Two phase entropy solutions for forward-backward parabolic problems

Introduction

Forward-backward parabolic equation

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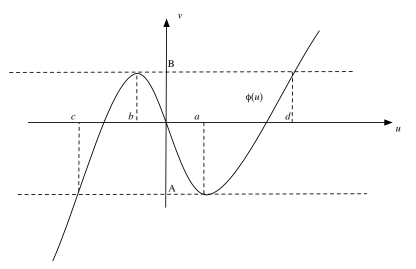
Example 1
Model of phase separation

$$\phi'(u) > 0 \text{ if } u \in (-\infty, b) \cup (a, \infty),\ \phi'(u) < 0 \text{ if } u \in (b, a);$$



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Model of image processing (1D), Perona–Malik equation

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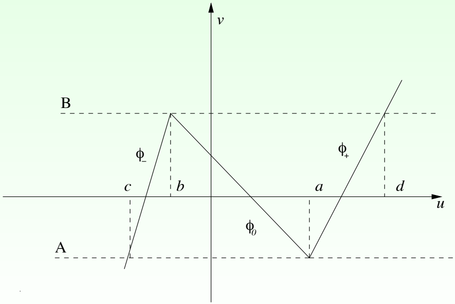
Hollig (Trans. Amer. Math. Soc. 83) ϕ piecewise linear, there are an infinite number of solutions of the Neumann boundary problem.



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Conservation law

$$u_t + \mathrm{div} f(u) = 0$$

is thought as limit, when ϵ goes to 0^+, of the parabolic approximation.

$$u_t + \mathrm{div} f(u) = \epsilon \Delta u.$$

Phase transition, Cahn–Hilliard equation

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Turbolent shear flow (1D), Barenblatt, Bertsch, Dal Passo, Ughi (SIAM J. Math. Anal. 1993)

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In the following $\delta = 0$, ϕ is of cubic type.

Novick Cohen-Pego (*Trans. Amer. Math. Soc.* 1991) study the viscosity problem

$$\begin{cases} u_t = \Delta v & \text{in} \quad \Omega \times (0,T] =: Q_T \\ \frac{\partial v}{\partial \nu} = 0 & \text{in} \quad \partial\Omega \times (0,T] \\ u = u_0 & \text{in} \quad \Omega \times \{0\}, \end{cases} \tag{2}$$

where

$$v := \phi(u) + \epsilon u_t \qquad (\epsilon > 0)\,, \tag{3}$$

is the *chemical potential*, $\Omega \subseteq \mathbb{R}^n$ is bounded , $\partial\Omega$ regular, $T > 0$.

Equation (2) can be rewritten

$$u_t = -\frac{1}{\epsilon}(I - (I - \epsilon\Delta)^{-1})\phi(u)$$

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Moreover $v = (I - \epsilon\Delta)^{-1}\phi(u)$.
Using the standard theory of ODE in the Banach spaces we have

Theorem

(Novick Cohen-Pego) Given $u_0 \in L^\infty(\Omega)$, $\epsilon > 0$ *there exists a unique solution* (u_ϵ, v_ϵ) *defined in* $(0, T_\epsilon)$, $u_\epsilon \in C^1([0, T_\epsilon), L^\infty(\Omega))$.

A priori estimates

For every $g \in C^1(\mathbb{R})$ such that $g' \geq 0$

$$G(u) = \int_0^u g(\phi(s))\, ds + c.$$

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Then

$$[G(u_\epsilon)]_t = \mathrm{div}\left[g(v_\epsilon)\nabla v_\epsilon\right] - g'(v_\epsilon)|\nabla v_\epsilon|^2 +$$

$$-\frac{1}{\epsilon}\left[g(\phi(u_\epsilon)) - g(v_\epsilon)\right]\left(\phi(u_\epsilon) - v_\epsilon\right).$$

Integrating in Ω and using boundary condition

$$\frac{d}{dt}\int_\Omega G(u_\epsilon(x,t))\, dx \leq 0$$

Existence of *invariant regions* for the problem (2).

Proposition

Let $I = [u_1, u_2]$ *such that*

$$\phi(u_1) \leq \phi(u) \leq \phi(u_2) \qquad \textit{for every } u \in [u_1, u_2]\,;$$

then I *is invariant for the problem (2). More precisely* $u_0(x) \in I \implies u_\epsilon(x, t) \in I$ *a.e. in* Q_{T_ϵ}.

