Singularly perturbed Cahn-Hilliard equations

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Nonconvex Evolution Problems

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Basics

- isotropic, isothermal, isobaric solid binary solution, free from imperfections, in a bdd domain $\Omega \subset \mathbb{R}^N$, $N \leq 3$
- **A** and **B** atoms
- molar volume independent of atom concentrations
- ρ (relative) concentration of **B** atoms

$$\boxed{\mathcal{F}(\rho) = \int_\Omega \left(\frac{\alpha}{2} |\nabla \rho|^2 + f(\rho) \right) dx}$$

- $\alpha > 0$ gradient energy coefficient
- f nonconvex potential (minima correspond to pure phases), e.g.,

$$f(\rho) = \rho^2 (\rho - 1)^2$$

Cahn-Hilliard equations

GOAL

modeling phase separation due to cooling processes

- mass flux

$$\boxed{\mathbf{J} = -\kappa \nabla \left(\frac{\delta \mathcal{F}}{\delta \rho} + \boldsymbol{\nu} \rho_t \right)}$$

- $\kappa > 0$ mobility, $\boldsymbol{\nu} \geq 0$ viscosity (A. Novick-Cohen 1988)
- mass conservation

$$\boxed{\rho_t + \nabla \cdot \mathbf{J} = 0}$$

C-H equation

$$\rho_t - \kappa \Delta(-\alpha \Delta \rho + \boldsymbol{\nu} \rho_t + f'(\rho)) = 0$$

Time relaxation of the mass flux: motivation

- oxides or glasses rapidly quenched
- instability in the early stage of the spinodal decomposition★
- local non-equilibrium phenomena
- atomic diffusion flux relaxation (P.Galenko *et al.*)

$$\begin{aligned} &\varepsilon \mathbf{J}_t + \mathbf{J} = -\kappa \nabla(-\alpha \Delta \rho + \boldsymbol{\nu} \rho_t + f'(\rho)) \\ &\rho_t + \nabla \cdot \mathbf{J} = 0 \end{aligned}$$

- $\varepsilon \in (0, 1]$ relaxation time
- good agreement of numerical simulations with experimental observations (e.g. P.Galenko & V.Lebedev, Phys. Letters A 2008)

★ phase transformation in which both phases have an equivalent symmetry, but they differ in composition

Differences and similarities

C-H equation with inertial term

$$\varepsilon\rho_{tt} + \rho_t - \kappa\Delta(-\alpha\Delta\rho + \nu\rho_t + f'(\rho)) = 0$$

- $\varepsilon = 0$ and/or $\nu > 0$: the solutions get smoother than the initial data for positive times
- $\varepsilon > 0$ and $\nu = 0$: there are no regularization effects
- the equation always describes a dissipative phenomenon (energy decreases as time increases)

Remark

Formally we have ($u = (-\Delta)^{-1}\rho$)

$$\varepsilon u_{tt} + u_t - \kappa\alpha\Delta^2 u + \nu\Delta u_t + f'(\Delta u) = 0$$

Some mathematical references on $\varepsilon > 0$

1D case

- A.Debussche, Asymptot. Anal. 1991
- Zheng S. & A.Milani, JDE 2005, Nonlin. Anal. 2004
- S.Gatti, M.G., A.Miranville & V.Pata, JMAA 2005
- A.Bonfoh, M.G. & A.Miranville, NoDEA 2010

3D case

- S.Gatti, M.G., A.Miranville & V.Pata, M^3AS 2005
- M.B.Kania, Colloq. Math. 2007, TMNA 2008
- A.Segatti, M^3AS 2007 ($\nu = 0$)
- M.G. & M. Pierre, M^3AS 2010 ($\nu = 0$, numerical analysis)

$N = 2, 3$: the non-viscous case ($\nu = 0$)

$$
\begin{cases}
\varepsilon \rho_{tt} + \rho_t - \Delta(-\Delta\rho + f'(\rho)) = 0, & \text{in } \Omega \times (0, \infty) \\
\rho = \Delta\rho = 0, & \text{on } \partial\Omega \times (0, \infty) \\
\rho(0) = \rho_0, \quad \rho_t(0) = \rho_1, & \text{in } \Omega
\end{cases}
$$

- $\kappa = \alpha = 1$
- $f \in C^2(\mathbb{R})$ s.t. $f'(0) = 0$
- $|f'(y)| \leq c(1 + |y|^p)$ (for simplicity: $p \in [0, 5]$ if $N = 3$)
- $f''(y) \geq -\lambda$ for some $\lambda > 0$
- $\liminf_{|y| \to \infty} \dfrac{f'(y)}{y} > -\lambda_1$ (λ_1: 1st eigenval. $-\Delta$ + hom. Dir. b.c.)

