Navier-Stokes equations on the flat cylinder with vorticity production on the boundary

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Motivation

Our study is inspired by the recent papers:

- Dinaburg, E., Li, D., Sinai, Ya.G.: A new boundary problem for the two dimensional Navier-Stokes system. J. Stat. Phys. **135**, 737–750 (2009)
- Dinaburg, E., Li, D., Sinai, Ya.G.: Navier-Stokes system on the flat cylinder and unit square with slip boundary conditions. Commun. Contemp. Math. **12**, 325–349 (2010)

Extending the classical works on the incompressible Navier-Stokes (NS) equations on the flat two-dimensional torus, they show how to obtain, by mainly elementary methods, deep results on the regularity of the solutions to the plane NS system with suitable boundary conditions.

Remarks

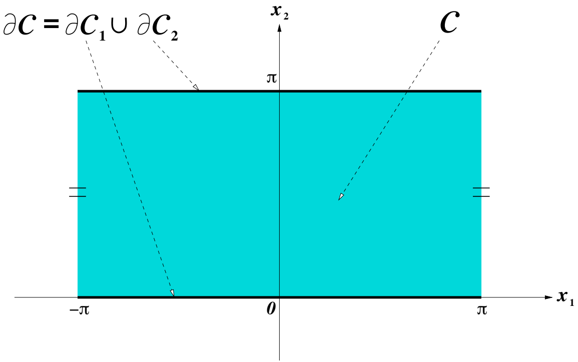
- The main point in their approach is a formulation of the NS system as an infinite set of coupled ordinary differential equations (ODE's) for the Fourier modes, so that they can apply the techniques available for such dynamical systems.
- The problem with the classical Dirichlet (no slip) boundary conditions is that the eigenfunctions of the Stokes operator (i.e. the linear part of the NS evolution) satisfying such conditions are not explicitely known, and we cannot write a manageable ODE's system.

Our plan

To study the case of Dirichlet (no slip) boundary conditions using the "vorticity production method"

- Batchelor, G.K.: An introduction to fluid dynamics. Cambridge Mathematical Library. Cambridge University Press, second paperback edition, Cambridge (1999)
- Chorin, A.J.: Numerical study of slightly viscous flow. J. Fluid Mech. **57**, 785–796 (1973)
- Benfatto, G., Pulvirenti, M.: Generation of vorticity near the boundary in planar Navier-Stokes flows. Comm. Math. Phys. **96**, 59–95 (1984)

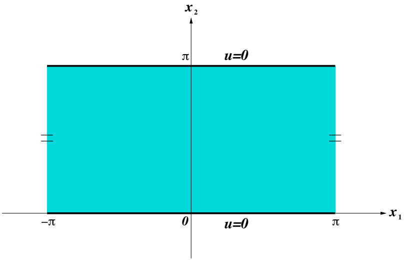
Our geometry: the flat cylinder $\mathcal{C}$



Incompressible NS system on $\mathcal{C}$

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \Delta u - \nabla p \\ \nabla \cdot u = 0 \\ u\big|_{t=0} = u^{(0)} \end{cases} \tag{1}$$

- $u = (u_1, u_2)$ is the velocity field, p is the pressure, the viscosity is taken equal to 1.
- Dirichlet (no-slip) boundary conditions: $u\big|_{\partial\mathcal{C}} = 0$:



Vorticity

- $x = (x_1, x_2) \in (-\pi, \pi] \times [0, \pi]$ (coordinates on $\mathcal{C}$).
- $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$ ("twisted gradient").
- The vorticity is defined as

$$\omega := \nabla^\perp \cdot u = \partial_{x_1} u_2 - \partial_{x_2} u_1.$$

Lemma

If $u \in C^1(\mathcal{C}; \mathbb{R}^2)$ is solenoidal, i.e. $\nabla \cdot u = 0$, satisfies the boundary conditions $u\big|_{\partial\mathcal{C}} = 0$, then $\int_{\mathcal{C}} \mathrm{d}x\, \omega(x) = 0$, and u can be represented as

$$u = \nabla^\perp \Delta_N^{-1} \omega,$$

where Δ_N is the Laplacian on $\mathcal{C}$ with zero Neumann boundary conditions.

