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Adaptive Finite Element Methods for Elliptic Optimal Control Problems

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Books on Adaptive Finite Element Methods

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P. Neittaanmäki and S. Repin; Reliable methods for mathematical modelling. Error control and a posteriori estimates. Elsevier, New York, 2004.

R. Verfürth; A Review of A Posteriori Estimation and Adaptive Mesh-Refinement Techniques. Wiley-Teubner, New York, Stuttgart, 1996.

Adaptive Finite Element Methods for Optimal Control Problems

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A. Schiela and A. Günther; Interior point methods in function space for state state constraints - inexact Newton and adaptivity.
ZIB-Report 09-01, Konrad-Zuse-Zentrum für Informationstechnik Berlin, 2009.

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SIAM J. Control Optim., 47, 1150-1177, 2008

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Adaptive Finite Element Methods for Control Constrained Optimal Elliptic Control Problems

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AFEM for Control Constrained Elliptic Optimal Control Problems

- Review of the a posteriori error analysis of adaptive finite element methods
- Distributed optimal control problem with control constraints
- Residual-type a posteriori error estimator
- Reliability and discrete local efficiency

The Loop in Adaptive Finite Element Methods (AFEM)

Adaptive Finite Element Methods (AFEM) consist of successive loops of the cycle

$$\textbf{SOLVE} \Longrightarrow \textbf{ESTIMATE} \Longrightarrow \textbf{MARK} \Longrightarrow \textbf{REFINE}$$

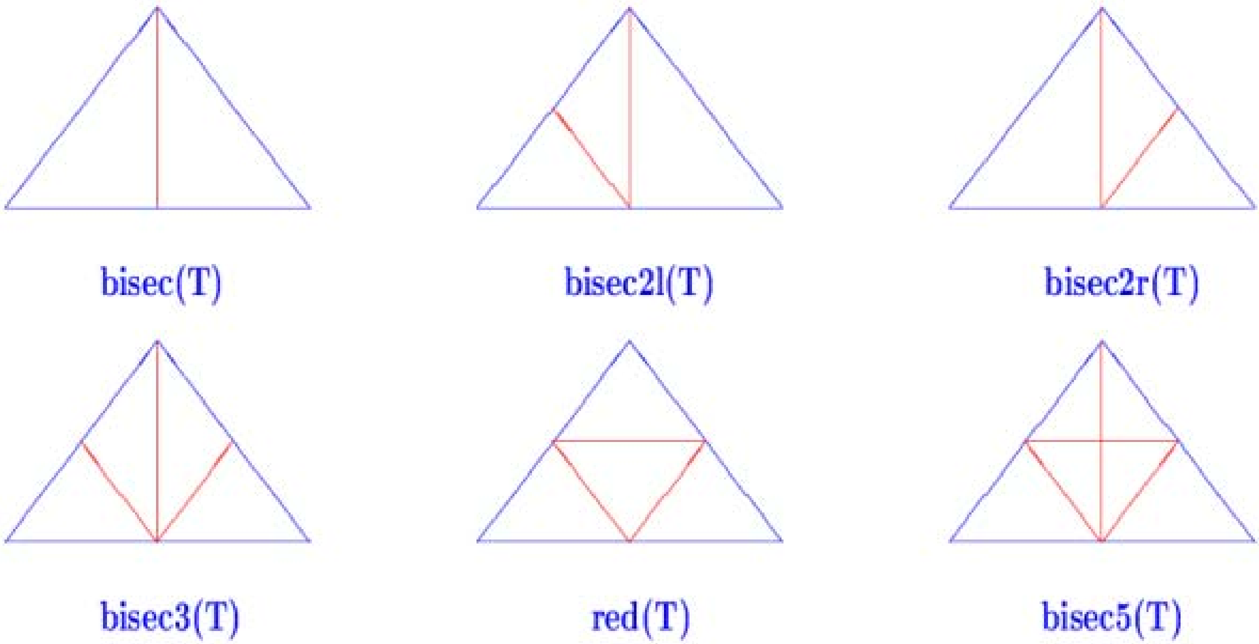
SOLVE: Multilevel iterative solvers and domain decomposition methods

ESTIMATE: Residual-type a posteriori error estimators
Hierarchical-type a posteriori error estimators
Error estimators based on local averaging
Goal oriented weighted dual approach

MARK: Heuristic strategies (max. error, averaged error)
Bulk criterion for AFEMs
[Binev/Dahmen/DeVore (2002), Carstensen/H. (2004),Dörfler (1996)]

REFINE: Bisection or 'red/green' refinement or combinations thereof

Possible Refinements of a Triangle in Step REFINE



A Posteriori Error Analysis of AFEM for Optimal Control Problems

(i) Unconstrained problems

R. Becker, H. Kapp, and R. Rannacher, Adaptive finite element methods for for optimal control of partial differential equations: Basic concepts. SIAM J. Control Optim. 39, 113–132, 2000

R. Becker, R. Rannacher, An optimal control approach to error estimation and mesh adaptation in finite element methods. Acta Num. 11, 1–101, 2001

(ii) Control constrained problems

W. Liu and N. Yan, A posteriori error estimates for distributed optimal control problems. Adv. Comp. Math. 15, 285–309, 2001

R. Li, W. Liu, H. Ma, and T. Tang, Adaptive finite element approximation for distributed elliptic optimal control problems. SIAM J. Control Optim. 41, 1321–1349, 2002

Model Problem (Distributed Elliptic Control Problem with Control Constraints)

Given a bounded domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\Gamma = \partial\Omega$, functions $f, u^d, y^d, \psi \in L^2(\Omega)$, and $\alpha > 0$, consider the distributed optimal control problem

$$\begin{array}{ll} \text{Minimize} & J(y,u) := \frac{1}{2}\,\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\,\|u - u^d\|_{0,\Omega}^2 \,, \\ \text{over} & (y,u) \in H_0^1(\Omega) \times L^2(\Omega) \,, \\ \text{subject to} & -\Delta\, y = f + u \,, \\ & u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\} \,. \end{array}$$

Optimality Conditions for the Distributed Control Problem

There exists an adjoint state $p \in H_0^1(\Omega)$ and an adjoint control $\sigma \in L^2(\Omega)$ such that the quadruple (y, p, u, σ) satisfies

$$\begin{aligned} a(y, v) &= (f + u, v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) , \\ a(p, v) &= -\,(y - y^d, v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) , \\ u &= u^d + \frac{1}{\alpha}\,(p - \sigma) , \\ \sigma &\in \partial I_K(u) , \end{aligned}$$

where $a(\cdot,\cdot)$ stands for the bilinear form

$$a(w, z) = \int_\Omega \nabla w \cdot \nabla z \, dx \quad , \quad w, z \in H_0^1(\Omega) ,$$

and $\partial I_K(u)$ denotes the subdifferential of the indicator function of the constraint set K.

Complementarity Conditions for the Optimal Control Problem

The inclusion $\sigma \in \partial I_K(u)$ can be equivalently stated as the variational inequality

$$(\sigma, u - w)_{0,\Omega} = (\alpha(u^d - u) + p, u - w)_{0,\Omega} \geq 0 \quad , \quad w \in K .$$

Consequently, σ and u satisfy the complementarity conditions

$$\sigma \in L^2_+(\Omega) \quad , \quad \psi - u \in L^2_+(\Omega) ,$$
$$(\sigma, \psi - u)_{0,\Omega} = 0 .$$

We define the active control set $\mathcal{A}(u)$ and the inactive control set $\mathcal{I}(u)$ according to

$$\mathcal{A}(u) := \max\{A \subset \Omega \mid u(x) = \psi(x) \text{ f.a.a. } x \in A\} ,$$
$$\mathcal{I}(u) := \bigcup_{\varepsilon > 0} B_\varepsilon , \; B_\varepsilon := \max\{B \subset \Omega \mid u(x) \leq \psi(x) - \varepsilon \text{ f.a.a. } x \in B\} .$$

and note that $\sigma(x) = 0$ f.a.a. $x \in \mathcal{I}(u)$.

Basic Definitions and Notations

$\mathcal{T}_H(\Omega)$: Shape regular simplicial triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$

$\mathcal{N}_H(\Omega)$: Set of interior vertices of the triangulation $\mathcal{T}_H(\Omega)$

$\mathcal{E}_H(\Omega)$: Set of interior edges of the triangulation $\mathcal{T}_H(\Omega)$

$\mathcal{N}_H(D) := \mathcal{N}_H(\Omega) \cap D \quad , \quad \mathcal{E}_H(D) := \mathcal{E}_H(\Omega) \cap D \quad , \quad D \subseteq \Omega$

$h_T := \mathrm{diam}(T)\, ,\ T \in \mathcal{T}_H(\Omega) \quad , \quad h_E :=$ length of $E \in \mathcal{E}_H(\Omega) \quad , \quad x_T :=$ center of gravity of T

$\omega_E := T_1 \cup T_2\, ,\ T_\nu \in \mathcal{T}_H(\Omega)\, ,\ 1 \leq \nu \leq 2\, ,\ E = T_1 \cap T_2 \in \mathcal{E}_H(\Omega)$

$A \lesssim B$: There exists $C \in \mathbb{R}_+$, independent of H, such that $A \leq C\,B$

$A \approx B \quad \Longleftrightarrow \quad A \lesssim B$ and $B \lesssim A$

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Finite Element Approximation of the Distributed Control Problem

Let $\mathcal{T}_H(\Omega)$ be a shape regular, simplicial triangulation of Ω and let

$$V_H := \{ v_H \in C(\Omega) \mid v_H|_T \in P_1(T) \, , \; T \in \mathcal{T}_H(\Omega) \, , \; v_H|_{\partial\Omega} = 0 \}$$

be the FE space of continuous, piecewise linear finite elements and

$$W_H := \{ w_H \in L^2(\Omega) \mid w_H|_T \in P_0(T) \, , \; T \in \mathcal{T}_H(\Omega) \}$$

the linear space of elementwise constants.
Consider the following FE Approximation of the distributed control problem

$$\begin{array}{ll} \text{Minimize} & J(y_H, u_H) := \frac{1}{2} \|y_H - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2} \|u_H - u^d\|_{0,\Omega}^2 \, , \\ \text{over} & (y_H, u_H) \in V_H \times W_H \, , \\ \text{subject to} & a(y_H, v_H) = (f + u_H, v_H)_{0,\Omega} \, , \; v_H \in V_H \, , \\ & u_H \in K_H := \{ w_H \in W_H \mid w_H|_T \leq \psi_H|_T \, , \; T \in \mathcal{T}_H(\Omega) \} \, . \end{array}$$

where $\psi_H \in W_H$ with $\psi_H|_T \leq \psi$ a.e. on $T \in \mathcal{T}_H(\Omega)$.

Optimality Conditions for the FE Discretized Control Problem

There exists an adjoint state $p_H \in V_H$ and an adjoint control $\sigma_H \in W_H$ such that the quadruple $(y_H, p_H, u_H, \sigma_H)$ satisfies

$$\begin{aligned} a(y_H, v_H) &= (f + u_H, v_H)_{0,\Omega} \quad , \quad v_H \in V_H \, , \\ a(p_H, v_H) &= -\, (y_H - y^d, v_H)_{0,\Omega} \quad , \quad v_H \in V_H \, , \\ u_H &= u_H^d + \frac{1}{\alpha} (M_H p_H - \sigma_H) \, , \\ \sigma_H &\in \partial I_{K_H}(u_H) \, , \end{aligned}$$

where $u_H^d \in W_H$, $u_H^d|_T := u_T^d$, $T \in \mathcal{T}_H(\Omega)$ and the operator $M_H : V_H \to W_H$ is given by

$$(M_H v_H)|_T := |T|^{-1} \int_T v_H \, dx \quad , \quad T \in \mathcal{T}_H(\Omega) \, .$$

Complementarity for the FE Discretized Control Problem

We define the discrete active control set $\mathcal{A}(\mathbf{u}_H)$ and the discrete inactive control set $\mathcal{I}(\mathbf{u}_H)$ by

$$\mathcal{A}(\mathbf{u}_H) := \bigcup \{ T \in \mathcal{T}_H(\Omega) \mid \mathbf{u}_H|_T = \psi_H|_T \} ,$$

$$\mathcal{I}(\mathbf{u}_H) := \bigcup \{ T \in \mathcal{T}_H(\Omega) \mid \mathbf{u}_H|_T < \psi_H|_T \} .$$

Then, the discrete complementarity conditions can be written as

$$\begin{aligned}
\boldsymbol{\sigma}_H|_T &\geq \mathbf{0} \quad , \quad T \in \mathcal{T}_H(\Omega) , \\
\boldsymbol{\sigma}_H|_T &= \mathbf{0} \quad , \quad T \in \mathcal{I}(\mathbf{u}_H) , \\
\boldsymbol{\sigma}_H|_T &= \alpha \, (\mathbf{u}_H^d - \psi_H)|_T + (\mathbf{M}_H \mathbf{p}_H)|_T \quad , \quad T \in \mathcal{A}(\mathbf{u}_H) .
\end{aligned}$$

The A Posteriori Error Estimator

The A Posteriori Error Estimator

The a posteriori error estimator for the global discretization error in the state, adjoint state, control and the adjoint control consists of

- element and edge residuals with respect to the equations for the state and the adjoint state,
- a complementarity consistency error.

The error analysis further involves

- data oscillations with respect to the data f , u^d , y^d and ψ of the problem.

Element and Edge Residuals for the State and the Adjoint State

(i) Element and edge residuals for the state $\mathbf{y}$

$$\eta_y := \Big(\sum_{T \in \mathcal{T}_H(\Omega)} \eta_{y,T}^2 + \sum_{E \in \mathcal{E}_H(\Omega)} \eta_{y,E}^2 \Big)^{1/2}$$

$$\eta_{y,T} := \underbrace{h_T \, \|f + u_H\|_{0,T}}_{\text{element residuals}} \, , \; T \in \mathcal{T}_H(\Omega) \quad , \quad \eta_{y,E} := \underbrace{h_E^{1/2} \, \|\nu_E \cdot [\nabla y_H]\|_{0,E}}_{\text{edge residuals}} \, , \; E \in \mathcal{E}_H(\Omega)$$

(ii) Element and edge residuals for the adjoint state $\mathbf{p}$

$$\eta_p := \Big(\sum_{T \in \mathcal{T}_H(\Omega)} \big((\eta_{p,T}^{(1)})^2 + (\eta_{p,T}^{(2)})^2 \big) + \sum_{E \in \mathcal{E}_H(\Omega)} \eta_{p,E}^2 \Big)^{1/2}$$

$$\eta_{p,T}^{(1)} := \underbrace{h_T \, \|y^d - y_H\|_{0,T}}_{\text{element residuals}}$$

$$, \; T \in \mathcal{T}_H(\Omega) \quad , \quad \eta_{p,E} := \underbrace{h_E^{1/2} \, \|\nu_E \cdot [\nabla p_H]\|_{0,E}}_{\text{edge residuals}} \, , \; E \in \mathcal{E}_H(\Omega)$$

$$\eta_{p,T}^{(2)} := \|M_H p_H - p_H\|_{0,T}$$

Complementarity Consistency Error

$$\eta_{ce} := \left(\sum_{T \in \mathcal{A}(u_H)} (\sigma_H, \psi - \psi_H)_{0,T} \right)^{1/2}$$

- measures the deviation of the discrete control constraint set K_H from its continuous counterpart K on $\mathrm{supp}(\sigma_H) = \mathcal{A}(u_H)$,
- vanishes in case of a constant obstacle ψ.

Data Oscillations in the Data $\mathbf{f}$, $\mathbf{u}^d$, $\mathbf{y}^d$, ψ

(i) Low order data oscillations in u^d and ψ

$$\mu_H(u^d) := \Big(\sum_{T \in \mathcal{T}_H(\Omega)} \mu_T(u^d)^2 \Big)^{1/2} \quad , \quad \mu_H(\psi) := \Big(\sum_{T \in \mathcal{T}_H(\Omega)} \mu_T(\psi)^2 \Big)^{1/2}$$

$$\mu_T(u^d) := \|u^d - u^d_H\|_{0,T} \quad , \quad \mu_T(\psi) := \|\psi - \psi_H\|_{0,T} \quad , \quad T \in \mathcal{T}_H(\Omega)$$

(ii) Higher order data oscillations in f and y^d

$$\mathrm{osc}_H(f) := \Big(\sum_{T \in \mathcal{T}_H(\Omega)} \mathrm{osc}_T(f)^2 \Big)^{1/2} \quad , \quad \mathrm{osc}_H(y^d) := \Big(\sum_{T \in \mathcal{T}_H(\Omega)} \mathrm{osc}_T(y^d)^2 \Big)^{1/2}$$

$$\mathrm{osc}_T(f) := h_T \, \|f - f_H\|_{0,T} \quad , \quad \mathrm{osc}_T(y^d) := h_T \, \|y^d - y^d_H\|_{0,T} \quad , \quad T \in \mathcal{T}_H(\Omega)$$

Reliability of the Error Estimator

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Reliability of the A Posteriori Error Estimator

Theorem Let (y, p, u, σ) be the solution of the distributed control problem and $(y_H, p_H, u_H, \sigma_H)$ be the finite element approximation with respect to the triangulation $\mathcal{T}_H(\Omega)$. Further, let η be the residual type error estimator and $\mu_H(u^d)$ the data oscillations in u^d.
Then, there exists a positive constant C, depending only on α , Ω and on the shape regularity of the triangulation $\mathcal{T}_H(\Omega)$ such that

$$\alpha^{1/2}\,|y-y_H|_{1,\Omega} + |p-p_H|_{1,\Omega} + \alpha\,\|u-u_H\|_{0,\Omega} + \|\sigma-\sigma_H\|_{0,\Omega} \le C\left(\eta + \mu_H(u^d)\right).$$

Proof of the reliability of the a posteriori error estimator

Since standard Galerkin orthogonality does not apply, we introduce the

- auxiliary state $y(u_H) \in H_0^1(\Omega)$ satisfying

$$a(y(u_H, v) = (f + u_H, v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \, ,$$

- auxiliary adjoint state $p(u_H) \in H_0^1(\Omega)$ satisfying

$$a(p(u_H, v) = (y^d - y(u_H), v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \, .$$

Then, the partial errors

$$y(u_H) - y_H \quad , \quad p(u_H) - p_H$$

can be estimated by standard techniques giving rise to the element and edge residuals $\eta_{y,T}$, $\eta_{y,E}$ and $\eta_{p,T}$, $\eta_{p,E}$

Important Tool in the Error Analysis: Intermediate State and Adjoint State

Given a discrete control $u_H \in W_H$, the intermediate state $y(u_H) \in H_0^1(\Omega)$ and the intermediate adjoint state $p(u_H) \in H_0^1(\Omega)$ are the unique solutions of the variational equations

$$
\begin{aligned}
a(y(u_H), v) &= (f + u_H, v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) , \\
a(p(u_H), v) &= -\, (y(u_H) - y^d, v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) .
\end{aligned}
$$

Lemma. Let $y(u_H)$ and $p(u_H)$ be the intermediate state and adjoint state. Then, we have

$$
(p - p(u_H), u - u_H)_{0,\Omega} \;=\; -\, \|y - y(u_H)\|_{0,\Omega}^2 \;\leq\; 0 .
$$

Proof: Obviously, there holds

$$
a(y - y(u_H), v_1) \;=\; (u - u_H, v_1)_{0,\Omega} \quad , \quad a(p - p(u_H), v_2) \;=\; (y(u_H) - y, v_2)_{0,\Omega} \quad , \quad v_1, v_2 \in H_0^1(\Omega) .
$$

The assertion follows readily by choosing $v_1 := p - p(u_H)$ and $v_2 := y - y(u_H)$.

