Hyperbolic Phase Transitions

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Non Convex Evolution Problems
30.11–03.12.2010

Outline

Introduction

1D Hyperbolic Conservation Laws

$$\boxed{\boxed{\partial_t u + \partial_x f(u) = 0}}$$

$t \in [0, +\infty[\qquad x \in \mathbb{R} \qquad u \in \Omega,\ \Omega \subseteq \mathbb{R}^n$

$f : \Omega \mapsto \mathbb{R}^n$ smooth

Introduction – The Riemann Problem

Riemann Problem

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Introduction – The Riemann Problem

Riemann Problem	+	Wave Front Tracking	=	Well Posedness (and all the rest)

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Introduction – The Riemann Problem

Physics
↓
Riemann Problem

+

Analysis
↓
Wave Front Tracking

=

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Fluids – The p–system (Lagrangian)

Conservation of Mass	$\partial_t \tau - \partial_x v = 0$
Conservation of Momentum	$\partial_t v + \partial_x p(\tau) = 0$
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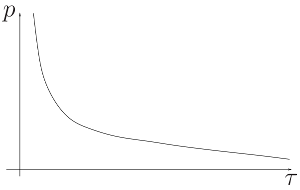

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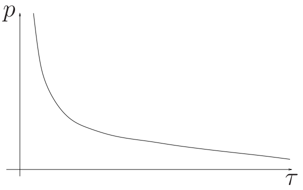

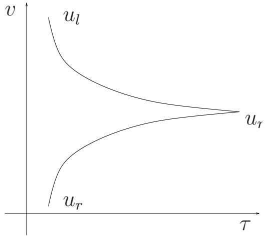

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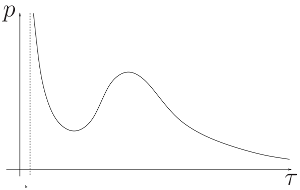

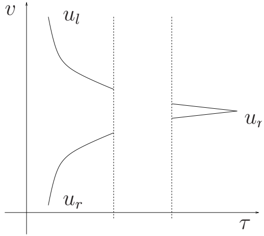

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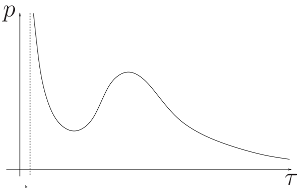

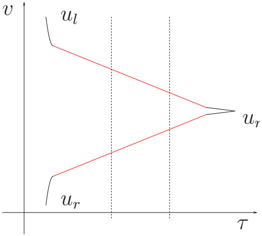

Fluids – Admissible Phase Boundaries

Phase boundary	$=$	jump discontinuity with side states in different phases
Ψ–phase boundary	$=$	phase boundary such that $\Psi(\text{left state}, \text{right state}) = 0$

Ψ is the kinetic relation

Fluids – Riemann Problem with Phase Transitions

A Lax solution to $\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = \begin{cases} u^l, \, x < 0 \\ u^r, \, x > 0 \end{cases} \end{cases}$ consists of a

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if u^l and u^r are in the same phase ⇒ a Lax 1–wave, a Ψ–phase boundary, a constant state, a Ψ–phase boundary and a Lax 2–wave

Fluids – For and Against

- $+$ Existence, Uniqueness, Continuous Dep., Stability
- $+$ Natural: Euler Equations $+$ Equation of State
- $+$ Elastodynamics, Deflagrations, Detonations, . . .
- $-$ Several space dimensions?
- $-$ Which Ψ? (talk by Zimmer!)

