Qualitative properties of coagulation equations.

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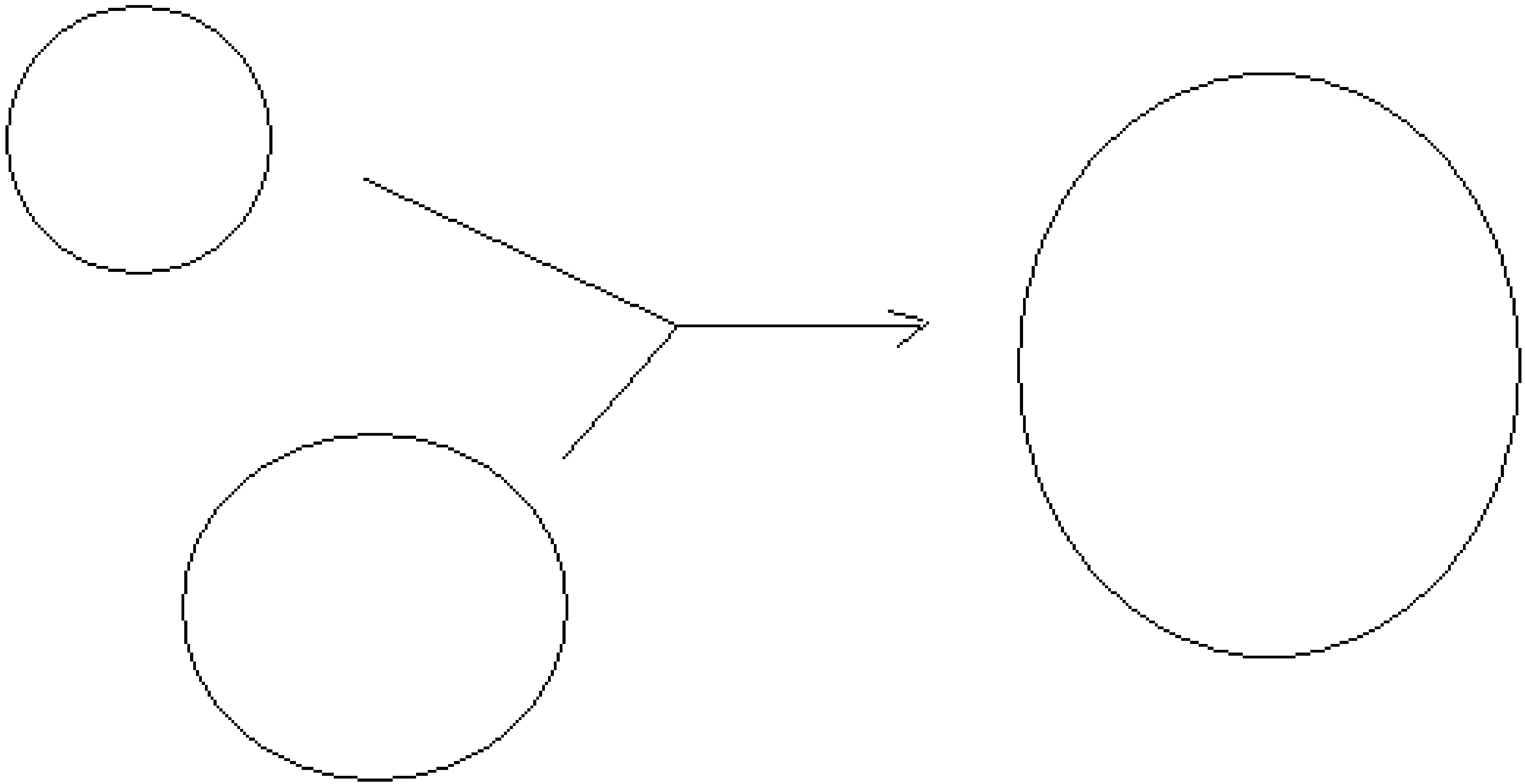
Coagulation



Model: Classical coagulation equation.

$$\frac{\partial f}{\partial t} = Q[f]$$

$$Q[f] = \frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)dy -$$
$$-\int_0^\infty K(x,y)f(x,t)f(y,t)dy$$

$x =$ particle size

$$(x,y) \to (x+y)$$

Rate $= K(x,y) = K(y,x)$

$f = f(x,t)$ (particle distribution)

(Discrete versions of the model=Measure valued solutions).

