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Diffeomorphic Matching and Dynamic Deformable Surfaces with Applications in 3D Medical Imaging

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based on joint work with

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Diffeomorphic Matching and Dynamic Deformable Surfaces

- Diffeomorphic matching in biomedical image processing
- Reproducing Kernel Hilbert Spaces (RKHS)
- Geometric surface matching distances
- Variational formulation of optimal diffeomorphic matching
- Discretization: Dirac measures and diffeomorphic point matching
- Numerical results: Matching snapshots of the mitral valve

Diffeomorphic Matching in Biomedical Image Processing

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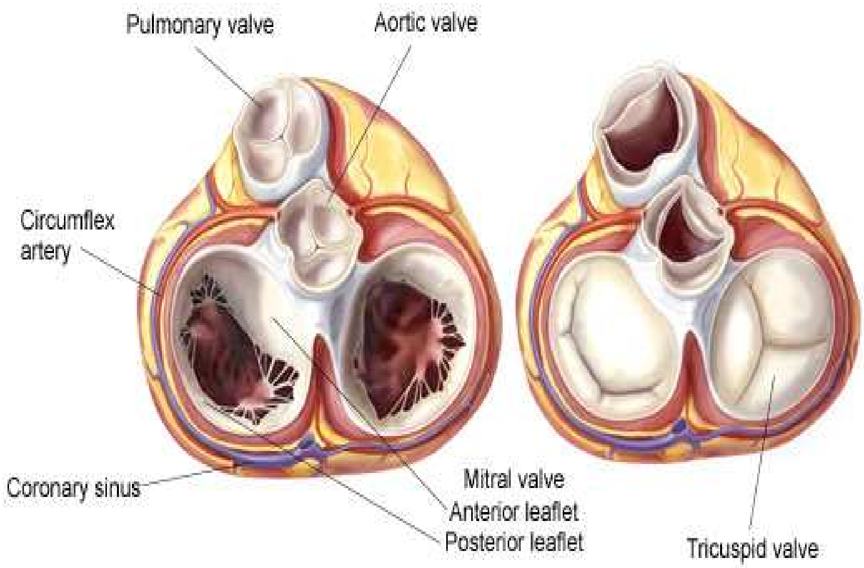
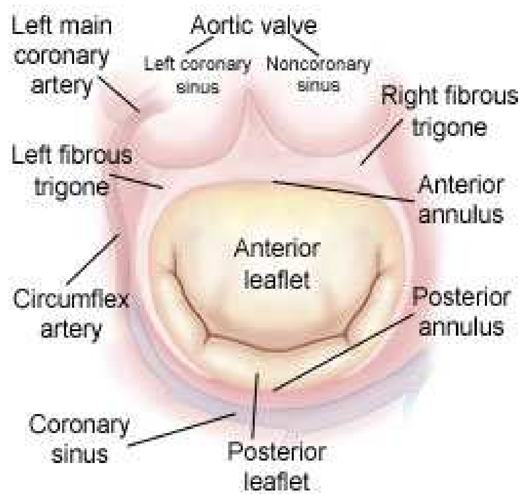
Optimal Diffeomorphic Matching of 3D Curves and Surfaces

Cardiovascular diseases often affect the mitral valve. Biomedical image processing provides the cardiologist with information about the degree of malfunctioning.





Valve anatomy of the human heart (left) and anatomy of the mitral valve (right)

Optimal Matching of Biomedical 3D Movies

Biomedical Data: Given a 3D movie of a deformable anatomical shape $S(t) \subset \mathbb{R}^3$, $t \in I := [0, T]$, biomedical techniques enable the extraction of snapshots $S_j := S(t_j)$ at specific time frames t_j , $0 \leq j \leq q$.

Mathematical Task: Find a family $F(\cdot, t) \in \mathrm{Diff}(\mathbb{R}^3)$, $t \in I$, of time dependent $\mathbb{R}^3$-diffeomorphisms

$$F(S_0; t_0) \ = \ S_0 \quad , \quad F(S_0; t_j) \ = \ \hat{S}_j \ , \ 1 \leq j \leq q \ ,$$

which map the initial shape S_0 onto shapes $\hat{S}_j$ at the time frames t_j such that for all $1 \leq j \leq q$ the shapes $\hat{S}_j$ are as close to S_j as possible.

Matching of Dynamic Deformable Surfaces: Previous Work

- Matching of two snapshots S_0 and S_1,
- Concepts based on diffeomorphic matching developed by Dupuis, Glaunès, Grenander, Miller, Mumford, Trouvè, Younes et al.,
- $F(\cdot, t) = F^{v_t}$, $t \in I$, generated by time dependent flow v_t

$$\begin{aligned} \partial_t F(\cdot, t) &= v_t(F(\cdot, t)) \ , \ t \in I \ , \\ F(\cdot, 0) &= Id \ , \end{aligned}$$

- Rigid constraint $F(S_0, t_1) = S_1$ replaced by soft constraint using suitably chosen geometric surface matching distances,
- Solution of the resulting optimization problem within a variational framework.

Literature on Diffeomorphic Matching

U. Grenander and M.I. Miller; Computational anatomy: an emerging discipline. Quart. Appl. Math. 56, 617-694, 1998

M.I. Miller and L. Younes; Group action, diffeomorphism and matching: A general framework. Int. J. Comp. Vis. 41, 61-84, 2001

M.F. Beg, M.I. Miller, A, Trouvé, and L. Younes; Computing large deformations metric mappings via geodesic flows of diffeomorphisms. Int. J. Comp. Vision 61, 139-157, 2005

H. Guo, A. Rangarajan, and S. Joshi; Diffeomorphic point matching. In: Handbook of Mathematical Models in Computer Vision, Springer, Berlin-Heidelberg-New York, pp. 205-219, 2006

J. Glaunès, A. Qiu, M.I. Miller, and L. Younes; Large deformation diffeomorphic metric curve mapping. Int. J. Comp. Vision 80, 317-336, 2008

Generalization to Arbitrarily Many Intermediary Snapshots

Given $q+1$ snapshots $S_j, 0 \leq j \leq q$, at time instants $t_j \in [0,1], 0 =: t_0 < t_1 < \cdots < t_q := 1$, find a time dependent family of diffeomorphisms $F(\cdot,t) \in \text{Diff}(\mathbb{R}^3), t \in [0,1]$, such that

$$\sum_{j=1}^{q} \text{dist}(F(S_0,t_j),S_j) \ \rightarrow \min \ ,$$

where $\text{dist}(\cdot,\cdot)$ is a geometric surface matching distance, and $F(\cdot,t) = F^{v_t}, t \in [0,1]$, is generated by a time dependent flow v_t according to

$$\begin{aligned} \partial_t F(\cdot,t) \ &= \ v_t(F(\cdot,t)) \ , \ \ t \in I \ , \\ F(\cdot,0) \ &= \ \text{Id} \ . \end{aligned}$$

Reproducing Kernel Hilbert Spaces

Reproducing Kernel Hilbert Spaces I

Let H be a Hilbert space of functions on $\mathbb{R}^d$ with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$.

A function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is said to be a reproducing kernel of H, if the following two conditions hold true:

$(RK)_1$ For every $x \in \mathbb{R}^d$, we have $K_x \in H$, where $K_x : \mathbb{R}^d \to \mathbb{C}$ is given by

$$K_x(y) \ = \ K(y,x) \quad , \quad y \in \mathbb{R}^d \ .$$

$(RK)_2$ For every $x \in \mathbb{R}^d$ and every $f \in H$ there holds

$$f(x) \ = \ (f, K_x)_H \quad , \quad x \in \mathbb{R}^d \ .$$

The kernel K is called Hermitian (positive definite), if for any finite set of points $\{y_1, \cdots, y_n\} \subset \mathbb{R}^d$ and any $\gamma_i \in \mathbb{C}, 1 \leq i \leq n$, there holds

$$\sum_{i,j=1}^{n} \bar{\gamma}_j \gamma_i \, K(y_j, y_i) \in \mathbb{R} \ (\in \mathbb{R}_+) \ .$$

The Hilbert space H is said to be a Reproducing Kernel Hilbert space (RKHS), if there exists a reproducing kernel on H.

Reproducing Kernel Hilbert Spaces II

Proposition 2 [Aronszajn] For any positive definite kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ there exists a uniquely determined RKHS H of functions on $\mathbb{R}^d$ with reproducing kernel K.
Any RKHS H with a positive definite kernel K is a Hilbert space of functions on $\mathbb{R}^d$ for which pointwise evaluations are continuous linear functionals.

