

Note that

- $\bar m = e^{-\bar u}/\left(\int_{Q_1} e^{-\bar u}\right)$ solves (MFG-ergo)(ii)
- the map

$$(x,t)\to(\bar\lambda(T-t)+\bar u(x,t)\ ,\ \bar m(x,t))$$

satisfies (MFG)(i-ii).

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In view of the HJ case, one expects :

- The convergence of u^T/T to $\bar{\lambda}$,
- the convergence of m^T to $\bar{m}$.
- the convergence of $u^T - T\bar{\lambda}$ to $\bar{u}$+constant

Heuristic arguments

The analogy with the HJ equation is misleading :

- (HJ) is a Cauchy problem

  ... while (MFG) system has an initial **and** a terminal condition.

  $\longrightarrow$ Where does the convergence take place ?

- (HJ) is a single equation with a **comparison principle**

  ... **no comparison** for the MFG **system**.

  $\longrightarrow$ no use of “semi-relaxed limit".

Heuristic arguments

Consequence

The proofs have to rely on energy estimates.

Two energy relations :

1. Hamiltonian structure (local case)
2. Main energy relation to compare (u^T, m^T) with $(\bar{u}, \bar{m})$.

The Hamiltonian structure (local case)

Set $\Phi(x, m) = \int_0^m F(x, \rho)\, d\rho$ and

$$\mathcal{E}(u, m) = \int_{Q_1} m \frac{1}{2}|Du|^2 + \langle Du, Dm \rangle - \Phi(x, m)\, dx$$

Lemma

(u^T, m^T) solution of (MFG) $\Leftrightarrow$ (u^T, m^T) satisfies the Hamiltonian system

$$\begin{cases} (i) & -\partial_t u^T = -\dfrac{\partial \mathcal{E}}{\partial m}(u^T, m^T) \\ (ii) & \partial_t m^T = -\dfrac{\partial \mathcal{E}}{\partial u}(u^T, m^T) \\ (iii) & m^T(0) = m_0, \; u^T(x, T) = G(x, m^T(T)) \end{cases}$$

In particular the energy $\mathcal{E}(u^T(t), m^T(t))$ is constant along the flow.

Main energy equality

Lemma (Lasry-Lions, 06)

For any $t \in [0, T]$

$$-\frac{d}{dt}\int_{Q_1}(u^T(t)-\bar{u})(m^T(t)-\bar{m})dx =$$

$$\int_{Q_1}\frac{(m^T(t)+\bar{m})}{2}|Du^T(t)-D\bar{u}|^2 + (F(x,m^T(t))-F(x,\bar{m}))(m^T(t)-\bar{m})$$

Proof : Multiply (MFG)(i)-(MFG-ergo)-(i) by $(m^T - \bar{m})$ and substract to (MFG)(ii)-(MFG-ergo)(ii) multiplied by $(u^T - \bar{u})$.

Why the convergence ?

We define the scaled functions on $\mathbb{R}^N \times [0,1]$:

$$v^T(x,t) := u^T(x,tT) \qquad ; \qquad \mu^T(x,t) := m^T(x,tT)$$

Integrate in time the main energy equality :

$$\int_0^1 \int_{Q_1} \frac{(\mu^T + \bar{m})}{2} |Dv^T - D\bar{u}|^2 + (F(x,\mu^T) - F(x,\bar{m}))(\mu^T - \bar{m}) \, dxdt$$
$$= -\frac{1}{T} \left[\int_{Q_1} (v^T - \bar{u})(\mu^T - \bar{m}) dx \right]_0^1$$

Assume F is increasing. If the RHS $\to 0$ as $T \to +\infty$, then

- $Dv^T \to D\bar{u}$,
- which should imply that $v^T \to \bar{u}$

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- $D\upsilon^T \to D\bar{u}$,
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Why the convergence ? (continued)

However,

- In the $RHS = -\frac{1}{T}\left[\int_{Q_1}(v^T - \bar{u})(\mu^T - \bar{m})dx\right]_0^1$ the quantity $v^T(0)$ is of order $\bar{\lambda}T$
- The main energy equality does not explain the behavior of $\frac{u^T}{T}$.