Classical results.

Explicitly solvable kernels:

$$K(x,y) = 1 \quad , \quad K(x,y) = x + y \quad , \quad K(x,y) = x \bullet y$$

Convolution kinetic equations with linear kernels. Laplace transform reduces the equations to first order hyperbolic models.

B. McLeod, M. H. Ernst-R. M. Ziff-E. M. Hendriks, M. Kreer-O. Penrose, ...

G. Menon, R. L. Pego.

General kernels: Asymptotic behaviour of the solutions:

- Tendency of the average radius to increase.
- Self-similarity.
- Behaviour depends on the growth of the kernel for large particles.

Homogeneous kernels:

$$K(\theta x, \theta y) = \theta^{\mu} K(x, y)$$

$\mu \leq 1$, Global existence

N. Fournier, P. Laurençot

$1 < \mu < 2$, Singularity formation in finite time.

I. Jeon, M. Escobedo, B. Perthame, S. Mischler.

$\mu > 2$, Instantaneous gelation.

J. R. Norris

Sublinear regime: Self-similar solutions.

Homogeneous kernels:

$$K(\theta x, \theta y) = \theta^{\mu} K(x,y) \ , \ 0 < \mu < 1$$

Mass-preserving self-similar solutions exist:

$$\frac{d}{dt}\int_0^{\infty} x f(x,t)dx = 0$$

$$f(x,t) = \frac{1}{t^{\frac{2}{1-\mu}}}\Phi(\xi) \quad , \quad \xi = \frac{x}{t^{\frac{1}{1-\mu}}}$$

$$-\frac{2}{1-\mu}\Phi(\xi) - \frac{1}{1-\mu}\xi\Phi'(\xi) = Q[\Phi](\xi)$$

N. Fournier-P. Laurençot.

M. Escobedo-S. Mischler-M. Rodríguez Ricard.

Sublinear regime ($0 < \mu < 1$): Self-similar solutions.

Open questions:

- Uniqueness of solutions.
- Stability.
- Asymptotic properties of the self-similar solutions.

Optimal estimates for the self-similar solutions in the sublinear regime ($0 < \mu < 1$).

Kernels:

$$K(x,y) = x^{\alpha}y^{\beta} + x^{\beta}y^{\alpha} \ , \ \alpha + \beta = 2\lambda = \mu \ , \ \alpha > 0 \ , \ \beta > 0$$

$$f(x,t) = \frac{1}{t^{\frac{2}{1-2\lambda}}}\Phi(\xi) \quad , \quad \xi = \frac{x}{t^{\frac{1}{1-2\lambda}}}$$

$$-\frac{2}{1-2\lambda}\Phi(\xi) - \frac{1}{1-2\lambda}\xi\Phi'(\xi) = Q[\Phi](\xi)$$

Formally balancing terms (P. G. J. van Dongen-M. H. Ernst):

$$\xi\Phi(\xi) \sim H_{\lambda}\xi^{-2\lambda} \ , \quad \xi \to 0^{+}$$

Estimates (M. Escobedo-S. Mischler).

$$\xi\Phi(\xi) \leq c_2\xi^{-(2\lambda+\varepsilon)} \ , \ \xi \to 0^{+} \ , \ \lim_{\xi\to 0^{+}} \inf[\xi\Phi(\xi)\xi^{(2\lambda-\varepsilon)}] = \infty \ , \ \varepsilon > 0$$

Optimal estimates (B. Niethammer-V):

$$c_1 \xi^{-2\lambda} \leq \xi \Phi(\xi) \leq c_2 \xi^{-2\lambda} \quad , \quad \xi \to 0^+ \quad , \quad c_1 > 0, \; c_2 > 0$$

Assumptions required on the kernel:

Homogeneity: $K(\theta x, \theta y) = \theta^{2\lambda} K(x,y) \quad , \quad 0 < \lambda < \frac{1}{2}$

Diagonal-dominated:

$$K(x,y) \leq K_0 (x^\alpha y^\beta + x^\beta y^\alpha) \quad , \quad \alpha > 0 \quad , \quad \beta > 0$$

$$\min_{\left[\frac{1}{4},1\right] \times \left[\frac{1}{4},1\right]} K(x,y) \geq k_0 > 0$$

Diagonal dominated kernels:

Coagulation is an event more likely for particles with comparable sizes.

