Explicit solutions of some Linear-Quadratic Mean Field Games

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Plan

- L-Q games with N players and ergodic cost
- Symmetric and almost identical players
- The limit as $N \to +\infty$
- Other limiting cases
  - Vanishing viscosity
  - Cheap control
  - Discounted cost and small discount
- Models of population distribution

References

We look for L-Q analogues of the results for stationary periodic problems in

- J.-M. Lasry, P.-L. Lions: Jeux à champ moyen. I. Le cas stationnaire. C. R. A. S. 2006
- J.-M. Lasry, P.-L. Lions: Jpn. J. Math. 2007

Related discounted L-Q problems were studied by

- M. Huang, P.E. Caines, R.P. Malhamé: Proc. IEEE Conf. 2003.
- M. Huang, P.E. Caines, R.P. Malhamé: IEEE Trans. Automat. Control 2007

Related models of population distribution are in

- O. Guéant: J. Math. Pures Appl. 2009
- O. Guéant, J.-M. Lasry, P.-L. Lions: Mean field games and applications , to appear.

L-Q games with N players and ergodic payoff

Control systems

$$dX_t^i = (A^i X_t^i - \alpha_t^i)dt + \sigma^i dt W_t^i, \quad X_0^i = x^i \in \mathbf{R}, \quad i = 1, \dots, N$$

W_t^i independent Brownian motions, $\alpha_t^i =$ control of i-th player, long-time-average cost functional:

$$J^i(X, \alpha^1, \dots, \alpha^N) := \lim_{T \to +\infty} \frac{1}{T} E\left[\int_0^T \frac{R_i}{2}(\alpha_t^i)^2 + F^i(X_t^1, \dots, X_t^N) dt\right],$$

with $R_i > 0$, quadratic running cost, for some reference position $\overline{X}_i$

$$F^i(x^1, \dots, x^N) := (X - \overline{X}_i)^T Q^i (X - \overline{X}_i)$$
$$= \sum_{j,k=1}^{N} q_{jk}^i (x^j - \overline{x}_i^j)(x^k - \overline{x}_i^k), \quad q_{ii}^i > 0,$$

An admissible strategy $\overline{\alpha}$ is a Nash equilibrium if

$$J^i(X,\overline{\alpha}) = \min_{\alpha^i} J^i(X,\overline{\alpha}^1,\dots,\overline{\alpha}^{i-1},\alpha^i,\overline{\alpha}^{i+1},\dots,\overline{\alpha}^N) \quad \forall\, i=1,\dots,N.$$

Admissible strategies are adapted controls such that $E[X_t^i]$, $E[(X_t^i)^2] \le C$, $\forall\, t>0$ and $\exists$ prob. measure m_{α^i}:

$$\lim_{T\to+\infty} \frac{1}{T} E\left[\int_0^T g\left(X_t^i\right)\, dt\right] = \int_{\mathbf{R}} g(x)\,dm_{\alpha^i}(x).$$

for any polynomial g, $\deg(g) \le 2$, locally uniformly in $x^i = X_0^i$.

Example: affine feedbacks

$$\alpha^i(x) = K^i x + c_i, \qquad K^i > A^i,$$

whose corresponding diffusion process is ergodic

$$dX_t^i = [(A^i - K^i)X_t^i - c_i]dt + \sigma^i dW_t^i.$$

The HJB-K PDEs

As in Lasry-Lions we set

$$H^i(x,p) = \frac{p^2}{2R_i} - A^i xp, \quad \nu^i := \frac{(\sigma^i)^2}{2},$$

$$f^i(x; m_1, \dots, m_N) := \int_{\mathbf{R}^{N-1}} F^i(x^1, \dots, x^{i-1}, x, x^{i+1}, \dots, x^N) \prod_{j \neq i} dm_j(x^j),$$

$$\begin{cases} -\nu^i v^i_{xx} + \frac{(v^i_x)^2}{2R_i} - A^i x v^i_x + \lambda_i = f^i(x; m_1, \dots, m_N), \quad i = 1, \dots, N \\ -\nu^i (m_i)_{xx} - \left(\frac{v^i_x}{R_i} m_i - A^i x m_i \right)_x = 0, \quad i = 1, \dots, N \\ \int_{\mathbf{R}} m_i(x) dx = 1, \quad m_i > 0 \text{ in } \mathbf{R}, \end{cases}$$

Cannot normalize v^i by $\int_{\mathbf{R}} v^i(x) dx = 0$ as in the periodic case!

