The thin film equation with backwards second order diffusion

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This lecture is dedicated

to the memory of

Roberta Dal Passo (1956-2007)

The thin film equation with "backwards" diffusion $(\nu = +1)$

(1) $u_t + \{u^n(u_{xxx} + \nu u^{m-n}u_x - Au^{M-n}u_x)\}_x = 0, \quad \nu = \pm 1, n > 0, M > m, A \geq 0,$

 describes

(i) the evolution of thin viscous films in the presence of gravity and thermo-capillary effects,

(ii) the thin film equation with a "porous media cutoff" of van der Waals forces.

We study (1) with $u_x(\pm a, t) = u_{xxx}(\pm a, t) = 0$ if $u(\pm a, t) > 0$, and $u_0 \in H^1([-a, a])$, proving

(a) Global existence of weak nonnegative solutions when $m - n > -2$ and $A > 0$ or $\nu = -1$, and when $-2 < m - n < 2$, $A = 0$, $\nu = 1$.

(b) Strong entropy solutions under the additional constraint: $m - n > -3/2$ if $\nu = 1$.

(c) Finite speed of propagation when $m > n/2$, if $0 < n < 2$, $\nu = 1$, or if $2 \leq n < 3$, $\nu = \pm 1$.

The thin film equation (Greenspan, 1978)

(2)
$$u_t + \{u^n(u_{xxx})\}_x = 0, \quad n > 0,$$

whose analysis was pioneered by Bernis & Friedman (1990), and

Beretta, Bertsch & Dal Passo (1995), Bertozzi & Pugh (1996), models

(i) the dynamics of thin viscous films with no slip boundary conditions with $n = 3$,

(ii) Hele-Shaw flow with $n = 1$.

Often (2) needs to be augmented with various lower order terms in order to take into account the presence of additional physical effects, and certain such equations shall be considered here. See Oron, Davis & Bankoff (1997) for a survey and review.

Some physical systems which we accommodate in our analysis:

(i) the evolution of thin viscous films in the presence of gravity and thermo-capillary effects

(3) $\qquad u_t + \{u^n(u_{xxx} + u^{m-n}u_x - Au^{M-n}u_x)\}_x = 0, \quad A \geq 0, 0 < n, m < M,$

as well as the more accurate variant of (3),

(4) $u_t + \{u^n(u_{xxx} + h'(u)u_x)\}_x = 0, \quad h'(u) = -\overline{\nu}G + \dfrac{B_1}{u(1 + B_2u)^2}, \quad G, B_1, B_2 > 0,$

where $\overline{\nu} = +1(-1)$ represents stabilizing (destabilizing) gravitational forces.

The value of n reflects the assumptions on the slip conditions at the interface of the thin film with the underlying substrate, with $n = 3$ modeling no slip and $0 < n < 3$ modeling various types of slip.

(ii) The equation

(5)
$$u_t + \{u^n(u_{xxx} + h'(u)u_x)\}_x = 0,$$

can describe

(a) the evolution of a thin viscous film with attractive polar forces, if $h(u) = -b_1 e^{-u/b_2}$ and $b_1, b_2 > 0$, and

(b) the evolution of a thin viscous film with attractive van der Waals forces, if $h(u) = Bu^{-b}$, when $b > 0$, $B < 0$, and B is a (negative) Hamaker constant.

When $\lim_{u \downarrow 0} h'(u) > 0 \, (< 0)$, $h(u)$ is said to represent limiting attractive (repulsive) forces.

Equation (5) can reflect a combination of attractive and repulsive forces.

The limiting power $m = \lim_{u \downarrow 0} uh'(u)/h(u)$ can assume both positive and negative values in modeling various forces.

(iii) The Hocherman-Rosenau equation (1993)

(6) $$u_t + \{f(u)u_{xxx} + g(u)u_x)\}_x = 0,$$

 was proposed as a generalization of the cylindrical Kuramoto-Sivashinsky equation, which models low Reynolds number two phase cylindrical flows.

Equation (6) with $f(u), g(u) > 0$ has been a prototype equation for studying the relative strength of the second and fourth order terms in determining blow up, Bertozzi & Pugh (1998).

Setting $f(u) = u^n$ and $g(u) = u^m$ or $g = h'(u)$ with $h'(u) > 0$ yields (5) with "limiting attractive forces."

Issues of interest regarding

(i) $u_t + \{u^n(u_{xxx})\}_x = 0,$

and

(ii) $u_t + \{u^n(u_{xxx} + \nu u^{m-n}u_x - Au^{M-n}u_x)\}_x = 0, \quad A \geq 0, \nu = \pm 1.$

- $u_0 > 0 \Rightarrow u(x,t) > 0$? $u_0 \geq 0 \Rightarrow u(x,t) \geq 0$?