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The MFG system

We work on the system

$$(MFG) \quad \left\{ \begin{array}{ll} (i) & -\partial_t u^T - \Delta u^T + \frac{1}{2}|Du^T|^2 = F(x, m^T) \\ (ii) & \partial_t m^T - \Delta m^T - \mathrm{div}(m^T Du^T) = 0 \\ (iii) & m^T(0) = m_0, \; u^T(x,T) = G(x, m^T(T)) \end{array} \right.$$

where F and G are nonlocal, non-decreasing and smoothing.

Namely, $F, G : \mathbb{R}^N \times \mathcal{P}_1 \to \mathbb{R}$, where $\mathcal{P}_1$ is the set of Borel probability measures on $\mathbb{R}^N/\mathbb{Z}^N$.

Assumptions

The maps $F, G : \mathbb{R}^N \times \mathcal{P}_1 \to \mathbb{R}$ satisfy :

1. (Regularity) F and G are Lipschitz continuous, and $F(\cdot, m)$ and $G(\cdot, m)$ are bounded in $\mathcal{C}^2_{loc}$ unif. in m.
2. (Periodicity) $F(\cdot, m)$ and $G(\cdot, m)$ are $\mathbb{Z}^N-$periodic.
3. (Monotonicity)
$$\int_{Q_1} (F(x,m) - F(x,m'))d(m-m')(x) \geq 0 \qquad \forall m, m' \in \mathcal{P}_1$$
and
$$\int_{Q_1} (G(x,m) - G(x,m'))d(m-m')(x) \geq 0 \qquad \forall m, m' \in \mathcal{P}_1$$
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Example

Let

$$F(x,m) = (\rho * m) \; * \; \rho(x)$$

where $\rho : \mathbb{R}^N \to \mathbb{R}$ is smooth and $\mathbb{Z}^N-$periodic.

Then F satisfies the above conditions.

Well-posedness of (MFG) and (MFG-ergo)

Under the above assumptions :

Theorem (Lasry-Lions, 2006)

- *(MFG) has a unique solution* (u^T, m^T).

  Moreover (u^T, m^T) *is smooth,* $\mathbb{Z}^N-$*periodic in space, with* $m^T > 0$.

- *Symmetrically, (MFG-ergo) has a unique solution* $(\bar\lambda, \bar u, \bar m)$ *(up to a constant for* $\bar u$*).*

  Moreover $(\bar u, \bar m)$ *is smooth,* $\mathbb{Z}^N-$*periodic, with* $\bar m = e^{-\bar u}/(\int_{Q_1} e^{-\bar u}) > 0$.

The convergence result

Recall the definition of the scaled functions on $\mathbb{R}^N \times [0,1]$:

$$\upsilon^T(x,t) := u^T(x,tT) \qquad ; \qquad \mu^T(x,t) := m^T(x,tT)$$

Theorem

As $T \to +\infty$,

- $\dfrac{\upsilon^T}{T}$ *converges uniformly to* $(1-t)\bar{\lambda}$ *in* $\mathbb{R}^N \times [0,1]$.
- μ^T *converges to* $\bar{m}$ *in* $L^p((0,1)\times Q_1)$, *for any* $p \geq 1$.

Ingredients of proof

Lemma

The map u^T is uniformly semi-concave in space.

In particular, u^T is uniformly Lipschitz continuous in space.

Proof : Comes from the regularity of $F(\cdot, m^T)$.

Ingredients of proof

Plugging the previous Lemma into the main energy equality gives :

Corollary

$$\int_0^T \int_{Q_1} \frac{(m^T + \bar{m})}{2} |Du^T - D\bar{u}|^2 + (F(x, m^T) - F(x, \bar{m}))(m^T - \bar{m}) \, dxdt \leq C$$

In particular

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_{Q_1} |Du^T - D\bar{u}|^2 \, dxdt = 0 ,$$

which is equivalent to

$$\lim_{T \to +\infty} \int_0^1 \int_{Q_1} |Dv^T - D\bar{u}|^2 \, dxdt = 0 ,$$

Ingredients of proof

Lemma

For any $p \geq 1$ there is a constant C_p such that

$$\int_0^T \int_{Q_1} (m^T)^p \leq C_p T \qquad \forall T \geq 1 \; .$$

Idea of proof : Multiply (MFG)(ii) by $(m^T)^p$ and integrate :

$$\begin{aligned}
&\int_0^T \int_{Q_1} p(m^T)^{p-1}|Dm^T|^2 \\
&\quad = -\left[\int_{Q_1} \frac{(m^T)^{p+1}}{p+1}\right]_0^T - \int_0^T \int_{Q_1} p(m^T)^p \langle Dm^T, Du^T \rangle \\
&\quad \leq C + C \int_0^T \int_{Q_1} (m^T)^p |Dm^T|
\end{aligned}$$