Remark: The result extends to the case $\omega \in L_2(\mathcal{C})$, except that solenoidality holds in the usual L_2-sense (orthogonality to the gradients).

The vorticity production method

Assuming sufficient smoothness, by taking the curl of both sides of the NS system (1),

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \Delta \omega, \\ \int_{\mathcal{C}} \mathrm{d}x\, \omega = 0, \\ \partial_{x_1} \Delta_N^{-1} \omega|_{\partial \mathcal{C}} = 0, \\ \omega|_{t=0} = \nabla^{\perp} \cdot u^{(0)}. \end{cases} \tag{2}$$

Remark: Since $u = \nabla^{\perp} \Delta_N^{-1} \omega$ then $u_1\big|_{\partial \mathcal{C}} = \partial_{x_2} \Delta_N^{-1} \omega\big|_{\partial \mathcal{C}} = 0$.

Equations (2) should be completed by the balance equation for the component along x_1 of the total momentum of the fluid, in absence of external forces, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{C}} \mathrm{d}x\, u_1(x,t) = \int_{\mathbb{T}} \mathrm{d}x_1 \left[\omega(x_1, 0, t) - \omega(x_1, \pi, t) \right]. \tag{3}$$

Remark: The pressure disappears, but the Dirichlet boundary conditions are replaced by the linear non-local condition $\partial_{x_1}\Delta_N^{-1}\omega|_{\partial\mathcal{C}} = 0$, which is difficult to handle.

Consider instead the formal problem,

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = \Delta_N\omega + f\delta_{\partial\mathcal{C}}, \\ \int_{\mathcal{C}} \mathrm{d}x\,\omega = 0, \\ \partial_{x_1}\Delta_N^{-1}\omega|_{\partial\mathcal{C}} = 0, \\ \omega|_{t=0} = \nabla^\perp\cdot u^{(0)}, \end{cases} \tag{4}$$

where the vorticity production at the boundary $f\delta_{\partial\mathcal{C}}$ is given by

$$f\delta_{\partial\mathcal{C}}(x,t) = f_1(x_1,t)\delta(x_2) + f_2(x_1,t)\delta(x_2-\pi),$$

and the functions $f_1(x_1,t)$, $f_2(x_1,t)$ should be determined such that $(4)_2$-$(4)_3$ and (3) are satisfied.

The precise meaning of equation $(4)_1$ is given by its mild integral version,

$$\begin{aligned}\omega(x,t) &= \int_{\mathcal{C}} \mathrm{d}y\, \mathrm{e}^{t\Delta_N}(x,y)\,\omega(y,0) \\ &\quad - \int_0^t \mathrm{d}s \int_{\mathcal{C}} \mathrm{d}y\, \mathrm{e}^{(t-s)\Delta_N}(x,y)(u\cdot\nabla\omega)(y,s) \\ &\quad + \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y_1\, \mathrm{e}^{(t-s)\Delta_N}(x,(y_1,0))\, f_1(y_1,s) \\ &\quad + \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y_1\, \mathrm{e}^{(t-s)\Delta_N}(x,(y_1,\pi))\, f_2(y_1,s), \qquad (5)\end{aligned}$$

where $\mathrm{e}^{t\Delta_N}(x,y)$ is the heat kernel on $\mathcal{C}$ with Neumann boundary conditions and $\omega(\cdot,0) = \nabla^\perp \cdot u^{(0)}$.

Remark: In general, the vorticity does not satisfy homogeneous Neumann boundary conditions in the classical sense.