- Solutions which "touchdown" (vanish)?

- Solutions with $u(x,t) = 0$ on a set of positive measure?

- Speed of propagation of support of the solution?

- Contact angles? (and regularity of solutions....).

- Global existence? or possibly blow up? or rupture?

- Special solutions: self-similar solutions, traveling waves, steady states.

For the thin film equation:

- Nonnegative data $\Rightarrow$ nonnegative solutions, $0 < n$ (energy/entropy estimates).

- Existence (nonexistence) of source type solutions, $0 < n < 3$ ($n \geq 3$).

  At the contact line, $(x_0 - x)^2$ for $0 < n < 3/2$, $(x_0 - x)^{3/n}$ for $3/2 < n < 3$.

- Existence of strong (C^1 for a.e. $t > 0$) nonnegative solutions (entropy estimates),

  with the regularity of source type solutions.

- Positivity properties depend strongly on n, "touchdown" possible for small n.

- Solutions become positive in finite time, and $u(x, t)_{t \to \infty} = \overline{u}$.

- The nonconstant steady states: $(x - a)^+ (b - x)^+$, which are not strong solutions.

- Finite speed of propagation of the support of strong solutions, $0 < n < 3$.

- Measure valued initial data, waiting time phenomena, higher spatial dimensions.

For: $u_t + \{u^n(u_{xxx} - u^{m-n}u_x)\}_x = 0$

- Nonnegative data $\Rightarrow$ nonnegative solutions, $0 < n$, $-1 < m$.

- Existence (nonexistence) of source type solutions, $0 < n < 3$ ($n \geq 3$), if $m = n + 2$.

  At the contact line, $(x_0 - x)^2$ for $0 < n < 3/2$, $(x_0 - x)^{3/n}$ for $3/2 < n < 3$.

- Existence of strong (C^1 for a.e. $t > 0$) nonnegative solutions,

  with the regularity of source type solutions.

- Solutions become positive in finite time, and $u(x, t)_{t \to \infty} = \overline{u}$.

- Touchdown steady states with non-zero contact angle exist if $m - n > -2$.

  Positivity, touchdown properties depend strongly on n.

- Finite speed of propagation $m > 0$, $1/8 < n < 2$, infinite speed of propagation if

  $m < 0$, $-2 < m - n < -3/2$.

- Waiting time phenomena, higher spatial dimensions.

For: $u_t + \{u^n(u_{xxx} + u^{m-n}u_x)\}_x = 0$

- Longstanding interest in rupture and blow up, ($m \geq n + 2;\ m \geq 3, n = 1$).

- Nonnegative data $\Rightarrow$ nonnegative solutions, $0 < n < 3,\ n \leq m < n + 2$,

  ($m \geq n$ as a conjectured "well-posedness condition").

- Existence (nonexistence) of source type solutions, $0 < n < 3$ ($n \geq 3$), if $m = n + 2$.

  At the contact line, $(x_0 - x)^2$ for $0 < n < 3/2$, $(x_0 - x)^{3/n}$ for $3/2 < n < 3$.

- Existence of strong (C^1 for a.e. $t > 0$) solutions, $0 < n < 3,\ n \leq m < n + 2$, with the regularity of (thin film) source type solutions.

- A rich set of steady states, with both zero and non-zero contact angle.

- Finite speed of propagation, $0 < n < 3,\ n \leq m < n + 2$,

- Existence of self-similar rupture solutions.

- Blow up self-similar solutions: ($0 < n < 3/2$, existence), ($n \geq 3/2$, non-existence).

Weak solutions

Let $\nu = \pm 1$, $0 < T < \infty$, and $u_0 \in H^1(\Omega)$, $u_0 \geq 0$, $u_0 \not\equiv 0$, and consider

$$(\mathbb{P}) \begin{cases} u_t + (u^n(u_{xxx} + \nu u^{m-n} u_x - A u^{M-n} u_x))_x = 0, \quad (x, t) \in Q_T, \\[2mm] u_x(\pm a, t) = u_{xxx}(\pm a, t) = 0 \text{ when } u(\pm a, t) \neq 0, \quad t \in (0, T), \\[2mm] u(x, 0) = u_0(x), \quad x \in \overline{\Omega}, \end{cases}$$

Definition 1 : $u \in \mathcal{C}^{0, 1/2, 1/8}(\overline{\Omega} \times [0, \infty)) \cap L^\infty([0, \infty); H^1(\Omega))$ *is a weak solution of* $(\mathbb{P})$ *if:*

(a) $u \in C^{4, 1}(P)$, $u \geq 0$, *and* $u(x, 0) = u_0(x)$, $x \in \overline{\Omega}$.