Quadratic-Gaussian solutions

Look for solutions with each v^i quadratic

$$v^i(x) = \frac{(x-\mu_i)^2}{2s_i} + \frac{R_i A^i x^2}{2},$$

and measures m_i Gaussian

$$m_i(x) = c_i \exp\left(-\frac{v^i(x)}{\nu^i R_i} + \frac{R_i A^i x^2}{2}\right) = \frac{1}{\sqrt{2\pi s_i \nu^i R_i}} \exp\left(-\frac{(x-\mu_i)^2}{2s_i \nu^i R_i}\right)$$

Now the unknowns are the $3N$ constants μ_i, s_i, λ_i.
A useful auxiliary matrix B is

$$B_{ii} := 2q^i_{ii} + R_i(A^i)^2, \quad B_{ij} := 2q^i_{ij} \quad i \neq j.$$

Theorem: Q-G solution of the $2N$ HJB-K system

If $\det B \neq 0$ then
i) there exists unique μ_i, s_i, λ_i such that the Q-G solution v^i, m_i solves the $2N$ HJB-K system and

$$s_i = \left(2q^i_{ii}R_i + (R_iA^i)^2\right)^{-1/2},$$

$$\mu = B^{-1}p, \quad p_i := 2q^i_{ii}\overline{x}^i_i + 2\sum_{j\neq i} q^i_{ij}\overline{x}^j_i,$$

ii) the affine feedback

$$\overline{\alpha}^i(x) = \frac{x-\mu_i}{s_iR_i} + A^i x, \quad i = 1, \dots, N,$$

is a Nash equilibrium point for all initial positions $X \in \mathbf{R}^N$ among the admissible strategies and $J^i(X, \overline{\alpha}) = \lambda_i$ for all X and i.

Proof

i) Plug the candidate solutions in the equation,

find 2nd degree polynomial and equate the coefficients,

solve $3N$ algebraic equations for the $3N$ unknown parameters.

ii) It's a verification theorem, the idea of proof is classical,
here must be careful with the unbounded terms...

Remark:
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A symmetry condition on the running cost

F^i symmetric with respect to the position of any two other players:

$$F^i(x^1, \dots, x^j, \dots, x^k, \dots, x^N) = F^i(x^1, \dots, x^k, \dots, x^j, \dots, x^N) \quad \forall j, k \neq i.$$

This is equivalent, $\forall j, k, l, m \neq i$, $l \neq j$, $k \neq l$, $k \neq m$,

$q^i_{ij} = q^i_{ik} =: \frac{\beta_i}{2}$ = primary cost of cross-displacement

$\overline{x}^j_i = \overline{x}^k_i =: r_i$ = reference position of the other players

$q^i_{jj} = q^i_{kk} =: \eta_i$ = secondary cost of self-displacement

$q^i_{lj} = q^i_{kl} = q^i_{km} =: \gamma_i$ = secondary cost of cross-displacement.