Evolution equation for the Fourier modes

We consider the Fourier version of the integral equation (5), formally obtained by taking the Fourier components of both sides of (5) in the basis of the eigenfunctions of Δ_N,

$$v_k(x) = \mathrm{e}^{\mathrm{i}k_1 x_1}\cos(k_2 x_2), \quad k = (k_1, k_2), \quad k_1 \in \mathbb{Z}, \quad k_2 \in \mathbb{Z}_+.$$

We get an infinite set of integral equations for the Fourier components $\omega_{k_1,k_2}(t)$, $k_1 \in \mathbb{Z}, k_2 \geq 0$,

$$\omega_{k_1,k_2}(t) = \mathrm{e}^{-k^2 t}\omega_{k_1,k_2}(0) + \int_0^t \mathrm{d}s\, \mathrm{e}^{-k^2(t-s)} \left\{ f_{\pm,k_1}(s) - N_{k_1,k_2}[\omega(s)] \right\}$$

with $+$ [resp. $-$] sign for k_2 even [resp. odd]

- $\omega_{k_1,k_2}(t) = \dfrac{1}{2\pi^2}\displaystyle\int_{\mathcal{C}} \mathrm{d}x\, \omega(x,t)\, \mathrm{e}^{-\mathrm{i}k_1 x_1}\cos(k_2 x_2),$
- $N_{k_1,k_2}[\omega(t)] = \dfrac{1}{2\pi^2}\displaystyle\int_{\mathcal{C}} \mathrm{d}x\, u(x,t)\cdot\nabla\omega(x,t)\, \mathrm{e}^{-\mathrm{i}k_1 x_1}\cos(k_2 x_2),$
- $f_{\pm,k_1}(t) = \dfrac{1}{2\pi^2}\displaystyle\int_{\mathbb{T}} \mathrm{d}x_1 \left[f_1(x_1,t) \pm f_2(x_1,t) \right] \mathrm{e}^{\mathrm{i}k_1 x_1}.$

Boundary conditions and constrains

Imposing the conditions,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}} \mathrm{d}x\, \omega(x,t) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}} \mathrm{d}x\, u_1(x,t) = \int_{\mathbb{T}} \mathrm{d}x_1 \left[\omega(x_1,0,t) - \omega(x_1,\pi,t)\right],$$

we get

$$f_{+,0}(t) \equiv f_{-,0}(t) \equiv 0.$$

Instead, the boundary condition $\partial_{x_1}\Delta_N^{-1}\omega\big|_{\partial\mathcal{C}} = 0$ gives the constrains,

$$\sum_{k_2,\pm} \frac{\omega_{k_1,k_2}(t)}{k^2} = 0, \qquad \forall\, k_1 \neq 0 \tag{6}$$

(here $\sum_{s,+} a_s = a_0 + 2\sum_{i\geq 1} a_{2i}$, $\sum_{s,-} a_s = 2\sum_{i\geq 1} a_{2i-1}$).

Equations for $f_{\pm,k_1}(t)$

By plugging the evolution equation into the constraints (6) we obtain a Volterra integral equation of the first kind for $f_{\pm,k_1}(t)$,

$$\sum_{k_2,\pm} \frac{1}{k^2} \int_0^t \mathrm{d}s \, \mathrm{e}^{-k^2(t-s)} \, f_{\pm,k_1}(s) = g_{\pm,k_1}[t;\omega], \quad k_1 \neq 0,$$

where

$$g_{\pm,k_1}[t;\omega] = \sum_{k_2,\pm} \frac{1}{k^2} \left\{ -\mathrm{e}^{-k^2 t} \omega_{k_1,k_2}(0) + \int_0^t \mathrm{d}s \, \mathrm{e}^{-k^2(t-s)} \, N_{k_1,k_2}[\omega(s)] \right\}.$$

The fundamental system

$$\begin{cases} \omega_{k_1,k_2}(t) = \mathrm{e}^{-k^2 t}\omega_{k_1,k_2}(0) \\ \qquad\qquad + \int_0^t \mathrm{d}s\, \mathrm{e}^{-k^2(t-s)} \Big\{ f_{\pm,k_1}(s) - N_{k_1,k_2}[\omega(s)] \Big\}, \\ \\ \sum_{k_2,\pm} \frac{1}{k^2} \int_0^t \mathrm{d}s\, \mathrm{e}^{-k^2(t-s)} f_{\pm,k_1}(s) = g_{\pm,k_1}[t;\omega], \quad (k_1 \neq 0), \end{cases} \tag{7}$$

and $\omega_{0,0}(t) = f_{+,0}(t) = f_{-,0}(t) = 0$.