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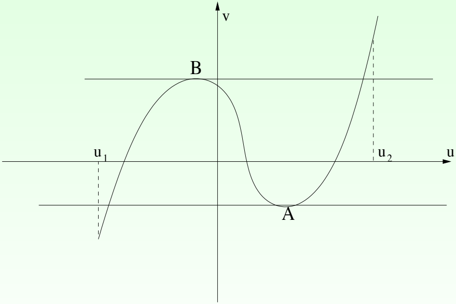

A priori estimates in L^∞ that do not depend on ϵ. Global existence.

A priori estimate

Using again

$$\frac{d}{dt}\int_{\Omega}[G(u_\epsilon)]\,dx = -\int_{\Omega} g'(v_\epsilon)|\nabla v_\epsilon|^2\,dx$$

$$-\int_{\Omega}\frac{1}{\epsilon}\left[g(\phi(u_\epsilon)) - g(v_\epsilon)\right]\,(\phi(u_\epsilon) - v_\epsilon)\,dx\,.$$

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and choosing $g(u) \equiv u$ we have

$$\iint_{Q_T}\left\{|\nabla v^\epsilon|^2 + \epsilon|\partial_t u^\epsilon|^2\right\}dxdt \leq C_2\,.$$

Entropy formulation

In analogy with conservation laws we characterize an entropy solution of problem

$$\begin{cases} u_t = \Delta\phi(u) & \text{in} \quad \Omega \times (0, T] = Q_T \\ \frac{\partial\phi(u)}{\partial\nu} = 0 & \text{in} \quad \partial\Omega \times (0, T] \\ u = u_0 & \text{in} \quad \Omega \times \{0\}, \end{cases} \tag{4}$$

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as that obtained as limit of the solutions of problem (2) when $\epsilon \to 0^+$.
For every $\epsilon > 0$ and $g \in C^1(\mathbb{R})$, $g' \geq 0$ we have

$$\iint_{Q_T} \Big\{ G(u^\epsilon)\psi_t - g(v^\epsilon)\nabla v^\epsilon \cdot \nabla\psi - g'(v^\epsilon)|\nabla v^\epsilon|^2 \psi \Big\} \geq 0 \tag{5}$$

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The idea is to pass in the limit in (5) to characterize an entropy solution of (4).

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Using previous a priori estimate we deduce that there exist two subsequence $\{u^{\epsilon_n}\}$, $\{v^{\epsilon_n}\}$ and a couple (u, v) $u \in L^\infty(Q_T)$, $v \in L^\infty(Q_T) \cap L^2((0,T); H^1(\Omega))$ such that for every $T > 0$:

$$u^{\epsilon_n} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(Q_T)\,,$$

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Unfortunately this is not enough to pass to the limit in (5).
Let $\nu_{(x,t)}$ a family of Young measures associate to $\{u^{\epsilon_n}\}$, then $f \in C(\mathbb{R})$:

$$f(u^{\epsilon_n}) \overset{*}{\rightharpoonup} \overline{f} \quad \text{in } L^\infty(Q_T)\,;$$

where

$$\overline{f}(x,t) := \int_{\mathbb{R}} f(\tau)\, d\nu_{(x,t)}(\tau) \qquad \text{for a.e. } (x,t) \in Q_T$$

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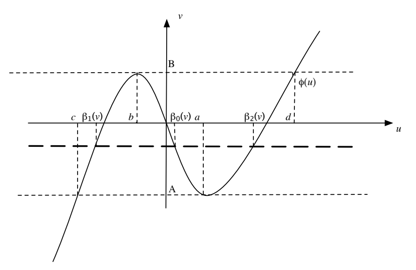
$$\nu_{(x,t)}(\tau) = \sum_{i=0}^{2} \lambda_i(x,t)\delta(\tau - \beta_i(v(x,t)))$$

a.e in Q_T, where $\beta_i(v)$, $i = 0, 1, 2$ are the three branches of the graph $v = \phi(u)$



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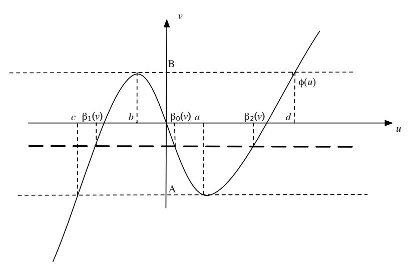

Moreover, $0 \leq \lambda_i \leq 1$ e $\sum_{i=0}^{2} \lambda_i(x,t) = 1$.

$$f(\lambda) = \lambda, \quad u(x,t) = \int_R \tau \, d\nu_{x,t}(\tau) = \sum_{i=0}^{2} \lambda_i(x,t)\beta_i(v(x,t)),$$

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and

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but in general $v \neq \phi(u)$.