Reliability of the A Posteriori Error Estimator

Lemma. Under the same assumptions as in the previous theorem there holds

$$\alpha|y-y_H|^2_{1,\Omega} + |p-p_H|^2_{1,\Omega} + \alpha^2\|u-u_H\|^2_{0,\Omega} + \|\sigma-\sigma_H\|^2_{0,\Omega} \lesssim$$

$$\lesssim |y_H-y(u_H)|^2_{1,\Omega} + |p_H-p(u_H)|^2_{1,\Omega} + \|M_H p_H - p_H\|^2_{0,\Omega} + (\sigma_H, \psi-\psi_H)_{0,\Omega} + \mu_H^2(u^d)\,.$$

Proof: In view of the equations for the adjoint controls σ and σ_H and using Young's inequality, we find

$$\alpha^2\,\|u-u_H\|^2_{0,\Omega} = \alpha\,(\sigma_H-\sigma, u-u_H)_{0,\Omega} + \alpha\,(p-p_H, u-u_H)_{0,\Omega} +$$
$$+\,\alpha\,(p_H - M_H p_H, u-u_H)_{0,\Omega} + \alpha^2(u_d - u^d_H, u-u_H)_{0,\Omega} \le \alpha\,(\sigma_H-\sigma, u-u_H)_{0,\Omega} +$$
$$+\,\alpha\,(p-p_H, u-u_H)_{0,\Omega} + \frac{\alpha^2}{4}\|u-u_h\|^2_{0,\Omega} + \frac{2}{\alpha^2}\|p_H - M_H p_H\|^2_{0,\Omega} + \frac{2}{\alpha^2}\mu_H(u^d)\,.$$

Using the complementarity conditions for $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_H$, we find

$$(\sigma_H - \sigma, u - u_H)_{0,\Omega} = \underbrace{(\sigma_H, u - \psi)_{0,\Omega}}_{\leq 0} + (\sigma_H, \psi - \psi_H)_{0,\Omega} + \underbrace{(\sigma_H, \psi_H - u_H)_{0,\Omega}}_{= 0} -$$

$$- \underbrace{(\sigma, u - \psi)_{0,\Omega}}_{= 0} - \underbrace{(\sigma, \psi - \psi_H)_{0,\Omega}}_{\geq 0} - \underbrace{(\sigma, \psi_H - u_H)_{0,\Omega}}_{\geq 0} \leq (\sigma_H, \psi - \psi_H)_{0,\Omega} .$$

Moreover, for the remaining term there holds

$$\alpha\, (p - p_H, u - u_H)_{0,\Omega} \leq \underbrace{\alpha\, (p - p(u_H), u - u_H)_{0,\Omega}}_{\leq 0} + \alpha\, (p(u_H) - p_H, u - u_H)_{0,\Omega} ,$$

whence by another application of Young's inequality

$$\alpha\, (p - p_H, u - u_H)_{0,\Omega} \leq \frac{\alpha^2}{4} \|u - u_H\|_{0,\Omega}^2 + \frac{c(\Omega)^2}{\alpha^2} |p(u_H) - p_H|_{1,\Omega}^2 .$$

Discrete Local Efficiency

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Discrete Local Efficiency of the Error Estimator

Theorem Suppose that $T \in \mathcal{T}_H(\Omega)$ is a refined element and that $(y_H, p_H, u_H, \boldsymbol{\sigma}_H)$ and $(y_h, p_h, u_h, \boldsymbol{\sigma}_h)$ are the finite element approximations w.r.t. the triangulations $\mathcal{T}_H(\Omega)$ and $\mathcal{T}_h(\Omega)$. Then there holds

$$\eta_{y,T}^2 \lesssim |y_H - y_h|_{1,T}^2 + h_T^2 \|u_H - u_h\|_{0,T}^2 + \operatorname{osc}_T^2(f) ,$$

$$(\eta_{p,T}^{(1)})^2 \lesssim |p_H - p_h|_{1,T}^2 + |y_H - y_h|_{1,T}^2 + \operatorname{osc}_T^2(y^d) .$$

Proof: Let $\varphi_h^a \in V_h$ be a nodal basis function associated with an interior point $a \in \mathcal{N}_H(T)$.
Then, with $z_h := (f_T + u_H)\varphi_h^a$ there holds

$$(*) \qquad \eta_{y,T}^2 \lesssim h_T^2 (f+u_h, z_h)_{0,T} + \Big(h_T^2 \|u_H - u_h\|_{0,T} + \operatorname{osc}_T(y^d) \Big) \|z_h\|_{0,T} + \operatorname{osc}_T^2(f) .$$

Since z_h is an admissible test function, we have $a|_T(y_h, z_h) = (f + u_h, z_h)_{0,T}$.
Moreover, Green's formula reveals $a|_T(y_H, z_h) = 0$, and hence

$$h_T^2 (f + u_h, z_h)_{0,T} = h_T^2 \, a(y_h - y_H, z_h) \leq h_T^2 \, |y_h - y_H|_{1,T} |z_h|_{1,T} .$$

Inserting into $(*)$ and using Young's inequality, gives the assertion. □

Discrete Local Efficiency of the Error Estimator

Theorem Suppose that $T \in \mathcal{T}_H(\Omega)$ is a refined element and that $(y_H, p_H, u_H, \boldsymbol{\sigma}_H)$ and $(y_h, p_h, u_h, \boldsymbol{\sigma}_h)$ are the finite element approximations w.r.t. the triangulations $\mathcal{T}_H(\Omega)$ and $\mathcal{T}_h(\Omega)$. Then there holds

$$\|M_H p_H - p_H\|_{0,T} \lesssim |p_H - p_h|_{1,T} + \alpha\, \|u_H - u_h\|_{0,T} + \|\boldsymbol{\sigma}_H - \boldsymbol{\sigma}_h\|_{0,T} + \alpha\, \|u_H^d - u_h^d\|_{0,T}\,.$$

Proof: The result follows readily by means of

$$M_H p_H - M_h p_h \;=\; \alpha\,(u_h - u_H) \;+\; \alpha\,(u_H^d - u_h^d) \;+\; \boldsymbol{\sigma}_H - \boldsymbol{\sigma}_h$$

and

$$\|M_h p_H - p_H\|_{0,T} \;\le\; \rho\, \|M_H p_H - p_H\|_{0,T}\,,$$

where $0 < \rho < 1$ only depends on the shape regularity of the triangulation. $\square$

Discrete Local Efficiency of the Error Estimator

Theorem Suppose that $E \in \mathcal{E}_H(\Omega)$ is a refined edge and that $(y_H, p_H, u_H, \sigma_H)$ and $(y_h, p_h, u_h, \sigma_h)$ are the finite element approximations w.r.t. the triangulations $\mathcal{T}_H(\Omega)$ and $\mathcal{T}_h(\Omega)$. Let further $\omega_E := T_1 \cup T_2$, where $T_\nu \in \mathcal{E}_H(\Omega), 1 \le \nu \le 2$, such that $E = T_1 \cap T_2$. Then there holds

$$\eta_{y,E}^2 \lesssim |y_H - y_h|_{1,\omega_E}^2 + h_T^2 \|u_H - u_h\|_{0,\omega_E}^2 + \eta_{y,\omega}^2 ,$$

$$\eta_{p,E}^2 \lesssim |p_H - p_h|_{1,\omega_E}^2 + |y_H - y_h|_{1,\omega_E}^2 + \eta_{p,E}^2 .$$

Proof: Let $\varphi_h^{\mathrm{mid}_E} \in V_h$ be the nodal basis function associated with $\mathrm{mid}(E) \in \mathcal{N}_h(\Omega)$.
Then, the function $z_h := [\nu_E \cdot \nabla y_H]\varphi_h^{\mathrm{mid}_E}$ satisfies

$$\|[\nu_E \cdot \nabla y_H]\|_{0,E}^2 \lesssim ([\nu_E \cdot \nabla y_H], z_h)_{0,T} \quad , \quad \|z_h\|_{0,E} \lesssim h_E^{1/2} \|[\nu_E \cdot \nabla y_H]\|_{0,\omega_E} \quad , \quad |z_h|_{1,T} \lesssim h_E^{-1/2} \|[\nu_E \cdot \nabla y_H]\|_{0,\omega_E} .$$

Since z_h is an admissible test function, we have $a|_{\omega_E}(y_h, z_h) = (f + u_h, z_h)_{0,\omega_E}$, and hence

$$\eta_{y,E}^2 = h_E \|[\nu_E \cdot \nabla y_H]\|_{0,E}^2 \lesssim h_E([\nu_E \cdot \nabla y_H], z_h)_{0,E} = h_E \Big(a|_{\omega_E}(y_H - y_h, z_h) + (u_H - u_h, z_h)_{0,\omega_E} + (f + u_H, z_h)_{0,\omega_E} \Big)$$

$$\lesssim h_E^{1/2} \|[\nu_E \cdot \nabla y_H]\|_{0,E} \Big(|y_H - y_h|_{1,\omega_E} + h_T \|u_H - u_h\|_{0,\omega_E} + \eta_{y,\omega_E} \Big) .$$

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Discrete Local Efficiency of the Error Estimator

Theorem Let $(y_H, p_H, u_H, \sigma_H)$ and $(y_h, p_h, u_h, \sigma_h)$ be the finite element approximations with respect to the triangulations $\mathcal{T}_H(\Omega)$ and $\mathcal{T}_h(\Omega)$, respectively. Then, for $T \subset \mathcal{A}_H(u_H)$ there holds

$$(\boldsymbol{\sigma}_H, \psi - \psi_H)_{0,T} \lesssim \left(|p_H - p_h|^2_{1,T} + \|u_H - u_h\|^2_{0,T} + \|\boldsymbol{\sigma}_H - \boldsymbol{\sigma}_h\|^2_{0,T} \right) + \boldsymbol{\mu}^2_T(u^d) + \boldsymbol{\mu}_T(\psi) .$$

Proof: Since $\psi_H = u_H$ for $T \subset \mathcal{A}_H(u_H)$, it follows that

$$(\boldsymbol{\sigma}_H, \psi - \psi_H)_{0,T} = (\boldsymbol{\sigma}_H, \psi - u_H)_{0,T} = (\boldsymbol{\sigma}_H, \psi - u_h)_{0,T} + (\boldsymbol{\sigma}_H, u_h - u_H)_{0,T} .$$

Using $(\boldsymbol{\sigma}_h, u_h - \psi_h)_{0,T} = 0$, for the first term on the right-hand side we obtain

$$(\boldsymbol{\sigma}_H, \psi - u_h)_{0,T} = (\boldsymbol{\sigma}_H, \psi - \psi_H)_{0,T} + (\boldsymbol{\sigma}_H, u_h - u_H)_{0,T} .$$

$$(\boldsymbol{\sigma}_H, u_h - u_H)_{0,T} = \alpha\, (u^d_H - u^d_h, u_h - u_H)_{0,T} + (\boldsymbol{\sigma}_h, u_h - u_H)_{0,T} + (M_H p_H - M_h p_h, u_h - u_H)_{0,T} + \alpha\, \|u_h - u_H\|^2_{0,T} .$$

On the other hand, for the second term we have

$$(\boldsymbol{\sigma}_H, \psi - u_h)_{0,T} = \underbrace{(\boldsymbol{\sigma}_H, \psi - u_H)_{0,T}}_{= (\boldsymbol{\sigma}_H, \psi - \psi_H)_{0,T}} - (\boldsymbol{\sigma}_H, u_h - u_H)_{0,T} .$$

Numerical Example: Distributed Control Problem with Control Constraints)

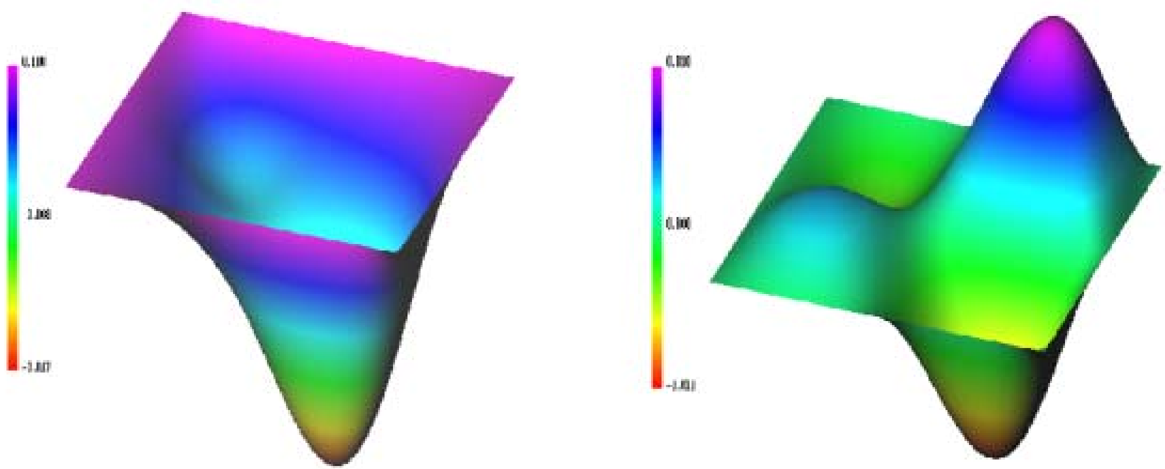
$$
\begin{aligned}
&\text{Minimize} \quad && J(y,u) := \frac{1}{2}\,\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\,\|u - u^d\|_{0,\Omega}^2 ,\\
&\text{over} && (y,u) \in H_0^1(\Omega) \times L^2(\Omega) ,\\
&\text{subject to} && -\Delta\, y = f + u ,\\
& && u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\} .
\end{aligned}
$$

Data:

$$y^d := \sin(2\pi x_1)\,\sin(2\pi x_2)\,\exp(2x_1)/6 \quad ,$$

$$\alpha := 0.01 \quad , \quad u^d := 0 \quad , \quad \psi := 0 \quad , \quad f := 0 \quad .$$

Numerical Results: Distributed Control Problem with Control Constraints I



Optimal state (left) and optimal adjoint state (right)

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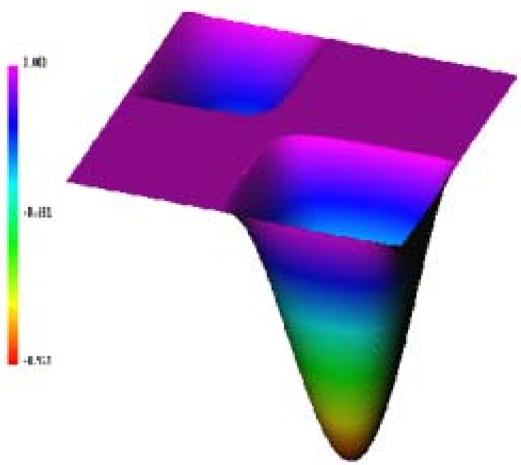
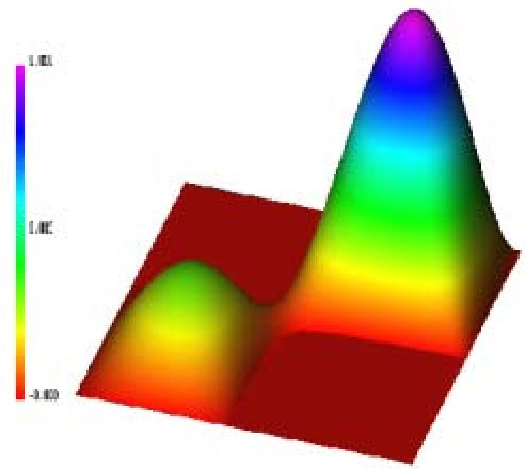
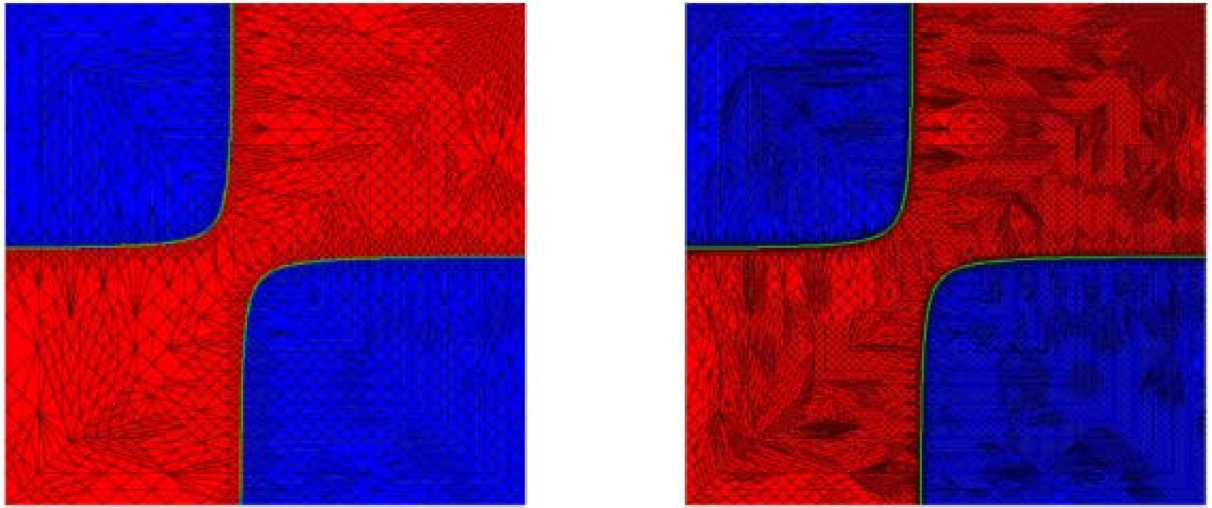
Numerical Results: Distributed Control Problem with Control Constraints I