A kernel K is said to be translation invariant, if for all $a \in \mathbb{R}^d$

$$K(x-a, y-a) \; = \; K(x,y) \quad , \quad x, y \in \mathbb{R}^d \; .$$

Proposition 3 [Bochner] A kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is a continuous positive definite translation invariant kernel, iff there exists a finite positive Borel measure μ on $\mathbb{R}^d$ such that

$$K(x,y) \; = \; \int_{\mathbb{R}^d} \exp(i(x-y)\cdot z) \; d\mu(z) \quad , \quad x, y \in \mathbb{R}^d \; .$$

Reproducing Kernel Hilbert Spaces III

A function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is called radial, if there exists a function r on $\mathbb{R}_+$ such that

$$K(x,y) = r(|x-y|) \quad , \quad x,y \in \mathbb{R}^d .$$

Proposition 4 [Schönberg] A radial function K with a continuous function r is a continuous positive definite translation invariant kernel, iff there exists a positive Borel measure μ on $\mathbb{R}_+$ such that

$$r(t) = \int_{\mathbb{R}_+} \exp(-st^2)\, d\mu(s) \quad , \quad t \in \mathbb{R}_+ .$$

Proposition 5 [Schönberg] Let H be an RKHS of vector valued functions on $\mathbb{R}^d$ with Gaussian kernel K, i.e., $r(t) = (2\pi)^{-d/2}\exp(-t^2/(2\sigma^2))$. If $f \in H$ with Jacobian $Df \in \mathbb{R}^{d \times d}$, then there holds

$$\|Df\|_F \leq \frac{d}{\sigma} \|f\|_H .$$

Geometric Surface Matching Distances

Geometric Matching Distances: Hausdorff Distance

The Hausdorff distance between two bounded subsets $S, S' \in \mathbb{R}^3$ is given by

$$D_H(S,S') \ := \ \max\Big(h(S,S'), h(S',S)\Big) \ ,$$

where the Hausdorff disparity $h(S,S')$ is defined by means of

$$h(S,S') \ := \ \max_{x \in S} \Big(\min_{x' \in S'} |x - x'| \Big) \ .$$

Remark: The Hausdorff distance is not smooth. Instead, we use

$$\widetilde{D}_H(S,S') \ := \ h_{sm}(S,S') + h_{sm}(S',S) \ ,$$

where $h_{sm}(S,S')$ refers to a smoothed Hausdorff disparity.

Geometric Matching Distances: Borel Measure Distance

- We denote by $\mathrm{BM}(\mathbb{R}^3)$ the linear space of bounded Borel measures on $\mathbb{R}^3$ equipped with the inner product

$$\langle \mu, \mu' \rangle_\Gamma := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Gamma(\mathrm{x}, \mathrm{x}') \, d\mu(\mathrm{x}) \, d\mu'(\mathrm{x}') \ ,$$

where $\Gamma(\cdot, \cdot)$ is a smooth, symmetric, and translation-invariant bounded positive definite kernel on $\mathbb{R}^3 \times \mathbb{R}^3$.

- We identify a bounded Borel subset $\mathrm{S} \subset \mathbb{R}^3$ with a measure $\mu_\mathrm{S} \in \mathrm{BM}(\mathbb{R}^3)$ induced on S by the Lebesgue measure of $\mathbb{R}^3$.

- The distance between bounded Borel subsets $\mathrm{S}, \mathrm{S}' \in \mathbb{R}^3$ is defined by means of

$$\mathrm{D}^2_\Gamma(\mathrm{S}, \mathrm{S}') := \|\mu_\mathrm{S} - \mu_{\mathrm{S}'}\|^2_\Gamma \ .$$

Variational Formulation of the Optimal Matching Problem

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Variational Formulation of the Optimal Matching Problem

Let $\mathcal{D}(\mathbf{I};\mathbf{V})$ be the space of all disparity functionals $\mathbf{D} : \mathbf{L}^2(\mathbf{I};\mathbf{V}) \to \mathbb{R}_+$ of the form

$$\mathrm{D}(\mathrm{v}) \;=\; \Phi(\mathrm{F}^{\mathrm{v}}(\cdot,\mathrm{t}_1),\cdots.\mathrm{F}^{\mathrm{v}}(\cdot,\mathrm{t}_q)) \ ,$$

where $\Phi : \mathrm{Diff}(\mathbb{R}^3)^q \to \mathbb{R}_+$ is a continuous function, and let $\mathbf{E} : \mathbf{L}^2(\mathbf{I};\mathbf{V}) \to \mathbb{R}_+$ be the energy functional

$$\mathrm{E}(\mathrm{v}) \;=\; \frac{1}{2}\int_0^1 \|\mathrm{v}_t\|_V^2 \, dt \ .$$

Optimization Problem: For $\mathbf{D} \in \mathcal{D}(\mathbf{I};\mathbf{V})$, find $\mathrm{v}^* \in \mathbf{L}^2(\mathbf{I};\mathbf{V})$ such that

$$(\mathrm{OP})_1 \qquad \mathrm{J}(\mathrm{v}^*) \;=\; \inf_{\mathrm{v}\in \mathrm{L}^2(\mathrm{I};\mathrm{V})} \mathrm{J}(\mathrm{v}) \quad , \quad \mathrm{J}(\mathrm{v}) \;:=\; \mathrm{E}(\mathrm{v}) + \lambda \mathrm{D}(\mathrm{v}) \ ,$$

subject to

$$(\mathrm{OP})_2 \qquad \partial_t \mathrm{F}(\cdot,\mathrm{t}) \;=\; \mathrm{v}_t(\mathrm{F}(\cdot,\mathrm{t})) \quad , \quad \mathrm{t} \in \mathrm{I} \ ,$$

$$\mathrm{F}(\cdot,0) \;=\; \mathrm{Id} \ .$$

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Existence of a Minimizing Diffeomorphic Flow

Theorem 1. Assume that the embedding $V \subset W^{s,2}(\mathbb{R}^3), s > 5/2,$ is continuous. Then, the optimal diffeomorphic matching problem $(OP)_1, (OP)_2$ has a solution $v^* \in L^2(I; V)$.

Proof. Let $\{v^n\}_{\mathbb{N}}$ be a minimizing sequence. Due to the boundedness of $\{v^n\}_{\mathbb{N}}$, there exist $\mathbb{N}' \subset \mathbb{N}$ and $v^* \in L^2(I; V)$ such that

$$\liminf_{n \to \infty} \|v^n\|_{L^2(I;V)} \leq \|v^*\|_{L^2(I;V)} .$$

Denoting by $F^n(\cdot, t), F^*(\cdot, t) \in \mathrm{Diff}(\mathbb{R}^3), t \in I,$ the unique flows solving $(OP)_2$ w.r.t. v^n, v^*, the main part of the proof is to show that

$$F^n(\cdot, t) \to F^*(\cdot, t) \quad (n \to \infty) \ , \ t \in I \ ,$$

uniformly on bounded subsets of $\mathbb{R}^3$. This implies $D(v^n) \to D(v^*)$ $(n \to \infty)$, and hence,

$$\liminf_{n \to \infty} J(v^n) \leq \lim_{n \to \infty} D(v^n) + \liminf_{n \to \infty} E(v^n) \leq D(v^*) + E(v^*) = J(v^*) \ ,$$

which allows to conclude.

Necessary Optimality Conditions

Theorem 2. In addition to the assumptions of Theorem 1 suppose that the functional $\Phi : C(\mathbb{R}^3)^q \to \mathbb{R}$ has Gâteaux derivatives $\partial_j \Phi \in M(\mathbb{R}^3), 1 \leq j \leq q$.

If $v^* \in L^2(I; V)$ is a solution of $(OP)_1, (OP)_2$, then there exists a family $p^* = p_t^*, t \in I$, of vector valued Borel measures on $I \times \mathbb{R}^3$ satisfying the jump process

$$(OP)_3 \qquad -\partial_t p_t^* - b_{v^*,t} p_t^* \; = \; 0 \quad , \quad t \in (t_{j-1}, t_j) \; ,$$

$$p^*_{t_q^+} = 0 \; , \; p^*_{t_j^-} = p^*_{t_j^+} + \lambda \partial_j \Phi(F^*(\cdot, t_j)) \; , \; 1 \leq j \leq q \; .$$

$$(OP)_4 \qquad p_t^* + \rho_{t,v^*} \; = \; 0 \quad , \quad t \in I \; ,$$

Here, $b_{v,t}$ is a Borel function of $D_v v_t(F^v(\cdot, t))$, and $\rho_{t,v}$ is a vector valued Borel measure with density $K v_t$.