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Superposition of phases, λ_i fraction of phase i.
Solution in the sense of measured valued solution.

Letting $\epsilon_n \to 0^+$ in the viscous entropy inequality

$$\iint_{Q_T} \Big\{ G(u^\epsilon)\psi_t - g(v^\epsilon)\nabla v^\epsilon \cdot \nabla\psi - g'(v^\epsilon)|\nabla v^\epsilon|^2\psi \Big\}\, dxdt +$$

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we have

$$\iint_{Q_T} \left\{ \overline{G}(u)\psi_t - g(v)\nabla v \cdot \nabla\psi - g'(v)|\nabla v^2|\psi \right\} dxdt +$$

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where $\overline{G}(u) = \sum_{i=0}^{2} \lambda_i G(\beta_i(v))$.

Entropy solution

Entropy solution

Definition

Given $u_0 \in L^\infty(\Omega)$ *an entropy solution of problem forward–backward (4) is given by the functions* $\lambda_i \in L^\infty(Q_T)$, $i = 0, 1, 2$, $u \in L^\infty(Q_T)$, $v \in L^\infty(Q_T) \cap L^2((0,T), H^1(\Omega))$. *Such that*

(i) $\sum_{i=0}^{2} \lambda_i = 1$, $\lambda_i \geq 0$, $u = \sum_{i=0}^{2} \lambda_i \beta_i(v)$

(ii) u and v satisfy (6) (weak solution)

(iii) u and v satisfy (7) for every $g \in C^1(\mathbb{R})$, $g' \geq 0$ *(entropy condition).*

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Perona–Malik, Smarrazzo (Discrete Contin. Dyn. Syst 2008)

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Perona–Malik, Smarrazzo (Discrete Contin. Dyn. Syst 2008)
Problems: Existence in a stronger sense? Uniqueness? Study of the evolution of the different phases.

Two phase entropy solution

Case $n = 1$. Let $\Omega = (-L, L)$, $u_0 \leq b$ in $(-L, 0)$, $u_0 \geq a$ in $(0, L)$, initial data in the two stable phases.

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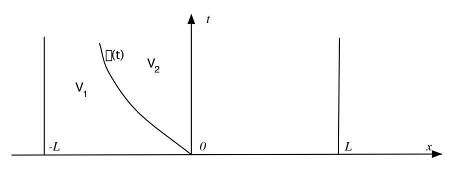
$V_1 := \{(x, t) \in Q_T \mid -L \leq x < \xi(t),\ t \in (0, T)\}$, $V_2 := Q_T \setminus \overline{V}_1$



An entropy solution is a triple of functions (ξ, u, v) such that :
(a) $\xi \in C^{\frac{3}{2}}([0, T])$, $\xi(0) = 0$, $\gamma(t) = \{(\xi(t), t) \,:\, t \in (0, T))\}$;
(b) u, v satisfy

$$u = \beta_i(v) \; in \; V_i \quad (i = 1, 2) \quad (v = \phi(u));$$

(c) $v(\cdot, t)$ continuous in $[-L, L]$, $v((\xi(\cdot), \cdot))$ continuous in $[0, T]$;
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$$\lim_{s \to 0^{\pm}} v_x(\xi(t) \pm s, t);$$

(e) $u_t = v_{xx}$ in the weak sense, entropy condition, boundary and initial condition.
$u_t = \phi(u)_{xx}$ in V_i,
$u, v : Q_T \to \boldsymbol{R}$ regular in $Q_T \setminus \gamma$.

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(Evans–Portilheiro Math. Models Methods Appl. Sci. (2004))
Let u, v, ξ a two phase entropy solution for the problem (2) then
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Analogy with the conditions for piecewise regular solution of scalar conservation laws.