The shape of the self-similar solutions is due to the balance between the coagulation process (increases particle size) and the transport term that arises due to the change of variables (reduces particle size).

The optimal estimates for the self-similar solutions mean that the particle distribution for a given (self-similar) size is due to particles with comparable sizes.

Precise formulation:

$$g(\xi) = \frac{\xi\Phi(\xi)}{1-2\lambda}$$

$$\frac{\partial}{\partial\xi}(\xi g(\xi)) = \frac{\partial}{\partial\xi}(J[g](\xi))$$

$$J[g](\xi) = \int_0^{\xi} g(\eta)d\eta \int_{\xi-\eta}^{\infty} \frac{K(\eta,\zeta)}{\zeta} g(\zeta)d\zeta$$

$$\xi g(\xi) = \int_0^{\xi} g(\eta)d\eta \int_{\xi-\eta}^{\infty} \frac{K(\eta,\zeta)}{\zeta} g(\zeta)d\zeta$$

$$g(\xi) \approx \xi^{-2\lambda} \ , \ \xi \to 0^{+} \ \ , \ K(\eta,\zeta) \approx (\eta)^{\lambda}(\zeta)^{\lambda}$$

Comparison with the case of Non-diagonal dominated kernels.

Example (N. Fournier-P. Laurençot, J. A. Cañizo-S. Mischler).

$$K(x,y) = x^{2\lambda} + y^{2\lambda} \quad , \quad 0 < \lambda < \frac{1}{2}$$

$$\xi g(\xi) = \int_0^{\xi} g(\eta)d\eta \int_{\xi-\eta}^{\infty} \frac{K(\eta,\zeta)}{\zeta} g(\zeta)d\zeta$$

$$\frac{\partial}{\partial \xi}(\xi g)(\xi) = g(\xi)\int_0^{\infty} \frac{K(\eta,\zeta)}{\zeta} g(\zeta)d\zeta - \int_0^{\xi} \frac{K(\eta,\xi-\eta)}{(\xi-\eta)} g(\eta)g(\xi-\eta)d\eta$$

Assuming fast decay of $g(\xi)$ as $\xi \to \infty$ we obtain that the leading terms that determine the asymptotics as $\xi \to 0^+$ are:

$$\frac{\partial}{\partial \xi}(\xi g)(\xi) = g(\xi) \int_0^\infty \frac{K(\eta,\zeta)}{\zeta} g(\zeta) d\zeta$$

$$g(\xi) \approx (\xi)^{-1+\sigma} \ , \ \xi \to 0^+ \ \ \sigma = \int_0^\infty \frac{K(\eta,\zeta)}{\zeta} g(\zeta) d\zeta > 0$$

The asymptotics depends on global properties of g. The distribution g is determined by interactions with much larger particles.

Oscillatory behaviour of self-similar solutions in the sublinear regime. (Diagonal dominated kernels not too far away from nondiagonal dominated ones).

Coagulation equation. Self-similar solutions:

$$\xi g(\xi) = \int_0^{\xi} g(\eta)d\eta \int_{\xi-\eta}^{\infty} \frac{K(\eta,\zeta)}{\zeta} g(\zeta)d\zeta$$

$$K(\eta,\zeta) = (\eta)^{\lambda}(\zeta)^{\lambda}$$

Oscillations of $g(\xi)$ as $\xi \to 0^+$ numerically observed by M. H. Lee and N. Filbet-P. Laurençot.