Main results

Theorem

Let $\omega_{k_1,k_2}(0)$ *satisfy*

$$\omega_{0,0}(0) = 0, \qquad \sum_{k_2,\pm} \frac{\omega_{k_1,k_2}(0)}{k^2} = 0, \quad \forall\, k_1 \neq 0,$$

and the inequalities

$$|\omega_{k_1,k_2}(0)| \leq \frac{D_0}{|k|^\alpha \left(1 + |k_1|^\beta\right)} \qquad \forall\, k_1 \in \mathbb{Z},\ k_2 \geq 0,\ k \neq (0,0), \qquad (8)$$

with $1 < \alpha < 2$, $\beta \geq 0$, *and some* $D_0 > 0$.
Then there exist real numbers $D_1, \nu > 0$ *(depending on* D_0, α, β*), and a unique continuous solution* $\{\omega_{k_1,k_2}(t);\ k_1 \in \mathbb{Z},\ k_2 \geq 0\}$ *to the system* (7) *which satisfies for all* $t \geq 0$ *the inequalities*

$$|\omega_{k_1,k_2}(t)| \leq \frac{D_1\, \mathrm{e}^{-\nu(1+|k_1|)t}}{|k|^\alpha \left(1 + |k_1|^\beta\right)} \qquad \forall\, k_1 \in \mathbb{Z},\ k_2 \geq 0,\ k \neq (0,0). \qquad (9)$$

Moreover, for each $t_0 > 0$ there is a constant $\tilde{D}_1 = \tilde{D}_1(D_0, t_0, \alpha, \beta)$ such that

$$|\omega_{k_1,k_2}(t)| \leq \frac{\tilde{D}_1 \, \mathrm{e}^{-\nu(1+|k_1|)t/2}}{k^2} \qquad \forall\, t \geq t_0. \tag{10}$$

Finally, the velocity field $u(x,t) := \nabla^\perp \Delta_N^{-1} \omega(x,t)$ associated to the vorticity

$$\omega(x,t) = \sum_{k_1 \in \mathbb{Z}} \omega_{k_1,0}(t)\, \mathrm{e}^{\mathrm{i}k_1 x_1} + 2 \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \geq 1} \omega_{k_1,k_2}(t)\, \mathrm{e}^{\mathrm{i}k_1 x_1} \cos(k_2 x_2) \tag{11}$$

is a weak solution to the original NS system.

That is: for each $T > 0$, $u \in L^2([0,T]; V)$, where V is the space of solenoidal vector fields in $H_0^1(\mathcal{C})^2$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{C}} \mathrm{d}x\, u \cdot \Phi + \sum_{i=1}^{2} \int_{\mathcal{C}} \mathrm{d}x\, \left[\partial_{x_i} u \cdot \partial_{x_i} \Phi + u_i (\partial_{x_i} u) \cdot \Phi\right] = 0 \qquad \forall\, \Phi \in V.$$

Comments

- The decay estimate (9) guarantees continuity of $\omega(x,t)$ and C^∞ regularity with respect to the periodic variable x_1 for any $t > 0$, up to the border $\partial\mathcal{C}$.
- The stronger estimate (10) only implies that $\partial_{x_2}\omega(\cdot,t)$ is in $L^2(\mathcal{C})$ for any $t > 0$. In fact, the vorticity possesses higher regularity:

Corollary

For each $t > 0$ the velocity field $u(x,t) := \nabla^\perp \Delta_N^{-1} \omega(x,t)$ is continuous and twice differentiable in x_1, x_2 up to the boundary $\partial\mathcal{C}$.

Remark: If $\alpha > \frac{3}{2}$ and $\alpha + \beta > 2$ then $u^{(0)} \in W^{2,2}(\mathcal{C})$, whence $u(x,t)$ is a classical solution of the NS system, see e.g. [Ladyzhenskaya, 1969].