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Choosing properly the function g we can select admissibility conditions for the interface $\gamma = (\xi(t), t)$

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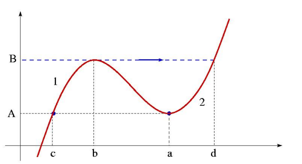
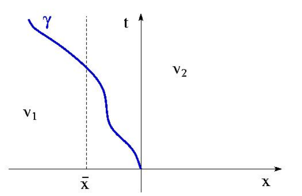
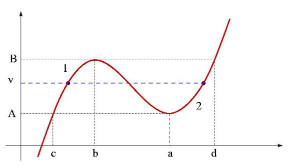
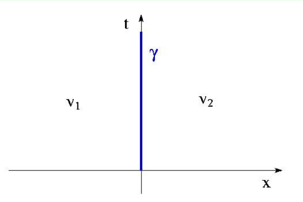
Condition for the phase change. We can pass from phase 1 to phase 2 only if $v = B$





If $v \in (A, B)$ phase does not change



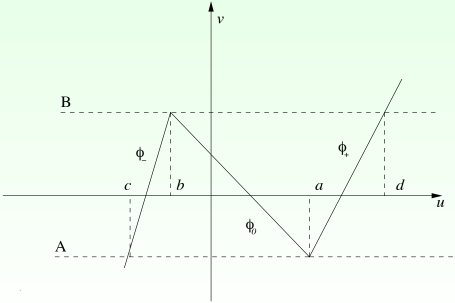


Existence and uniqueness

$$\phi(u) = \begin{cases} \phi_-(u) & \text{if} \quad u \le b \\ \phi_0(u) & \text{if} \quad b < u < a \\ \phi_+(u) & \text{if} \quad u \ge a\,, \end{cases}$$

where

$$\phi_\pm(u) := \alpha_\pm\, u + \beta_\pm\,, \qquad \phi_0(u) := \frac{A(u-b) - B(u-a)}{a-b}\,.$$



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$$F := \Big(|u_1 - u_2|, \mathrm{sgn}(u_1 - u_2)(-{v_1}_x + {v_2}_x) \Big)$$

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Formally we obtain:

$$\mathrm{div} F := |u_1 - u_2|_t + \big[\mathrm{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})\big]_x$$

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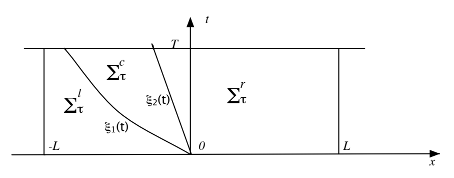
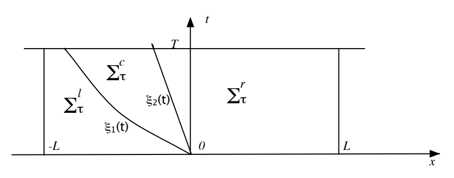
Integrating in Q_T we have :

$$\iint_{Q_T} \Big\{ |u_1 - u_2|_t + \big[\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})\big]_x \Big\} dxdt \leq 0 \qquad (8)$$


t
T
Σ_τ^c
Σ_τ^l
$\xi_2(t)$
Σ_τ^r
$\xi_1(t)$
-L
0
L
x



we have

$$\int_{-L}^{L} |u_1(x,\tau) - u_2(x,\tau)|\, dx \tag{9}$$

$$\leq \int_0^\tau \{-[|u_1 - u_2|]_1\, \xi_1' + [\mathrm{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})]_1\}\, dt$$

$$+ \int_0^\tau \{-[|u_1 - u_2|]_2\, \xi_2' + [\mathrm{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})]_2\}\, dt\,.$$

where $[h]_i \equiv [h]_i(t) := h^{i,+}(t) - h^{i,-}(t)$ is the jump along the interface γ_i of a function h
$(h^{i,\pm}(t) := \lim_{\eta\to 0} h(\xi_i(t) \pm \eta, t)$ $(i = 1, 2;\ t \in [0, T]))$.