Abeyaratne, Knowles: Archive for Rational Mechanics and Analysis, 1991
Colombo, Corli: SIAM Journal of Mathematical Analysis, 1999
LeFloch: Archive for Rational Mechanics and Analysis, 1993

Fluids – Other Hyperbolic Approach

$$\left\{\begin{array}{l}\partial_t \tau - \partial_x v = 0\\ \partial_v v + \partial_x p(\tau, \lambda) = \varepsilon\, \partial^2_{xx} v\\ \partial_t \lambda = \dfrac{a}{\varepsilon} \left(p(\tau,\lambda) - p_{\mathrm{eq}}\right) \lambda \left(\lambda - 1\right) + b\, \varepsilon\, \partial^2_{xx} \lambda\end{array}\right.$$

λ vapour fraction in the fluid

Fluids – Other Hyperbolic Approach

$$\begin{cases} \partial_t \tau - \partial_x v = 0 \\ \partial_v v + \partial_x p(\tau, \lambda) = 0 \\ \partial_t \lambda = 0 \qquad \text{with } \lambda = 0, 1 \text{ or } p = p_{\text{eq}} \end{cases}$$

λ vapour fraction in the fluid

Corli, Fan: SIAM Journal Applied Mathematics, 2004

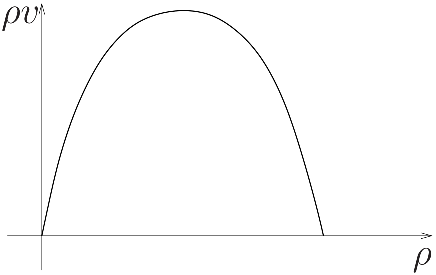
Amadori, Corli: SIAM Journal of Mathematical Analysis, 2008

Traffic Flow

Lighthill-Whitham and Richards model

- Cars are conserved
- The car speed is a function of the car density $\rho \in [0, R]$

$$\partial_t \rho + \partial_x \left[\rho \cdot v(\rho)\right] = 0$$



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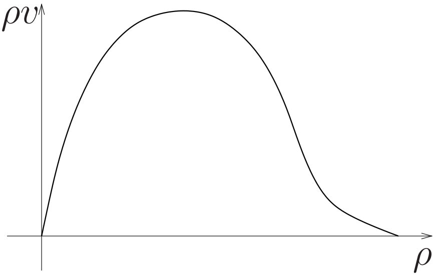

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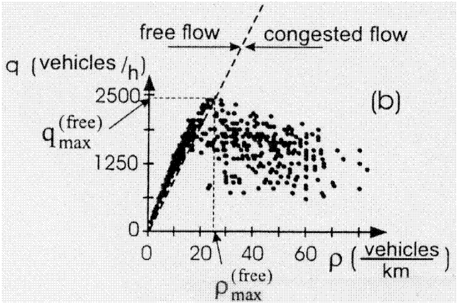

Kerner, in *Traffic and Granular Flow '99*

Phase Transitions in Traffic Flow – I

Free phase: $(\rho, q) \in \mathcal{F}$

$$\partial_t \rho + \partial_x [\rho \cdot v] = 0$$

$$v = v_f(\rho)$$

Congested phase: $(\rho, q) \in \mathcal{C}$

$$\begin{cases} \partial_t \rho + \partial_x [\rho \cdot v] = 0 \\ \partial_t q + \partial_x \left[(q - q_*) \cdot v\right] = 0 \end{cases}$$

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The initial data is in a single phase	$\Rightarrow$	the solution will always remain in the same phase

$$\Downarrow$$

$\mathcal{F}$ and $\mathcal{C}$ invariant domains

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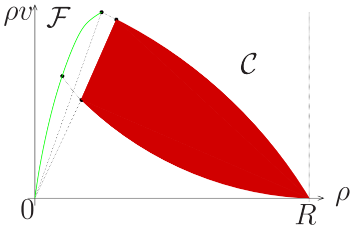

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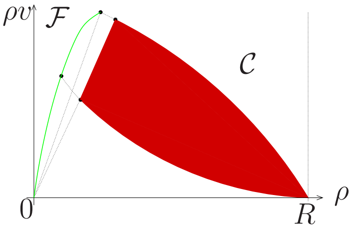