- We cannot expect a better decay with respect to k_2 since the vorticity does not satisfy Neumann boundary conditions in the classical sense.
- However, the recent paper:

  Dinaburg, E., Li, D., Sinai, Ya.G.: Navier-Stokes system on the unit square with no slip boundary condition. J. Stat. Phys. **141**, 342–358 (2010)

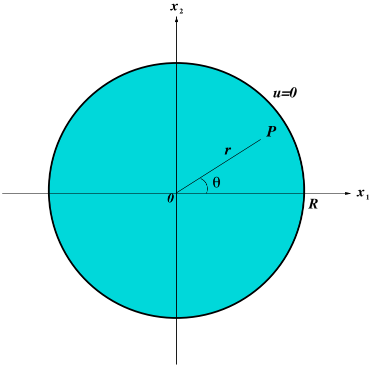
  indicates that a decay faster than an inverse square for the vorticity modes is still not guaranteed by choosing a basis of functions which satisfy the full boundary conditions.

Concluding remarks

- We proved that a kind of asymmetric regularization of the solutions takes place. If the vorticity modes at time $t = 0$ decay slower than an inverse square, then for any $t > 0$ they decay exponentially fast in the periodic direction, but only as an inverse square in the other direction.
- A similar picture for the decay of the Fourier modes was first shown to hold for a plane NS problem with different boundary conditions in

  Dinaburg, E., Li, D., Sinai, Ya.G.: A new boundary problem for the two dimensional Navier-Stokes system. J. Stat. Phys. **135**, 737–750 (2009)
- Our results thus corroborate the opinion of those authors that an exponential decay of Fourier modes is exceptional: for a generic NS boundary value problems in a bounded plane region, the decay is only power-like, with a power depending on the geometry of the domain.

Future work:



Now Δ_N has eigenvalues/eigenfunctions $\{\lambda_{k,j}; v_{k,j}(r,\theta)\}$, $j,k \in \mathbb{Z}_+$, where

$$\lambda_{k,j} = -\frac{\left[\mu_j^{(k)}\right]^2}{R^2}, \qquad v_{k,j}(r,\theta) := \mathrm{e}^{\mathrm{i}k\theta} J_k\left(\mu_j^{(k)} \frac{r}{R}\right),$$

where J_k is the Bessel function of order k and $\mu_j^{(k)}$ are the positive roots of $J_k'(x) = 0$.

Sketch of the proof

Recall the system is

$$\begin{cases} \omega_{k_1,k_2}(t) = \mathrm{e}^{-k^2 t}\omega_{k_1,k_2}(0) \\ \qquad\qquad + \int_0^t \mathrm{d}s\, \mathrm{e}^{-k^2(t-s)} \Big\{ f_{\pm,k_1}(s) - N_{k_1,k_2}[\omega(s)] \Big\}, \\ \\ \sum_{k_2,\pm} \frac{1}{k^2} \int_0^t \mathrm{d}s\, \mathrm{e}^{-k^2(t-s)}\, f_{\pm,k_1}(s) = g_{\pm,k_1}[t;\omega], \quad (k_1 \neq 0), \end{cases}$$

and $\omega_{0,0}(t) = f_{+,0}(t) = f_{-,0}(t) = 0$.

Step 1. (Preliminary lemma for $f_{\pm,k_1}$)

The Volterra equation of the first kind for the unknown function $a(t)$,

$$\sum_{k_2,\pm} \frac{1}{k^2} \int_0^t \mathrm{d}s\, \mathrm{e}^{-k^2(t-s)}\, a(s) = b(t), \qquad k_1 \neq 0,$$

$b(t)$ a bounded differentiable function, $b(0) = 0$, has solution

$$a(t) = \int_0^t \mathrm{d}s\, G^{\pm}_{k_1}(t-s)\, b'(s) + \int_0^t \mathrm{d}s\, H^{\pm}_{k_1}(t-s)\, b(s).$$

Here, denoting by $\Gamma(\cdot)$ the Euler Gamma function, $G^{\pm}_{k_1}$ is given by

$$G^{\pm}_{k_1}(t) := \frac{2}{\pi} k_1 \left[\tanh\left(\frac{\pi}{2} k_1\right)\right]^{\pm 1} \left[\delta(t) + \frac{\mathrm{e}^{-k_1^2 t}}{\sqrt{t}} \sum_{n=1}^{4} \frac{d_{\pm}(k_1)^n}{\Gamma(n/2)}\, t^{(n-1)/2}\right],$$

and $H^{\pm}_{k_1}(t)$ is a continuous function such that, for each $0 < \gamma < 1$,

$$H^{\pm}_{k_1}(t) \leq B_\gamma\, |k_1|^3\, \exp\left[-(1-\gamma)\, k_1^2 t\right],$$

with B_γ a positive constant.