Optimal control (left) and optimal adjoint control (right)

Numerical Results: Distributed Control Problem with Control Constraints I



Grid after 6 (left) and 8 (right) refinement steps ($\Theta_i = 0.6, 1 \leq i \leq 5$)

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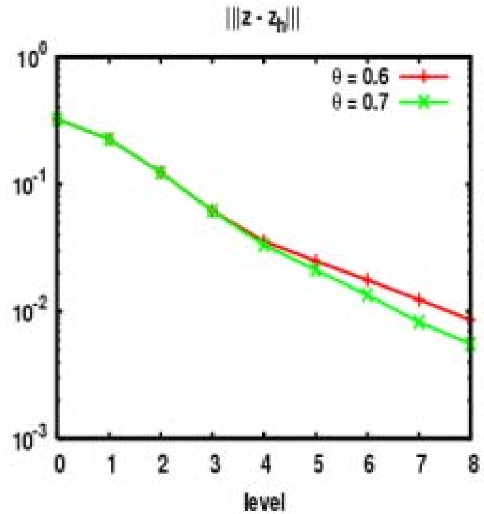
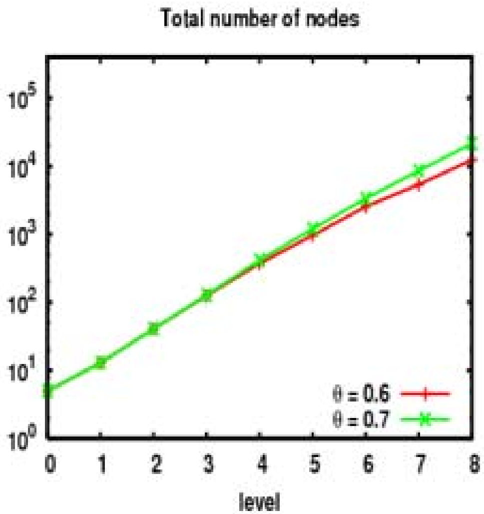
Numerical Results: Distributed Control Problem with Control Constraints I





Error reduction in $|||z - z_H|||$ and total number of nodes ($\Theta_i = 0.6/0.7$)

Numerical Results: Distributed Control Problem with Control Constraints I

l	N_{dof}	$\|\|\|z - z_H\|\|\|$	$\|y - y_H\|_1$	$\|p - p_H\|_1$	$\|\|u - u_H\|\|_0$	$\|\|\sigma - \sigma_H\|\|_0$
0	5	3.24e-01	3.63e-02	3.28e-02	2.52e-01	2.80e-03
1	13	2.27e-01	1.95e-02	1.48e-02	1.91e-01	2.11e-03
2	41	1.24e-01	1.35e-02	1.36e-02	9.59e-02	1.06e-03
3	126	6.19e-02	6.85e-03	7.86e-03	4.68e-02	5.09e-04
4	374	3.57e-02	3.93e-03	4.41e-03	2.65e-02	3.67e-04
5	968	2.50e-02	2.63e-03	2.75e-03	1.88e-02	2.50e-04
6	2553	1.77e-02	1.91e-03	2.32e-03	1.33e-02	1.56e-04
7	5396	1.24e-02	1.30e-03	1.66e-03	9.33e-03	1.16e-04
8	12318	8.60e-03	9.21e-04	1.16e-03	6.45e-03	7.48e-05

Total error, errors in the state, adjoint state, control, adjoint control ($\Theta_i = 0.7$)

Numerical Results: Distributed Control Problem with Control Constraints I

l	N_{dof}	η_y	η_p	$osc_h(y^d)$
0	5	2.57e-01	4.16e-01	2.83e-01
1	13	1.04e-01	2.04e-01	1.12e-01
2	41	7.95e-02	1.09e-01	2.58e-02
3	126	5.16e-02	6.49e-02	7.12e-03
4	374	3.15e-02	4.10e-02	2.77e-03
5	968	2.13e-02	2.79e-02	1.22e-03
6	2553	1.56e-02	1.92e-02	4.58e-04
7	5396	1.06e-02	1.33e-02	1.87e-04
8	12318	7.56e-03	9.45e-03	8.48e-05

Components of the error estimator and data oscillations ($\Theta_i = 0.7$)

Numerical Results: Distributed Control Problem with Control Constraints I

l	N_{dof}	$\eta_{y,T}$	$\eta_{y,E}$	$\eta_{p,T}^{(1)}$	$\eta_{p,T}^{(2)}$	$\eta_{p,E}$
0	5	1.22e-01	6.11e-02	1.36e-01	8.65e-04	6.11e-02
1	13	9.72e-03	6.07e-03	3.27e-02	5.33e-04	1.17e-02
2	41	3.15e-03	2.15e-03	7.16e-03	1.31e-04	3.04e-03
3	126	1.07e-03	8.33e-04	2.01e-03	3.89e-05	1.10e-03
4	374	3.69e-04	2.95e-04	6.58e-04	1.33e-05	3.86e-04
5	968	1.44e-04	1.19e-04	2.52e-04	5.22e-06	1.59e-04
6	2553	6.12e-05	5.42e-05	1.06e-04	2.21e-06	7.30e-05
7	5396	2.71e-05	2.55e-05	4.70e-05	9.82e-07	3.45e-05
8	12318	1.21e-05	1.22e-05	2.09e-05	4.36e-07	1.68e-05

Average values of the local estimators ($\Theta_i = 0.7$)

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Numerical Results: Distributed Control Problem with Control Constraints I

l	N_{dof}	$osc(y^d)$	$M_{fb,T}$	$M_{\eta,E}$	$M_{\eta,T}$	$M_{osc,T}$
0	5	1.33e-01	0.0	75.0	0.0	0.0
1	13	2.53e-02	68.8	0.0	0.0	0.0
2	41	2.83e-03	42.2	1.1	6.2	0.0
3	126	4.47e-04	25.9	5.9	12.1	0.0
4	374	9.47e-05	16.3	11.4	10.3	0.0
5	968	2.53e-05	12.0	13.1	5.4	0.1
6	2553	7.18e-06	9.9	12.5	7.2	0.0
7	5396	2.28e-06	8.6	12.3	9.3	0.0
8	12318	6.90e-07	7.5	11.6	8.4	0.0

Average values of the data oscillations, bulk criterion ($\Theta_i = 0.7$)

Numerical Results: Distributed Control Problem with Control Constraints II

$$\begin{aligned} &\text{Minimize} && J(y,u) := \frac{1}{2}\,\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\,\|u - u^d\|_{0,\Omega}^2 \\ &\text{over} && (y,u) \in H_0^1(\Omega) \times L^2(\Omega) \\ &\text{subject to} && -\Delta y = f + u \quad \text{in } \Omega\,, \\ & && u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\} \end{aligned}$$

Data: $\Omega := (0,1)^2 \quad , \quad y^d := 0 \quad , \quad u^d := \hat{u} + \alpha^{-1}(\hat{\sigma} + \Delta^{-2}\hat{u})\,,$

$$\psi := \begin{cases} (x_1 - 0.5)^8 & , \quad (x_1,x_2) \in \Omega_1, \\ (x_1 - 0.5)^2 & , \quad \text{otherwise} \end{cases} \quad , \quad \alpha := 0.1 \quad , \quad f := 0$$

$$\hat{u} := \begin{cases} \psi & , \quad (x_1,x_2) \in \Omega_1 \cup \Omega_2, \\ -1.01\,\psi & , \quad \text{otherwise} \end{cases} \quad , \quad \hat{\sigma} := \begin{cases} 2.25\,(x_1 - 0.75) \cdot 10^{-4} & , \quad (x_1,x_2) \in \Omega_2, \\ 0 & , \quad \text{otherwise} \end{cases} \quad ,$$

$$\Omega_1 := \{(x_1,x_2) \in \Omega \mid \Big((x_1 - 0.5)^2 + (x_2 - 0.5)^2\Big)^{1/2} \le 0.15\} \quad , \quad \Omega_2 := \{(x_1,x_2) \in \Omega \mid x_1 \ge 0.75\}\,.$$

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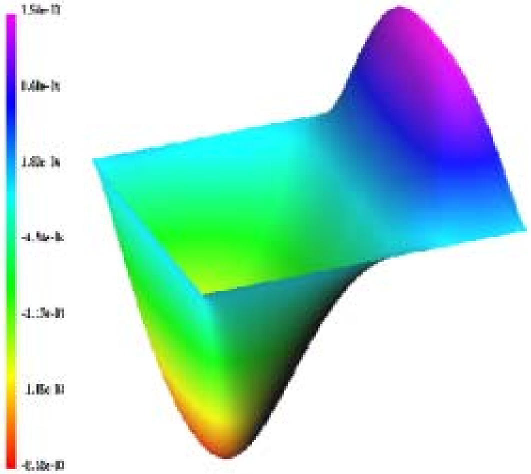
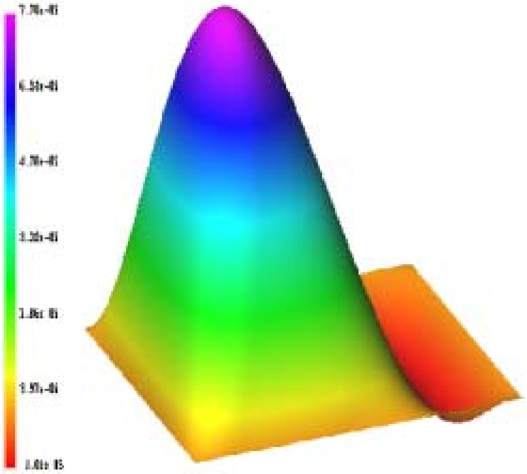
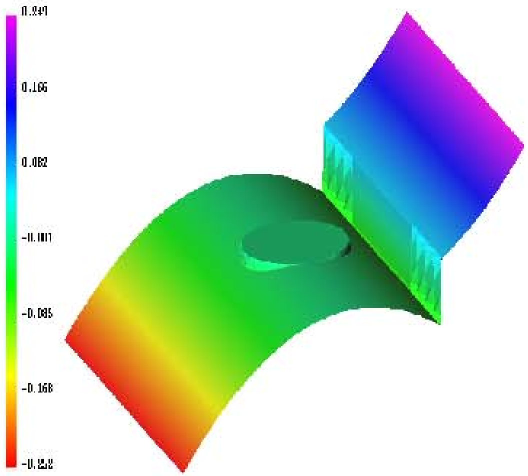
Numerical Results: Distributed Control Problem with Control Constraints II

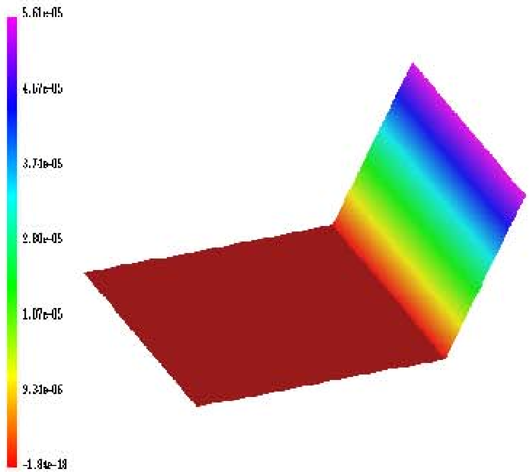




Optimal state (left) and optimal adjoint state (right)

Numerical Results: Distributed Control Problem with Control Constraints II





Optimal control (left) and optimal adjoint control (right)

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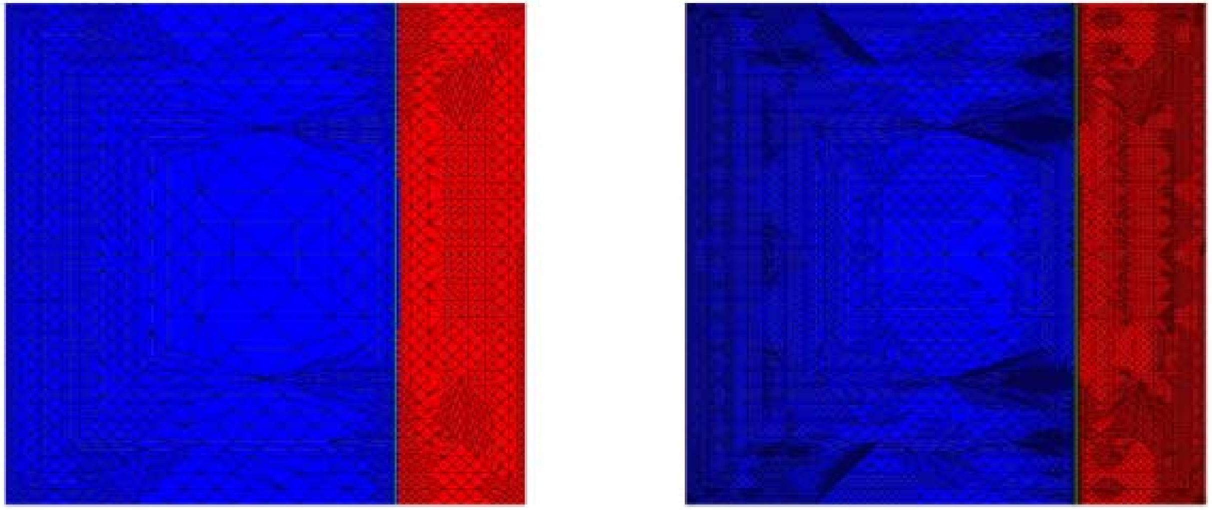
Numerical Results: Distributed Control Problem with Control Constraints II



Grid after 6 (left) and 8 (right) refinement steps ($\Theta_i = 0.7, 1 \le i \le 5$)

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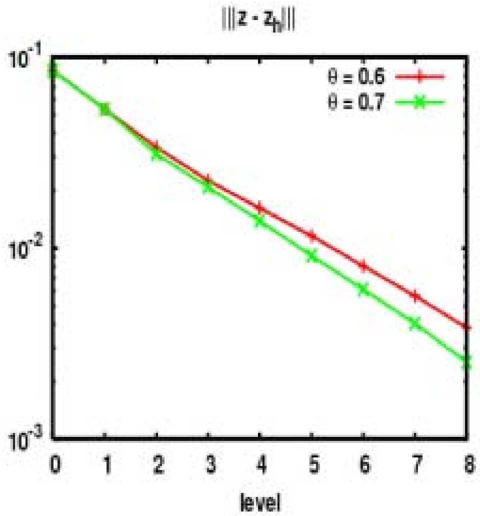
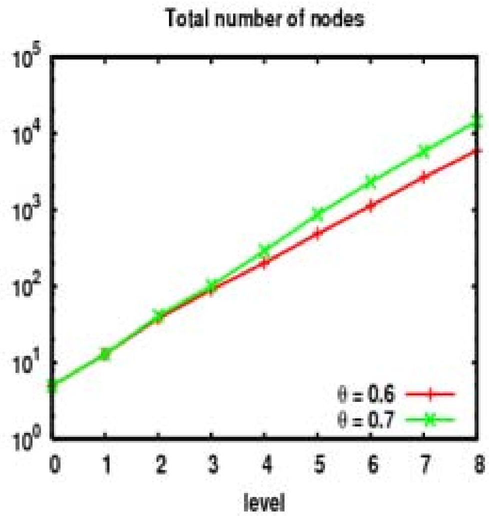
Numerical Results: Distributed Control Problem with Control Constraints II





Error reduction in $|||z - z_H|||$ and total number of nodes ($\Theta_i = 0.6/0.7$)

Numerical Results: Distributed Control Problem with Control Constraints II

l	N_{dof}	$\|\|\|z - z_H\|\|\|$	$\|y - y_H\|_1$	$\|p - p_H\|_1$	$\|\|u - u_H\|\|_0$	$\|\|\sigma - \sigma_H\|\|_0$
0	5	8.50e-02	9.31e-03	1.87e-04	7.55e-02	1.31e-05
1	13	5.35e-02	6.87e-03	1.05e-04	4.66e-02	8.86e-06
2	41	3.12e-02	3.84e-03	6.04e-05	2.73e-02	4.62e-06
3	102	2.09e-02	2.39e-03	4.11e-05	1.84e-02	2.28e-06
4	291	1.39e-02	1.58e-03	2.94e-05	1.23e-02	1.38e-06
5	873	9.14e-03	9.71e-04	1.93e-05	8.15e-03	8.35e-07
6	2325	6.08e-03	6.14e-04	1.21e-05	5.46e-03	5.52e-07
7	5813	4.04e-03	3.97e-04	7.56e-06	3.63e-03	3.68e-07
8	14513	2.53e-03	2.60e-04	5.19e-06	2.26e-03	2.32e-07

Total error, errors in the state, adjoint state, control, adjoint control ($\Theta_i = 0.7$)

Numerical Results: Distributed Control Problem with Control Constraints II

l	η_y	η_p	η_{ce}	$\mu_H(u^d)$	$\mu_H(\psi)$
0	8.91e-02	1.35e-03	1.51e-03	7.00e-02	6.30e-02
1	6.81e-02	7.82e-04	4.15e-04	4.77e-02	3.93e-02
2	4.03e-02	4.60e-04	3.56e-04	2.64e-02	2.06e-02
3	2.34e-02	3.76e-04	2.65e-04	1.83e-02	1.34 e-02
4	1.39e-02	2.53e-04	2.13e-04	1.21e-02	8.63e-03
5	8.49e-03	1.70e-04	1.62e-04	8.33e-03	5.50e-03
6	5.50e-03	1.07e-04	1.32e-04	5.74e-03	3.40e-03
7	3.57e-03	6.91e-05	1.07e-04	4.22e-03	2.40e-03
8	2.32e-03	4.64e-05	8.50e-05	3.08e-03	1.57e-03

Components of the error estimator and data oscillations ($\Theta_i = 0.7$)