We also set

$q_i := q^i_{ii}$ = primary cost of self-displacement

$h_i := \overline{x}^i_i$ = preferred own position (happy state)

Now each F^i involves only 6 parameters and can be written as

$$(V_i) \qquad F^i(x^1,\dots,x^N) = V_i\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](x^i)$$

for $\quad V_i : \{\text{prob. measures}\} \to \{\text{quadratic polynomials}\}$

$$\begin{aligned} V_i[m](x) := q_i(x-h_i)^2 & \\ & + \beta_i(x-h_i)(N-1)\int_{\mathbf{R}}(y-r_i)\,dm(y) \\ & + \gamma_i\left((N-1)\int_{\mathbf{R}}(y-r_i)\,dm(y)\right)^2 \\ & + (\eta_i-\gamma_i)(N-1)\int_{\mathbf{R}}(y-r_i)^2\,dm(y). \end{aligned}$$

Viceversa, F^i of the form (V_i) implies F^i symmetric.

Almost Identical players

We say the players are A. I. if F^i are symmetric and the players have the same

- control system, i.e., $A^i = A$ and $\sigma^i = \sigma$ (so $\nu^i = \nu > 0$) for all i,

- cost of the control, i.e., $R_i = R > 0$ for all i,

- reference positions, i.e., $h_i = h$ and $r_i = r$ for all i,

- primary costs of displacement, i.e., $q_i = q > 0$ and $\beta_i = \beta$ for all i.

Secondary costs of displacement γ_i, η_i may still depend on i.

Theorem: identically distributed Q-G solutions

Almost Identical players and $\quad 2q + RA^2 \neq \beta(1-N) \quad \Longrightarrow$

the unique μ , $s > 0$, λ_i, $i = 1, \ldots, N$ such that

$$v^i(x) = v(x) := \frac{(x-\mu)^2}{2s} + \frac{RAx^2}{2}$$

$$m_i(x) = m(x) := \frac{1}{\sqrt{2\pi s\nu R}} \exp\left(-\frac{(x-\mu)^2}{2s\nu R}\right)$$

solve the $2N$ HJB-K equations are

$$s = \left(2qR + R^2A^2\right)^{-1/2}, \quad \mu = \frac{2qh + r\beta(N-1)}{2q + \beta(N-1) + RA^2},$$

$$\lambda_i = \frac{\nu}{s} + \nu RA - \frac{\mu^2}{2Rs^2} + qh^2 - h\beta(N-1)(\mu - r)$$
$$+ \gamma_i(N-1)(N-2)(\mu - r)^2 + \eta_i(N-1)(s\nu R + (\mu - r)^2).$$

The limit as $N \to +\infty$

Assume the parameters A, ν, R, h, r independent of N and the coefficients of $F^{i,N}$ scale, as $N \to +\infty$

$$q^N \to \overline{q}, \qquad \beta^N \sim \frac{\overline{\beta}}{N}, \qquad \eta_i^N \sim \frac{\overline{\eta}}{N}, \qquad \gamma_i^N \sim \frac{\overline{\gamma}}{N^2}$$

Then for any prob. measure m on $\mathbf{R}$, $\forall\, i$,

$$V_i^N[m](x) \to \overline{V}[m](x) \quad \text{locally uniformly in } x,$$

$$\overline{V}[m](x) := \overline{q}(x-h)^2 + \overline{\beta}(x-h)\int_{\mathbf{R}}(y-r)\,dm(y) + \overline{\gamma}\left(\int_{\mathbf{R}}(y-r)\,dm(y)\right)^2 + \overline{\eta}\int_{\mathbf{R}}(y-r)^2\,dm(y).$$

The Mean Field Equations

$$
\text{(MFE)} \quad \begin{cases} -\nu v_{xx} + \frac{(v_x)^2}{2R} - Axv_x + \lambda = \overline{V}[m](x) \quad \text{in } \mathbf{R}, \\ -\nu m_{xx} - \left(\frac{v_x}{R} m - Axm\right)_x = 0 \quad \text{in } \mathbf{R}, \\ \min\left[v(x) - \frac{RAx^2}{2}\right] = 0, \quad \int_{\mathbf{R}} m(x)dx = 1, \quad m > 0 \text{ in } \mathbf{R}. \end{cases}
$$

Remark: the normalization of v is different from $\int v\,dx = 0$ of the periodic case.