Asymptotics:

$$g(\xi) \sim K_{1,\lambda}(\xi)^{-2\lambda} + K_{2,\lambda}(\xi)^{-2\lambda+a}\cos(b\log(\xi)+\delta) \ , \ \ \xi \to 0^+$$

Construction of oscillatory self-similar solutions of the coagulation equation. ($\lambda \to 0^+$).

Formal argument (J. B. McLeod, B. Niethammer, V).

Rigorous construction (B. Niethammer, V).

$$f(x,t) = \frac{1}{t^{\frac{2}{1-2\lambda}}}\Phi(\xi) \quad , \quad \xi = \frac{x}{t^{\frac{1}{1-2\lambda}}}$$

$$\Phi(\xi) = \frac{H_\lambda}{(1-2\lambda)} H(X) \ , \ H_\lambda = \frac{\lambda}{B(1-\lambda, 1-\lambda)} \ , \ \xi = e^X$$

$$U(X) = (1-\lambda)\int_{-\infty}^{X} H(Y) e^{(1-\lambda)(Y-X)} dY \ , \ V(X) = \lambda \int_X^{\infty} H(Z) e^{\lambda(X-Y)} dZ$$

Volterra-like integro-differential equation:

$$\frac{dU}{dX} = -(1-\lambda)U + (1-\lambda)H$$

$$\frac{dV}{dX} = \lambda V - \lambda H$$

$$H = \frac{H_\lambda}{\lambda(1-\lambda)} UV + I[H]$$

$$I[H](X) = H_\lambda \int_{-\infty}^{X} dY e^{(1-\lambda)(Y-X)} H(Y) \int_{\log(e^X - e^Y)}^{X} dZ e^{-\lambda(Z-X)} H(Z)$$

Approximations of the solutions if $\lambda \to 0^+$.

(a) U, $\frac{V-1}{\sqrt{\lambda}}$, H of order one:

$$H(X) \approx U(X)V(X) \quad , \quad V = 1 + \sqrt{\lambda}\,\omega \ , \ X = \sqrt{\lambda}\,\tau$$

$$\frac{dU}{d\tau} = U\omega \ , \ \frac{dV}{d\tau} = 1 - U$$

Conserved quantity:

$$-\log(U) + (U-1) + \frac{\omega^2}{2}$$

(Periodic solutions).

A more detailed asymptotics:

$$\frac{dU}{d\tau} = U\omega + \sqrt{\lambda}\, U(U-1) \;\; , \;\; \frac{dV}{d\tau} = 1 - U + \sqrt{\lambda}\, \omega(1-U)$$

"Adiabatic" increase of the energy.

(b) $\lambda U,\ \lambda H,\ (V-1)$ of order one:

Coupled ODE Integro-Differential Equation behaviour.

(c) Shooting argument.

Remark: The limit $\lambda \to 0^+$ corresponds to the convergence of the diagonal dominated kernel to the non-diagonal dominated kernel.

Oscillations seem to be due to the interaction of particles with much bigger ones. (A clear particle interpretation still missing).

Kernels that dot yield interaction with particles having sizes different from themselves:

$$K(x,y) = (x)^{1+2\lambda}\delta(x-y)$$

$$\partial_t f(x,t) = \frac{1}{4}\left(\frac{x}{2}\right)^{1+2\lambda}\left(f\left(\frac{x}{2}\right)\right)^2 - x^{1+2\lambda}(f(x))^2$$

Self-similar solutions. (Mass conserving and not mass conserving cases).

$$f(x,t) = t^{-1+(1+2\lambda)\beta}\Phi(\xi) \quad , \quad \xi = \frac{x}{t^\beta}$$

$$\beta\xi^2\Phi(\xi) = \int_{\frac{\xi}{2}}^{\xi} s^{2+2\lambda}(\Phi(s))^2 ds + (1-2\lambda)(\beta-\beta_*)\int_0^{\xi} s\Phi(s)ds$$

$$\beta_* = \frac{1}{1-2\lambda}$$

Mass-preserving case (F. Levyraz) ($\beta = \beta_*$). Exponential decay.

General case (B. Niethammer-V). Power law behaviour as $\xi \to \infty$. Tail solutions. Previously known only for the explicit kernels $K(x,y) = 1$ and $K(x,y) = x + y$. (G. Menon, R. L. Pego).