Numerical Results: Distributed Control Problem with Control Constraints II

l	$\eta_{y,T}$	$\eta_{y,E}$	$\eta_{p,T}^{(1)}$	$\eta_{p,T}^{(2)}$	$\eta_{p,E}$	$\eta_{ce,T}$
0	3.15e-02	4.02e-03	4.18e-04	1.97e-05	3.34e-04	1.51e-03
1	1.01e-02	4.21e-05	1.22e-04	2.19e-06	1.43e-05	2.27e-04
2	2.52e-03	1.43e-05	3.05e-05	6.38e-07	1.43e-05	8.69e-05
3	9.61e-04	5.94e-06	1.17e-05	2.69e-07	5.94e-06	3.16e-05
4	3.24e-04	1.33e-04	3.95e-06	9.53e-08	2.13e-06	1.35e-05
5	1.06e-04	4.73e-05	1.27e-06	3.14e-08	7.42e-07	5.31e-06
6	9.13e-05	1.94e-05	4.70e-07	1.19e-08	3.02e-07	2.69e-06
7	1.57e-05	8.20e-06	1.87e-07	4.76e-09	1.28e-07	1.38e-06
8	6.36e-06	3.56e-06	7.55e-08	1.94e-09	5.37e-08	6.96e-07

Average values of the local estimators ($\Theta_i = 0.7$)

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Numerical Results: Distributed Control Problem with Control Constraints II

l	$\mu_T(u^d)$	$\mu_T(\psi)$	$M_{fb,T}$	$M_{\eta,E}$	$M_{\eta,T}$	$M_{\mu,T}$
0	3.49e-02	3.09e-02	75.0	0.0	0.0	0.0
1	1.04e-02	3.09e-02	37.5	5.0	12.5	0.0
2	2.70e-03	2.26e-03	18.8	4.5	14.1	0.0
3	1.09e-03	8.87e-04	14.0	11.1	24.4	1.6
4	3.83e-04	3.11e-04	9.1	13.1	32.1	0.0
5	1.29e-04	1.04e-04	5.8	14.0	28.2	0.0
6	4.98e-05	3.98e-05	4.3	12.6	25.6	0.4
7	2.04e-05	1.62e-05	3.4	13.5	26.1	0.2
8	8.34e-06	6.53e-06	2.7	16.0	28.0	0.2

Average values of the data oscillations, bulk criterion ($\Theta_i = 0.7$)

Adaptive Finite Element Methods for State Constrained Optimal Elliptic Control Problems

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Summer School 'Optimal Control of PDEs'

Cortona, July 12-17, 2010

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

C O N T E N T S

I. State Constrained Optimal Control of Elliptic PDEs

- Optimality conditions and finite element discretization

II. Residual-Type a posteriori error estimators

- Element/edge residuals, data oscillations, and consistency error
- Reliabilty and local efficiency of the error estimator
- Lavrentiev regularization: Mixed control-state constraints

III. Goal Oriented Dual Weighted Approach

- Primal-dual weighted error terms in the state and the adjoint state
- Consistency error and the structure of the Lagrange multiplier

Literature on State-Constrained Optimal Control Problems

M. Bergounioux, K. Ito, and K. Kunisch (1999)
M. Bergounioux, M. Haddou, M. Hintermüller, and K. Kunisch (2000)
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E. Casas (1986)
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K. Deckelnick and M. Hinze (2006)
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C. Meyer and F. Tröltzsch (2006)
C. Meyer, U. Prüfert, and F. Tröltzsch (2005)
U. Prüfert, F. Tröltzsch, and M. Weiser (2004)
A. Rösch and F. Tröltzsch (2006)

Model Problem (Distributed Elliptic Control Problem with State Constraints)

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, and let $A : V \to H^{-1}(\Omega)$, $V := \{v \in H^1(\Omega) \mid v|_{\Gamma_d} = 0\}$, be the linear second order elliptic differential operator $Ay := -\Delta y + cy$, $c \geq 0$, with $c > 0$ or $\mathrm{meas}(\Gamma_D) > 0$. Assume that Ω is such that for each $v \in L^2(\Omega)$ the solution y of $Ay = u$ satisfies $y \in W^{1,r}(\Omega) \cap V$ for some $r > 2$. Moreover, let $u^d, y^d \in L^2(\Omega)$, and $\psi \in W^{1,r}(\Omega)$ such that $\psi|_{\Gamma_D} > 0$ be given functions and let $\alpha > 0$ be a regularization parameter.

Consider the state constrained distributed elliptic control problem

$$
\begin{aligned}
&\text{Minimize} && J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u - u^d\|^2_{0,\Omega} \,, \\
&\text{subject to} && Ay = u \text{ in } \Omega \,, \quad y = 0 \text{ on } \Gamma_D \,, \quad \nu \cdot \nabla y = 0 \text{ on } \Gamma_N \,, \\
& && Iy \in K := \{v \in C(\overline{\Omega}) \mid v(x) \leq \psi(x) \,,\ x \in \overline{\Omega}\} \,.
\end{aligned}
$$

where I stands for the embedding operator $W^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$.

The Reduced Optimal Control Problem

We introduce the control-to-state map

$$G : L^2(\Omega) \to C(\overline{\Omega}) \quad , \quad y = Gu \ \text{ solves } \ Ay + cy = u \ .$$

We assume that the following Slater condition is satisfied

$$\text{(S)} \quad \text{There exists } \ v_0 \in L^2(\Omega) \ \text{ such that } \ Gv_0 \in \operatorname{int}(K) \ .$$

Substituting $y = Gu$ allows to consider the reduced control problem

$$\inf_{u \in U_{ad}} J_{red}(u) \ := \ \frac{1}{2} \, \|Gu - y^d\|_{0,\Omega}^2 \ + \ \frac{\alpha}{2} \, \|u - u^d\|_{0,\Omega}^2 \ ,$$

$$U_{ad} \ := \ \{v \in L^2(\Omega) \mid (Gv)(x) \le \psi(x) \ , \ x \in \overline{\Omega}\} \ .$$

Theorem (Existence and uniqueness). The state constrained optimal control problem admits a unique solution $y \in W^{1,r}(\Omega) \cap K$.

Optimality Conditions for the State Constrained Optimal Control Problem

Theorem. There exists an adjoint state $p \in V_s := \{v \in W^{1,s}(\Omega) \mid v_{\Gamma_D} = 0\}$, where $1/r + 1/s = 1$, and a multiplier $\boldsymbol{\sigma} \in \mathcal{M}_+(\Omega)$ such that

$$
\begin{aligned}
(\nabla y, \nabla v)_{0,\Omega} + (cy, v)_{0,\Omega} &= (u, v)_{0,\Omega} \quad , \quad v \in V \ , \\
(\nabla p, \nabla w)_{0,\Omega} + (cp, w)_{0,\Omega} &= (y - y^d, w)_{0,\Omega} + \langle \boldsymbol{\sigma}, x \rangle \quad , \quad w \in V_r \ , \\
p + \alpha(u - u^d) &= 0 \ , \\
\langle \boldsymbol{\sigma}, y - \psi \rangle &= 0 \ .
\end{aligned}
$$

Proof: The reduced problem can be written in unconstrained form as

$$\inf_{v \in L^2(\Omega)} \hat{J}(v) \ := \ J_{red}(v) \ + \ (I_K \circ G)(v) \ ,$$

where I_K stands for the indicator function of the constraint set K. The Slater condition and subdifferential calculus give rise to the optimality condition

$$0 \in \partial \hat{J}(u) \ = \ J'_{red}(u) + \partial(I_K \circ G)(u) = J'_{red}(u) + G^* \circ \partial I_K(Gu) \ .$$

Hence, there exists $\boldsymbol{\sigma} \in \partial I_K(Gu)$ such that

$$(y(u) - y^d, y(v))_{0,\Omega} + \alpha(u - u^d, v)_{0,\Omega} + (G^* \boldsymbol{\sigma}, v)_{0,\Omega} = 0 \quad , \quad v \in L^2(\Omega) \ .$$

We define $\bar{\boldsymbol{\sigma}} := G^* \boldsymbol{\sigma}$ as a regularization of $\boldsymbol{\sigma}$ and introduce $\bar{p} \in V$ as the solution of

$$(\nabla \bar{p}, \nabla v)_{0,\Omega} + (c\bar{p}, v)_{0,\Omega} \ = \ (y(u) - y^d, v)_{0,\Omega} \quad , \quad v \in V \ .$$

We set $p := \bar{p} + \bar{\boldsymbol{\sigma}}$. Since $\boldsymbol{\sigma} \in \mathcal{M}(\Omega)$, we have $\bar{\boldsymbol{\sigma}} \in V_s$ [Casas;1986] whence $p \in V_s$.

Finite Element Approximation

Let $\mathcal{T}_\ell(\Omega)$ be a simplicial triangulation of Ω and let

$$V_\ell \;:=\; \{\, v_\ell \in C(\overline{\Omega}) \mid v_\ell|_T \in P_1(T)\;,\; T \in \mathcal{T}_\ell(\Omega)\;,\; v_\ell|_{\Gamma_D} = 0 \,\}$$

be the FE space of continuous, piecewise linear functions.

Let $u_\ell^d \in V_\ell$ be some approximation of u^d, and let ψ_ℓ be the V_ℓ-interpoland of ψ.

Consider the following FE Approximation of the state constrained control problem

$$\begin{aligned}
&\text{Minimize} && J_\ell(y_\ell, u_\ell) \;:=\; \frac{1}{2}\,\|y_\ell - y^d\|_{0,\Omega}^2 \;+\; \frac{\alpha}{2}\,\|u_\ell - u_\ell^d\|_{0,\Omega}^2\;,\\
&\text{over} && (y_\ell, u_\ell) \in V_\ell \times V_\ell\;,\\
&\text{subject to} && (\nabla y_\ell, \nabla v_\ell)_{0,\Omega} + (c y_\ell, v_\ell)_{0,\Omega} \;=\; (u_\ell, v_\ell)_{0,\Omega}\;,\; v_\ell \in V_\ell\;,\\
& && y_\ell \in K_\ell \;:=\; \{v_\ell \in V_\ell \mid v_\ell(x) \le \psi_\ell(x)\;,\; x \in \overline{\Omega}\}\;.
\end{aligned}$$

Since the constraints are point constraints associated with the nodal points, the discrete multipliers are chosen from

$$\mathcal{M}_\ell \;:=\; \{\mu_\ell \in \mathcal{M}(\Omega) \mid \mu_\ell = \sum_{a \in \mathcal{N}_\ell(\Omega \cup \Gamma_N)} \kappa_a \delta_a\;,\; \kappa_a \in \mathbb{R}\}\;.$$

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Optimality Conditions for the FE Discretized Optimal Control Problem

Theorem 2. There exists a discrete adjoint state $p_\ell \in V_\ell$, and a discrete multiplier $\boldsymbol{\sigma}_\ell$ with $\boldsymbol{\sigma}_\ell \in \mathcal{M}_\ell \cap \mathcal{M}_+(\Omega)$ such that

$$\begin{aligned}
(\nabla y_\ell, \nabla v_\ell)_{0,\Omega} + (c y_\ell, v_\ell)_{0,\Omega} &= (u_\ell, v_\ell)_{0,\Omega} \quad , \quad v_\ell \in V_\ell \, , \\
(\nabla p_\ell, \nabla w_\ell)_{0,\Omega} + (c p_\ell, w_\ell)_{0,\Omega} &= (y_\ell - y^d, w_\ell)_{0,\Omega} + \langle \boldsymbol{\sigma}_\ell, w_\ell \rangle \quad , \quad w_\ell \in V_\ell \, , \\
p_\ell + \alpha(u_\ell - u_\ell^d) &= 0 \, , \\
\langle \boldsymbol{\sigma}_\ell, y_\ell - \psi_\ell \rangle &= 0 \, .
\end{aligned}$$

We introduce a regularized discrete multiplier $\bar{\boldsymbol{\sigma}}_\ell \in V_\ell$ according to

$$(\nabla \bar{\boldsymbol{\sigma}}_\ell, \nabla v_\ell)_{0,\Omega} + (c \bar{\boldsymbol{\sigma}}_\ell, v_\ell)_{0,\Omega} = \langle \boldsymbol{\sigma}_\ell, v_\ell \rangle_{0,\Omega} \quad , \quad v_\ell \in V_\ell \, ,$$

and a regularized discrete adjoint state $\bar{p}_\ell \in V_\ell$ as the solution of

$$(\nabla \bar{p}_\ell, \nabla w_\ell)_{0,\Omega} + (c \bar{p}_\ell, w_\ell)_{0,\Omega} = (y_\ell - y^d, w_\ell)_{0,\Omega} \quad , \quad w_\ell \in V_\ell \, .$$

As in the continuous regime, we have $p_\ell = \bar{p}_\ell + \bar{\boldsymbol{\sigma}}_\ell$.

Residual Type A Posteriori Error Estimator

The error estimator $\eta_\ell := \eta_\ell(y) + \eta_\ell(\overline{p})$ consists of element and edge residuals for the state and the modified adjoint state:

$$\eta_\ell(y) \; := \; \Big(\sum_{T \in \mathcal{T}_\ell(\Omega)} \eta_T^2(y) \; + \sum_{E \in \mathcal{E}_\ell(\Omega)} \eta_E^2(y) \Big)^{1/2} \quad , \quad \eta_\ell(\overline{p}) \; := \; \Big(\sum_{T \in \mathcal{T}_\ell(\Omega)} \big((\eta_\ell^2(\overline{p}) \; + \sum_{E \in \mathcal{E}_\ell(\Omega)} \eta_E^2(\overline{p})\big)\Big)^{1/2}$$

(i) Element and edge residuals for the state y

$$\eta_T(y) \; := \; \underbrace{h_T \, \|c y_\ell \; - \; u_\ell\|_{0,T}}_{\text{element residuals}} \; , \quad T \in \mathcal{T}_\ell(\Omega) \quad , \quad \eta_E(y) \; := \; \underbrace{h_E^{1/2} \, \|\nu_E \cdot [\nabla y_\ell]\|_{0,E}}_{\text{edge residuals}} \; , \quad E \in \mathcal{E}_\ell(\Omega)$$

(ii) Element and edge residuals for the modified adjoint state $\overline{p}$

$$\eta_T(\overline{p}) \; := \; \underbrace{h_T \, \|c \overline{p}_\ell - (y_\ell - y^d)\|_{0,T}}_{\text{element residuals}} \, , \quad T \in \mathcal{T}_\ell(\Omega) \quad , \quad \eta_E(\overline{p}) \; := \; \underbrace{h_E^{1/2} \, \|\nu_E \cdot [\nabla \overline{p}_\ell]\|_{0,E}}_{\text{edge residuals}} \, , \quad E \in \mathcal{E}_\ell(\Omega) \, .$$

Data Oscillations in the Data u^d and y^d

Data oscillations occur in the shift control u^d and in the desired state y^d:

$$\mathrm{osc}_\ell \;:=\; \left(\mathrm{osc}_\ell^2(u^d) \;+\; \mathrm{osc}_\ell^2(y^d)\right)^{1/2}$$

(i) Low order data oscillations in u^d

$$\mathrm{osc}_\ell(u^d) \;:=\; \Big(\sum_{T\in\mathcal{T}_\ell(\Omega)} \mathrm{osc}_T(u^d)^2\Big)^{1/2} \quad , \quad \mathrm{osc}_T(u^d) \;:=\; \|u^d - u_\ell^d\|_{0,T}$$

(ii) Higher order data oscillations in y^d

$$\mathrm{osc}_\ell(y^d) \;:=\; \Big(\sum_{T\in\mathcal{T}_\ell(\Omega)} \mathrm{osc}_T(y^d)^2\Big)^{1/2} \quad , \quad \mathrm{osc}_T(y^d) \;:=\; h_T\,\|y^d - y_H^d\|_{0,T}$$

Bulk Criteria and Refinement Strategy

Given universal constants Θ_i , $1 \le i \le 4$, choose a set of edges $\mathcal{M}_{\eta,E} \subset \mathcal{E}_\ell(\Omega)$ and sets of elements $\mathcal{M}_{\eta,T}, \mathcal{M}_{osc,y^d}, \mathcal{M}_{osc,u^d} \subset \mathcal{T}_\ell(\Omega)$ such that

$$\Theta_1 \sum_{E \in \mathcal{E}_\ell(\Omega)} (\eta_{y,E}^2 + \eta_{\bar{p},E}^2) \le \sum_{E \in \mathcal{M}_{\eta,E}} (\eta_{y,E}^2 + \eta_{\bar{p},E}^2) ,$$

$$\Theta_2 \sum_{T \in \mathcal{T}_\ell(\Omega)} (\eta_{y,T}^2 + \eta_{\bar{p},T}^2) \le \sum_{E \in \mathcal{M}_{\eta,T}} (\eta_{y,T}^2 + \eta_{\bar{p},T}^2) ,$$

$$\Theta_3 \sum_{T \in \mathcal{T}_\ell(\Omega)} \mathrm{osc}_T^2(y^d) \le \sum_{E \in \mathcal{M}_{osc,y^d}} \mathrm{osc}_T^2(y^d) \quad , \quad \Theta_4 \sum_{T \in \mathcal{T}_\ell(\Omega)} \mathrm{osc}_T^2(u^d) \le \sum_{E \in \mathcal{M}_{osc,u^d}} \mathrm{osc}_T^2(u^d) .$$

We set $\mathcal{M}_T := \mathcal{M}_{\eta,T} \cup \mathcal{M}_{osc,y^d} \cup \mathcal{M}_{osc,u^d}$ and refine an element $T \in \mathcal{T}_\ell(\Omega)$ by bisection, if $T \in \mathcal{M}_T$.
An element $E \in \mathcal{E}_\ell(\Omega)$ is refined by bisection, if $E \in \mathcal{M}_{\eta,E}$.
Moreover, all elements $T \in \mathcal{T}_\ell(\Omega)$ that share at least one edge with the discrete free boundary are refined by bisection.