Step 2. (Local solutions)

Introduce the norm

$$\|\omega\|_{\alpha,\beta,t} := \sup_{s\in[0,t]} \sup_{k_1\in\mathbb{Z}} \sup_{k_2\geq 0} |\omega_{k_1,k_2}(s)| \, \mathrm{e}^{(1+|k_1|)s/4} \, |k|^{\alpha} \, \left(1+|k_1|^{\beta}\right),$$

so that the hypothesis on the initial datum read

$$\omega_{0,0}(0)=0, \quad \|\omega\|_{\alpha,\beta,0} \leq D_0, \quad \sum_{k_2,\pm} \frac{\omega_{k_1,k_2}(0)}{k^2} = 0, \quad \forall\, k_1 \neq 0.$$

Then (fixed point theorem):

- There exist $T_0 = T_0(D_0,\alpha,\beta)$ and $D_2 = D_2(D_0,\alpha,\beta)$ such that there is a unique continuous solution $\{\omega_{k_1,k_2}(t);\, k_1\in\mathbb{Z},\, k_2\geq 0\}$, $t\in[0,T_0]$, which satisfies $\|\omega\|_{\alpha,\beta,t} \leq D_2$.
- If D_0 is sufficiently small, the corresponding solution is global in time and $\|\omega\|_{\alpha,\beta,t} \leq D_2$ is valid for any $t \geq 0$.

Step 3. (A priori bounds)

Local solutions with the weaker norm,

$$\|\omega\|_{\alpha,t} := \sup_{s\in[0,t]} \sup_{k_1\in\mathbb{Z}} \sup_{k_2\geq 0} |\omega_{k_1,k_2}(s)|\ |k|^\alpha \quad (1<\alpha<2).$$

do exist and the corresponding velocity field $u(x,t) := \nabla^\perp \Delta_N^{-1}\omega(x,t)$, $t\in[0,T_0]$, is a weak solution to the NS system.
This implies a priori bounds on the energy $\mathcal{U}$ and enstrophy $\mathcal{E}$:

$$\mathcal{U}(t) \leq \mathcal{U}(0)\,\mathrm{e}^{-t} \qquad \mathcal{E}(t) \leq\ E_0\,\mathrm{e}^{-\sigma t} \qquad \forall\, t\in[0,T_0],$$

where

$$\mathcal{U}(t) := \sum_{\substack{(k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}_+\\ k\neq(0,0)}} \frac{\left|\omega_{k_1,k_2}(t)\right|^2}{k^2}, \qquad \mathcal{E}(t) := \sum_{(k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}_+} \left|\omega_{k_1,k_2}(t)\right|^2$$

($E_0, \sigma > 0$ are constants depending on $\mathcal{E}(0)$, $\mathcal{U}(0)$).

Step 4. (Global solutions)

- By the a priori bounds, any local solution $\{\omega_{k_1,k_2}(t)\}$, $t \in [0,T]$, such that $\|\omega\|_{\alpha,T} < \infty$ extends uniquely to a global solution.

- Moreover, if

$$|\omega|_{\alpha,t} := \sup_{k_1 \in \mathbb{Z}} \sup_{k_2 \geq 0} |\omega_{k_1,k_2}(t)|\ |k|^\alpha$$

  then $|\omega|_{\alpha,t} \to 0$ exponentially fast as $t \to +\infty$.

- Using $|\omega|_{\alpha,t} \to 0$, the theorem on local existence of solutions with the stronger norms $\|\omega\|_{\alpha,\beta,t}$ can be applied with $\beta = 0$ to time intervals $[t, t+\bar{T}]$ for a suitable $\bar{T}$. From this one easily get the main estimate,

$$|\omega_{k_1,k_2}(t)| \leq \frac{D_1\, \mathrm{e}^{-\nu(1+|k_1|)t}}{|k|^\alpha \big(1+|k_1|^\beta\big)},$$

  with an appropriate choice of $D_1, \nu > 0$.