Existence

Auxiliary problems:
Moving boundary problem
for every $C \in [A, B]$ let $\kappa_- \in (-\infty, b]$, $\kappa_+ \in [a, \infty)$ defined by

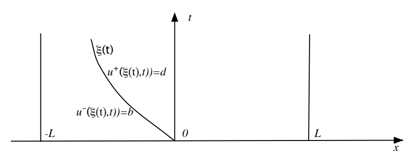
$$\alpha_- \kappa_- + \beta_- = \alpha_+ \kappa_+ + \beta_+ = C\,. \tag{10}$$

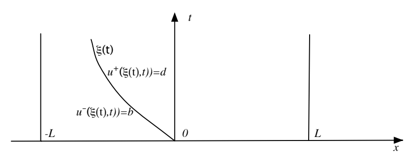
Definition

Let $C \in [A, B]$. A couple of functions $\xi = \xi(t)$, $u = u(x, t)$ is solution of the moving boundary problem if it satisfies the following conditions
(i) $\xi \in C^{\frac{3}{2}}([0, \tau])$, $\xi(0) = 0$;
(ii) $u_t = \alpha_\pm u_{xx}$ *in* $A^\pm_\tau := \{(x, t) \in \mathbb{R} \times (0, \tau) \mid \pm (x - \xi(t)) > 0\}$
(iii) for every $t \in (0, \tau]$ *we have:*

$$u(\xi(t)^\pm, t) = \kappa_\pm \; (\phi(u(\xi(t), t) = C)\,, \tag{11}$$

$$\xi'(t) = -\frac{\alpha_+ \, u_x(\xi(t)^+, t) - \alpha_- \, u_x(\xi(t)^-, t)}{\kappa_+ - \kappa_-}\,. \tag{12}$$

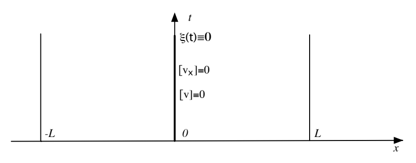

t
ξ(t)
u+(ξ(t),t))=d
u−(ξ(t),t))=b
-L
0
L
x



Steady boundary problem ($\xi' \equiv 0$)

Definition

u is a solution of the steady boundary problem if it satisfies
i) $u_t = \alpha_\pm u_{xx}$ *in* $\mathbb{R}^\pm \times (0, \tau)$;
ii)$\alpha_- u(0-, t) + \beta_- = \alpha_+ u(0+, t) + \beta_+$;
iii) $\alpha_- u_x(0-, t) = \alpha_+ u_x(0+, t)$.



Theorem

(Mascia, T., Tesei) Suppose that one of the following conditions is satisfies
i) $\alpha_- u_0(0-) + \beta_- = \alpha_+ u_0(0+) + \beta_+ \in (A, B)$;
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The moving boundary problem gives an entropy solution if $\phi(u(\xi(t), t) = B$ $(\kappa_- = b, \kappa_+ = d)$ and $\xi' \leq 0$ or $\phi(u(\xi(t), t) = A$ $(\kappa_- = c, \kappa_+ = a)$ and $\xi' \geq 0$.

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We can extend in time the solution until a first time τ such that $\xi'(\tau) = 0$ and $\phi(u(\xi(\tau), \tau)) = A$ or B.

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Proposition

(T. Siam J. Mat. Anal.) Let u_0 such that $\phi(u_0(0)) = A$ or B and $\alpha_- u_0'(0-) - \alpha_+ u_0'(0+) = 0$. If the function $h_0(z) = \alpha_+ u_0'(2\sqrt{\alpha_+} z) - \alpha_- u_0'(-2\sqrt{\alpha_-} z)$ has a given sign in a right interval of 0 then there exists $\tau > 0$ such that the two phase problem has solution in $\mathbb{R} \times (0, \tau)$.

Theorem

(T.) Let (ξ, u) *be a solution of the two phase problem in* Q_τ*. Let* $t_1 < \tau$ *such that in* (t_1, τ) *the solution is given by the solution either of the moving boundary problem or of the steady boundary problem. Then there exists* $t_2 > \tau$ *such that the solution of the two phase problem can be extended in* $(0, t_2)$*.*

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Proof We have to analyze the function $h(z,\tau) = \alpha_+ u_x(2\sqrt{\alpha_+}z + \xi(\tau), \tau) - \alpha_- u_x(-2\sqrt{\alpha_-}z + \xi(\tau), \tau)$ in a right interval of 0.

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We prove that the solution is analytical in the space variable until the interface then function h has a sign.

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Let $N(t)$ the number of disjoint intervals in which $u(\cdot, t)$ is convex. Then $N(t) \leq N(s) + 1$ for every $s \leq t$.

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