Discretization: Dirac Measures and Diffeomorphic Point Matching

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Discretization: Dirac Measures and Diffeomorphic Point Matching

- We discretize the snapshots $S_j, 0 \leq j \leq q$, and the dynamically deformed surfaces $\hat{S}_j = F^v(S_0, t_j)$ by point sets

$$X_j \;=\; \{x_1^j, \cdots, x_{N_j}^j\} \quad , \quad \hat{X}_j \;=\; F^v(X_0, t_j) \;=\; \{F^v(x_1^0, t_1), \cdots, F^v(x_{N_0}^0, t_j)\} \; .$$

- We denote by $x_n(t) = F^v(x_n^0, t_j)$, $x_n(0) = x_n^0, 1 \leq n \leq N_0$, the trajectories emanating from x_n^0, i.e., the solutions of the initial value problems

$$\frac{d}{dt} x_n(t) \;=\; v_t(x_n(t)) \; , \; t \in [0,1] \quad , \quad x_n(0) \;=\; x_n^0 \; .$$

- We approximate the Borel measures associated with S_j and $\hat{S}_j$ by weighted sums of Dirac measures

$$\mu_{S_j} \;=\; \sum_{m=1}^{N_j} b_m^j \; \delta_{x_m^j} \quad , \quad \mu_{\hat{S}_j} \;=\; \sum_{n=1}^{N_0} a_n \; \delta_{x_n(t_j)} \quad , \quad 1 \leq j \leq q \; .$$

Discretization: Dirac Measures and Diffeomorphic Point Matching

- Setting $x(t) = (x_1(t), \cdots, x_{N_0}(t))^T, t \in (0,1)$, the disparity cost funtional reads

$$D(v) \ = \ \sum_{j=1}^{q} \lambda_j \ D_j(x(t_j)) \quad , \quad D_j(x(t_j)) \ := \ \|\mu_{S_j} - \mu_{\hat{S}_j}\|^2_{K_{\sigma_j}} \ ,$$

  where $K_{\sigma_j}, 1 \leq j \leq q$, are appropriately chosen radial Gaussian kernels.

- We approximate the flow v_t by a linear combination of $K_{x_n(t)}, 1 \leq n \leq N_0$,

$$v_t(x) \ = \ \sum_{n=1}^{N_0} K_{\sigma_0}(x_n(t), x) \ \alpha_n(t) \quad , \quad x \in \mathbb{R}^3 \ .$$

  It follows that

$$\|v_t\|^2_V \ = \ \sum_{n=1}^{N_0} \sum_{n'=1}^{N_0} K_{\sigma_0}(x_n(t), x_{n'}(t)) \ \alpha_n^T(t) \alpha_n(t) \quad , \quad t \in [0,1] \ .$$

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The Discrete Optimization Problem

Setting $\alpha(t) = (\alpha_1(t), \cdots, \alpha_{N_0}(t))^T \in \mathbb{R}^{dN_0}, t \in (0,1)$, and

$$A(x(t)) \;:=\; \Big(K_{\sigma_0}(x_n(t), x_{n'}(t)) I_d\Big)_{n,n'=1}^{N_0} \in \mathbb{R}^{dN_0 \times dN_0} \;,$$

the discrete optimization problem reads:

Discrete Optimization Problem: Find $\alpha^* \in L^2(I; \mathbb{R}^{dN_0})$ and $x^*(t)$ such that

$$(DOP)_1 \qquad J(\alpha^*) \;=\; \inf_{\alpha} J(\alpha)\;,\; J(\alpha) := \frac{1}{2}\int_0^1 \alpha(t)^T A(x(t))\alpha(t)\; dt + \sum_{j=1}^{q} \lambda_j D_j(x(t_j))\;,$$

subject to

$$(DOP)_2 \qquad \frac{d}{dt}\, x^*(t) \;=\; A(x^*(t))\; \alpha^*(t) \quad , \quad t \in I\;,$$

$$x^*(0) \;=\; x^0\;.$$

Existence of a Solution and Necessary Optimality Conditions

Theorem 3. The discrete optimization problem $(\mathrm{DOP})_1, (\mathrm{DOP})_2$ has a solution $\alpha^* = \alpha^*(t), t \in I$. If $x^* = x^*(t), t \in I$, is the associated trajectory, there exists a function $p^* = p^*(t), t \in I$, which solves the final time problem

$$(\mathrm{DOP})_3 \qquad -\frac{d}{dt}\, p^*(t) \;=\; B(x^*(t),\alpha^*(t))^T \left(p^*(t) + \frac{1}{2}\,\alpha^*(t)\right) \,,\; t \in (t_{j-1},t_j) \,,$$

$$p^*(t_q^+) = 0 \,,\; p^*(t_j^-) = p^*(t_j^+) + \lambda_j \,\nabla D_j(x^*(t_j)) \,,\; 1 \le j \le q \,,$$

$$(\mathrm{DOP})_4 \qquad A(x^*(t))(\alpha^*(t) + p^*(t)) \;=\; 0 \,,\; t \in I \,,$$

where the matrix $B(x^*(t),\alpha^*(t)) \in \mathbb{R}^{dN_0 \times dN_0}$ is given by

$$B(x^*(t),\alpha^*(t)) \;=\; \nabla_x (A(x^*(t),\alpha^*(t))) \,.$$

Fully Discrete Optimization Problem

Fully Discrete Optimal Diffeomorphic Matching Problem I

For the discretization in time of the optimality system $(\mathrm{DOP})_2 - (\mathrm{DOP})_4$ we introduce

$$\Delta_I \;:=\; \bigcup_{j=1}^{q} \Delta_{I_j} \;,\; \Delta_{I_j} \;:=\; \{t_{j-1} =: t^{L_{j-1}} < t^{L_{j-1}+1} < \cdots < t^{L_j} := t_j\} \;,$$

where $\Delta_{I_j}, 1 \leq j \leq q$, are subpartitions of $I_j := [t_{j-1}, t_j]$.
Setting $\Delta t^{\ell} := t^{\ell+1} - t^{\ell}, 0 =: L_0 \leq \ell \leq L := L_q,$, the discretized optimality system reads

$$(\mathrm{DOC})_1 \qquad \frac{x^{\ell+1} - x^{\ell}}{\Delta t^{\ell}} \;=\; A(x^{\ell}\alpha^{\ell} \;,\; L_0 \leq \ell \leq L \;,$$

$$x^0 \;=\; x^{(0)} \;,$$

$$(\mathrm{DOC})_2 \qquad \frac{p^{(\ell-1)^+} - p^{\ell^-}}{\Delta t^{\ell-1}} \;=\; B(x^{\ell}, \alpha^{\ell})^T (p^{\ell^-} + \alpha^{\ell}/2) \;,\; \ell = L_j, \cdots, L_{j-1}+1 \;,$$

$$p^{L_q^+} = 0 \;,\; p^{L_j^-} = p^{L_j^+} + \lambda_j \nabla D_j(x^{L_j}) \;,\; 1 \leq j \leq q \;,$$

$$(\mathrm{DOC})_3 \qquad A(x^{\ell})(\alpha^{\ell} + p^{\ell^+}) \;,\; L_0 \leq \ell \leq L-1 \;.$$

Fully Discrete Optimal Diffeomorphic Matching Problem II

Theorem 4. Let J_{Δ_I} be the discrete objective functional

$$J_{\Delta_I}(\alpha) \;=\; \frac{1}{2}\sum_{\ell=0}^{L-1} \Delta t^{\ell}\,(\alpha^{\ell})^T A(x^{\ell})\alpha^{\ell} \;+\; \sum_{j=1}^{q} \lambda_j D_j(x^{L_j})\ .$$

The discrete optimality system $(DOC)_1 - (DOC)_3$ represents the first order necessary optimality conditions for the discrete optimization problem

$$\min_{\alpha} J_{\Delta_I}(\alpha)\ ,$$

subject to

$$\frac{x^{\ell+1} - x^{\ell}}{\Delta t^{\ell}} \;=\; A(x^{\ell})\alpha^{\ell} \quad , \quad L_0 \le \ell \le L-1\ ,$$

$$x^0 \;=\; x^{(0)}\ .$$

Corollary. Let (x^*, p^*, α^*) with $x^* = \{x_*^{\ell}\}_{\ell=0}^{L}$ etc. satisfy $(DOC)_1 - (DOC)_3$. Then, we have

$$0 \;=\; \nabla J_{\Delta_I}(\alpha^*) \;=\; \{g^{\ell}\}_{\ell=0}^{L-1} \quad , \quad g^{\ell} \;=\; A(x_*^{\ell})(\alpha_*^{\ell} + p_*^{\ell})\ .$$

Matching Algorithm: Continuation in the Regularization Parameter

Role of the regularization parameters: For simplicity, we assume $\lambda_j = \lambda > 0, 1 \le j \le q$. The regularization parameter provides a balance between the matching quality and the regularizing kinetic energy. The larger λ, the more emphasis is on the matching quality.

Problem: The gradient method does not converge for large λ, in particular, if the initial iterate is not close to a local minimum.

Remedy: Continuation in the regularization parameter. This results in an inner/outer iteration with outer iterations in λ and inner iterations featuring the gradient method with Armijo line search. A termination criterion for the outer iterations is

$$D_j := \kappa \Big(\sum_{n=1}^{N_0} (d_n^j)^2 \Big)^{1/2} < \vartheta \quad , \quad d_n^j := \min_{1 \le m \le N_j} |x_n(t_j) - x_m(t_j)| \ ,$$

where $\vartheta > 0$ is a given threshold and $0 < \kappa \le 1$ (e.g., $\kappa = 0.9$).