Remark

Other b.c. can be considered (e.g., no-flux or periodic b.c.)

Existence of an energy solution

- $A = -\Delta : \mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$
- $\mathcal{D}(A^{k/2}) \approx H^k(\Omega)$ for any $k \in \mathbb{Z}$

Energy

$$\mathcal{E}_\varepsilon(\rho^0, \rho^1) = \frac{1}{2}\left[\varepsilon\|A^{-1/2}\rho^1\|^2 + \|A^{1/2}\rho^0\|^2\right] + f(\rho^0)$$

Theorem

Let $(\rho_0, \rho_1) \in \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{-1/2})$. *Then,* $\exists$ *a function* $\rho \in L^\infty_{loc}(0,\infty;\mathcal{D}(A^{1/2})) \cap W^{1,\infty}_{loc}(0,\infty;\mathcal{D}(A^{-1/2}))$ *s.t.*

$$\begin{aligned} &\varepsilon\rho_{tt} + \rho_t + A(A\rho + f'(\rho)) = 0, \quad \text{in } (0,\infty) \\ &\rho(0) = \rho_0, \qquad \rho_t(0) = \rho_1 \end{aligned}$$

$$\sup_{t\geq 0} \mathcal{E}_\varepsilon(\rho(t), \rho_t(t)) + \frac{1}{2}\int_0^\infty \|A^{-1/2}\rho_t(\tau)\|^2 d\tau \leq C$$

The case $N = 2$

M.G., G.Schimperna & S.Zelik (CPDE 2009)

- uniqueness of energy solutions
- energy identity
- semigroup on $\mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{-1/2})$ with global attractor
- convergence to single equilibria (f real analytic)
- semigroup on $\mathcal{D}(A^{3/2}) \times \mathcal{D}(A^{1/2})$ with (smooth) global attractor
- existence of exponential attractors
- existence (and uniqueness) of weak sols. (no dissipative estimates)

$N = 2$: energy solutions

Theorem

If $p \in [0,3]$, then the energy solution ρ is unique and satisfies the energy identity

$$\boxed{\frac{d}{dt}\mathcal{E}_{\varepsilon}(\rho(t), \rho_t(t)) + \frac{1}{2}\|A^{-1/2}\rho_t(t)\|^2 = 0}$$

and

$$\boxed{\rho \in C^0_{loc}([0,\infty); \mathcal{D}(A^{1/2})) \cap C^1_{loc}([0,\infty); \mathcal{D}(A^{-1/2}))}$$

Thus we can define a s-continuous semigroup $\Sigma_{\varepsilon}(t)$ on $\mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{-1/2})$ which possesses the global attractor $\mathbb{A}^{en}_{\varepsilon}$

$N = 2$: uniqueness of energy solutions

[adaptation from V.I.Sedenko, Dokl. Akad. Nauk. SSSR '91]

Let v^n be the (unique) Galerkin-solution to ($\varepsilon = 1$)

$$A^{-1} v^n_{tt} + A^{-1} v^n_t + A v^n + P_n f'(v^n) = 0$$

with

$$v^n(0) = P_n \rho_0, \qquad v^n_t(0) = P_n \rho_1$$

set $\rho^n = P_n \rho$ and observe that $\theta^n = \rho^n - v^n$ solves

$$A^{-1} \theta^n_{tt} + A^{-1} \theta^n_t + A \theta^n + P_n[f'(\rho^n) - f'(v^n)] = P_n[f'(\rho) - f'(\rho^n)]$$

with

$$\theta^n(0) = 0, \qquad \theta^n_t(0) = 0$$

$N = 2$: uniqueness of energy solutions

Multiplying by $A^{-1}\theta_t^n$ we get

$$\frac{d}{dt}\left[\|A^{-1}\theta_t^n\|^2 + \|\theta^n\|^2\right] + 2\|\theta^n\|^2$$
$$= 2(P_n[f'(\rho) - f'(\rho^n)], A^{-1}\theta_t^n) - 2(P_n[f'(\rho^n) - f'(v^n)], A^{-1}\theta_t^n)$$

observe now that

$$(P_n[f'(\rho) - f'(\rho^n)], A^{-1}\theta_t^n) \leq C\|A^{-1/2}[f'(\rho) - f'(\rho^n)]\|$$
$$\leq C\|\rho - \rho^n\| \leq C_0 n^{-1/2}$$

