# Nonconvex Evolution Problems

Rome, November 30 – December 3, 2010

## Program

### Tuesday, November 30th, 2010, afternoon

13.30 – 14.45 **Welcome lunch**

14.45 – 15.30

speaker Nicholas ALIKAKOS (University of Athens, Greece)

**Heteroclinic traveling waves of gradient diffusion systems**.

15.30 – 16.15

speaker Johannes ZIMMER (University of Bath, United Kingdom)

**Ill-posed problem in nonlinear elasticity: Macro and micro**.

16.15 – 16.45 **Coffee break**

16.45 – 17.30

speaker Giovanni BELLETTINI ("Tor Vergata" Università di Roma, Italy)

**Remarks on a Perona-Malik type equation:**
**Convergence of discrete approximations for large times**.

### Wednesday, December 1st, 2010, morning

10.00 – 10.45

speaker Amy NOVICK-COHEN (Technion - Israel Institute of Technology, Israel)

**The thin film equation with lower order destabilizing forcing**.

10.45 – 11.30

speaker Andrea TERRACINA ("Sapienza" Università di Roma, Italy)

**Two phase entropy solutions for forward-backward parabolic problems**.

11.30 – 12.00 **Coffee break**

12.00 – 12.45

speaker Juan José Lopez VELÀZQUEZ (Universidad Complutense de Madrid, Spain)

**Qualitative properties of solutions of coagulation equations**.

## Wednesday, December 1st, 2010, afternoon

12.45 – 14.30 **Lunch**

14.30 – 15.15

speaker Alain MIRANVILLE (Université de Poitiers, France)

**The Cahn-Hilliard equation with dynamic boundary conditions**.

15.15 – 16.00

speaker Maurizio GRASSELLI (Politecnico di Milano, Italy)

**Singularly perturbed Cahn-Hilliard equations**.

## Thursday, December 2nd, 2010, morning

10.00 – 10.45

speaker Mark A. PELETIER (Technische Universiteit Eindhoven, The Netherlands)

**Passing to the limit in the Wasserstein gradient-flow formulation**.

10.45 – 11.30

speaker Paolo BUTTÀ ("Sapienza" Università di Roma, Italy)

**Navier-Stokes equations on the flat cylinder with vorticity production on the boundary**.

11.30 – 12.00 **Coffee break**

12.00 – 12.45

speaker Ulisse STEFANELLI (IMATI - CNR, Italy)

**A new variational view at Lagrangian mechanics**.

## Thursday, December 2nd, 2010, afternoon

12.45 – 14.30 **Lunch**

14.30 – 15.15

speaker Philippe LeFLOCH (Université de Paris VI and CNRS, France)

**Self-similar vanishing viscosity-capillarity limits**.

15.15 – 16.00

speaker Rinaldo M. COLOMBO (Università di Brescia, Italy)

**Phase transitions in hyperbolic models**.

## Friday, December 3rd, 2010, morning

10.00 – 10.45

speaker Lorenzo BERTINI ("Sapienza" Università di Roma, Italy)

**Boundary effects in the gradient theory of phase transitions**.

10.45 – 11.30

speaker Giorgio FUSCO (Università dell'Aquila, Italy)

**Remarks on the dynamics of some forward-backward parabolic equations**.

11.30 – 12.00 **Coffee break**

12.00 – 12.45

speaker Felix OTTO (Universität Bonn, Germany)

**Pattern formation and Partial Differential Equations**.

## End of the Workshop

# Titles, abstracts & schedule

## Nicholas D. Alikakos

University of Athens, Greece

title **Heteroclinic traveling waves of gradient diffusion systems**.

abstract We establish existence of a traveling wave to a parabolic gradient system connecting two minima of the potential.
This is joint work with N. Katzourakis. It is based on previous joint work with G. Fusco.