Matching Algorithm: Inner/Outer Iterative Scheme

Step 1: Initialization

Choose thresholds $\theta > 0, \vartheta > 0$, as well as $\gamma > 1$ for continuation and $0 < \kappa \leq 1$.

Step 2: Initialization of the outer iteration

Choose initial value λ_0 and set $\nu := 0$.

Step 3: Initialization of the inner iteration

Compute $\alpha_\nu^{(0)}$ by an appropriate initialization and set $\mu := 0$.

Step 4: Gradient method with Armijo line search

Step 4.1: Set $\mu := \mu + 1$ and compute $\alpha_\nu^{(\mu)}$ by gradient descent with Armijo line search.

Step 4.2: If the termination criterion $|\nabla J(\alpha_\nu^{(\mu)})| < \theta |\nabla J(\alpha_\nu^{(0)})|$ is satisfied, go to Step 5. Otherwise, go to Step 4.1.

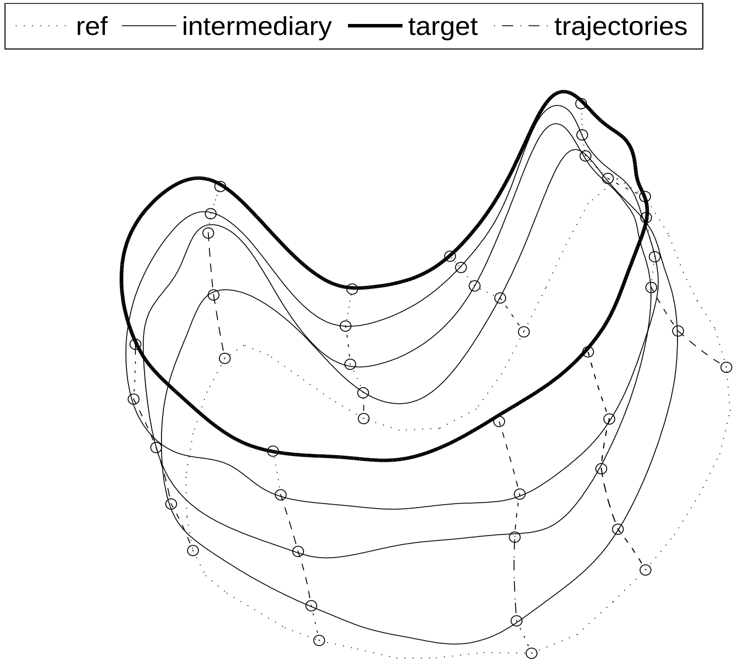
Step 5: Termination of the outer iteration

If the termination criterion $D_j < \vartheta, 1 \leq j \leq q$, is satisfied, stop the algorithm. Otherwise, set $\nu := \nu + 1, \alpha_\nu^{(0)} := \alpha_{\nu-1}^{(\mu)}, \ \lambda_\nu := \gamma \lambda_{\nu-1}$, and go to Step 4.

Numerical Results:

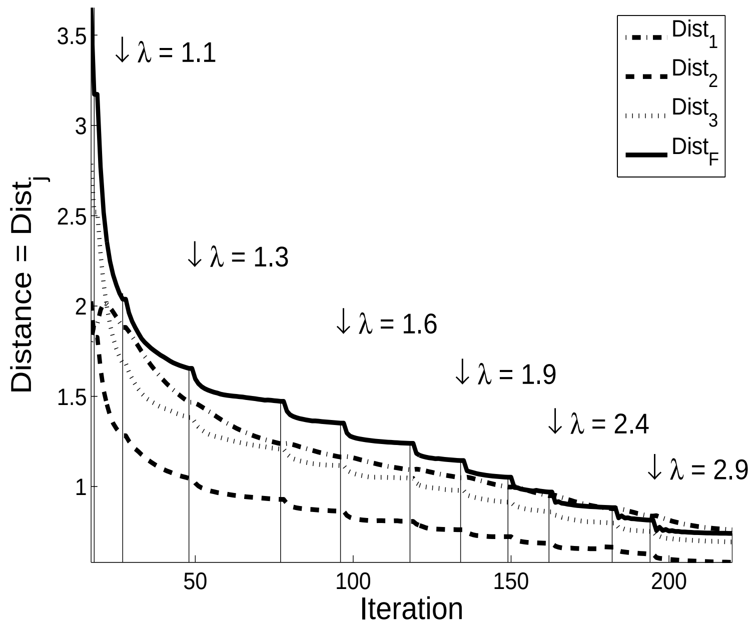
Matching Mitral Annulus Snapshots

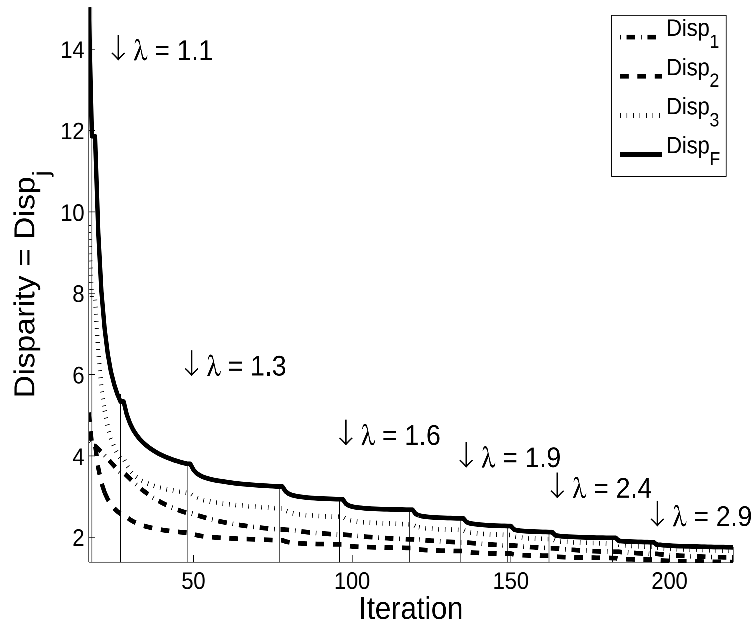
Diffeomorphic Matching of Multiple Annulus Snapshots



Matching Multiple Snapshots of the Mitral Annulus at t = 1,3,5,7,10

Diffeomorphic Matching of Multiple Annulus Snapshots: Hausdorff Matching





Convergence history: Accuracy indicators (l.) and Hausdorff disparities (r.)

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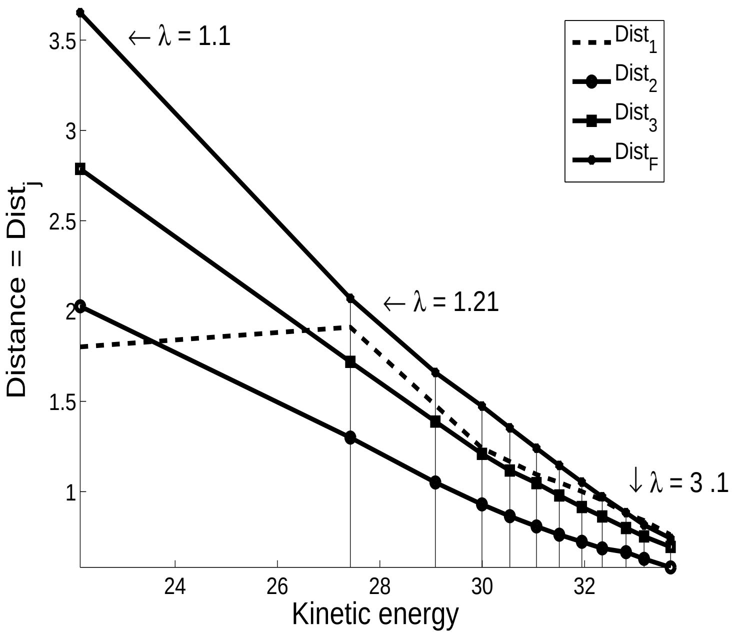
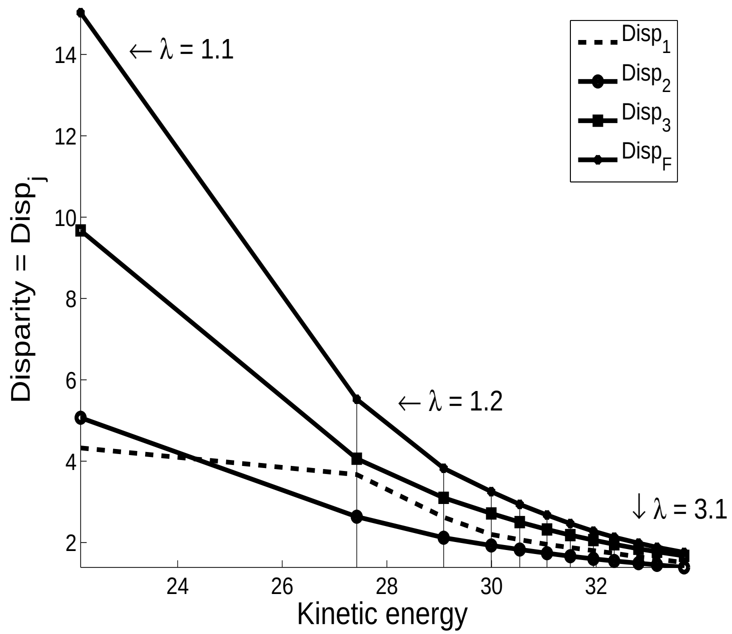
Diffeomorphic Matching of Multiple Annulus Snapshots: Hausdorff Matching





Pareto frontiers: Accuracy indicators (l.) and Hausdorff disparities (r.)