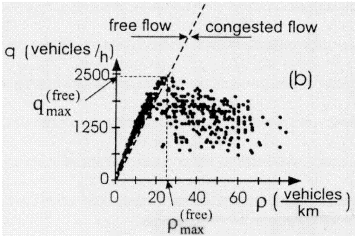

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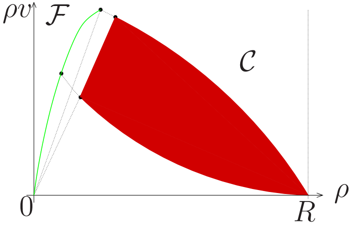

Riemann Problem:
u^l and u^r in the same phase
$\Downarrow$
Lax solution in that phase

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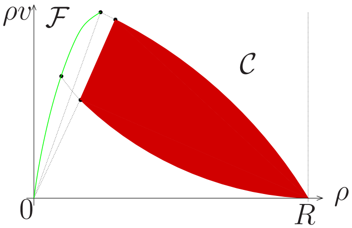

Riemann Problem:
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solution with phase boundary

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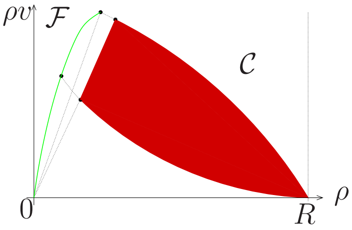

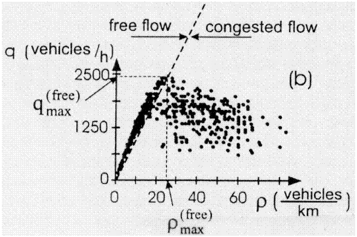

Colombo: SIAM Journal Applied Mathematics, 2002
Colombo, Goatin, Priuli: Nonlinear Analysis, 2007
Colombo, Goatin, Piccoli: J. Hyperbolic Differential Equations, 2010

Phase Transitions in Traffic Flow – II

LWR model: $\partial_t \rho + \partial_x (\rho\, v) = 0 \quad$ with $\quad v = v(\rho)$

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f is $\mathbf{C}^{0,1}$

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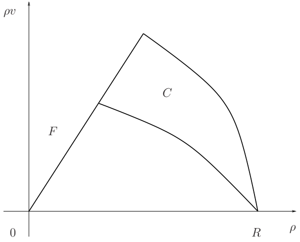
$$\begin{aligned} \mathcal{F} &= \text{free phase} \\ &= \left\{(\rho, \rho w) \colon v = V_{\max}\right\} \\ \mathcal{C} &= \text{congested phase} \\ &= \left\{(\rho, \rho w) \colon v < V_{\max}\right\} \end{aligned}$$



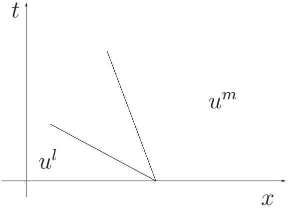
Colombo, Marcellini, Rascle: SIAM Journal Applied Mathematics, 2010
Blandin, Work, Goatin, Piccoli, Bayen: SIAM Journal Applied Mathematics, to appear

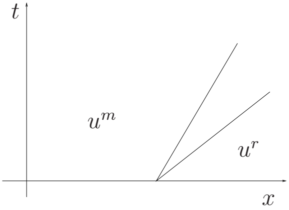
Phase Transition – How to select them?

Mass Conservation and Consistency

Phase Transition – How to select them?