Non-oscillatory asymptotics near the origin:

$$\Phi(\xi) \sim K_{1,\lambda}(\xi)^{-(1+2\lambda)} - K_{2,\lambda}(\xi)^{-(1+2\lambda)+a} \quad , \quad \xi \to 0^+ \quad , \quad a > 0$$

Oscillations are due to interaction with much larger particles.

Superlinear regime: Gelation.

$$\frac{\partial f}{\partial t} = Q[f]$$

$$Q[f] = \frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)dy -$$

$$-\int_0^\infty K(x,y)f(x,t)f(y,t)dy$$

$$K(x,y) = (x \cdot y)^{\frac{\lambda}{2}} \quad , \quad \lambda > 1$$

Formal mass conservation:

$$\frac{d}{dt}\left(\int_0^\infty xf(x,t)dx\right) = 0$$

However, explicit solutions for the kernel $K(x,y) = x \bullet y$ show that there are solutions of the corresponding equation satisfying:

$$\frac{d}{dt}\left(\int_0^\infty xf(x,t)dx\right) < 0 \quad , \quad t > T$$

(J. B. McLeod, M. H. Ernst-R. M. Ziff-E. M. Hendriks).

(Gelation).

Non-explicit kernels (I. Jeon, M. Escobedo.B. Perthame-S. Mischler).

Goals: (M. Escobedo, V).

- Obtain classical solutions exhibiting ”loss of mass”.
- Derive detailed asymptotics on how the loss of mass takes place.
- Develop more robust tools allowing to describe gelling solutions in cases where explicit solutions are not available.
- To understand equations exhibiting particle fluxes.

Construction of classical solutions of the coagulation equations exhibiting loss of mass at infinity for the kernels $K(x,y) = (x \cdot y)^{\frac{\lambda}{2}}, \ \lambda > 1.$ (Asymptotics $J(t)x^{-\left(\frac{3+\lambda}{2}\right)}$ as $x \to \infty$).

Three steps:

- Study of the solutions of the linearized problem near the solution $f_s(x) = x^{-\left(\frac{3+\lambda}{2}\right)}$.
- Well-posedness theory for the linearized problem near bounded initial data $f_0(x)$ behaving as $x^{-\left(\frac{3+\lambda}{2}\right)}$ as $x \to \infty$.
- Analysis of the nonlinear problem by means of a fixed point argument.

Step 1:

(a) The function $f_s(x) = x^{-\left(\frac{3+\lambda}{2}\right)}$ is a stationary solution that describes a constant flux of particles from $x = 0$ to $x = \infty$.

Conservative form of the equation (H. Tanaka, S. Inaba, K. Nakaza). Weak formulation of the problem:

$$\frac{\partial}{\partial t}(xf) + \frac{\partial}{\partial x}(j) = 0$$

$$\frac{d}{dt}\left(\int_{R_1}^{R_2} xf dx\right) + j(R_2) - j(R_1) = 0$$

$$j = j(f) = \int_0^x \int_{x-y}^{\infty} yK(y,z)f(y)f(z)dydz$$

$$j(f_s)(R) = K \text{ for any } R > 0$$

Linearization near f_s :

$$f = f_s + g$$

$$\frac{\partial g}{\partial t} = L[g]$$

$$L[g] = \int_0^{\frac{x}{2}} \Big((x-y)^{\frac{\lambda}{2}} f_s(x-y) - x^{\frac{\lambda}{2}} f_s(x) \Big) y^{\frac{\lambda}{2}} g(y)dy +$$

$$+ \int_0^{\frac{x}{2}} \Big((x-y)^{\frac{\lambda}{2}} g(x-y) - x^{\frac{\lambda}{2}} g(x) \Big) y^{-\frac{3}{2}} dy -$$

$$- x^{-\frac{3}{2}} \int_{\frac{x}{2}}^{\infty} y^{\frac{\lambda}{2}} g(y)dy - 2\sqrt{2}\, x^{\frac{\lambda-1}{2}} g(x)$$

It is possible to find a representation formula for the solutions. (Carleman, Balk-Zakharov).