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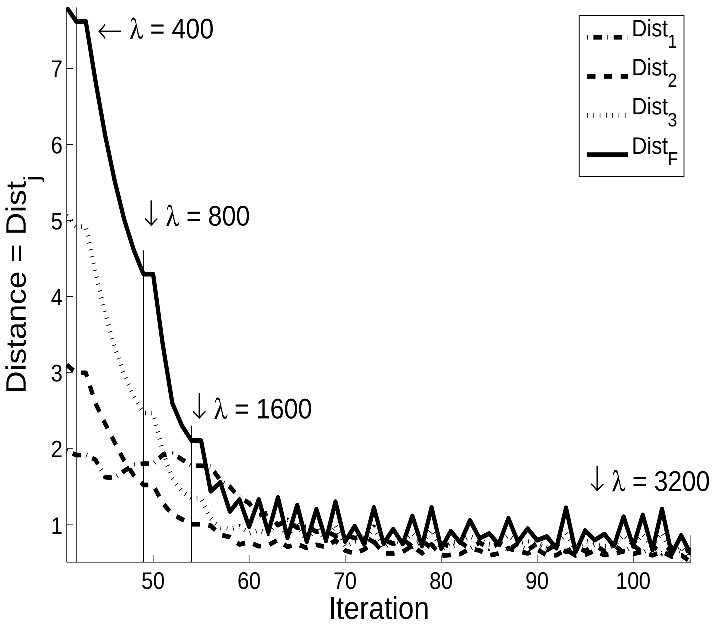
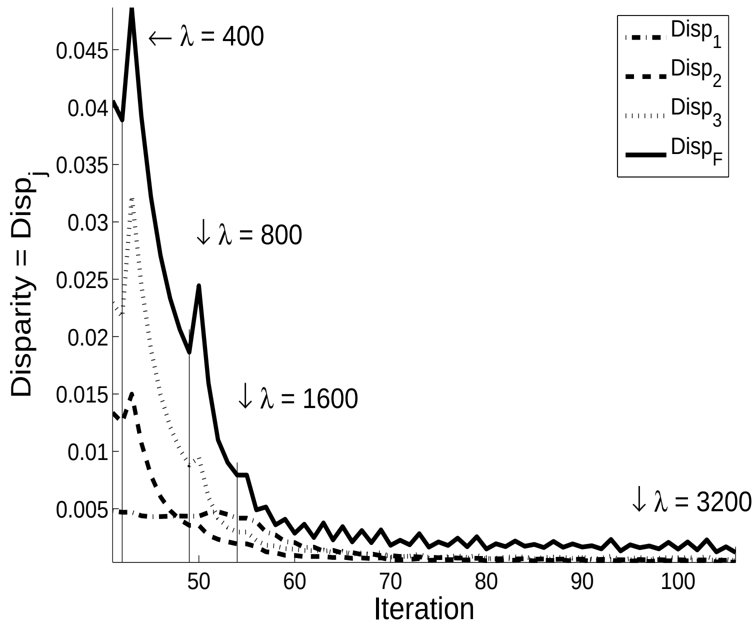
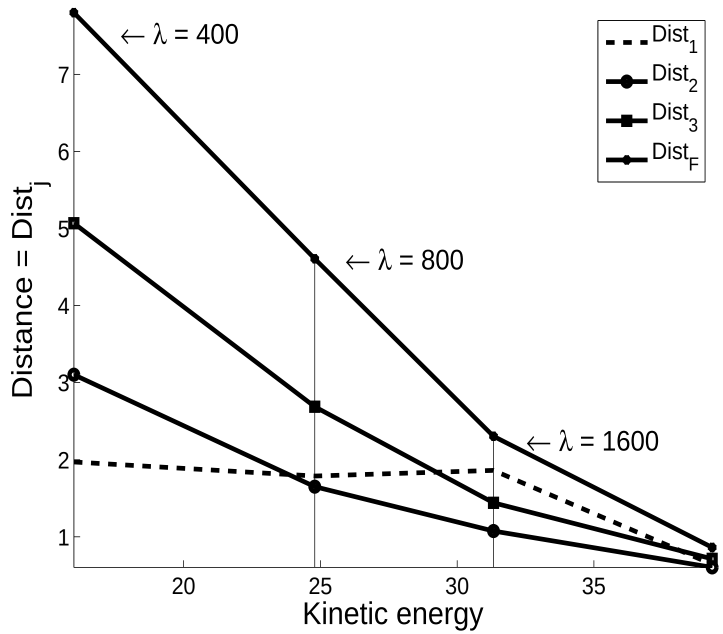
Diffeomorphic Matching of Multiple Annulus Snapshots: Measure Matching

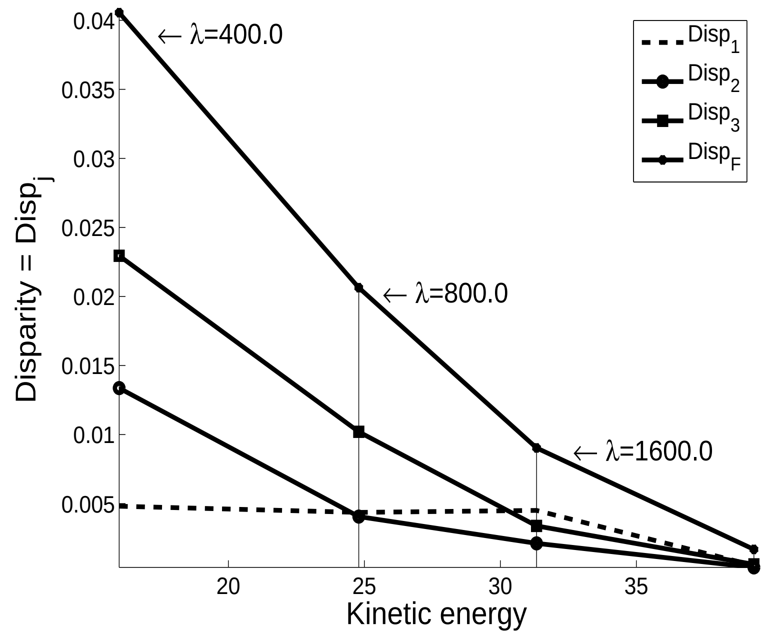




Convergence history: Accuracy indicators (l.) and measure matching disp. (r.)

Diffeomorphic Matching of Multiple Annulus Snapshots: Measure Matching



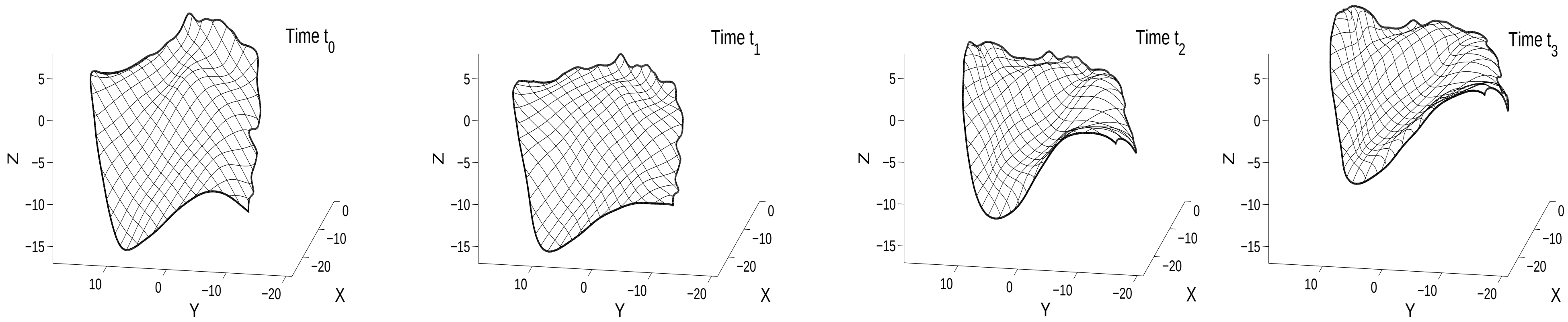


Pareto frontiers: Accuracy indicators (l.) and measure matching disp. (r.)