Mass Conservation and Consistency





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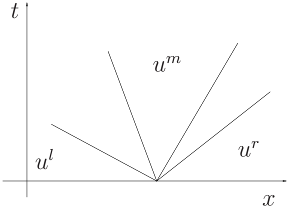
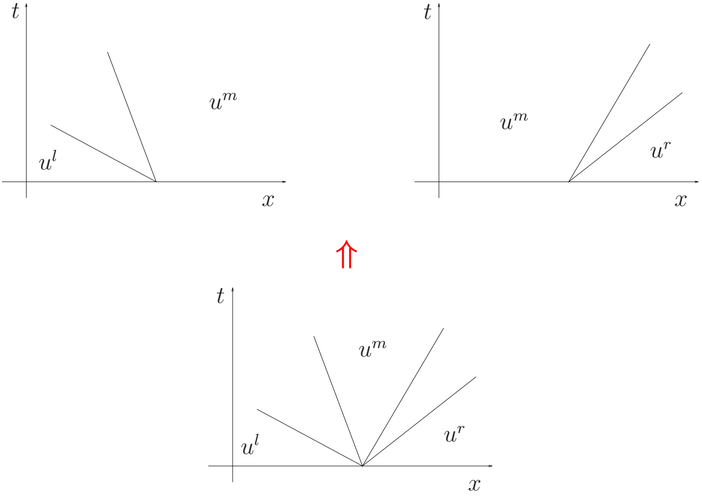
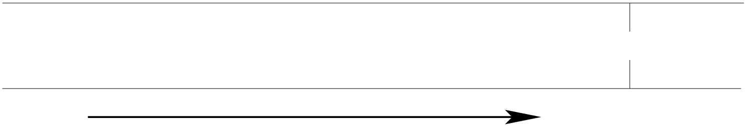


Phase Transition – How to select them?

Mass Conservation and **Consistency**



Crowd Dynamics



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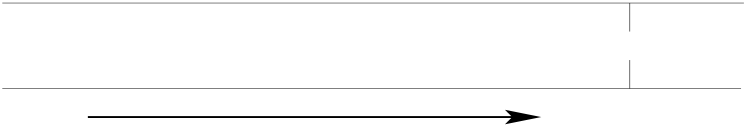

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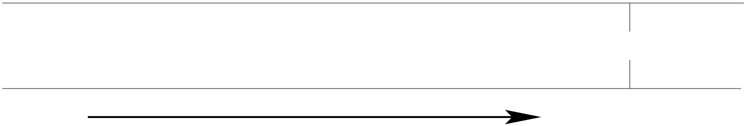

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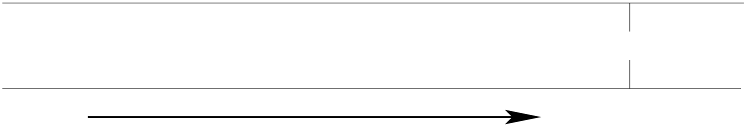
Panic

Crowd Dynamics



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Panic $\Leftarrow$ overcompression

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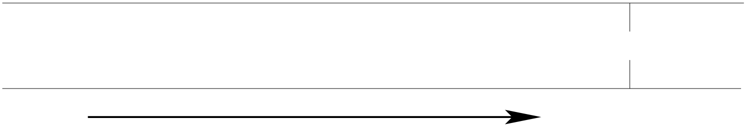

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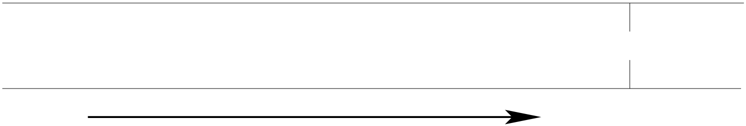
Panic $\Leftarrow$ overcompression $\Leftarrow$ panic states

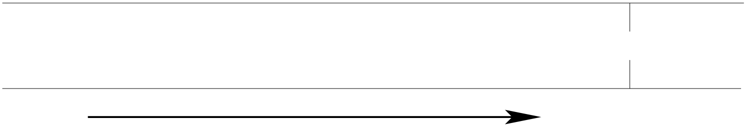
Crowd Dynamics



Pedestrians conserved + $v = v(\rho)$ $\Rightarrow$ $\partial_t \rho + \partial_x(\rho\, v) = 0$