$$\hat{G}(t,\xi) = -\frac{\sqrt{2}}{\sqrt{\pi}\, i(\lambda - 1)} \cdot$$

$$\int_{\mathrm{Im}(y)=\beta_0} \frac{V(\xi)}{V(y)} t^{\frac{2i(\xi-y)}{\lambda-1}} \Gamma\left(-\frac{2i(\xi - y)}{\lambda - 1}\right) dy$$

$$\beta_0 \in \left(\frac{3}{2}, 2\right)$$

$$G(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathrm{Im}(\xi)=a} \hat{G}(t,\xi) e^{i\xi x} d\xi \ , \ a \in \left(\frac{3}{2}, 2\right)$$

$$V(\eta) = -V\left(\eta + \frac{\lambda - 1}{2} i\right) \Phi\left(\eta + \frac{\lambda - 1}{2} i\right)$$

for $\mathrm{Im}(\eta) \in (\frac{3}{2}, 2)$ (plus integrability conditions).

Delay equation in the complex plane:

$$V(\eta) = -V\left(\eta + \frac{\lambda - 1}{2} i\right) \Phi\left(\eta + \frac{\lambda - 1}{2} i\right)$$

for $\mathrm{Im}(\eta) \in (\frac{3}{2}, 2)$

$$\Phi(\eta) = -\frac{2\sqrt{\pi}\,\Gamma(i\eta + 1 + \frac{\lambda}{2})}{\Gamma(i\eta + \frac{1}{2} + \frac{\lambda}{2})}$$

Solvable using Wiener-Hopf method.

Consequences: Detailed understanding of the fundamental solution of the linearized problem.

$$\frac{\partial g}{\partial t} = L[g] \ , \quad g(0,x;x_0) = \delta(x - x_0)$$

$$g(t,x,x_0) = \frac{1}{x_0} g\left(t x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1 \right)$$

where:

$$L[g] = \int_0^{\frac{x}{2}} \left((x-y)^{\frac{\lambda}{2}} f_s(x-y) - x^{\frac{\lambda}{2}} f_s(x) \right) y^{\frac{\lambda}{2}} g(y) dy +$$

$$+ \int_0^{\frac{x}{2}} \left((x-y)^{\frac{\lambda}{2}} g(x-y) - x^{\frac{\lambda}{2}} g(x) \right) y^{-\frac{3}{2}} dy -$$

$$- x^{-\frac{3}{2}} \int_{\frac{x}{2}}^{\infty} y^{\frac{\lambda}{2}} g(y) dy - 2\sqrt{2}\, x^{\frac{\lambda-1}{2}} g(x)$$

Asymptotics of g :

$$g(t,x,1) \sim t^{\frac{2}{\lambda-1}} \varphi(\sigma) \quad , \quad t >> 1$$

$$\sigma = t^{\frac{2}{\lambda-1}} x$$

$$\varphi(\sigma) \sim \left\{ \begin{array}{cc} a_1 \sigma^{-\frac{3}{2}} & , \quad \sigma \to 0 \\ a_2 \sigma^{-\frac{3+\lambda}{2}} & , \quad \sigma \to \infty \end{array} \right\}$$

$$g(t,x,1) \sim a_3 t x^{-\frac{3}{2}} \quad , \quad t << 1 \quad , \quad x \leq \frac{1}{2}$$

$$g(t,x,1) \sim a_3 t x^{-\frac{3+\lambda}{2}} \quad , \quad t << 1 \quad , \quad x \geq \frac{3}{2}$$

Fundamental solution for short times near the Dirac mass.

$$g(t,x,1) \sim t^{-2}\Psi\left(\frac{x-1}{t^2}\right) \ , \ x = O(1) \ , \ t \to 0$$

$$\Psi(\chi) = \frac{2}{\pi}\frac{e^{-\frac{\pi}{\chi^{3/2}}}}{\chi^{3/2}} \ , \ \chi > 0$$

$$\Psi(\chi) = 0 \ , \ \chi \le 0$$

Smoothing effects:

$$g(t,\bullet,1) \in C^\infty(\mathrm{R}^+) \ , \ t > 0$$

Step 2:

- Replace the singular solution $x^{-\frac{3+\lambda}{2}}$ by a function $f_0(x)$ behaving in the same way for $x \to \infty$, but bounded near infinity.