$N = 2$: uniqueness of energy solutions

Hence we deduce

$$\frac{d}{dt}\left[\|A^{-1}\theta_t^n\|^2 + \|\theta^n\|^2\right] \leq C_0 n^{-1/2} + C\|f'(\rho^n) - f'(v^n)]\|\|A^{-1}\theta_t^n\|$$

we now recall (H.Brézis & T.Gallouet, Nonlinear Analysis 1980)

$$\|w\|_{L^\infty(\Omega)} \leq C\left(1 + \sqrt{\ln(1 + \|w\|_{H^2(\Omega)})}\right)$$
$$\forall\, w \in H^2(\Omega)\ s.t.\ \|w\|_{H^1(\Omega)} \leq 1$$

so that

$$\begin{aligned} &\|f'(\rho^n) - f'(v^n)]\| \\ &\leq C(1 + \|\rho^n\|_{L^\infty}^2 + \|v^n\|_{L^\infty}^2)\|\theta^n\| \\ &\leq C\left(1 + \ln(1 + \|\rho^n\|_{H^2}) + \ln(1 + \|v^n\|_{H^2})\right)\|\theta^n\| \leq C_1 \ln n\|\theta^n\| \end{aligned}$$

$N = 2$: uniqueness of energy solutions

Thus we have

$$\frac{d}{dt}\left[\|A^{-1}\theta_t^n\|^2 + \|\theta^n\|^2\right] \leq C_0 n^{-1/2} + C_1 \ln n \|A^{-1}\theta_t^n\|\|\theta^n\|$$

and the Gronwall's Lemma yields

$$\|A^{-1}\theta_t^n(t)\|^2 + \|\theta^n(t)\|^2 \leq C n^{-1/2} n^{t/C_1}$$

choose, e.g., $t^* = \frac{C_1}{4}$ and we have, as $n \to \infty$,

$$\|A^{-1}\theta_t^n(t)\|^2 + \|\theta^n(t)\|^2 \to 0, \qquad \forall\, t \in [0, t^*]$$

this entails the uniqueness on $[0, t^*]$, the argument can be repeated to reach any given final time $T > 0$

$N = 2$: quasi-strong solutions

- $f \in C^{3,1}_{loc}(\mathbb{R})$ s.t. $f'(0) = 0$
- $|f'''(y)| \leq c(1 + |y|)$
- $\exists\, \xi > 0$ s.t. $f'(y)y \geq -\xi,\ \forall\, y \in \mathbb{R}$

Theorem

If $(\rho_0, \rho_1) \in \mathcal{D}(A^{3/2}) \times \mathcal{D}(A^{1/2})$, then the unique energy solution ρ is s.t.

$$\rho \in C^0_{loc}([0,\infty); \mathcal{D}(A^{3/2})) \cap C^1_{loc}([0,\infty); \mathcal{D}(A^{1/2}))$$

Corollary

$\exists$ *semigroup $S_\epsilon(t)$ on $\mathcal{D}(A^{3/2}) \times \mathcal{D}(A^{1/2})$*

$N = 2$: global attractor for quasi-strong solutions

Theorem

$S_\varepsilon(t)$ has the global attractor $\mathbb{A}_\varepsilon$ bdd in $H^4 \times H^2$

Remark

Trajectories originated from $\mathbb{A}_\varepsilon$ are strong solutions

Remark

Clearly we have $\mathbb{A}_\varepsilon \subseteq \mathbb{A}_\varepsilon^{en}$, but is it true that

$$\mathbb{A}_\varepsilon \equiv \mathbb{A}_\varepsilon^{en} \ ?$$

$N = 2$: exponential attractors

Theorem

$(\mathcal{D}(A^{3/2}) \times \mathcal{D}(A^{1/2}), S_\varepsilon(t))$ possesses an exponential attractor $\mathbb{E}_\varepsilon$ which is bdd in $H^4 \times H^2$. More precisely, $\mathbb{E}_\varepsilon$ is a compact invariant set with finite fractal dimension s.t., $\forall$ bdd set $B \subset \mathcal{D}(A^{3/2}) \times \mathcal{D}(A^{1/2})$, $\exists\, C_B > 0$ and $k_B > 0$ s.t.

$$\boxed{\operatorname{dist}(S_\varepsilon(t)B, \mathbb{E}_\varepsilon) \le C_B e^{-k_B t}}$$

where

$$\operatorname{dist}(B_1, B_2) = \sup_{b_1 \in B_2} \inf_{b_2 \in B_2} \|b_1 - b_2\|_{\mathcal{D}(A^{3/2}) \times \mathcal{D}(A^{1/2})}$$

Remark

$\mathbb{A}_\varepsilon \subset \mathbb{E}_\varepsilon$ *has finite fractal dimension*

$N = 2$: weak solutions

Theorem

If $(\rho_0, \rho_1) \in \mathcal{D}(A) \times L^2(\Omega)$, *then the unique energy solution* ρ *is s.t.*

$$\rho \in C^0_{loc}([0,\infty); \mathcal{D}(A)) \cap C^1_{loc}([0,\infty); L^2(\Omega))$$

thus a semigroup can be defined on $\mathcal{D}(A) \times L^2(\Omega)$; however, is it dissipative?