Auxiliary States and Consistency Error

The error analysis involves an auxiliary state $y(u_\ell) \in V$ and an auxiliary adjoint state $\bar{p}(y_\ell) \in V$ which are given as follows

$$(\nabla y(u_\ell), \nabla v)_{0,\Omega} + (cy(u_\ell), v)_{0,\Omega} = (u_\ell, v)_{0,\Omega} \quad , \quad v \in V ,$$
$$(\nabla \bar{p}(y_\ell), \nabla v)_{0,\Omega} + (c\bar{p}(y_\ell, v)_{0,\Omega} = (y_\ell - y^d, v)_{0,\Omega} \quad , \quad v \in V .$$

We also introduce an auxiliary discrete state $y_\ell(u) \in V_\ell$ according to

$$(\nabla y_\ell(u), \nabla v_\ell)_{0,\Omega} + (cy_\ell(u), v_\ell)_{0,\Omega} = (u, v_\ell)_{0,\Omega} \quad , \quad v_\ell \in V_\ell .$$

In general, neither $y(u_\ell) \in K$ nor $y_\ell(u) \in K_\ell$. Therefore, there is a consistency error

$$e_c(u, u_\ell) := \max(\langle \sigma_\ell, y_\ell(u) - \psi_\ell \rangle + \langle \sigma, y(u_\ell) - \psi \rangle, 0) .$$

Since $e_c(u, u_\ell) = 0$ for $u = u_\ell$, we define

$$\tilde{e}_c(u, u_\ell) := \begin{cases} e_c(u, u_\ell)/\|u - u_\ell\|_{0,\Omega} , & u \neq u_\ell \\ 0 , & u = u_\ell \end{cases} .$$

Reliability of the A Posteriori Error Estimator

Theorem. For the errors $e_y := y - y_\ell$ in the state, $e_p := p - p_\ell$, $e_{\bar{p}} := \bar{p} - \bar{p}_\ell$ in the adjoint and modified adjoint state, and $e_u := u - u_\ell$ in the control there holds

$$\|e_y\|_{1,\Omega} + \|e_{\bar{p}}\|_{1,\Omega} + \|e_p\|_{0,\Omega} + \|e_u\|_{0,\Omega} \preceq \eta_\ell + \mathrm{osc}_\ell(u^d) + \tilde{e}_c(u, u_\ell) .$$

Proof: We will show that for $e_y, e_{\bar{p}}$ and e_u there holds

$$\|e_y\|_{1,\Omega} + \|e_{\bar{p}}\|_{1,\Omega} + \|e_u\|_{0,\Omega} \preceq \eta_\ell + \mathrm{osc}_\ell(u^d) + \tilde{e}_c(u, u_\ell) .$$

Since $p + \alpha(u - u^d) = 0$ and $p_\ell + \alpha(u_\ell - u^d_\ell)$, we have $e_p = \alpha((u^d - u^d_\ell) - e_u)$ whence

$$\|e_p\|_{0,\Omega} \preceq \|e_u\|_{0,\Omega} + \mathrm{osc}_\ell(u^d)) .$$

Reliability of the A Posteriori Error Estimator (Cont'd)

Lemma. For the errors $e_y := y - y_\ell$ in the state, $e_{\bar p} := \bar p - \bar p_\ell$ in the modified adjoint state, and $e_u := u - u_\ell$ in the control there holds

$$\|e_y\|_{1,\Omega} + \|e_{\bar p}\|_{1,\Omega} + \|e_u\|_{0,\Omega} \preceq \eta_\ell + \mathrm{osc}_\ell(u^d) + \tilde e_c(u, u_\ell) \, .$$

Proof: Straightforward estimation from above results in

$$\|e_y\|_{1,\Omega} \le \|y - y(u_\ell)\|_{1,\Omega} + \underbrace{\|y(u_\ell) - y_\ell\|_{1,\Omega}}_{\preceq \eta_\ell(y)} \quad , \quad \|e_{\bar p}\|_{1,\Omega} \le \|\bar p - \bar p(y_\ell)\|_{1,\Omega} + \underbrace{\|\bar p(y_\ell) - \bar p_\ell\|_{1,\Omega}}_{\preceq \eta_\ell(\bar p)} \, .$$

On the other hand, using the ellipticity of the differential operator

$$\|y - y(u_\ell\|_{1,\Omega} \preceq \|e_u\|_{0,\Omega} \quad , \quad \|\bar p - \bar p(y_\ell)\|_{1,\Omega} \preceq \|e_u\|_{0,\Omega} + \underbrace{\|y(u_\ell) - y_\ell\|_{1,\Omega}}_{\preceq \eta_\ell(y)} \, .$$

Hence, it remains to estimate the L^2-error $\|e_u\|_{0,\Omega}$ in the control.

Reliability of the A Posteriori Error Estimator (Cont'd)

Proof (Cont'd): Using $p = \alpha(u^d - u)$ and $p_\ell = \alpha(u^d_\ell - u_\ell)$ as well as $p = \bar{p} + \bar{\sigma}$ and $p_\ell = \bar{p}_\ell + \bar{\sigma}_\ell$

$$\|e_u\|^2_{0,\Omega} = \underbrace{(e_u, u^d - u^d_\ell)_{0,\Omega}}_{=:\ I} + \underbrace{\alpha^{-1}(e_u, \bar{p}_\ell - \bar{p}(y_\ell))_{0,\Omega}}_{=:\ II} + \underbrace{\alpha^{-1}(e_u, \bar{p}(y_\ell) - \bar{p})_{0,\Omega}}_{=:\ III} + \underbrace{\alpha^{-1}(e_u, \bar{\sigma}_\ell - \bar{\sigma})_{0,\Omega}}_{=:IV} .$$

By Young's inequality, for the two first terms we obtain

$$|I| \leq \frac{1}{8}\,\|e_u\|^2_{0,\Omega} + 2\,\mathrm{osc}^2_\ell(u^d) \quad , \quad |II| \leq \frac{\alpha}{8}\,\|e_u\|^2_{0,\Omega} + \frac{2}{\alpha}\,\underbrace{\|\bar{p}_\ell - \bar{p}(y_\ell)\|^2_{1,\Omega}}_{\preceq\ \eta_\ell(\bar{p})} .$$

For the third term, we use the duality between the state and the auxiliary adjoint state equations in the continuous and discrete regime. Setting $v := \bar{p}(y_\ell) - \bar{p}$ and $v = y(u_\ell) - y$:

$$III = (e_u, \bar{p}(y_\ell) - \bar{p})_{0,\Omega} = (y - y_\ell, y(u_\ell) - y)_{0,\Omega} = -\|y - y(u_\ell)\|^2_{0,\Omega} + (y(u_\ell) - y_\ell, y(u_\ell) - y)_{0,\Omega}$$

$$\leq \|y(u_\ell) - y_\ell\|_{1,\Omega}\|y - y(u_\ell\|_{1,\Omega} \leq C\|y(u_\ell) - y_\ell\|_{1,\Omega}\|e_u\|_{0,\Omega} \leq \frac{\alpha}{8}\,\|e_u\|^2_{0,\Omega} + \frac{2C^2}{\alpha}\,\eta^2_\ell(y) .$$

Reliability of the A Posteriori Error Estimator (Cont'd)

Proof (Cont'd): The estimate of the fourth term essential relies on the complementary conditions and results in the consistency error due to the mismatch between the continuous and discrete free boundaries. Using the auxiliary states $y(u_\ell)$ and $y_\ell(u)$, we obtain

$$\begin{aligned}
(e_u, \bar{\sigma}_\ell - \bar{\sigma})_{0,\Omega} \;&=\; (\nabla(y_\ell(u) - y_\ell, \nabla\bar{\sigma}_\ell)_{0,\Omega} \;+\; (c(y_\ell(u) - y_\ell), \bar{\sigma}_\ell)_{0,\Omega} \\
&\quad - (\nabla(y - y(u_\ell), \nabla\bar{\sigma})_{0,\Omega} \;-\; (c(y - y(u_\ell)), \bar{\sigma})_{0,\Omega} \;= \\
&= \langle \sigma_\ell, y_\ell(u) - \psi_\ell\rangle \;+\; \underbrace{\langle \sigma_\ell, , \psi_\ell - y_\ell\rangle}_{=\,0} + \langle \sigma, y(u_\ell) - \psi\rangle \;+\; \underbrace{\langle \sigma, \psi - y\rangle}_{=\,0} \\
&\le \|e_u\|_{0,\Omega}\, \tilde{e}_c(u, u_\ell) \;\le\; \frac{\alpha}{8}\, \|e_u\|^2_{0,\Omega} \;+\; \frac{2}{\alpha}\, \tilde{e}^2_c(u, u_\ell)\;.
\end{aligned}$$

Collecting the estimates for terms $\mathrm{I} - \mathrm{IV}$ gives the assertion.

Efficiency of the A Posteriori Error Estimator

Theorem. For the errors $e_y := y - y_\ell$ in the state, $e_{\bar{p}} := \bar{p} - \bar{p}_\ell$ in the regularized adjoint state, and $e_u := u - u_\ell$ in the control there holds

$$\eta_\ell \; - \; \mathrm{osc}_\ell(y^d) \; \preceq \; \|e_y\|_{1,\Omega} \; + \; \|e_{\bar{p}}\|_{1,\Omega} \; + \; \|e_u\|_{0,\Omega} \; .$$

Proof: The assertion follows by standard arguments from the a posteriori error analysis of adaptive finite element methods.

Numerical Results: Distributed Control Problem with State Constraints I

$$\text{Minimize} \quad J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u - u^d\|^2_{0,\Omega} \quad \text{over } (y,u) \in H^1_0(\Omega) \times L^2(\Omega)$$

$$\text{subject to} \quad -\Delta y = u \text{ in } \Omega \quad , \quad y \in K := \{v \in H^1_0(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}$$

Data: $\Omega := (-2,+2)^2$, $y^d(r) := y(r) + \Delta p(r) + \sigma(r)$, $u^d(r) := u(r) + \alpha^{-1} p(r)$,
$\psi := 0$, $\alpha := 0.1$,

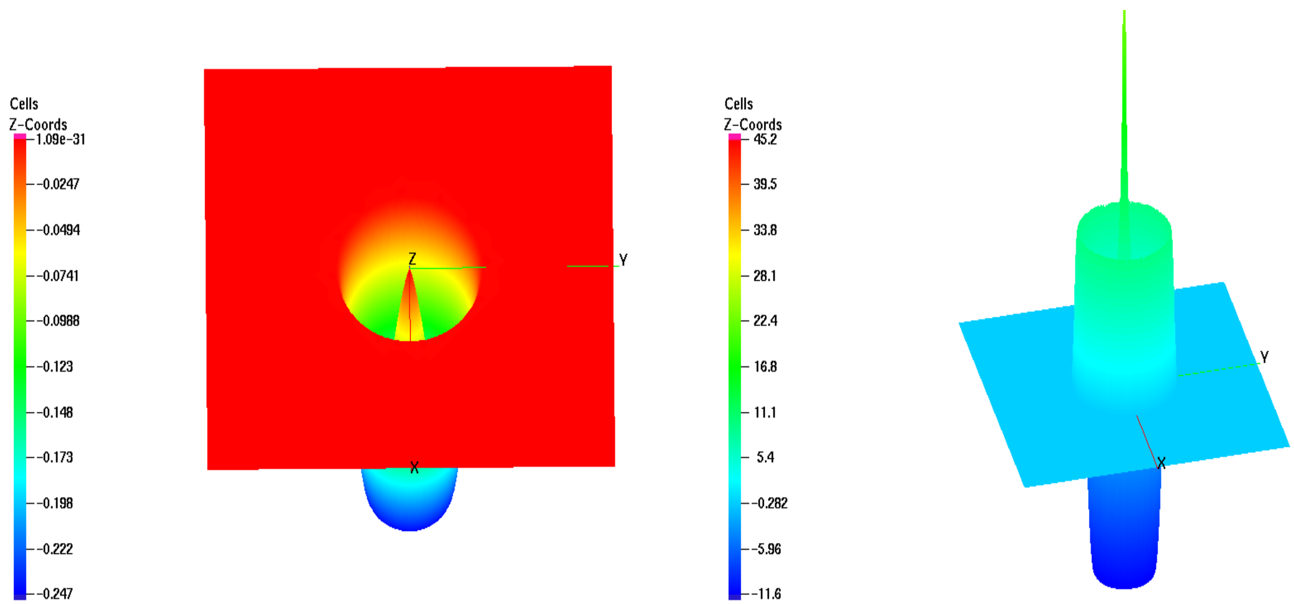
where $y(r), u(r), p(r), \sigma(r)$ is the solution of the problem:

$$y(r) := -r^{4/3} + \gamma_1(r) \ , \ u(r) = -\Delta y(r) \ , \ p(r) = \gamma_2(r) + r^4 - \frac{3}{2} r^3 + \frac{9}{16} r^2 \ , \ \sigma(r) := \begin{cases} 0.0 \ , & r < 0.75 \\ 0.1 \ , & \text{otherwise} \end{cases}$$

$$, \qquad \gamma_1 := \begin{cases} 1 \ , & r < 0.25 \\ -192(r-0.25)^5 + 240(r-0.25)^4 - 80(r-0.25)^3 + 1 \ , & 0.25 < r < 0.75 \\ 0 \ , & \text{otherwise} \end{cases} ,$$

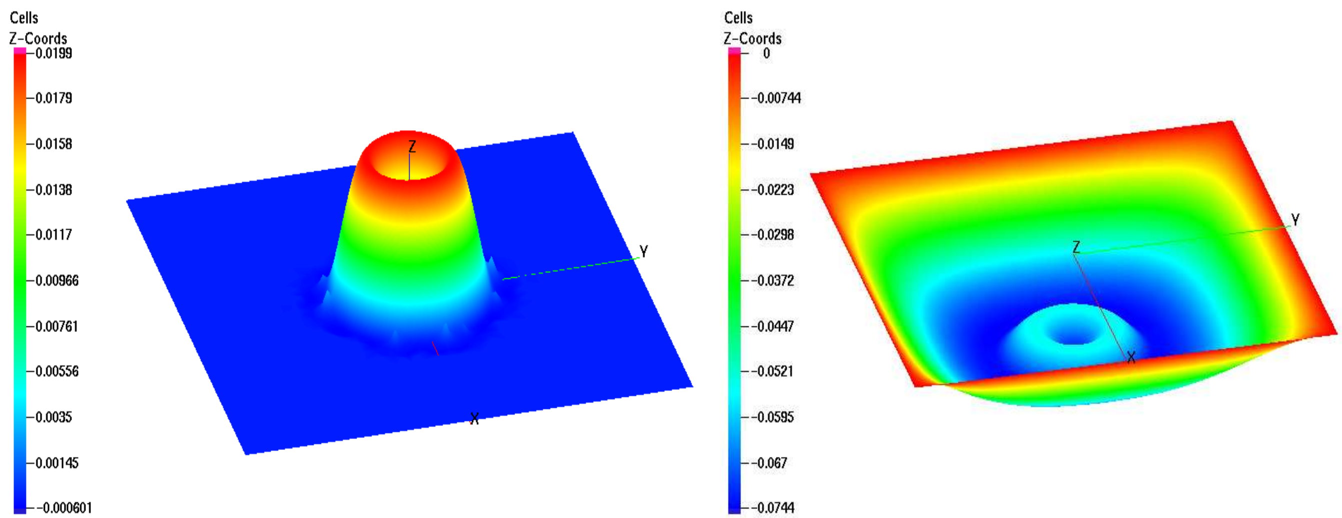
$$\gamma_2 := \begin{cases} 1 \ , & r < 0.75 \\ 0 \ , & \text{otherwise} \end{cases} .$$

Numerical Results: Distributed Control Problem with State Constraints I



Optimal state (left) and optimal control (right)

Numerical Results: Distributed Control Problem with State Constraints I



Optimal adjoint state (left) and regularized adjoint state (right)

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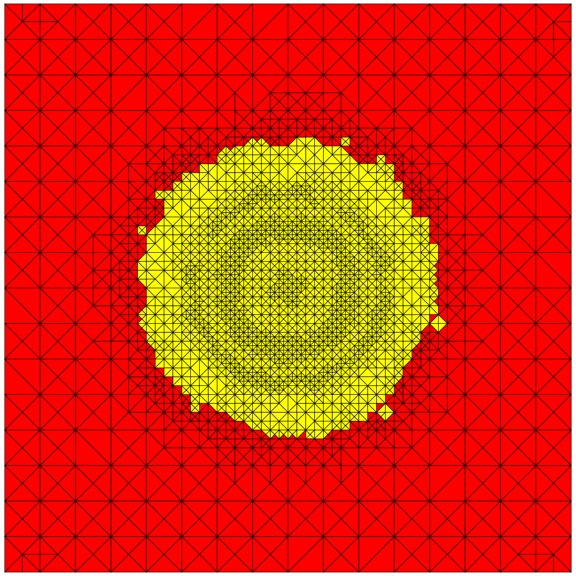
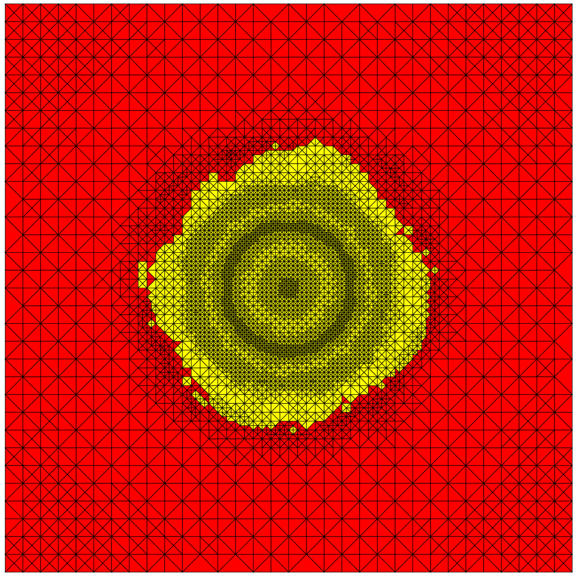
Numerical Results: Distributed Control Problem with State Constraints I





Adaptively refined grid after 12(left) and 14 (right) iterations

Numerical Results: Distributed Control Problem with State Constraints I

ℓ	N_{dof}	$\|z - z_\ell\|$	$\|y - y_\ell\|_1$	$\|u - u_\ell\|_0$	$\|p - p_\ell\|_0$	$\|\overline{p} - \overline{p}_\ell\|_1$
0	5	2.48e+01	2.13e+00	2.11e+01	9.45e-01	7.54e-01
1	13	2.58e+01	1.51e+00	2.37e+01	2.06e+00	6.74e-01
2	41	1.46e+01	1.02e+00	1.35e+01	1.28e-01	1.06e-01
4	105	1.02e+01	7.34e-01	9.41e+00	9.54e-02	7.88e-02
6	244	6.58e+00	5.41e-01	6.01e+00	4.78e-02	6.02e-02
8	532	3.47e+00	2.80e-01	3.18e+00	3.92e-02	4.53e-02
10	1147	2.09e+00	1.74e-01	1.91e+00	2.36e-02	3.44e-02
12	2651	1.39e+00	1.03e-01	1.29e+00	1.81e-02	2.02e-02
14	6340	1.04e+00	6.32e-02	9.74e-01	1.22e-02	1.17e-02