Panic $\Leftarrow$ overcompression $\Leftarrow$ panic states $\Leftarrow$ transition to panic

Crowd Dynamics



Pedestrians conserved $+$ $\boxed{v = v(\rho)}$ $\Rightarrow$ $\boxed{\partial_t \rho + \partial_x (\rho\, v) = 0}$

Panic $\Leftarrow$ overcompression $\Leftarrow$ panic states $\Leftarrow$ transition to panic

1. Introduce overcompressed (panic) states.
2. Modify the speed law.
3. Modify the evolution

Crowd Dynamics

1. Introduce overcompressed (panic) states.

Crowd Dynamics

1. Introduce overcompressed (panic) states.

Extend $\rho \in [0, R]$ to $\rho \in [0, R_*]$

Panic $\Leftrightarrow \rho \in \,]R, R_*]$

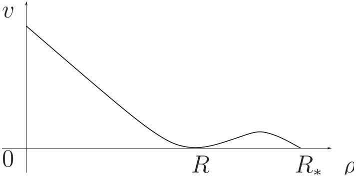
Crowd Dynamics

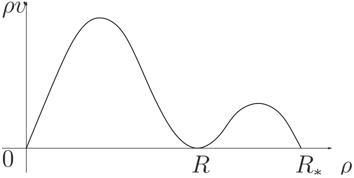
1. Introduce overcompressed (panic) states.
2. Modify the speed law.

Crowd Dynamics

1. Introduce overcompressed (panic) states.
2. Modify the speed law.

Extend the speed law → new fundamental diagram





Crowd Dynamics

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Crowd Dynamics

1. Introduce overcompressed (panic) states.
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$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 \\ \rho(0,x) = \begin{cases} \rho_l & x < 0 \\ \rho_r & x > 0 \end{cases} \end{cases}$$

Crowd Dynamics

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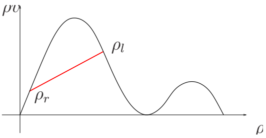
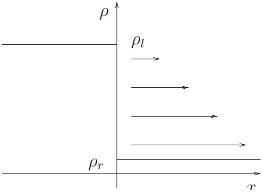
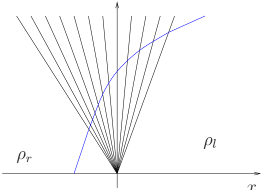
$\boxed{\rho_l > \rho_r}$







Crowd Dynamics

1. Introduce overcompressed (panic) states.
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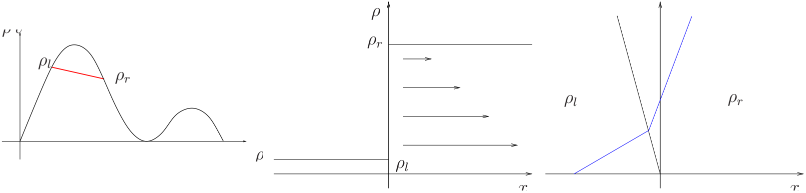
$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 \\ \rho(0,x) = \begin{cases} \rho_l & x < 0 \\ \rho_r & x > 0 \end{cases} \end{cases}$$

$\boxed{\rho_l < \rho_r,\ \rho_l \text{ small},\ \rho_r - \rho_l \text{ small}}$

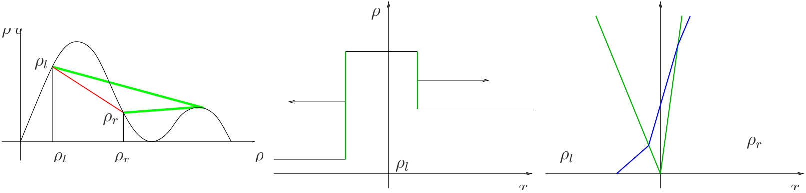


Crowd Dynamics

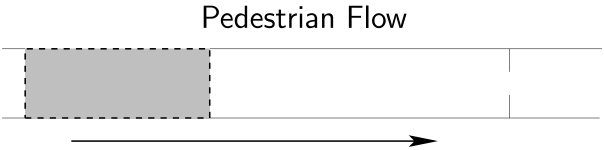
1. Introduce overcompressed (panic) states.
2. Modify the speed law.
3. Modify the evolution