Uniqueness

$\overline{V}$ is monotone increasing in $L^2 \iff \overline{\beta} \geq 0$
so in this case there is at most one solution of MFEs.

Meaning: $\overline{\beta} > 0$ if imitation is costly, $\overline{\beta} < 0$ if imitation is rewarding.

Q-G solution of the MFEs

$2\overline{q} + RA^2 \neq -\overline{\beta} \qquad \Longrightarrow \qquad$ the MFEs has exactly one solution v, m, λ of the Q-G form

$$v(x) := \frac{(x-\overline{\mu})^2}{2\overline{s}} + \frac{RAx^2}{2}, \quad m(x) := \frac{1}{\sqrt{2\pi\overline{s}\nu R}} \exp\left(-\frac{(x-\overline{\mu})^2}{2\overline{s}\nu R}\right),$$

given by

$$\overline{s} = \left(2\overline{q}R + R^2A^2\right)^{-1/2}, \quad \overline{\mu} = \frac{2\overline{q}h + r\overline{\beta}}{\overline{\beta} + 2\overline{q} + RA^2},$$

$$\lambda = \frac{\nu}{\overline{s}} + \nu RA - \frac{\overline{\mu}^2}{2R\overline{s}^2} + \overline{q}h^2 - h\overline{\beta}(\overline{\mu} - r) + (\overline{\gamma} + \overline{\eta})(\overline{\mu} - r)^2 + \overline{\eta}\overline{s}\nu R$$

Convergence Theorem

Call v^N, m^N, λ_i^N the identically distributed Q-G solutions of the $2N$ HJB-K equations for N players. Then $2\overline{q} + RA^2 \neq -\overline{\beta} \implies$ as $N \to +\infty$,

- $v^N \to v$ in $C^2_{loc}(\mathbf{R})$
- $m^N \to m$ in $C^k(\mathbf{R})$ for all k,
- $\lambda_i^N \to \lambda$ for all i,

where v, m, λ is the Q-G solution of the MFEs.

Remark: in the critical case $2\overline{q} + RA^2 = -\overline{\beta}$

- no Q-G solution if $2\overline{q}h \neq -\overline{\beta}r$,
- a continuum of Q-G solutions if $2\overline{q}h = -\overline{\beta}r$.

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The vanishing viscosity limit for N players

In the system of $2N$ HJB-K equations let $\nu^j = (\sigma^j)^2/2 \to 0$ for the j-th player. From the explicit formulas we see

$$v^j,\ \overline{\alpha}^j,\ \mu_j,\ s_j \ \text{ independent of } \ \nu^j, \qquad m_j \to \delta_{\mu_j}$$

and the j-th Kolmogorov equation $\quad -\left(\frac{v_x^j}{R_j} m_j - A^j x m_j\right)_x = 0 \quad$ is satisfied in the sense of distributions.

If all $\nu^i = (\sigma^i)^2/2 \to 0,\ i = 1, \ldots, N$, the limit $2N$ HJB-K equations are of first order and the Quadratic-Dirac solutions give the value of a deterministic game for the same feedback Nash equilibrium

$$\overline{\alpha}^i(x) = \frac{x - \mu_i}{s_i R_i} + A^i x.$$

The vanishing viscosity limit in the MFEs

Letting $\nu \to 0$ in the Mean Field equations we have $v, \overline{s}, \overline{\mu}$ independent of ν, $m \to \delta_{\overline{\mu}}$ and get a Quadratic-Dirac solution of the first order MFEs

$$\begin{cases} \frac{(v_x)^2}{2R} - Axv_x + \lambda = \overline{V}[m](x) \\ -\left(\frac{v_x}{R} m - Axm\right)_x = 0 \\ \min\left[v(x) - \frac{RAx^2}{2}\right] = 0, \quad \int_{\mathbf{R}} m(x)dx = 1, \quad m > 0. \end{cases}$$

Rmk: the limits $\nu \to 0$ and $N \to \infty$ commute.