(b) $u_x(x, t) = u_{xxx}(x, t) = 0$ *when* $u(x, t) \neq 0$, *for* $(x, t) \in \partial\Omega \times (0, \infty)$,

(c) $J \equiv u^n(u_{xxx} + \nu u^{m-n} u_x - A u^{M-n} u_x) \in L^2(P)$,

(d) for all $\phi \in \mathrm{Lip}(\overline{\Omega} \times (0, \infty))$ *with compact support,*

$$\int_Q u\,\phi_t\,dxdt + \int_P u^n\,(u_{xxx} + \nu u^{m-n} u_x - A u^{M-n} u_x)\,\phi_x\,dxdt = 0.$$

Theorem 1 *There exists a weak solution to* $(\mathbb{P})$ *in the sense of Definition 1 for:*

(i) $\nu = -1$, $0 < n$, $0 \le A$, $n - 2 < m < M$,   *(ii)* $\nu = 1$, $0 < n$, $0 < A$, $n - 2 < m < M$,

(iii) $\nu = 1$, $0 < n$, $0 = A$, $n - 2 < m < n + 2$.

Proof.

- We set: $f_\epsilon(s) = \dfrac{|s|^{n+4}}{\epsilon |s|^n + s^4}$.

- We set: $u_{0\epsilon} \in \mathcal{C}^{4,\,\lambda}(\bar{\Omega})$, $u_0 + \epsilon^\theta \le u_{0\epsilon} \le u_0 + 1$, $\theta \in (0, 2/5]$,

$$u'_{\epsilon 0}(\pm a) = u'''_{\epsilon 0}(\pm a) = 0, \quad u_{\epsilon 0} \to u_0 \text{ in } H^1((-a,\, a)) \text{ as } \epsilon \to 0.$$

- Existence of a unique maximal positive solution, $u_\epsilon \in \mathcal{C}^{4,\,\lambda,\,\lambda/4}\left(\bar{\Omega} \times [0,\, \tau_\epsilon)\right)$, $\tau_\epsilon > 0$,

- Testing with $\phi \equiv 1$ (*mass conservation*): $\overline{u_\epsilon}(t) = \overline{u_{0\epsilon}}$.

- Testing with $-u_{\epsilon x x} - h(u_\epsilon)$ (*energy estimate*):

$$\int_\Omega \left[\frac{1}{2} u_{\epsilon x}^2 - H(u_\epsilon)\right] dx + \int_{Q_t} f_\epsilon(u_\epsilon)(u_{\epsilon xxx} + h'(u_\epsilon)u_{\epsilon x})^2 \, dx \, dt = \int_\Omega \left[\frac{1}{2} u_{0\epsilon x}^2 - H(u_{0\epsilon})\right] dx.$$

- $\|u_\epsilon\|_{\mathcal{C}^{0,\,1/2,\,1/8}(\overline{Q_t})} \le c_1$.

- $\epsilon s^{-4} \le \frac{1}{f_\epsilon(s)}, \quad 0 < s, \quad (a)$.

- *A basic entropy estimate* :

  Let $G_\epsilon(s) = -\int_s^{\tilde{A}} g_\epsilon(r) \, dr, \quad g_\epsilon(s) = -\int_s^{\tilde{A}} \frac{dr}{f_\epsilon(r)}$, then

  $$\int_\Omega G_\epsilon(u_\epsilon) \, dx + \int_{Q_t} \frac{1}{2} u_{\epsilon xx}^2 \, dx \, dt \le \frac{c_2}{\epsilon} \int_{Q_t} G_\epsilon(u_\epsilon(x,\,t)) \, dx \, dt + c_3 t + \int_\Omega G_\epsilon(u_{0\epsilon}) \, dx.$$

- $\int_\Omega G_\epsilon(u_{0\epsilon}) \, dx \le c_4$, by (a) and construction of initial conditions.

- $\int_\Omega G_\epsilon(u_\epsilon(x,\,t)) \, dx \le D_\epsilon(t) < \infty, \quad t \in [0,\,\tau_\epsilon)$.

- Positivity of solutions $\Rightarrow$ global existence.

Strong entropy-energy solutions

- **Entropy estimates**: Let $\zeta \in \mathcal{C}^4([-a, a])$ with support in $(-a, a)$ and $\zeta \geq 0$, or $\zeta = 1$,

  and let $G_\epsilon(s) = \frac{\epsilon s^{\alpha+n-3}}{(\alpha+n-4)(\alpha+n-3)} + \frac{s^{\alpha+1}}{\alpha(\alpha+1)}$, where $\alpha \in (1/2 - n, 2 - n) \setminus \{0, -1\}$.