**14.45, Tuesday, November 30, 2010**.

## Giovanni Bellettini

"Tor Vergata" Università di Roma, Italy

title **Remarks on a Perona-Malik type equation: Convergence of discrete approximations for large times**.

abstract Using an asymptotic expansion argument, we discuss a rigorous convergence result of a semidiscrete Perona-Malik type equation as the grid size goes to zero, in a suitable large time scale. Despite the fact that the original equation is forward-backward, a comparison argument based on the construction of suitable sub/supersolutions guarantees the convergence to a limit system of ordinary differential equations.

**16.45, Tuesday, November 30, 2010**.

## Lorenzo Bertini

“Sapienza" Università di Roma, Italy

title **Boundary effects in the gradient theory of phase transitions**.

abstract We consider the van der Waals free energy functional in a bounded interval with inhomogeneous Dirichlet boundary conditions imposing the two stable phases at the endpoints. We compute the asymptotic free energy cost, as the length of the interval diverges, of shifting the interface from the midpoint. We then discuss the effect of thermal fluctuations by analyzing the so- called $\phi^4_1$ measure with Dobrushin boundary conditions. In particular, we obtain a non-trivial limit in a suitable scaling in which the length of the interval diverges and the temperature vanishes. The limiting state is not translation invariant and describes a localized interface. This result can be seen as the probabilistic counterpart of the variational convergence of the associated excess free energy.

**10.00, Friday, December 3, 2010**.

## Paolo Buttà

"Sapienza" Università di Roma, Italy

title **Navier-Stokes equations on the flat cylinder with vorticity production on the boundary**.

abstract I present a recent result obtained in collaboration with C. Boldrighini. We consider the incompressible two-dimensional Navier-Stokes system on a flat cylinder with the usual Dirichlet boundary conditions for the velocity field. We take into account the boundary conditions by adding a vorticity production at the boundary, and formulate the problem as an infinite system of ordinary differential equations for the natural Fourier components of the vorticity. Under some general assumptions on the initial data, we prove existence and uniqueness of the solution, as well as equivalence to the original Navier-Stokes system. We show that the decay of the Fourier modes is exponential for any positive time in the periodic direction, but it is only as an inverse square in the other direction.

**10.45, Thursday, December 2, 2010**.

## Rinaldo M. Colombo

Università di Brescia, Italy

title **Phase transitions in hyperbolic models**.

abstract An increasing quantity of phenomena is being classified under the terms "Phase Transition". Conservation laws provide models for several of these phenomena. The present talk will consider analytical results obtained in models for

- liquid-vapor phase transitions;
- combustion and deflagration/detonation;
- vehicular traffic;
- crowd dynamics.

In spite of the deep physical differences, the same analytical framework applies to all these situations.

**15.15, Thursday, December 2, 2010**.

## Giorgio Fusco

University of L'Aquila, Italy

title **Remarks on the dynamics of some forward-backward parabolic equations**.

abstract If $\phi : \mathbb{R} \to \mathbb{R}$ is a non-convex energy density, the initial-boundary value problem

for the equation

$$\begin{cases} u_t = \frac{1}{2}(\phi(u_x))_x, & x \in (0,1), \\ u(\cdot,0) = u_0, \\ +BC. \end{cases} \qquad (1)$$

is not well posed. In the attempt to define a notion of solution for equation (1) we consider, for $0 < \epsilon << 1$, the regularization:

$$u_t = -\varepsilon^2 u_{xxxx} + \frac{1}{2}(\phi(u_x))_x,$$

of problem (1) and, motivated by numerical experiments, we reinterpret equation (1) as a suitable free-boundary problem which is well defined for any given initial datum.

**10.45, Friday, December 3, 2010**.

## Maurizio Grasselli

Politecnico di Milano, Italy

title **Singularly perturbed Cahn-Hilliard equations**.

abstract In order to describe fast separation processes in certain binary solutions (e.g., glasses), some physicists have proposed to modify the classical Cahn-Hilliard equation. The resulting nonlinear evolution equation is characterized by the presence of a second-order time derivative of the order parameter multiplied by a (small) inertial coefficient. In the non-viscous case, the additional term changes the nature of the equation and instantaneous regularization effects are lost. In one spatial dimension, this modified Cahn-Hilliard equation has been analyzed by a number of authors. Here I intend to illustrate some recent results in two and three dimensions.