Other singular limits

- Cheap control : can take the limit as $R_i \to 0$ of solutions to $2N$ HJB-K equations for N players. Again
  - $m_i \to$ Dirac mass,
  - limits $R \to 0$ and $N \to \infty$ commute.

- Large cost of cross-displacement : if

$$\lim_{N\to\infty} N|\beta^N| = +\infty$$

instead of $\overline{\beta}$, the term

$$\beta^N(N-1)(x-h)\int_{\mathbf{R}}(y-r)\,dm(y)$$

is singular but the solutions have a Q-G limit with $\overline{\mu} = r$.

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Discounted cost functional and vanishing discount

For $\rho > 0$, infinite horizon and discounted running cost:

$$J^i = E\left[\int_0^{+\infty}\left(\frac{R_i}{2}(\alpha_t^i)^2 + F^i(X_t^1,\ldots,X_t^N)\right)e^{-\rho t}dt\right].$$

For quadratic F^i find again Q-G solutions of $2N$ HJB-K equations where λ_i is replaced by ρv^i. Can show that

- as $\rho \to 0$ $\quad \rho v_\rho^i \to \lambda_i, \quad m_i^\rho \to m_i,$

  $v_\rho^i(x) - v_\rho^i(0) + (\mu_i^\rho)^2/2s_i^\rho \to v^i(x)$ and the limit solves the $2N$ HJB-K equations of the ergodic games,
- as $N \to +\infty$ and $\rho > 0$ fixed the Q-G solutions converge to the MFEs with 1st equation

$$-\nu v_{xx} + (v_x)^2/2R - Axv_x + \rho v = \overline{V}[m](x)$$

- limits $\rho \to 0$ and $N \to \infty$ commute.

Guéant studied a model with discounted cost functional

$$E\left[\int_0^{+\infty}\left(\frac{|\alpha_t^i|^2}{2}+\overline{q}|X_t^i-h|^2-\log m(X_t^i)\right)e^{-\rho t}dt\right], \qquad \overline{q}\geq 0,$$

and therefore the first MF equation is ($A=0$, $R=1$)

$$-\nu v_{xx}+(v_x)^2/2+\rho v=\overline{q}|x-h|^2-\log m(x)$$

- V is NOT integral operator, not deriving from limit of N players;
- V decreasing in m models a population whose agents wish to resemble their peers, imitation is rewarding;
- MFEs have one Q-G solution of mean $h \,\forall\, \overline{q}>0$,
- a continuum of Q-G solutions with any mean μ if $\overline{q}=0$.

L-Q model with $A = 0, R = 1, r = h$ has first MF equation

$$-\nu v_{xx} + (v_x)^2/2 + \lambda = \overline{q}(x-h)^2 + \overline{\beta}(x-h)\int_{\mathbf{R}}(y-r)\,dm(y) + \overline{\gamma}\left(\int_{\mathbf{R}}(y-r)\,dm(y)\right)^2 + \overline{\eta}\int_{\mathbf{R}}(y-r)^2\,dm(y)$$

- if $\beta > 0$ imitation is costly, uniqueness of solution,
- if $\beta < 0$ imitation is rewarding as in the LOG model,
- MFEs have one Q-G solution of mean h $\quad \forall\, \overline{q} > 0$ and $\overline{q} \neq -\beta/2$,
- a continuum of Q-G solutions with any mean μ if $\overline{q} = -\beta/2$: same bifurcation phenomenon at different critical value,
- $Var(m) \to +\infty$ as $\overline{q} \to 0$, m tends to uniform distribution.

Developments

- *$d-$dimensional control system* for each player, $d \geq 1$: need o solve matrix Riccati equations; the MFEs are PDEs instead of ODEs.
- Interacting populations with L-Q costs, to be compared with the log-model of Guéant.

Thanks for your attention !

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