Total error, errors in the state, control, and adjoint state ($\Theta_i = 0.7$)

Numerical Results: Distributed Control Problem with State Constraints I

ℓ	N_{dof}	$\eta_\ell(y)$	$\eta_\ell(\bar{p})$	$osc_\ell(u^d)$	$osc_\ell(y^d)$	$\tilde{e}_c(u, u_\ell)$
0	5	3.95e+01	7.05e+00	1.48e+01	4.99e+00	5.14e-02
1	13	2.19e+01	2.04e+00	1.37e+01	5.42e-01	0.00e+00
2	41	9.83e+00	8.10e-01	1.36e+01	6.22e-01	8.48e-02
4	105	3.67e+00	4.35e-01	9.42e+00	3.32e-01	0.00e+00
6	244	1.63e+00	2.60e-01	5.99e+00	1.11e-01	0.00e+00
8	532	1.17e+00	1.69e-01	3.17e+00	4.47e-02	0.00e+00
10	1147	7.72e-01	1.22e-01	1.90e+00	2.17e-02	0.00e+00
12	2651	4.71e-01	7.37e-02	1.29e+00	9.27e-03	0.00e+00
14	6340	2.93e-01	4.55e-02	9.74e-01	4.62e-03	0.00e+00

Components of the error estimator, data oscillations, and the consistency error

Numerical Results: Distributed Control Problem with State Constraints I

ℓ	N_{dof}	$M_{fb,T}$	$M_{\eta,E}$	$M_{\eta,T}$	M_{osc,y^d}	M_{osc,u^d}
0	5	100.0	75.0	75.0	75.0	75.0
1	13	100.0	20.0	18.8	18.8	18.8
2	41	43.8	13.6	12.5	6.2	7.8
4	105	22.2	4.7	16.7	8.3	13.9
6	244	29.7	8.0	12.2	8.5	10.7
8	532	16.0	8.1	16.1	6.5	4.2
10	1147	11.6	8.3	19.8	4.0	1.4
12	2651	8.7	10.0	25.9	2.0	0.1
14	6340	5.1	11.0	28.3	1.2	0.1

Percentages of elements/edges marked by the bulk criteria

Numerical Results: Distributed Control Problem with State Constraints II

$$\text{Minimize} \qquad J(y,u) \;:=\; \frac{1}{2}\,\|y-y^d\|_{0,\Omega}^2 \;+\; \frac{\alpha}{2}\,\|u-u^d\|_{0,\Omega}^2 \quad \text{over} \;\; (y,u) \in H^1(\Omega)\times L^2(\Omega)$$

$$\text{subject to} \qquad -\;\Delta y \;+\; cy \;=\; u \quad \text{in}\;\Omega \quad , \quad y \in K \;:=\; \{v \in H^1(\Omega) \mid v \le \psi \;\text{a.e. in}\;\Omega\}$$

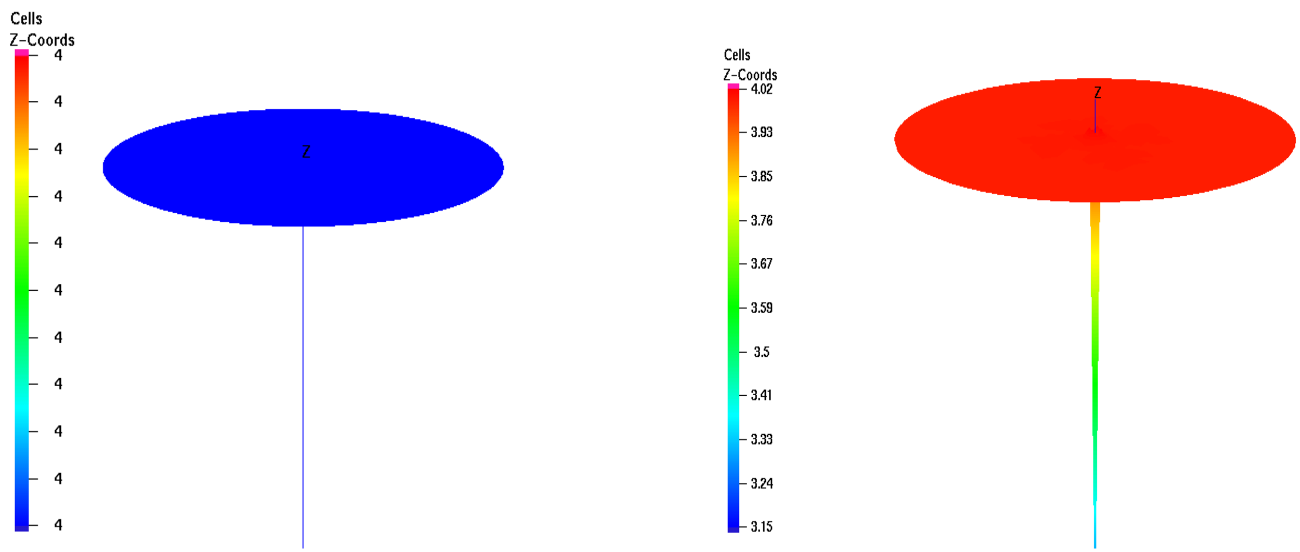
Data: $\Omega \;=\; B(0,1) \;:=\; \{x=(x_1,x_2)^T \mid x_1^2+x_2^2<1\} \quad , \quad y^d(r) \;:=\; 4+\frac{1}{\pi}-\frac{1}{4\pi}r^2+\frac{1}{2\pi}\ln(r)$,

$$u^d(r) \;:=\; 4+\frac{1}{4\pi}r^2-\frac{1}{2\pi}\ln(r) \quad , \quad \psi \;:=\; 4+r \quad , \quad \alpha \;:=\; 1 \;.$$

The solution $y(r), u(r), p(r), \sigma(r)$ of the problem is given by

$$y(r) \;\equiv\; 4 \quad , \quad u(r) \;\equiv\; 4 \quad , \quad p(r) \;=\; \frac{1}{4\pi}r^2-\frac{1}{2\pi}\ln(r) \quad , \quad \sigma(r) \;=\; \delta_0 \;.$$

Numerical Results: Distributed Control Problem with State Constraints II



Optimal state (left) and optimal control (right)

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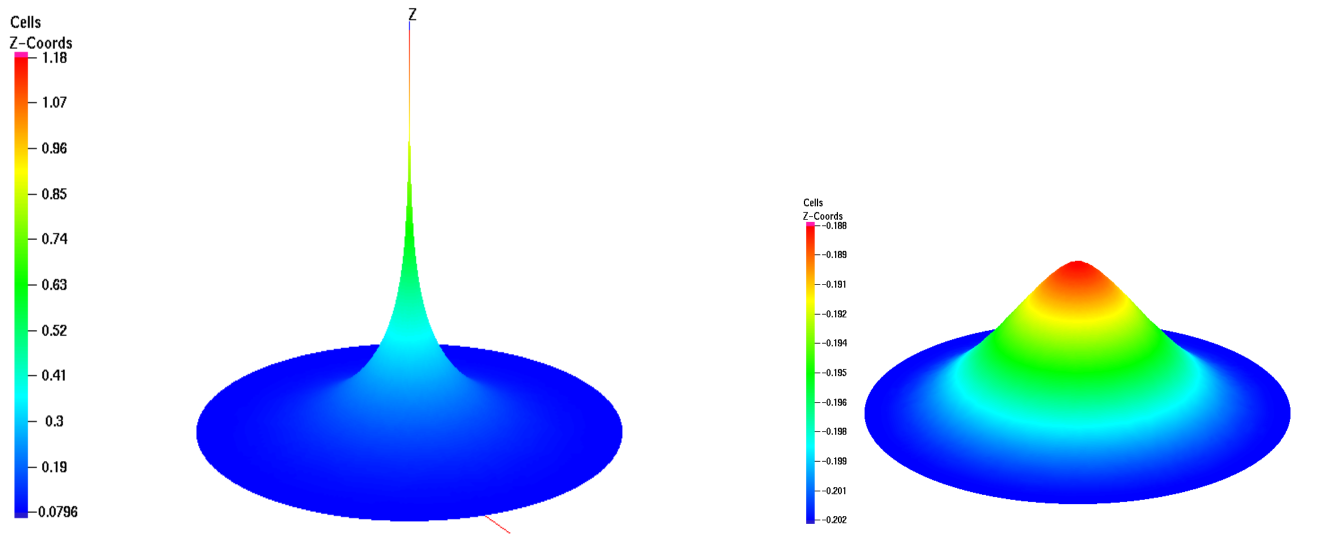
Numerical Results: Distributed Control Problem with State Constraints II



Optimal adjoint state (left) and regularized adjoint state (right)

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Numerical Results: Distributed Control Problem with State Constraints II

ℓ	N_{dof}	$\|z - z_\ell\|$	$\|y - y_\ell\|_1$	$\|u - u_\ell\|_0$	$\|p - p_\ell\|_0$	$\|\bar{p} - \bar{p}_\ell\|_1$
0	5	1.55e-01	1.20e-02	1.43e-01	6.46e-02	3.81e-02
1	13	1.13e-01	8.51e-03	1.04e-01	3.73e-02	1.74e-02
2	41	7.39e-02	4.43e-03	6.95e-02	1.86e-02	9.01e-03
4	73	5.96e-02	2.30e-03	5.73e-02	1.00e-02	7.36e-03
6	121	3.60e-02	1.79e-03	3.42e-02	7.41e-03	6.11e-03
8	243	2.10e-02	1.07e-03	1.99e-02	4.13e-03	4.02e-03
10	604	1.18e-02	4.02e-04	1.14e-02	1.95e-03	2.43e-03
12	1621	6.55e-03	1.60e-04	6.39e-03	9.26e-04	1.52e-03
14	3991	3.62e-03	6.81e-05	3.55e-03	4.55e-04	8.79e-04

Total error, errors in the state, control, and adjoint state ($\Theta_i = 0.7$)

Numerical Results: Distributed Control Problem with State Constraints II

ℓ	N_{dof}	$\eta_\ell(y)$	$\eta_\ell(\overline{p})$	$osc_\ell(u^d)$	$osc_\ell(y^d)$	$\tilde{e}_c(u, u_\ell)$
0	5	1.91e-01	1.38e-01	1.73e-01	1.36e-01	0.00e+00
1	13	7.32e-02	7.62e-02	1.29e-01	4.36e-02	0.00e+00
2	41	2.45e-02	3.83e-02	8.14e-02	1.26e-02	0.00e+00
4	73	1.02e-02	2.54e-02	5.95e-02	7.78e-03	0.00e+00
6	121	3.11e-03	1.97e-02	3.56e-02	4.96e-03	0.00e+00
8	243	9.10e-04	1.32e-02	2.06e-02	1.87e-03	0.00e+00
10	604	2.59e-04	8.07e-03	1.17e-02	8.27e-04	0.00e+00
12	1621	7.22e-05	4.75e-03	6.54e-03	3.16e-04	0.00e+00
14	3991	2.01e-05	2.89e-03	3.62e-03	1.41e-04	0.00e+00

Components of the error estimator, data oscillations, and the consistency error

Numerical Results: Distributed Control Problem with State Constraints II

ℓ	N_{dof}	$M_{fb,T}$	$M_{\eta,E}$	$M_{\eta,T}$	M_{osc,y^d}	M_{osc,u^d}
0	5	100.0	75.0	75.0	75.0	75.0
1	13	25.0	30.0	25.0	25.0	31.2
2	41	6.3	27.3	7.8	12.5	6.2
4	73	6.5	10.8	25.8	17.7	5.6
6	121	3.9	7.5	30.0	16.4	3.4
8	243	1.8	11.3	31.7	21.4	1.6
10	604	0.7	15.2	37.9	17.1	0.5
12	1621	0.3	8.1	39.8	15.9	0.2
14	3991	0.1	8.7	47.7	9.8	0.1

Percentages of elements/edges marked by the bulk criteria

Lavrentiev Regularization: Mixed Control-State Constraints

Introduce a regularization parameter $\varepsilon > 0$ and consider the mixed control-state constrained optimal control problem

$$\text{Minimize} \quad J(y^\varepsilon, u^\varepsilon) := \frac{1}{2}\|y^\varepsilon - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u^\varepsilon - u^d\|_{0,\Omega}^2 ,$$

$$\text{subject to} \quad Ay^\varepsilon = u^\varepsilon \text{ in } \Omega \quad , \quad y^\varepsilon = 0 \text{ on } \Gamma_D \quad , \quad \nu \cdot \nabla y^\varepsilon = 0 \text{ on } \Gamma_N ,$$

$$\varepsilon u^\varepsilon + y^\varepsilon \in K := \{v \in L^2(\Omega) \mid v(x) \le \psi(x) , \text{ f.a.a. } x \in \Omega\} .$$

Theorem (Optimality conditions). The optimal solution $(y^\varepsilon, u^\varepsilon) \in V \times L^2(\Omega)$ is characterized by the existence of an adjoint state $p^\varepsilon \in V$ and a multiplier $\sigma^\varepsilon \in L^2_+(\Omega)$ such that

$$(\nabla y^\varepsilon, \nabla v)_{0,\Omega} + (cy^\varepsilon, v)_{0,\Omega} = (u^\varepsilon, v)_{0,\Omega} \quad , \quad v \in V ,$$

$$(\nabla p^\varepsilon, \nabla v)_{0,\Omega} + (cp^\varepsilon, v)_{0,\Omega} = (y^\varepsilon - y^d, v)_{0,\Omega} + (\sigma^\varepsilon, v)_{0,\Omega} \quad , \quad v \in V ,$$

$$p^\varepsilon + \alpha\,(u^\varepsilon - u^d) + \varepsilon\sigma^\varepsilon = 0 \quad , \quad (\sigma^\varepsilon, \varepsilon u^\varepsilon + y^\varepsilon - \psi)_{0,\Omega} = 0 .$$

Features of the Lavrentiev Regularization

Since the Lavrentiev regularization formally represents a control constrained optimal control problem, numerical solution techniques for the control constrained case can be employed.

Does this also hold true for the a posteriori error estimation?

Apply the a posteriori error estimator from [Hintermüller/H./Iliash/Kieweg] to the mixed control-state constrained problem:

$$\eta_T(y^\varepsilon) \; = \; h_T \; \|cy_\ell^\varepsilon - u_\ell^\varepsilon\|_{0,T} \quad , \quad \eta_T(p^\varepsilon) \; = \; h_T \; \|cp_\ell^\varepsilon - (y_\ell^\varepsilon - y^d) - \sigma_\ell^\varepsilon\|_{0,T} \; ,$$

$$\eta_E(y^\varepsilon) \; = \; h_E^{1/2} \; \|\nu_E \cdot [\nabla y_\ell^\varepsilon]\|_{0,E} \quad , \quad \eta_E(p^\varepsilon) \; = \; h_T \; \|\nu_E \cdot [\nabla p_\ell^\varepsilon]\|_{0,E} \; .$$

Since we are interested in the approximation of the solution (y, u, p) of the state-constrained problem by $(y^\varepsilon, u^\varepsilon, p^\varepsilon)$ as $\varepsilon \to 0$, but p lacks smoothness and $\sigma \notin L^2_+(\Omega)$, we have to proceed by appropriate regularizations.

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Regularization of the Multiplier and of the Adjoint State

We define a regularized multiplier $\bar{\sigma}^\varepsilon \in V$ and a regularized adjoint state $\bar{p}^\varepsilon \in V$ as the solution of

$$(\nabla\bar{\sigma}^\varepsilon, \nabla v)_{0,\Omega} + (c\bar{\sigma}^\varepsilon, v)_{0,\Omega} = (\sigma^\varepsilon, v)_{0,\Omega} \quad , \quad v \in V ,$$
$$(\nabla\bar{p}^\varepsilon, \nabla v)_{0,\Omega} + (c\bar{p}^\varepsilon, v)_{0,\Omega} = (y^\varepsilon - y^d, v)_{0,\Omega} \quad , \quad v \in V .$$

In the discrete regime, we define $\bar{\sigma}_\ell^\varepsilon \in V_\ell$ and $\bar{p}_\ell^\varepsilon \in V_\ell$ analogously.
This gives rise to the following element and edge residuals

$$\eta_T(y^\varepsilon) = h_T \, \|cy_\ell^\varepsilon - u_\ell^\varepsilon\|_{0,T} \quad , \quad \eta_T(\bar{p}^\varepsilon) = h_T \, \|c\bar{p}_\ell^\varepsilon - (y_\ell^\varepsilon - y^d)\|_{0,T} ,$$
$$\eta_E(y^\varepsilon) = h_E^{1/2} \, \|\nu_E \cdot [\nabla y_\ell^\varepsilon]\|_{0,E} \quad , \quad \eta_E(\bar{p}^\varepsilon) = h_T \, \|\nu_E \cdot [\nabla \bar{p}_\ell^\varepsilon]\|_{0,E} .$$

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Error Analysis of the Mixed Control-State Optimal Control Problem

Theorem (Reliability of the Estimator). For the errors $e(y^\varepsilon) := y^\varepsilon - y_\ell^\varepsilon$ in the state, $e(\bar{p}^\varepsilon) := \bar{p}^\varepsilon - \bar{p}_\ell^\varepsilon$ in the regularized adjoint state, and $e(u^\varepsilon) := u^\varepsilon - u_\ell^\varepsilon$ in the control there holds uniformly in $\varepsilon > 0$

$$\|e(y^\varepsilon)\|_{1,\Omega} + \|e(u^\varepsilon)\|_{0,\Omega} + \|e(\bar{p}^\varepsilon)\|_{1,\Omega} \preceq$$

$$\preceq \eta_\ell + \mathrm{osc}_\ell(y^d) + \mathrm{osc}_\ell(u^d) + \mathrm{osc}_\ell(\psi) + \tilde{e}_c(u^\varepsilon, u_\ell^\varepsilon) + \tilde{e}_c(\psi, \psi_\ell) .$$

Compared to the case of pure state constraints, the error analysis involves data oscillations in ψ and an additional consistency error

$$\mathrm{osc}_T(\psi) := \|\psi - \psi_\ell\|_{0,T} \quad , \quad \tilde{e}_c(\psi, \psi_\ell) := \begin{cases} e_c(\psi, \psi_\ell)/\|\psi - \psi_\ell\|_{0,\Omega} \, , & \psi \neq \psi_\ell \\ 0 \, , & \psi = \psi_\ell \end{cases} ,$$

$$e_c(\psi, \psi_\ell) := \max((\sigma^\varepsilon - \sigma_\ell^\varepsilon, \psi - \psi_\ell)_{0,\Omega}, 0) .$$

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Numerical Results: Mixed Control-State Constraints II

ℓ	N_{dof}	$\|y - y_\ell\|_1$	$\|u - u_\ell\|_0$	$\|p - p_\ell\|_0$	$\mathcal{M}_{\eta,E}$	$\mathcal{M}_{\eta,T}$	$\mathcal{M}_{osc_1,T}$	$\mathcal{M}_{osc_2,T}$
0	5	8.62e-03	5.33e-02	1.48e-01	75.0	75.0	75.0	75.0
2	41	7.09e-03	6.51e-02	2.88e-02	28.4	9.4	6.2	12.5
4	73	4.52e-03	5.56e-02	1.50e-02	11.9	25.8	5.6	17.7
6	121	1.48e-03	3.40e-02	8.05e-03	7.5	30.0	3.4	16.4
8	243	4.75e-04	1.99e-02	4.21e-03	11.3	31.7	1.6	21.4
10	603	1.54e-04	1.14e-02	1.96e-03	15.3	37.9	0.5	17.2
12	1618	4.95e-05	6.42e-03	1.91e-03	8.1	39.8	0.2	15.8
14	3989	2.28e-05	3.59e-03	6.58e-04	8.8	47.5	0.1	9.7

Errors in the state, control, and adjoint state ($\varepsilon = 10^{-4}$, $\Theta_i = 0.7$)

Goal-Oriented Adaptivity in State Constrained Optimal Control Problems

Under the same assumptions as before, consider the state constrained distributed elliptic control problem

$$
\begin{aligned}
\text{Minimize} \qquad & J(y,u) := \frac{1}{2}\,\|y - y^d\|_{0,\Omega}^2 \;+\; \frac{\alpha}{2}\,\|u\|_{0,\Omega}^2 \;, \\
\text{subject to} \qquad & Ay = u \text{ in } \Omega \;,\; y = 0 \text{ on } \Gamma_D \;,\; \nu \cdot \nabla y = 0 \text{ on } \Gamma_N \;, \\
& Iy \in K := \{ v \in C(\overline{\Omega}) \mid v(x) \le \psi(x) \;,\; x \in \overline{\Omega} \} \;.
\end{aligned}
$$

and its finite element approximation with respect to a shape regular family of simplicial triangulations $\mathcal{T}_\ell(\Omega)$ of the computational domain Ω.
The goal is to provide a reliable error estimate in a quantity of interest which is chosen here as the objective functional $J(y,u)$.