Proof of the Convergence Theorem

Recall the notations :

$$\upsilon^T(x,t) := u^T(x,tT) \qquad ; \qquad \mu^T(x,t) := m^T(x,tT)$$

Step 1 : Convergence of μ^T

- L^p bounds on $m^T \Rightarrow$ weak convergence of μ^T to some μ. (up to subsequences)
- Since $D\upsilon^T \to D\bar{u}$ in L^2 and μ^T solves

  $$\frac{\partial_t \mu^T}{T} - \Delta\mu^T - \mathrm{div}(\mu^T D\upsilon^T) = 0 \ ,$$

  μ satisfies

  $$-\Delta\mu - \mathrm{div}\,(\mu D\bar{u}) = 0 \qquad \text{in } \mathbb{R}^N \times (0,1) \ .$$
- Uniqueness of the solution of MFG-ergo(ii) $\Rightarrow \upsilon = \bar{m}$.

Proof of the Convergence Theorem

Step 2 : convergence of $\upsilon^T(\cdot,t)/T$

Fix t and integrate (MFG)(i) over $Q_1 \times [t,1]$:

$$\frac{1}{T}\left(\int_{Q_1} \upsilon^T(t)dx - \int_{Q_1} Gdx\right) + \frac{1}{2}\int_t^1\int_{Q_1} |D\upsilon^T|^2 dxds = \int_t^1\int_{Q_1} Fdxds$$

where

$$\int_t^1\int_{Q_1} F(x,\mu^T) - \frac{|D\upsilon^T|^2}{2}dxds \;\to\; \int_t^1\int_{Q_1} F(x,\bar{m}) - \frac{|D\bar{u}|^2}{2}dxds = (1-t)\bar{\lambda}$$

So

$$\frac{1}{T}\int_{Q_1} \upsilon^T(t)dx \;\to\; (1-t)\bar{\lambda}$$

Conclusion by Lipschitz estimates on $\upsilon^T(\cdot,t)$.

Outline

The MFG system in the local case

We now work on the system

$$(MFG)\quad \left\{\begin{array}{ll} (i) & -\partial_t u^T - \Delta u^T + \frac{1}{2}|Du^T|^2 = F(x, m^T(x,t)) \\ (ii) & \partial_t m^T - \Delta m^T - \mathrm{div}(m^T Du^T) = 0 \\ (iii) & m^T(0) = m_0,\ u^T(x,T) = G(x) \end{array}\right.$$

where $F : \mathbb{R}^N \times [0, +\infty)$ is local and increasing and $G : \mathbb{R}^N \to \mathbb{R}$ does not depend on m.

Two regimes of convergence

1. Exponential convergence under strong monotony condition
2. Convergence under mild monotonicity condition

Assumptions on the data (strong monotony)

- $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, $\mathbb{Z}^N-$periodic in x, and there is $\gamma > 0$ with

$$F(x,s) - F(x,t) \geq \gamma(s-t) \qquad \forall s \geq t,\ \forall x \in \mathbb{R}^N .$$

- $m_0 : \mathbb{R}^N \to \mathbb{R}$ is smooth, $\mathbb{Z}^N-$periodic, $m_0 > 0$ and $\int_{Q_1} m_0 = 1$.
- $G : \mathbb{R}^N \to \mathbb{R}$ is $\mathbb{Z}^N-$periodic and $\mathcal{C}^2$.

Well-posedness of (MFG) and (MFG-ergo)

Under the above assumptions :

- There exists a unique weak solution for general MFG systems with local interactions (Lasry-Lions, '86)
- Under the above conditions, this solution is $\mathcal{C}^2$.
- For the model case (MFG), a constructive proof can be found in Guéant '11.

The convergence result (strong monotony)

Set $\langle u^T(t)\rangle = \int_{Q_1} u^T(y,t)dy$.