**15.15, Wednesday, December 1, 2010**.

## Philippe LeFloch

Université de Paris VI and CNRS, France

title **Self-similar vanishing viscosity-capillarity limits**.

abstract In this lecture, I will discuss a system of conservation laws arising in liquid-vapor phase dynamics when physical viscosity and capillarity effects are taken into account. Within the class of self-similar solutions to the Riemann problem, uniform total variation bounds have been now established, which allow to deduce new existence results. The analysis covers, both, the hyperbolic and the hyperbolic-elliptic regimes and applies to arbitrarily large Riemann data. This study is motivated by the theory of kinetic relations, introduced for the selection of nonclassical undercompressive shock waves.

**14.30, Thursday, December 2, 2010**.

## Alain Miranville

Université de Poitiers, France

title **The Cahn-Hilliard equation with dynamic boundary conditions**.

abstract Our aim in this talk is to discuss issues (well-posedness, asymptotic behavior) related with the Cahn-Hilliard equation with dynamic boundary conditions. Such boundary conditions have recently been proposed in order to account for the wall effects in confined systems. In particular, we are interested in the case of thermodynamically relevant logarithmic potentials.

**14.30, Wednesday, December 1, 2010**.

## Amy Novick-Cohen

Technion - Israel Institute of Technology, Israel

title **The thin film equation with lower order destabilizing forcing**.

abstract Often the thin film equation

$$u_t + \{u^n(u_{xxx})\}_x = 0 \tag{1}$$

needs to be augmented by lower order forcing terms in order to take into account the presence of various additional physical effects. In accommodating, for example, gravity and thermo-capillarity effects, attractive polar and van der Waals forces, the resultant equation can be written as

$$u_t + \{u^n(u_{xxx} + h'(u))\}_x = 0, \tag{2}$$

where $h'(u) > 0$; which is endowed with the underlying non-convex energy

$$E = \int_\Omega \left\{ \frac{1}{2}\, u_x^2 - \int^u h(\bar{u})\, d\bar{u} \right\} dx.$$

In this work we focus primarily on the particular case in which $h'(u) = u^{m-n} - A\, u^{M-n}$, with $0 < n, m < M$, and $A \geq 0$. For (2), we obtain an assortment of new results regarding the existence of weak, strong, entropy, and energy solutions, limiting regularity, and finite speed of propagation. These results are discussed and compared with previous known results and conjectures in the literature.
Joint work with Andrey Shishkov.

**10.00, Wednesday, December 1, 2010**.

## Felix Otto

University of Bonn, Germany

title **Pattern formation and Partial Differential Equations**.

abstract In three specific examples, we shall demonstrate how the theory of partial differential equations (PDEs) relates to pattern formation in nature: Spinodal decomposition and the Cahn-Hilliard equation, Rayleigh-Bénard convection and the Boussinesq approximation, rough crystal growth and the Kuramoto-Sivashinsky equation.

These examples from different applications have in common that only a few physical mechanisms, which are modeled by simple-looking evolutionary PDEs, lead to complex patterns. These mechanisms will be explained, numerical simulation shall serve as a visual experiment. Numerical simulations also reveal that generic solutions of these deterministic equations have stationary or self-similar statistics that are independent of the system size and of the details of the initial data.

We show how PDE methods, i. e. a priori estimates, can be used to understand some aspects of this universal behavior. In case of the Cahn-Hilliard equation, the method makes use of its gradient flow structure and a property of the energy landscape. In case of the Boussinesq equation, a "driven gradient flow", the background field method is used. In case of the Kuramoto-Sivashinsky equation, that mixes conservative and dissipative dynamics, the method relies on a new result on Burgers' equation.

**12.00, Friday, December 3, 2010**.

## Mark A. Peletier

Technische Universiteit Eindhoven, The Netherlands

title **Passing to the limit in the Wasserstein gradient-flow formulation**.

abstract The Wasserstein gradient-flow structure describes a large number of parabolic, diffusive systems. This structure has been used to derive many properties of such systems, such as well-posedness, stability, and large-time behaviour. Here we focus on the use of the gradient-flow structure to prove convergence.