Numerical Results:

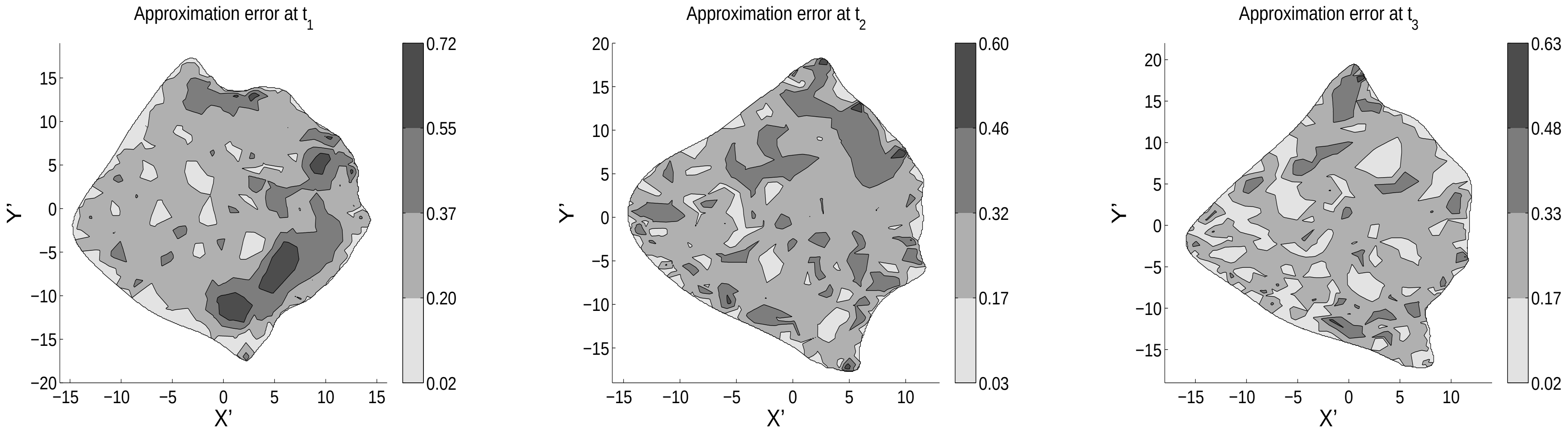
Matching Anterior Leaflet Snapshots

Diffeomorphic Matching of Multiple Anterior Leaflet Snapshots



Matching four snapshots of the anterior leaflet at instants 0,1,5,10

Diffeomorphic Matching of Multiple Anterior Leaflet Snapshots



Matching errors between computed deformations and snapshots

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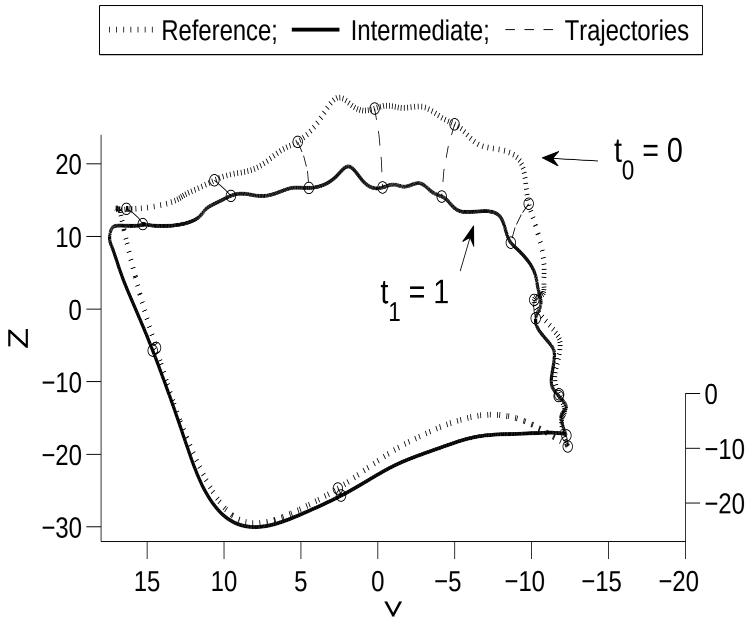
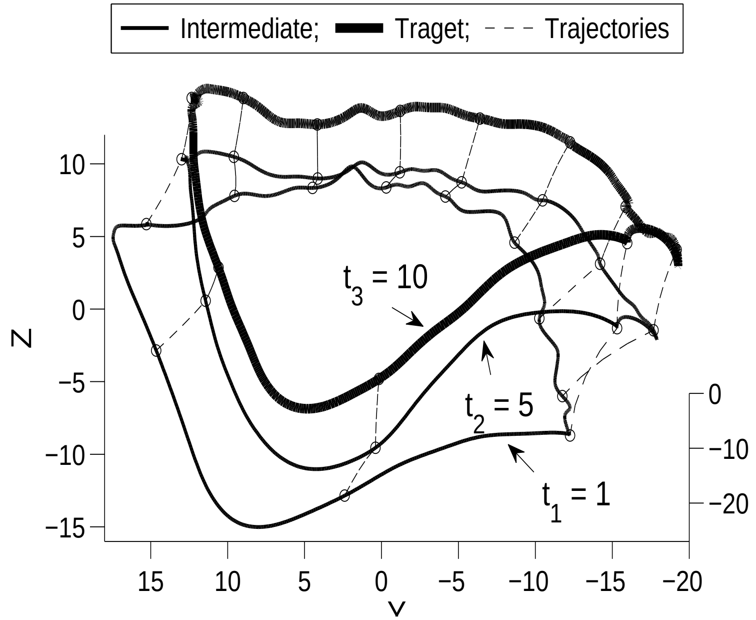
Diffeomorphic Matching of Multiple Anterior Leaflet Snapshots





Matching the anterior leaflet boundary: Instants 0,1 (l.) and 1,5,10 (r.)

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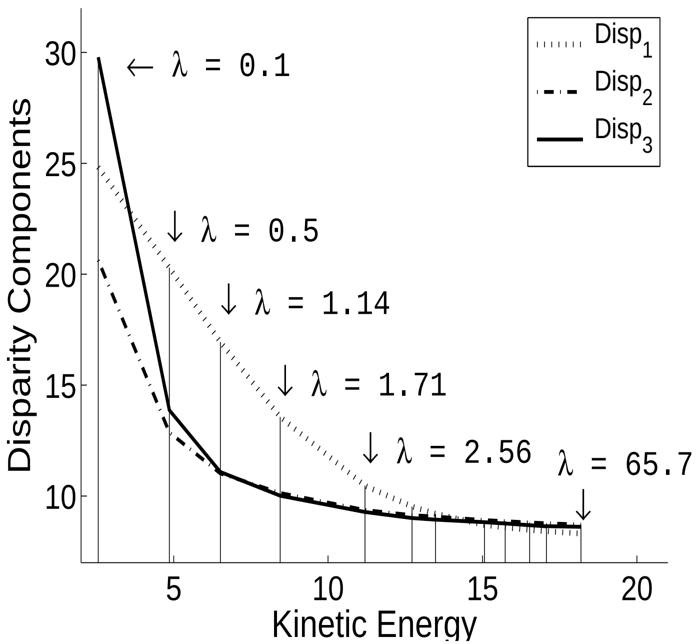
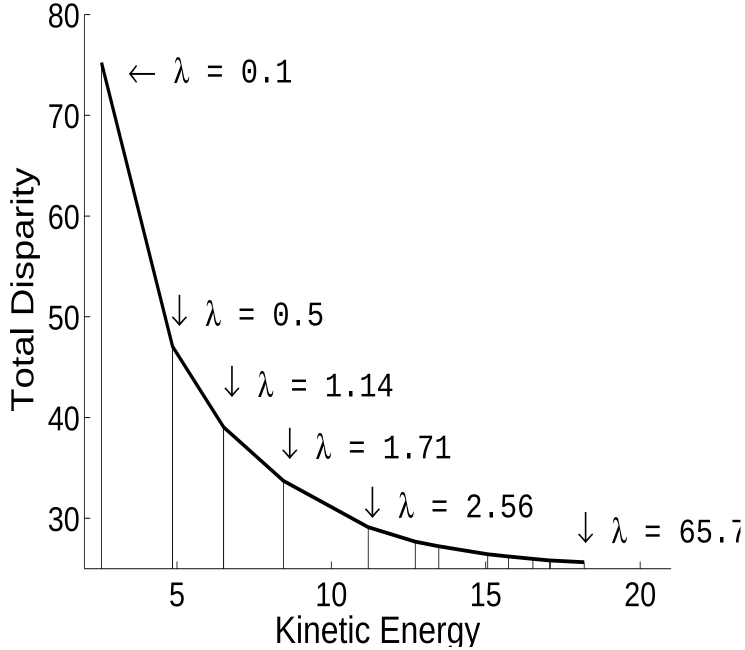
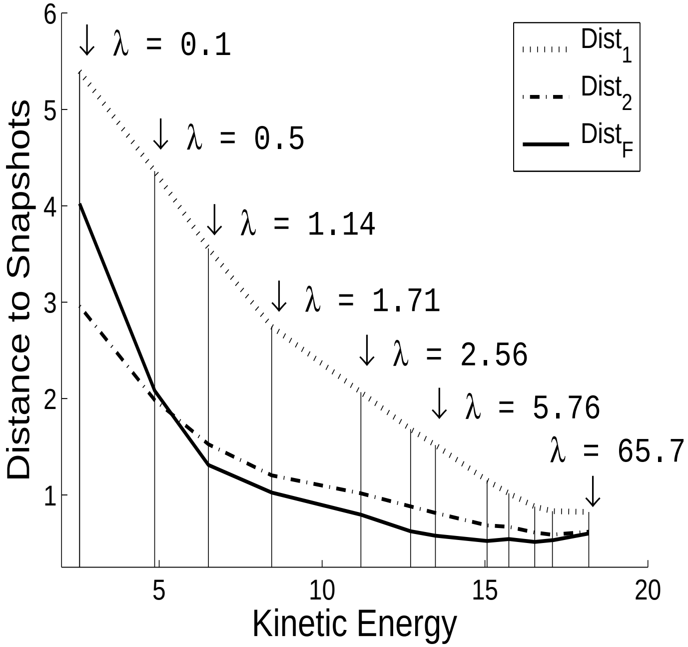
Diffeomorphic Matching of Multiple Anterior Leaflet Snapshots