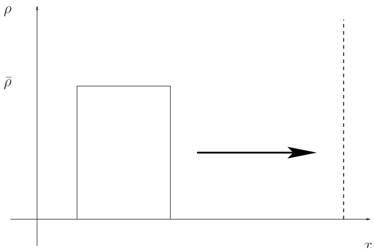
NonClassical Shocks
$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 \\ \rho(0,x) = \begin{cases} \rho_l & x < 0 \\ \rho_r & x > 0 \end{cases} \end{cases}$$

$\rho_l < \rho_r$, ρ_l LARGE, $\rho_r - \rho_l$ LARGE



Crowd Dynamics – Panic Lowers Exit Efficiency

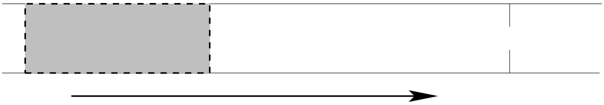


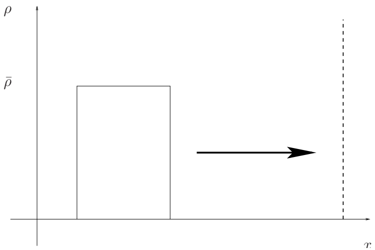


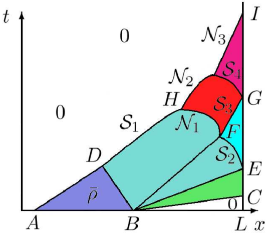
Riemann Problems
+
Wave Front Tracking

Crowd Dynamics – Panic Lowers Exit Efficiency

Pedestrian Flow

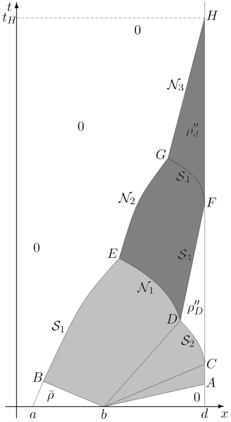


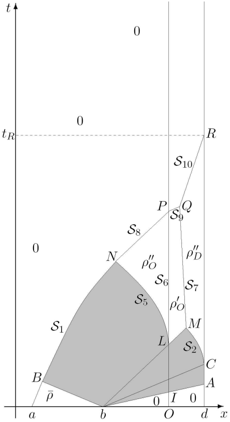




Crowd Dynamics – Braess' paradox

An obstacle may improve the outflow!

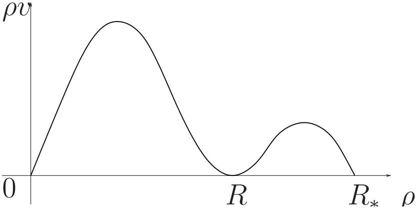




Colombo, Rosini: M2AS, 2005

Colombo, Rosini: Nonlinear Analysis RWA, 2008

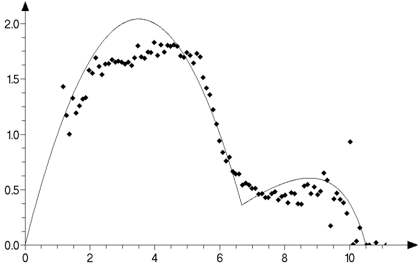
1D Pedestrian Flow – Experimental confirmation



Colombo, Rosini: M2AS, 2005

1D Pedestrian Flow – Experimental confirmation

Experimental data:



Helbing, Johansson, Al-Abideen: Physical Review E, 2007