Extending ideas of Stefanelli and Serfaty, we use the Wasserstein gradient-flow structure to prove convergence in a singularly perturbed diffusion problem. Our test problem arises from the interpretation of chemical reactions as diffusion in a potential landscape, initiated by Wigner and Kramers in the 1930's. In this interpretation a reaction event corresponds to the escape of the diffusing particle from one potential well into another. In earlier work (with Savaré and Veneroni) we proved the convergence of this system in the limit of high activation energy to the corresponding reaction-diffusion system — but without making use of the Wasserstein gradient-flow structure.

In this talk I revisit the result, and reprove it using the Wasserstein gradient-flow structure. The method has some interesting aspects, such as relatively weak compactness requirements, a somewhat surprising limit, and a tight connection to stochastic particle systems. In addition it has the potential for wide applicability among the broad class of Wasserstein gradient flows.

**10.00, Thursday, December 2, 2010**.

## Ulisse Stefanelli

IMATI - CNR, Italy

title **A new variational view at Lagrangian mechanics**.

abstract I shall present a new tool toward the variational resolution of (a suitable approximating version of) Lagrange's equations. In particular, by restricting to a finite-dimensional yet nonconvex evolution, I provide a result in the direction of a conjecture proposed by De Giorgi.

**12.00, Thursday, December 2, 2010**.

## Andrea Terracina

"Sapienza" Università di Roma, Italy

title **Two phase entropy solutions for forward-backward parabolic problems**.

abstract In this talk we will discuss an entropy formulation of a forward-backward parabolic problem. This formulation was obtained by Plotnikov using a third order pseudoparabolic approximation of the initial problem. This approximation is motivated in the context of phase transition models where the diffusion function is of cubic type. In particular we will analyze the case in which the entropy solution takes values only in the stable phases corresponding to the intervals in which the diffusion is increasing. We will present some uniqueness and existence results obtained in collaboration with C. Mascia and A. Tesei. Moreover we will examine some qualitative properties of the entropy solution.

**10.45, Wednesday, December 1, 2010**.

## Juan José Lopez Velàzquez

Universidad Complutense de Madrid, Spain

title **Qualitative properties of solutions of coagulation equations**.

abstract In this talk I will discuss several qualitative properties of the solutions of Smoluchowsky coagulation equations both in the cases in which mass conservation takes place, and also in cases in which loss of mass occurs. In particular, the

existence of classical solutions in the gelling case, as well as the oscillatory asymptotics of the particle distributions for small particles will be discussed and the existence of fat tail solutions for a class of nonexplicitly solvable kernels.
The results of this talk have been proved in collaboration with M. Escobedo, J. B. McLeod and B. Niethammer.

**12.00, Wednesday, December 1, 2010**.

## Johannes Zimmer

University of Bath, United Kingdom

title **Ill-posed problem in nonlinear elasticity: Macro and micro**.

abstract The equations of elasticity in one space dimension, $u_{tt} = \sigma(u_x)_x$, become ill-posed if the potential energy density is nonconvex, or equivalently if $\sigma$ is non-monotone. This complication necessarily arises in the theory of so-called martensitic phase transitions, which are diffusionless solid-solid transformations with where several stable phases can coexist.
Different regularisations of this ill-posed problem have been proposed; we will here focus on so-called kinetic relations, which relate the velocity of a moving interface to a driving force. Phenomenological kinetic relations have been proposed, but a natural question is whether they can in simple situations be derived from first principles, namely atomistic considerations.
To investigate this question, we study the simplest one-dimensional chain model of martensitic materials, where neighbouring atoms are coupled by a spring with bi-quadratic potential. We prove the existence of travelling waves and discuss a microscopic ill-posednedness that raises, namely the non-uniqueness of microscopic solutions. This non-uniqueness will be discussed in light of the macroscopic kinetic relation.

**15.30, Tuesday, November 30, 2010**.