The case $N = 3$

M.G., G.Schimperna, A.Segatti & S.Zelik (JEE 2009)

- $(\rho_0, \rho_1) \in \mathcal{D}(A) \times L^2(\Omega)$: existence and uniqueness for ε small (no growth assumptions on f)
- construction of a dynamical system with a (smooth) global attractor for weak solutions (ε small)
- existence of exponential attractors

open problem: no uniqueness for energy solution

Trajectory attractors and their smoothness

M.G., G.Schimperna & S.Zelik (Nonlinearity 2010)

- existence of the trajectory attractor $\mathcal{A}_{\varepsilon}^{tr}$ ($N = 3$, supercritical nonlinearities)
- any complete bdd trajectory is a (unique) strong solution backward in time
- energy solutions are exponentially asymptotically smooth (ε small enough if $N = 3$)

Backward regularity

Set

$$\mathcal{X}_0^{\varepsilon} = \left\{(u,v) \in D(A^{1/2}) \times \sqrt{\varepsilon} D(A^{-1/2}) \;:\; u \in L^{p+4}(\Omega)\right\}$$

where $p \geq 0$ is the growth exponent of $f^{(3)}$, then denote by

$$\mathcal{K}_{\varepsilon} \subset L^{\infty}(\mathbb{R}; \mathcal{X}_0^{\varepsilon})$$

the set of all complete energy solutions obtained as a limit of a Galerkin scheme

Theorem

If $(u, u_t) \in \mathcal{K}_{\varepsilon}$, then $\exists T = T_u$ s.t.
$(u, u_t) \in C_b((-\infty, T]; D(A^2) \times D(A))$ and is unique and uniformly bdd

Backward regularity

main idea

- using the dissipative integral

$$\int_{-\infty}^{\infty} \|u_t(\tau)\|^2_{D(A^{-1/2})} d\tau < \infty$$

  we construct a smooth function u_σ using stationary states (which are smooth)
- u_σ is close to u on $(-\infty, T]$ for some T
- u is smooth on $(-\infty, T]$ for some T

consequences

- $N = 2 : \mathbb{A}_\varepsilon \equiv \mathbb{A}_\varepsilon^{en}$
- $N = 3$: the trajectory attractor consists of strong solutions (ε small enough)

Exponential attraction of energy solutions

$N = 3$

Theorem

$\exists \varepsilon_0 > 0$ s.t. $\forall \varepsilon \in (0, \varepsilon_0]$, $\exists$ a family $\{\mathcal{M}_\varepsilon\}$ of exponential attractors for weak solutions and $\mathcal{M}_\varepsilon$ exponentially attracts any energy solution (u, u_t) w.r.t. $\mathcal{X}_0^\varepsilon$-metric

$N = 2$

Theorem

$\exists$ a family $\{\mathcal{M}_\varepsilon\}$ of exponential attractors for quasi-strong solutions and $\mathcal{M}_\varepsilon$ exponentially attracts any energy solution (u, u_t) w.r.t. $\mathcal{X}_0^\varepsilon$-metric

Theorem

$\exists$ absorbing set in the space of weak solutions

Past history effects

Pioneering work by A. Novick-Cohen (since 1994)

$$\rho_t = \int_0^\infty k_\varepsilon(s)\Delta\mu(t-s)ds, \quad \mu = -\Delta\rho + \nu\rho_t + f'(\rho)$$

- taking $k_\varepsilon(s) = \varepsilon^{-1}e^{-s/\varepsilon}$ equation can be reduced to the previous differential form
- M.Conti & G.Mola, M^{2}AS 2009 ($3D$, $\nu > 0$, standard b.c.)
- M.Conti & M.Coti Zelati, Nonlin. Anal. 2010 ($2D$, $\nu = 0$, standard b.c.)
- C.Cavaterra, C.G.Gal & M.G., Asymptot. Anal. 2010 ($3D$, $\nu > 0$, dynamic b.c.)

Some open issues

singularly perturbed CH eq.

- $N = 3$: uniqueness of energy solutions
- singular potentials for small ε
- spatially nonlocal potentials

CH eq. with past history effects ($\nu = 0$)

- $2D$: asymptotic behavior of energy sols for any ε
- $3D$: well-posedness and analysis of the dynamical system for small ε?
- robustness w.r.t. ε