Theorem

1. $$\|u^T(t) - \langle u^T(t)\rangle - \bar{u}\|_{L^1(Q_1)} \leq \frac{C}{T-t}\left(e^{-\kappa(T-t)} + e^{-\kappa t}\right)$$
2. $$\|m^T(t) - \bar{m}\|_{L^1(Q_1)} \leq \frac{C}{t}\left(e^{-\kappa(T-t)} + e^{-\kappa t}\right)$$
3. $$\left\|\frac{u^T(t)}{T} - \bar{\lambda}\left(1 - \frac{t}{T}\right)\right\|_{L^1(Q_1)} \leq C\frac{\ln(T)}{T} + C\frac{e^{-\kappa T}}{\sqrt{T}}$$

for any $t \in (0,T)$ (for some $\kappa > 0$)

Comments

- We have exponential convergence for $u^T(t) - \langle u^T(t) \rangle$ and m^T.
- Far from 0, the convergence estimate for $\frac{u^T}{T}$ is of order $\frac{\ln(T)}{T}$.

Main ingredient of proof

Lemma

There exists $\sigma > 0$ *such that*

$$\int_{\delta T}^{(1-\delta)T} \int_{Q_1} |Du^T(x,t) - D\bar{u}(x)|^2 + |m^T(x,t) - \bar{m}(x)|^2 \, dxdt \leq Ce^{-\sigma \delta T}$$

for every $\delta > 0$.

Proof of the Lemma

Proof. Set $\tilde{u}^T(x,t) = u^T(x,t) - \langle u^T(t)\rangle$ and

$$\varphi(t) = \int_{Q_1} (\tilde{u}^T(t) - \bar{u}(t))(m^T(t) - \bar{m}(t))\, dx$$

From the main energy equality : $\displaystyle \frac{d}{dt}\varphi(t) =$

$$-\int_{Q_1} \left\{ \frac{m^T + \bar{m}}{2} |D(u^T - \bar{u})|^2 + \Big(F(x,m^T) - F(x,\bar{m})\Big)(m^T - \bar{m}) \right\} dx$$

where, since $\bar{m} > 0$ and by Poincaré–Wirtinger inequality :

$$\int_{Q_1} \frac{m^T + \bar{m}}{2} |D(u^T - \bar{u})|^2 \geq c_0 \int_{Q_1} (\tilde{u}^T - \bar{u})^2\, dx$$

and by our assumption on F :

$$\Big(F(x,m^T) - F(x,\bar{m}^T)\Big)(m^T - \bar{m}) \geq \gamma (m^T - \bar{m})^2$$

Proof of the Lemma

So

$$\frac{d}{dt}\varphi(t) \;\leq\; -\sigma\int_{Q_1}\left\{(\tilde{u}^T-\bar{u})^2+(m^T-\bar{m})^2\right\}\,dx$$

$$\leq\; -2\sigma\left|\int_{Q_1}(\tilde{u}^T-\bar{u})(m^T-\bar{m})\right| \;=\; -2\sigma\,|\varphi(t)|$$

On concludes by using the Hamiltonian structure of the problem which gives

$$\varphi(0)\leq C \qquad \text{and} \qquad \varphi(T)\geq -C\,.$$

Convergence under mild monotony condition

Assume now that F is just increasing. Then

Theorem

As $T \to +\infty$,

1. $v^T(\cdot, t)/T$ *converges to* $t \to (1-t)\bar{\lambda}$ *in* $L^2(Q_1)$ *for any* $t \in [0,1]$,
2. $v^T - \int_{Q_1} v^T(t)$ *converges to* $\bar{u}$ *in* $L^2(Q_1 \times (0,1))$,
3. μ^T *converges to* $\bar{m}$ *in* $L^p(Q_1 \times (0,1))$,
   for any $p < \frac{N+2}{N}$ *if* $N > 2$ *and for any* $p < 2$ *if* $N = 2$.

Conclusion and open problems

In this preliminary work, we show that, as $T \to +\infty$

- $\frac{u^T(\cdot, t)}{T}$ converges to $\bar{\lambda}(T - t)$,
- m^T converges to $\bar{m}$.
- $u^T - \int_{Q_1} u^T$ converges to $D\bar{u}$.

However

- the behavior of $u^T - \bar{\lambda}(T - t)$ is not yet understood,
- one would expect exponential convergence in the nonlocal case (analysis of the linearized system)
- the approach seems to work for more general MFG...

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