The resulting linearized problem is not easy to solve.

It does not have smoothing effects.

It is not easy to control the contribution of the integral terms at infinity unless some control of the asymptotics of the solutions as $x \to \infty$ is available.

It can be treated as a ”perturbation” of the explicitly solvable problem mentioned above.

Linearization near $f_0(x)$. Continuity method.

$$\frac{\partial g}{\partial t} = (1-\theta)L_{x^{-\frac{3+\lambda}{2}}}[g] + \theta L_{f_0(x)}[g]$$

$$\theta \in [0,1]$$

A priori estimates in a weighted Sobolev norm in the (x,t) variable.

Natural rescaling for the time variable as $x \approx R >> 1$:

$$t \approx \frac{1}{R^{\frac{\lambda-1}{2}}}$$

Analogies with Schauder method for parabolic equations.

Difference: The operator $L_{f_0(x)}$ does not have smoothing effects for any $x < \infty$, but it behaves like the $\frac{1}{2}$ derivative as $x \to \infty$.

Step 3: Nonlinear terms.

General strategy:

$$f_t = Q[f] = \frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)\,dy -$$
$$-\int_0^\infty K(x,y)f(x,t)f(y,t)\,dy$$
$$f(x,0) = f_0(x)$$

Linearization around the initial data (corrected by a function $\lambda(t)$).

$$f = \lambda(t) f_0(x) + h(x,t) \ , \quad f_0(x) \sim \frac{C}{x^{\frac{3+\lambda}{2}}} \ \text{as} \ x \to \infty \ ,$$

$$\lambda(0) = 1 \ , \ h(x,0) = 0$$

$$h_\tau = L_{f_0}[h] + \frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau) Q[f_0] - \Lambda_\tau(\tau) f_0(x)$$

$$d\tau = \lambda(t) dt \ , \quad h(x,0) = 0$$

Fixed point argument.

$$h_\tau = L_{f_0}[h] + \frac{Q[\tilde{h}]}{\Lambda(\tau)} + \Lambda(\tau)Q[f_0] - \Lambda_\tau(\tau)f_0(x)$$

$$h(x,0) = 0$$

Solvability of the linear problem works if the source is bounded as $\frac{1}{x^{2'+\delta}}$ for large x. This requires $\tilde{h} \leq \frac{C}{x^{\frac{3+\lambda}{2}+\delta}}$ for large x.

Asymptotics of h as $x \to \infty$:

$$h(x,\tau) \sim \left[G\left[\tau;\tilde{h},\Lambda\right] - \int_0^\tau a(\tau-s)\Lambda_\tau(s)ds \right] x^{-\frac{3+\lambda}{2}} + O\left(x^{-\frac{3+\lambda}{2}-\delta}\right) \ , \ \ x \to \infty$$

Choice of Λ :

$$G[\tau;h,\Lambda] - \int_0^\tau a(\tau-s)\Lambda_\tau(s)ds = 0$$

This determines $\Lambda = \Lambda\left[\tau;\tilde{h}\right]$. Solving then the linearized equation we obtain a mapping:

$$\tilde{h} \to h = T\left[\tilde{h}\right]$$

T is a contractive operator.

(Some technical difficulties: Hölder regularity of the function $G\left[\bullet;\tilde{h},\Lambda\right]$). (It works like a parabolic equation).

CONCLUSIONS

- Classical theory for the coagulation equations was much restricted to the study of explicitly solvable kernels.
- However, new results and methods, which have been obtained in the last decades, are beginning to point towards a rigorous general theory for these equations, even in nonexplicitly solvable cases.
- The results obtained so far exhibit several analogies with the theory of nonlinear diffusion, although the nonlocal interactions often make the rigorous analysis more involved.
- There are many points of contact between these equations and the theory of stochastic processes. Often, particle interpretations, even at the heuristic level provide interesting insight in the equations and the type of estimates expected.