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Error Representation in Goal-Oriented Adaptivity

We introduce the Lagrangian $\mathcal{L} : V^r \times L^2(\Omega) \times V^s \times \mathcal{M}_+(\bar{\Omega}) \to \mathbb{R}$ according to

$$\mathcal{L}(y,u,p,\boldsymbol{\sigma}) := J(y,u) \ + \ (\nabla y, \nabla p)_{0,\Omega} \ - \ (u,p)_{0,\Omega} \ + \ \langle \boldsymbol{\sigma}, y - \psi \rangle \ ,$$

and refer to $\mathcal{L}_\ell : V_\ell \times V_\ell \times V_\ell \times (\mathcal{M}_\ell \cap \mathcal{M}_+(\bar{\Omega})) \to \mathbb{R}$ as its discrete counterpart.

Theorem. Let $(x,\boldsymbol{\sigma}) \in X \times \mathcal{M}_+(\bar{\Omega}), x := (y,u,p) \in X := V^r \times L^2(\Omega) \times V^s$, and $(x_\ell,\boldsymbol{\sigma}_\ell) \in X_\ell \times (\mathcal{M}_\ell \cap \mathcal{M}_+(\bar{\Omega})), x_\ell := (y_\ell,u_\ell,p_\ell) \in X_\ell := V_\ell \times V_\ell \times V_\ell$, be the solutions of the continuous and discrete problem, respectively. Then, we have

$$J(y,u) - J_\ell(y_\ell,u_\ell) = -\frac{1}{2}\nabla_{xx}\mathcal{L}(x_\ell - x, x_\ell - x) + \langle \boldsymbol{\sigma}, y_\ell - \psi \rangle + \mathrm{osc}_\ell^{(1)}(x_\ell) \ ,$$

where the oscillation term $\mathrm{osc}_\ell^{(1)}(x_\ell)$ is given by

$$\mathrm{osc}_\ell^{(1)}(x_\ell) := (y_\ell - y^d, y_\ell^d - y^d)_{0,\Omega} + \mathrm{osc}_\ell^2(y^d) \ .$$

Error Representation in Goal-Oriented Adaptivity (Cont'd)

Remark 1: In the unconstrained case, i.e., $\sigma = \sigma_\ell = 0$, the error representation reduces to

$$J(y,u) - J_\ell(y_\ell, u_\ell) =$$

$$= \nabla_x \mathcal{L}(x_\ell, \sigma_\ell)(x - x_\ell - \Delta x_\ell) + \frac{1}{2}(y^d - y_\ell^d, y - y_\ell)_{0,\Omega} + \mathrm{osc}_\ell(x_\ell) \ ,$$

which corresponds to Proposition 4.1 in [Becker/Kapp/Rannacher].

Remark 2: The contribution $\langle \sigma, y_\ell - \psi \rangle$ can be rewritten as

$$\langle \sigma, y_\ell - \psi \rangle \ = \ \langle \sigma, y_\ell - \psi_\ell \rangle \ + \ \langle \sigma, \psi_\ell - \psi \rangle$$

and thus represents the primal-dual weighted mismatch in complementarity and due to the approximation of ψ by ψ_ℓ.

Error Representation in Goal-Oriented Adaptivity (Cont'd)

Theorem. With interpolation operators $\mathcal{I}_\ell^p : V^p \to V_\ell, 1 < p < \infty$ there holds

$$J(y,u) - J_\ell(y_\ell,u_\ell) =$$

$$= -\frac{1}{2}\Big\{ (\nabla p_\ell, \nabla(\mathcal{I}_\ell^r y - y))_{0,\Omega} + (cp_\ell - (y_\ell - y_\ell^d), \mathcal{I}_\ell^r y - y)_{0,\Omega} - \langle \sigma_\ell, \mathcal{I}_\ell^r y - y\rangle +$$

$$+ (\nabla y_\ell, \nabla(\mathcal{I}_\ell^s p - p))_{0,\Omega} + (cy_\ell - u_\ell, \mathcal{I}_\ell^r p - p)_{0,\Omega} \Big\} +$$

$$+ \underbrace{\big[\langle \sigma, y_\ell - \psi\rangle + \langle \sigma_\ell, \psi_\ell - y\rangle \big]}_{\text{primal-dual mismatch}} + \underbrace{\frac{1}{2}(y^d - y_\ell^d, y - y_\ell)_{0,\Omega} + \text{osc}_\ell^{(1)}(x_\ell)}_{=:\ \text{osc}_\ell^{(2)}(x_\ell)} .$$

We note that the terms in brackets $\{\cdots\}$ represent the primal-dual residuals.

Goal Oriented Dual Weighted Residuals for Control and State Constrained Optimal Elliptic Control Problems

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The Loop in Adaptive Finite Element Methods (AFEM)

Adaptive Finite Element Methods (AFEM) consist of successive loops of the cycle

SOLVE $\Longrightarrow$ ESTIMATE $\Longrightarrow$ MARK $\Longrightarrow$ REFINE

SOLVE: Numerical solution of the FE discretized problem

ESTIMATE: Residual and hierarchical a posteriori error estimators
Error estimators based on local averaging
Goal oriented weighted dual approach
Functional type a posteriori error bounds

MARK: Strategies based on the max. error or the averaged error
Bulk criterion for AFEMs

REFINE: Bisection or 'red/green' refinement or combinations thereof

CONTENTS

- Representation of the error in the quantity of interest
- Primal-Dual Weighted Residuals
- Primal-Dual Mismatch in Complementarity
- Primal-Dual Weighted Data Oscillations
- Numerical Results

State Constrained Elliptic Control Problems

Literature on State-Constrained Optimal Control Problems

M. Bergounioux, K. Ito, and K. Kunisch (1999)
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M. Bergounioux and K. Kunisch (2002)
E. Casas (1986)
E. Casas and M. Mateos (2002)
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M. Hintermüller and K. Kunisch (2007)
K. Kunisch and A. Rösch (2002)
C. Meyer and F. Tröltzsch (2006)
C. Meyer, U. Prüfert, and F. Tröltzsch (2005)
U. Prüfert, F. Tröltzsch, and M. Weiser (2004)
H./M. Kieweg (2007)
A. Günther, M. Hinze (2007)
O. Benedix, B. Vexler (2008)
M. Hintermüller/H. (2008)
W. Liu, W. Gong and N. Yan (2008)

Model Problem (Distributed Elliptic Control Problem with State Constraints)

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, and let $A : V \to H^{-1}(\Omega)$, $V := \{v \in H^1(\Omega) \mid v|_{\Gamma_d} = 0\}$, be the linear second order elliptic differential operator $Ay := -\Delta y + cy$, $c \geq 0$, with $c > 0$ or $\mathrm{meas}(\Gamma_D) > 0$. Assume that Ω is such that for each $v \in L^2(\Omega)$ the solution y of $Ay = u$ satisfies $y \in W^{1,r}(\Omega) \cap V$ for some $r > 2$. Moreover, let $u^d, y^d \in L^2(\Omega)$, and $\psi \in W^{1,r}(\Omega)$ such that $\psi|_{\Gamma_D} > 0$ be given functions and let $\alpha > 0$ be a regularization parameter.

Consider the state constrained distributed elliptic control problem

$$\begin{aligned} &\text{Minimize} && J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u - u^d\|^2_{0,\Omega} \;, \\ &\text{subject to} && Ay = u \text{ in } \Omega \;,\quad y = 0 \text{ on } \Gamma_D \;,\quad \nu \cdot \nabla y = 0 \text{ on } \Gamma_N \;, \\ & && Iy \in K := \{v \in C(\overline{\Omega}) \mid v(x) \leq \psi(x) \;,\; x \in \overline{\Omega}\} \;. \end{aligned}$$

where I stands for the embedding operator $W^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$.

The Reduced Optimal Control Problem

We introduce the control-to-state map

$$G : L^2(\Omega) \to C(\overline{\Omega}) \quad , \quad y = Gu \ \text{ solves } \ Ay + cy = u \ .$$

We assume that the following Slater condition is satisfied

$$(S) \quad \text{There exists } \ v_0 \in L^2(\Omega) \ \text{ such that } \ Gv_0 \in \text{int}(K) \ .$$

Substituting $y = Gu$ allows to consider the reduced control problem

$$\inf_{u \in U_{ad}} J_{red}(u) \ := \ \frac{1}{2} \, \|Gu - y^d\|^2_{0,\Omega} \ + \ \frac{\alpha}{2} \, \|u - u^d\|^2_{0,\Omega} \ ,$$

$$U_{ad} \ := \ \{v \in L^2(\Omega) \mid (Gv)(x) \le \psi(x) \ , \ x \in \overline{\Omega}\} \ .$$

Theorem (Existence and uniqueness). The state constrained optimal control problem admits a unique solution $y \in W^{1,r}(\Omega) \cap K$.

Optimality Conditions for the State Constrained Optimal Control Problem

Theorem. There exists an adjoint state $p \in V^s := \{v \in W^{1,s}(\Omega) \mid v_{\Gamma_D} = 0\}$, where $1/r + 1/s = 1$, and a multiplier $\sigma \in \mathcal{M}_+(\Omega)$ such that

$$
\begin{aligned}
(\nabla y, \nabla v)_{0,\Omega} + (cy, v)_{0,\Omega} &= (u, v)_{0,\Omega} \quad , \quad v \in V^s \; , \\
(\nabla p, \nabla w)_{0,\Omega} + (cp, w)_{0,\Omega} &= (y - y^d, w)_{0,\Omega} + \langle \sigma, w \rangle \quad , \quad w \in V^r \; , \\
p + \alpha(u - u^d) &= 0 \; , \\
\langle \sigma, y - \psi \rangle &= 0 \; .
\end{aligned}
$$

Proof. The reduced problem can be written in unconstrained form as

$$\inf_{v \in L^2(\Omega)} \widehat{J}(v) \; := \; J_{red}(v) \; + \; (I_K \circ G)(u)$$

where I_K stands for the indicator function of the constraint set K. The Slater condition and and subdifferential calculus tell us

$$\partial\Big(I_K \circ G\Big)(u) \; = \; G^* \circ \partial I_K(Gu) \; .$$

The optimality condition then reads

$$0 \in \partial\widehat{J}(u) \; = \; J'_{red}(u) \; + \; G^* \circ \partial I_K(Gu) \; .$$

Hence, there exists $\sigma \in \partial I_K(Gu)$ such that

$$\Big(\underbrace{G^*(Gu - y^d + \sigma)}_{=:\; p} + \alpha(u - u^d), v\Big)_{0,\Omega} \; = \; 0 \quad , \quad v \in L^2(\Omega) \; .$$

Since $\sigma \in \mathcal{M}(\Omega)$, PDE regularity theory implies $p \in W^{1,s}(\Omega), 1/s + 1/r = 1$.

Finite Element Approximation

Let $\mathcal{T}_\ell(\Omega)$ be a simplicial triangulation of Ω and let

$$V_\ell := \{ v_\ell \in C(\overline{\Omega}) \mid v_\ell|_T \in P_1(T) \ , \ T \in \mathcal{T}_\ell(\Omega) \ , \ v_\ell|_{\Gamma_D} = 0 \}$$

be the FE space of continuous, piecewise linear functions.

Let $u_\ell^d \in V_\ell$ be some approximation of u^d, and let ψ_ℓ be the V_ℓ-interpoland of ψ.

Consider the following FE Approximation of the state constrained control problem

$$\begin{aligned}
&\text{Minimize} && J_\ell(y_\ell, u_\ell) := \frac{1}{2} \|y_\ell - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2} \|u_\ell - u_\ell^d\|_{0,\Omega}^2 \ , \\
&\text{over} && (y_\ell, u_\ell) \in V_\ell \times V_\ell \ , \\
&\text{subject to} && (\nabla y_\ell, \nabla v_\ell)_{0,\Omega} + (c y_\ell, v_\ell)_{0,\Omega} = (u_\ell, v_\ell)_{0,\Omega} \ , \ v_\ell \in V_\ell \ , \\
& && y_\ell \in K_\ell := \{ v_\ell \in V_\ell \mid v_\ell(x) \le \psi_\ell(x) \ , \ x \in \overline{\Omega} \} \ .
\end{aligned}$$

Since the constraints are point constraints associated with the nodal points, the discrete multipliers are chosen from

$$\mathcal{M}_\ell := \{ \mu_\ell \in \mathcal{M}(\Omega) \mid \mu_\ell = \sum_{a \in \mathcal{N}_\ell(\Omega \cup \Gamma_N)} \kappa_a \delta_a \ , \ \kappa_a \in \mathbb{R} \} \ .$$

Representation of the Error in the Quantity of Interest

Primal-Dual Weighted Error Representation I

We set $X := V^r \times L^2(\Omega) \times V^s$ as well as $X_\ell := V_\ell \times V_\ell \times V_\ell$ and introduce the Lagrangians $\mathcal{L} : X \times \mathcal{M}(\Omega) \to \mathbb{R}$ as well as $\mathcal{L}_\ell : X_\ell \times \mathcal{M}_\ell \to \mathbb{R}$ according to

$$\mathcal{L}(x, \boldsymbol{\sigma}) \ := \ J(y, u) \ + \ (\nabla y, \nabla p)_{0,\Omega} \ - \ (u, p)_{0,\Omega} \ + \ \langle \boldsymbol{\sigma}, y - \psi \rangle \ ,$$

$$\mathcal{L}_\ell(x_\ell, \boldsymbol{\sigma}_\ell) \ := \ J_\ell(y_\ell, u_\ell) \ + \ (\nabla y_\ell, \nabla p_\ell)_{0,\Omega} \ - \ (u_\ell, p_\ell)_{0,\Omega} \ + \ \langle \boldsymbol{\sigma}_\ell, y_\ell - \psi_\ell \rangle \ ,$$

where $x := (y, u, p)$ and $x_\ell := (y_\ell, u_\ell, p_\ell)$.

Then, the optimality conditions can be stated as

$$\nabla_x \mathcal{L}(x, \boldsymbol{\sigma})(\varphi) \ = \ 0 \quad , \quad \varphi \in X \ ,$$

$$\nabla_x \mathcal{L}_\ell(x_\ell, \boldsymbol{\sigma}_\ell)(\varphi_\ell) \ = \ 0 \quad , \quad \varphi_\ell \in X_\ell \ .$$

Primal-Dual Weighted Error Representation II

Theorem. Let $(x,\sigma) \in X$ and $(x_\ell,\sigma_\ell) \in X_\ell$ be the solutions of the continuous and discrete optimality systems, respectively. Then, there holds

$$J(y,u) - J_\ell(y_\ell,u_\ell) \;=\; -\frac{1}{2}\nabla_{xx}\mathcal{L}(x_\ell - x, x_\ell - x) \;+\; \langle \sigma, y_\ell - \psi\rangle \;+\; \mathrm{osc}_\ell^{(1)} \;,$$

where the data oscillations $\mathrm{osc}_\ell^{(1)}$ are given by

$$\mathrm{osc}_\ell^{(1)} \;:=\; \sum_{T\in\mathcal{T}_\ell(\Omega)} \mathrm{osc}_T^{(1)} \;,$$

$$\mathrm{osc}_T^{(1)} \;:=\; (y_\ell - y_\ell^d, y_\ell^d - y^d)_{0,T} \;+\; \frac{1}{2}\,\|y^d - y_\ell^d\|_{0,T}^2 \;+\; \alpha\,(u_\ell - u_\ell^d, u_\ell^d - u^d)_{0,T} \;+\; \frac{\alpha}{2}\,\|u^d - u_\ell^d\|_{0,T}^2 \;.$$

Remark: In the unconstrained case, i.e., $\sigma = \sigma_\ell = 0$, the above result reduces to the error representation in [Becker, Kapp, and Rannacher (2000)].