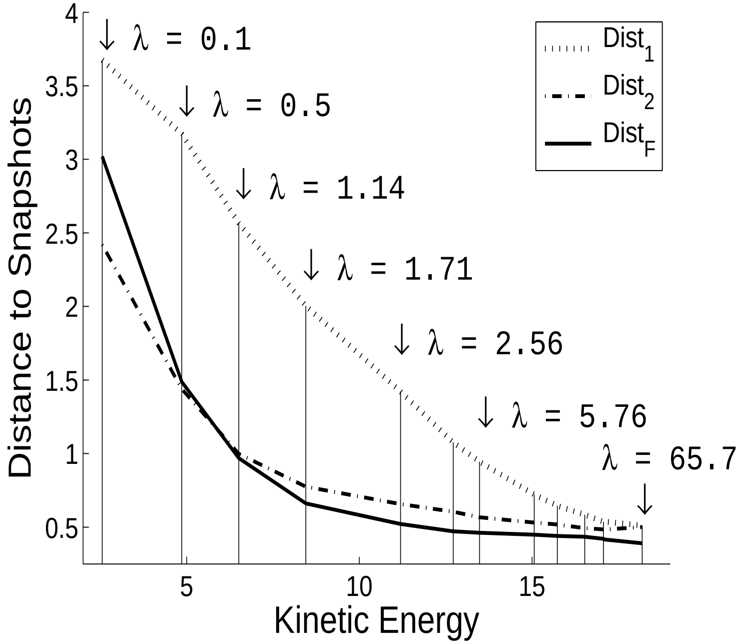




Pareto frontiers: separate Hausdorff disparities (l.), global Hausdorff disp. (r.)

Diffeomorphic Matching of Multiple Anterior Leaflet Snapshots





Pareto frontiers: max. distances to snapshots (l.), 90 % of max. dist. (r.)

Numerical Results:

Matching Posterior Leaflet Snapshots

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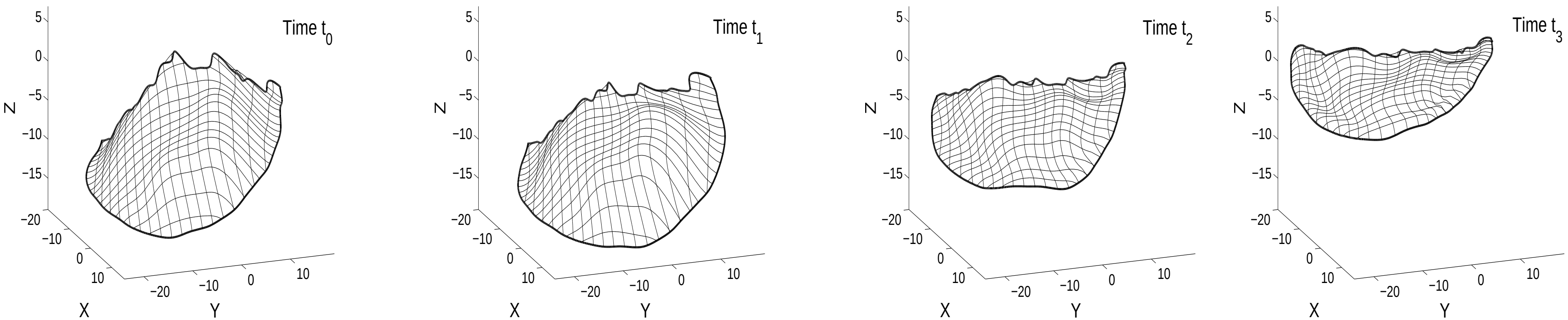
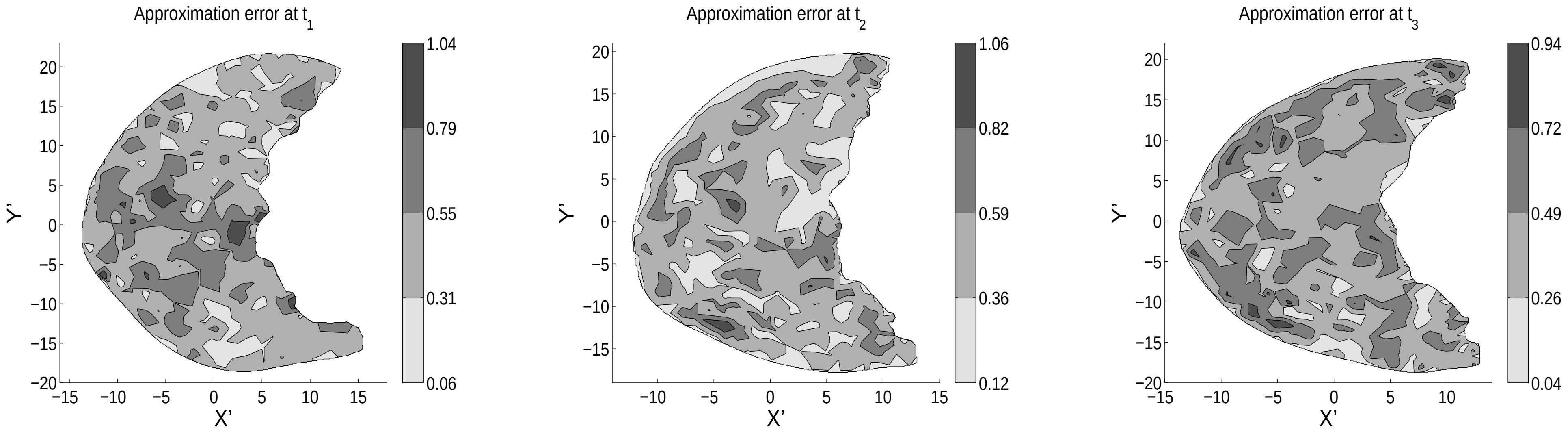
Diffeomorphic Matching of Multiple Posterior Leaflet Snapshots