- **Testing with** $\zeta^4 G'_\epsilon(u_\epsilon)$:

Theorem 2 (The unstable case.) *Let* $\nu = 1$, $0 \leq A$, $0 < n$, $-\frac{3}{2} < m - n$, $m - n < 2$ *if* $A = 0$,
and $m < M$ *if* $0 < A$.

$i)$ *Let* $\alpha_0 = \alpha_0(\zeta \equiv 1)$, $\alpha_1 = \max\{\alpha_0, -2m + n - 1\}$, *and* $\beta_1 = \frac{3}{n+\alpha_1+1}$, *and suppose that*
$\alpha_0 < 2 - n$. *Then* $u^{1/\beta}(\cdot, t) \in \mathcal{C}^1([-a, a])$, *for all* $\beta \in (0, \beta_1)$ *for almost every* $t > 0$.

$ii)$ *For* ζ *as above, let* $\alpha_0 = \alpha_0(\zeta)$ *and* $\alpha_2 = \max\{\alpha_0, -2m + n - 1, -m - 1\}$, *and*
suppose that $\alpha_0 < 2 - n$. *Then, for any* $\alpha \in (\alpha_2, 2 - n)/\{0, -1\}$ *and for any* γ *satisfying*

$$\frac{t + 1 - \sqrt{(t - 2)(1 - 2t)}}{3} < \gamma < \frac{t + 1 + \sqrt{(t - 2)(1 - 2t)}}{3},$$

$$\frac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 u^{1+\alpha}(x,\,T)\,dx + A \int_{Q_T} \zeta^4 u^{\alpha+M-1} u_x^2\,dx\,dt +$$

$$c_1 \left[\int_P \zeta^4 u^{\alpha+n-2\gamma+1}(u^\gamma)_{xx}^2\,dx\,dt + \int_{Q_T} \zeta^4 u^{\alpha+n-3} u_x^4\,dx\,dt \right] \leq$$

$$c_2 \int_{Q_T} (|\zeta_x|^4 + |\zeta \zeta_{xx}|^2) u^{n+\alpha+1}\,dx\,dt + c_3 \int_{Q_T} |(\zeta^3 \zeta_x)_x| u^{\alpha+m+1}\,dx\,dt +$$

$$c_4 \int_{Q_T} \zeta^4 u^{\alpha+2m-n+1}\,dx\,dt + \frac{1}{\alpha(\alpha+1)} \int_\Omega \zeta^4 {u_0}^{\alpha+1}\,dx.$$

If $0 < n < 2$ ("strong slippage"), the local entropy estimates can be used to prove finite speed propagation for the strong solutions. However, if $2 \leq n < 3$ ("weak slippage"), these local entropy estimates are insufficient. In this latter case, to demonstrate the finite speed of propagation property, we rely on certain local energy estimates . . .

Theorem 3 *Let $\nu = \pm 1$, $0 \leq A$, $2 \leq n < 3$, $m - \frac{2}{3}n > -\frac{2}{3}$ if $2 \leq n < \frac{5}{2}$, and $m - n > -\frac{3}{2}$ if $\frac{5}{2} \leq n < 3$, with the additional constraints that $m < M$ if $A > 0$, and $m < n + 2$ if $A = 0$ and $\nu = 1$. Then the strong solutions satisfy the following local energy estimate*

$$\int_{\Omega} \zeta^6 |u_x(x,T)|^2 \, dx + d_{10} \int_{Q_T} \zeta^6 \left((u^{\frac{n+2}{6}})_x^6 + (u^{\frac{n+2}{3}})_{xx}^3 + (u^{\frac{n+2}{2}})_{xxx}^2 \right) dx \, dt +$$

$$d_{10} \int_{Q_T \cap \{u>0\}} \zeta^6 u^n u_{xxx}^2 \, dx \, dt \leq \int_{\Omega} \zeta^6 |u_{0x}(x)|^2 \, dx +$$

$$d_9 \int_{Q_T} u^{n+2} (|\zeta_x|^6 \, dx \, dt + |\zeta \zeta_{xx}|^3) \, dx \, dt -$$

$$\int_{Q_T \cap \{u>0\}} (\nu u^m u_x - A u^M u_x)(u_x \zeta^6)_{xx} \, dx \, dt,$$

where $\zeta(x)$ is arbitrary nonnegative function from $C^4([-a, a])$.

Thank you for your attention,

and

Best wishes for 5571!

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