Matching four snapshots of the posterior leaflet at instants 0,1,5,10

Diffeomorphic Matching of Multiple Posterior Leaflet Snapshots



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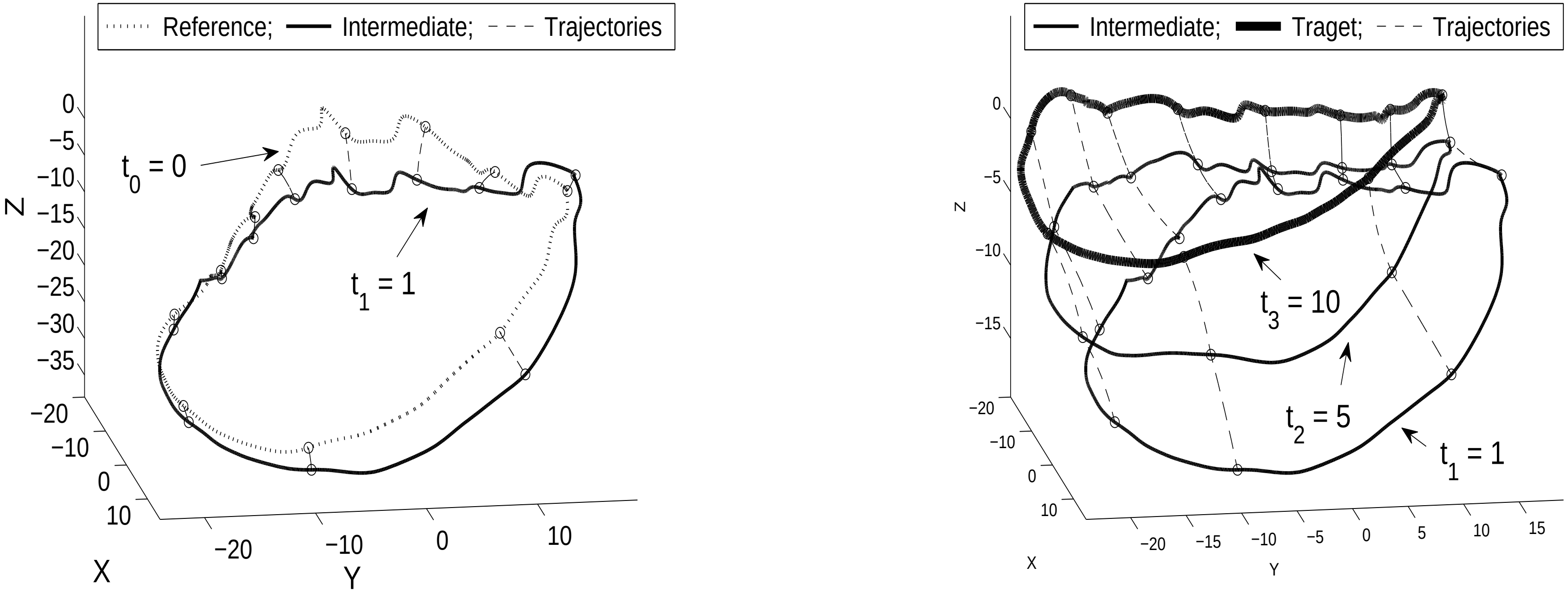
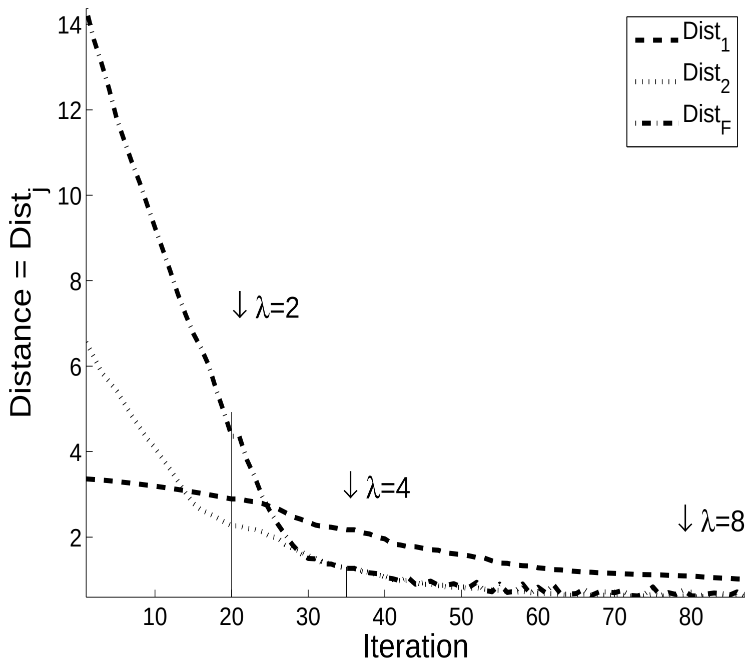
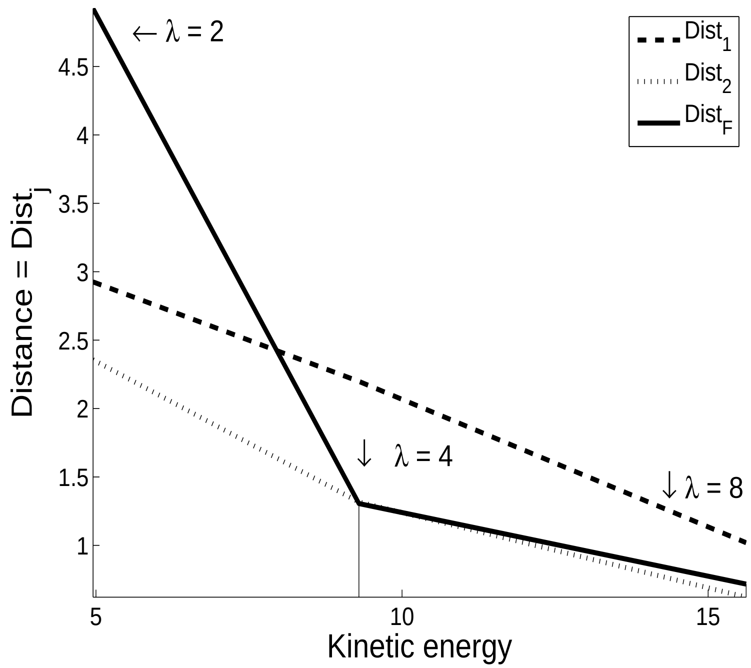
Diffeomorphic Matching of Multiple Posterior Leaflet Snapshots



Matching the posterior leaflet boundary: Instants 0,1 (l.) and 1,5,10 (r.)

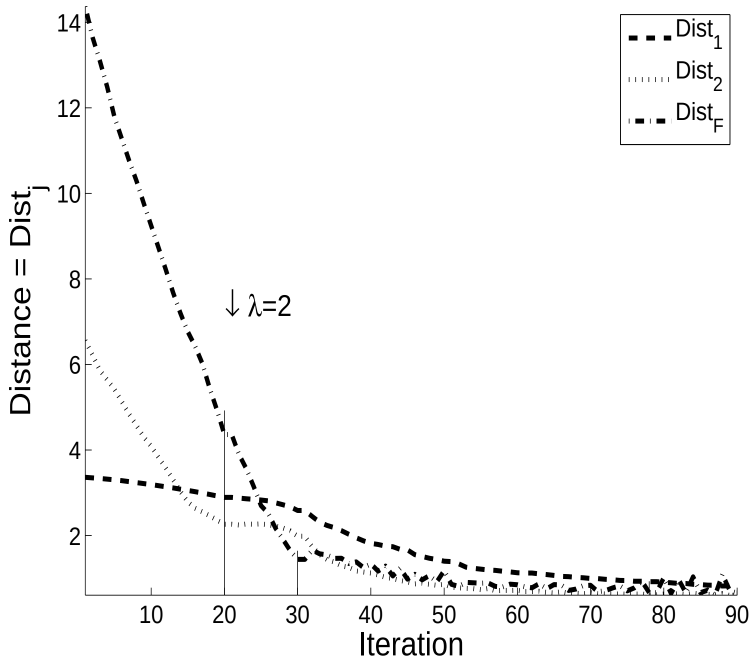
Diffeomorphic Matching of Multiple Posterior Leaflet Snapshots

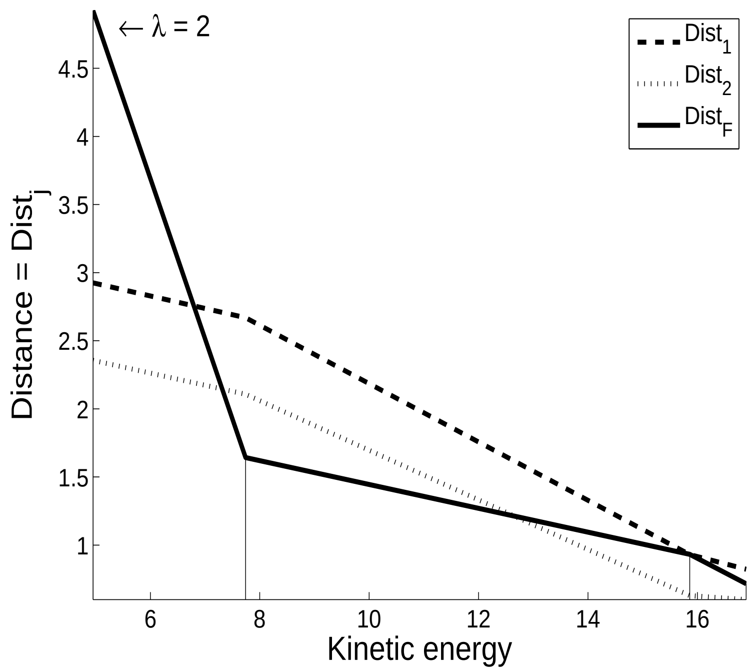




Geometric accuracy (l.), Pareto frontiers (r.) for equally chosen weights

Diffeomorphic Matching of Multiple Posterior Leaflet Snapshots





Geometric accuracy (l.), Pareto frontiers (r.) for dynamically adjusted weights