Interpolation Operators (State Constraints)

We introduce interpolation operators

$$i_\ell^y : V^{\bar{r}} \to V_\ell \ , \ r > \bar{r} > 2 \quad , \quad i_\ell^p : V^{\bar{s}} \to V_\ell \ , \ 0 < \bar{s} < s < 2 \ ,$$

such that for all $y \in V^r$ and $p \in V^s$ there holds

$$\Big(h_T^{r(t-1)} \ \|i_\ell^y y - y\|_{t,r,T}^r \Big)^{1/r} \lesssim \|y\|_{1,r,D_T} \ , \ 0 \le t \le 1 \ ,$$

$$\Big(h_T^{-r} \ \|i_\ell^y y - y\|_{0,r,T}^r \ + h_T^{-r/2} \ \|i_\ell^y y - y\|_{0,r,\partial T}^r \Big)^{1/r} \lesssim \|y\|_{1,r,D_T} \ ,$$

$$\Big(h_T^{-s} \ \|i_\ell^p p - p\|_{0,s,T}^s \ + h_T^{-s/2} \ \|i_\ell^p p - p\|_{0,s,\partial T}^s \Big)^{1/s} \lesssim h_T \ \|p\|_{1,s,D_T} \ ,$$

where $D_T := \{T' \in \mathcal{T}_\ell(\Omega) \mid \mathcal{N}_\ell(T') \cap \mathcal{N}_\ell(T) \neq \emptyset\}$.

Primal-Dual Weighted Error Representation III

Theorem. Under the assumptions of the previous Theorem let $\mathrm{i}_\ell^{\mathrm{z}}, \mathrm{z} \in \{\mathrm{y}, \mathrm{p}\}$, be the interpolation operators introduced before. Then, there holds

$$J(y,u) - J_\ell(y_\ell, u_\ell) \;=\; -(r(i_\ell^y y - y) \;+\; r(i_\ell^p p - p)) \;+\; \mu_\ell(x, \sigma) \;+\; \mathrm{osc}_\ell^{(1)} \;+\; \mathrm{osc}_\ell^{(2)} \;,$$

where $r(i_\ell^y y - y)$ and $r(i_\ell^p p - p)$ stand for the primal-dual weighted residuals

$$r(i_\ell^y y - y) \;:=\; \frac{1}{2}\,\big((y_\ell - y_\ell^d, i_\ell^y y - y)_{0,\Omega} \;+\; (\nabla(i_\ell^y y - y, \nabla p_\ell)_{0,\Omega} \;+\; \langle \sigma_\ell, i_\ell^y y - y\rangle\big) \;,$$

$$r(i_\ell^p p - p) \;:=\; \frac{1}{2}\,\big((\nabla(i_\ell^p p - p, \nabla y_\ell)_{0,\Omega} \;-\; (u_\ell, i_\ell^p p - p)_{0,\Omega}\big) \;.$$

Moreover, $\mu_\ell(x, \sigma)$ represents the primal-dual mismatch in complementarity

$$\mu_\ell(x, \sigma) \;:=\; \frac{1}{2}\,\big(\langle \sigma, y_\ell - \psi\rangle \;+\; \langle \sigma_\ell, \psi_\ell - y\rangle\big) \;,$$

and $\mathrm{osc}_\ell^{(2)}$ are further oscillation terms

$$\mathrm{osc}_\ell^{(2)} \;:=\; \sum_{T \in \mathcal{T}_\ell(\Omega)} \mathrm{osc}_T^{(2)} \quad , \quad \mathrm{osc}_T^{(2)} \;:=\; \frac{1}{2}\big((y^d - y_\ell^d, y_\ell - y)_{0,T} + \alpha\, (u^d - u_\ell^d, u_\ell - u)_{0,T}\big) \;.$$

Primal-Dual Weighted Residuals

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Primal-Dual Weighted Residuals

Theorem. The primal-dual residuals can be estimated according to

$$|r(i_\ell^y y - y)| \le C \sum_{T\in\mathcal{T}_\ell(\Omega)} \Big(\omega_T^y \rho_T^y + \omega_T^\sigma \rho_T^\sigma\Big) \quad , \quad |r(i_\ell^p p - p)| \le C \sum_{T\in\mathcal{T}_\ell(\Omega)} \omega_T^p \rho_T^p \ .$$

Here, ρ_T^y and ρ_T^p are L^r-norms and L^s-norms of the residuals associated with the state and the adjoint state equation

$$\rho_T^y := \Big(\|u_\ell\|_{0,r,T}^r + h_T^{-r/2} \, \|\frac{1}{2}\nu\cdot[\nabla y_\ell]\|_{0,r,\partial T}^r \Big)^{1/r} \ ,$$

$$\rho_T^p := \Big(\|y_\ell - y_\ell^d\|_{0,s,T}^s + h_T^{-s/2} \, \|\frac{1}{2}\nu\cdot[\nabla p_\ell]\|_{0,s,\partial T}^s \Big)^{1/s} \ .$$

The corresponding dual weights ω_T^y and ω_T^p are given by

$$\omega_T^y := \Big(\|i_\ell^p p - p\|_{0,s,T}^s + h_T^{s/2} \, \|i_\ell^p p - p\|_{0,s,\partial T}^s \Big)^{1/s} \ ,$$

$$\omega_T^p := \Big(\|i_\ell^y y - y\|_{0,r,T}^r + h_T^{r/2} \, \|i_\ell^y y - y\|_{0,r,\partial T}^r \Big)^{1/r} \ .$$

The residual ρ_T^σ and its dual weight ω_T^σ are given by

$$\rho_T^\sigma := n_a^{-1} \sum_{a\in\mathcal{N}_\ell(T)} \kappa_a \quad , \quad \omega_T^\sigma := \|i_\ell^y y - y\|_{2/r+\varepsilon,r,T} \ , \ 0 < \varepsilon < (r-2)/r \ .$$

Primal-Dual Mismatch in Complementarity

Primal-Dual Mismatch in Complementarity

The primal-dual mismatch $\mu_\ell(x,\boldsymbol{\sigma})$ can be made partially a posteriori in the following two particular cases (cf. [Bergounioux/Kunisch (2003)]):

Regular Case

The active set $\mathcal{A}$ is the union of a finite number of mutually disjoint, connected sets $\mathcal{A}_i$, $1 \le i \le m$, with $C^{1,1}$-boundary.

$$p|_{\mathcal{I}} \in H^2(\mathcal{I}) \ , \ p|_{int(\mathcal{A})} \in H^2(int(\mathcal{A}))$$

$$-\Delta p = y^d - y \text{ in } \mathcal{I} \ , \ p = -\alpha\Delta\psi \text{ in } \mathcal{A}$$

$$\boldsymbol{\sigma}_{\mathcal{A}} = \begin{cases} 0 & \text{on } \mathcal{I} \\ y^d - \psi - \alpha\Delta^2\psi & \text{on } \mathcal{A} \end{cases}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathcal{A}} + \boldsymbol{\sigma}_{\mathcal{F}},$$

$$\boldsymbol{\sigma}_{\mathcal{F}} = -\frac{\partial p|_{\mathcal{I}}}{\partial \boldsymbol{\nu}_{\mathcal{I}}} + \alpha\frac{\partial\Delta\psi}{\partial\boldsymbol{\nu}_{\mathcal{A}}}$$

Nonregular Case

The active set $\mathcal{A}$ is a Lipschitzian curve that divides Ω into two connected components Ω_+ and Ω_-.

$$(\nabla p, \nabla w)_{0,\Omega} = (y^d - y, w) - \langle\boldsymbol{\sigma}, w\rangle \ , \ w \in V^r$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathcal{A}} := \boldsymbol{\nu}_{\mathcal{A}} \cdot \nabla p|_{\mathcal{A}_+} - \boldsymbol{\nu}_{\mathcal{A}} \cdot \nabla p|_{\mathcal{A}_-}$$

Primal-Dual Mismatch in Complementarity

The primal-dual mismatch in complementarity has the representations

$$\mu_\ell|_{\mathcal{I}\cap\mathcal{I}_\ell} = \frac{1}{2}\,(\sigma_{\mathcal{F}}, y_\ell - \psi)_{0,\mathcal{F}\cap\mathcal{I}_\ell} + \frac{1}{2}\sum_{a\in\mathcal{N}_\ell(\mathcal{F}_\ell\cap\mathcal{I})} \kappa_a (y_\ell - y)(a),$$

$$\mu_\ell|_{\mathcal{I}\cap\mathcal{A}_\ell} = \frac{1}{2}\,(\sigma_{\mathcal{F}}, \psi_\ell - \psi)_{0,\mathcal{F}\cap\mathcal{A}_\ell} + \frac{1}{2}\sum_{a\in\mathcal{N}_\ell(\mathcal{I}\cap\mathcal{A}_\ell)} \kappa_a (\psi_\ell - y)(a),$$

$$\mu_\ell|_{\mathcal{A}\cap\mathcal{I}_\ell} = \frac{1}{2}\,(\sigma_{\mathcal{F}}, y_\ell - \psi)_{0,\mathcal{F}\cap\mathcal{I}_\ell} + \frac{1}{2}\,(y^d - \psi - \alpha\Delta^2\psi, y_\ell - \psi)_{0,\mathcal{A}\cap\mathcal{I}_\ell},$$

$$\mu_\ell|_{\mathcal{A}\cap\mathcal{A}_\ell} = \frac{1}{2}\,(\sigma_{\mathcal{F}}, \psi_\ell - \psi)_{0,\mathcal{F}\cap\mathcal{A}_\ell} + \frac{1}{2}(y^d - \psi - \alpha\Delta^2\psi, \psi_\ell - \psi)_{0,\mathcal{A}\cap\mathcal{A}_\ell}\;.$$

Hence, we need appropriate approximations of the continuous coincidence set $\mathcal{A}$, the continuous non-coincidence set $\mathcal{I}$, the continuous free boundary $\mathcal{F}$, and of $\sigma_{\mathcal{F}}$.

Primal-Dual Mismatch in Complementarity (State Constraints)

The coincidence set $\mathcal{A}$ and the non-coincidence set $\mathcal{I}$ will be approximated by

$$\hat{\mathcal{A}}_\ell := \bigcup \{T \in \mathcal{T}_\ell \mid \chi_\ell^{\mathcal{A}}(x) \ \geq \ 1 - \kappa h \text{ for all } x \in T\} \ ,$$
$$\hat{\mathcal{I}}_\ell := \bigcup \{T \in \mathcal{T}_\ell \mid \chi_\ell^{\mathcal{A}}(x) \ \leq \ 1 - \kappa h \text{ for some } x \in T\} \ ,$$

where

$$\chi_\ell^{\mathcal{A}} := I - \frac{\psi - i_\ell^y y_\ell}{\gamma h^r + \psi - i_\ell^y y_\ell} \quad , \quad 0 < \gamma \leq 1 \ , \ r > 0 \ .$$

Note that for $T \subset \mathcal{A}$ we have

$$\|\chi(\mathcal{A}) - \chi_\ell^{\mathcal{A}}\|_{0,T} \ \leq \ \min\left(|T|^{1/2}, \gamma^{-1} h^{-r} \|y - i_\ell^y y\|_{0,T}\right) \ \to \ 0 \quad \text{for} \quad \|y - i_\ell^y y\|_{0,T} = O(h^q) \ , \ q > r \ .$$

Moreover, $\boldsymbol{\sigma}_{\mathcal{F}}$ will be approximated by

$$\boldsymbol{\sigma}_{\hat{\mathcal{F}}_\ell} := \begin{cases} -\nu_{\hat{\mathcal{I}}_\ell} \cdot \nabla p_\ell|_{\hat{\mathcal{I}}_\ell} + \alpha \ \nu_{\hat{\mathcal{A}}_\ell} \cdot \nabla \Delta \psi & , \quad E \in \partial \mathcal{T}_\ell(\hat{\mathcal{A}}) \cap \partial \mathcal{T}_\ell(\hat{\mathcal{I}}) \\ \nu_{\hat{\mathcal{A}}_\ell} \cdot \nabla p_\ell|_{\hat{\mathcal{A}}_{\ell,+}} - \nu_{\hat{\mathcal{A}}_\ell} \cdot \nabla p_\ell|_{\hat{\mathcal{A}}_{\ell,-}} & , \quad E \in \mathcal{E}_\ell(\hat{\mathcal{A}}) \setminus (\partial \mathcal{T}_\ell(\hat{\mathcal{A}}) \cap \partial \mathcal{T}_\ell(\hat{\mathcal{I}})) \end{cases} \ .$$

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Primal-Dual Mismatch in Complementarity (State Constraints)

The primal-dual mismatch in complementarity can be estimated from above as follows:

$$|\mu_\ell|_{\mathcal{I}\cap\mathcal{I}_\ell}| \le \hat{\mu}_\ell^{(1)} + \hat{\mu}_\ell^{(2)} \;,\; |\mu_\ell|_{\mathcal{I}\cap\mathcal{A}_\ell}| \le \hat{\mu}_\ell^{(1)} + \hat{\mu}_\ell^{(3)} \;,\; |\mu_\ell|_{\mathcal{A}\cap\mathcal{I}_\ell}| \le \hat{\mu}_\ell^{(1)} + \hat{\mu}_\ell^{(4)} \;,\; |\mu_\ell|_{\mathcal{A}\cap\mathcal{A}_\ell}| \le \hat{\mu}_\ell^{(1)} + \hat{\mu}_\ell^{(5)} \;.$$

where

$$\hat{\mu}_\ell^{(1)} := \sum_{\mathrm{E}\in\mathcal{E}_\ell(\hat{\mathcal{F}}_\ell)} \hat{\mu}_\mathrm{E}^{(1)} \quad , \quad \hat{\mu}_\mathrm{E}^{(1)} := \frac{1}{2}\|\sigma_{\hat{\mathcal{F}}_\ell}\|_{0,\mathrm{E}}\|\mathrm{y}_\ell - \psi\|_{0,\mathrm{E}} \;,$$

$$\hat{\mu}_\ell^{(2)} := \sum_{\mathrm{E}\in\mathcal{E}_\ell(\mathcal{F}_\ell\cap\hat{\mathcal{I}}_\ell)} \hat{\mu}_\mathrm{E}^{(2)} \quad , \quad \hat{\mu}_\mathrm{E}^{(2)} := \frac{1}{2}\sum_{\mathrm{a}\in\mathcal{N}_\ell(\mathrm{E})} |(\mathrm{y}_\ell - \mathrm{i}_\ell^\mathrm{y}\mathrm{y}_\ell)(\mathrm{a})| \;\kappa_\mathrm{a} \;,$$

$$\hat{\mu}_\ell^{(3)} := \sum_{\mathrm{T}\in\mathcal{T}_\ell(\hat{\mathcal{I}}_\ell\cap\mathcal{A}_\ell)} \hat{\mu}_\mathrm{T}^{(3)} \quad , \quad \hat{\mu}_\mathrm{T}^{(3)} := \frac{1}{2}\sum_{\mathrm{a}\in\mathcal{N}_\ell(\mathrm{T})} |\mathrm{y}_\ell - \mathrm{i}_\ell^\mathrm{y}\mathrm{y}_\ell)(\mathrm{a})| \;\kappa_\mathrm{a}, \;,$$

$$\hat{\mu}_\ell^{(4)} := \sum_{\mathrm{T}\in\mathcal{T}_\ell(\hat{\mathcal{A}}_\ell\cap\mathcal{I}_\ell)} \hat{\mu}_\mathrm{T}^{(4)} \quad , \quad \hat{\mu}_\mathrm{T}^{(4)} := \frac{1}{2}\|\mathrm{y}^\mathrm{d} - \psi - \alpha\Delta^2\psi\|_{0,\mathrm{T}} \;\|\mathrm{y}_\ell - \psi\|_{0,\mathrm{T}} \;,$$

$$\hat{\mu}_\ell^{(5)} := \sum_{\mathrm{T}\in\mathcal{T}_\ell(\hat{\mathcal{A}}_\ell\cap\mathcal{A}_\ell)} \hat{\mu}_\mathrm{T}^{(5)} \quad , \quad \hat{\mu}_\mathrm{T}^{(5)} := \frac{1}{2}\|\mathrm{y}^\mathrm{d} - \psi - \alpha\Delta^2\psi\|_{0,\mathrm{T}} \;\|\psi_\ell - \psi\|_{0,\mathrm{T}} \;.$$

Primal-Dual Weighted Data Oscillations

Primal-Dual Weighted Data Oscillations

The data oscillations $\mathrm{osc}_\ell^{(2)}$ as given by

$$\mathrm{osc}_\ell^{(2)} := \sum_{T \in \mathcal{T}_\ell(\Omega)} \mathrm{osc}_T^{(2)} \quad , \quad \mathrm{osc}_T^{(2)} := \frac{1}{2}\Big((y^d - y_\ell^d, y_\ell - y)_{0,T} + \alpha \, (u^d - u_\ell^d, u_\ell - u)_{0,T} \Big) ,$$

can be estimated from above according to

$$\mathrm{osc}_\ell^{(2)} \preceq \sum_{T \in \mathcal{T}_\ell(\Omega)} \widehat{\mathrm{osc}}_T^{(2)} \quad , \quad \widehat{\mathrm{osc}}_T^{(2)} := \hat{\omega}_T^p \, \|u^d - u_\ell^d\|_{0,T} \; + \; \hat{\omega}_T^y \, \|y^d - y_\ell^d\|_{0,T} \; + \; \alpha \|u^d - u_\ell^d\|_{0,T}^2 ,$$

where the weights $\hat{\omega}_T^p$ and $\hat{\omega}_T^y$ are given by

$$\hat{\omega}_T^p := \|i_\ell^p p_\ell - p_\ell\|_{0,T} \quad , \quad \hat{\omega}_T^y := \|i_\ell^y y_\ell - y_\ell\|_{0,T} .$$

State Constraints: Numerical Results

Numerical Results: Distributed Control Problem with State Constraints I

$$\text{Minimize} \quad J(y,u) := \frac{1}{2}\,\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\,\|u - u^d\|_{0,\Omega}^2 \quad \text{over} \ (y,u) \in H_0^1(\Omega) \times L^2(\Omega)$$

$$\text{subject to} \quad -\Delta y = u \ \text{ in } \Omega \quad , \quad y \in K := \{v \in H_0^1(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}$$

Data: $\Omega := (-2,+2)^2 \quad , \quad y^d(r) := y(r) + \Delta p(r) + \sigma(r) \quad , \quad u^d(r) := u(r) + \alpha^{-1} p(r) \ ,$

$\psi := 0 \quad , \quad \alpha := 0.1 \ ,$

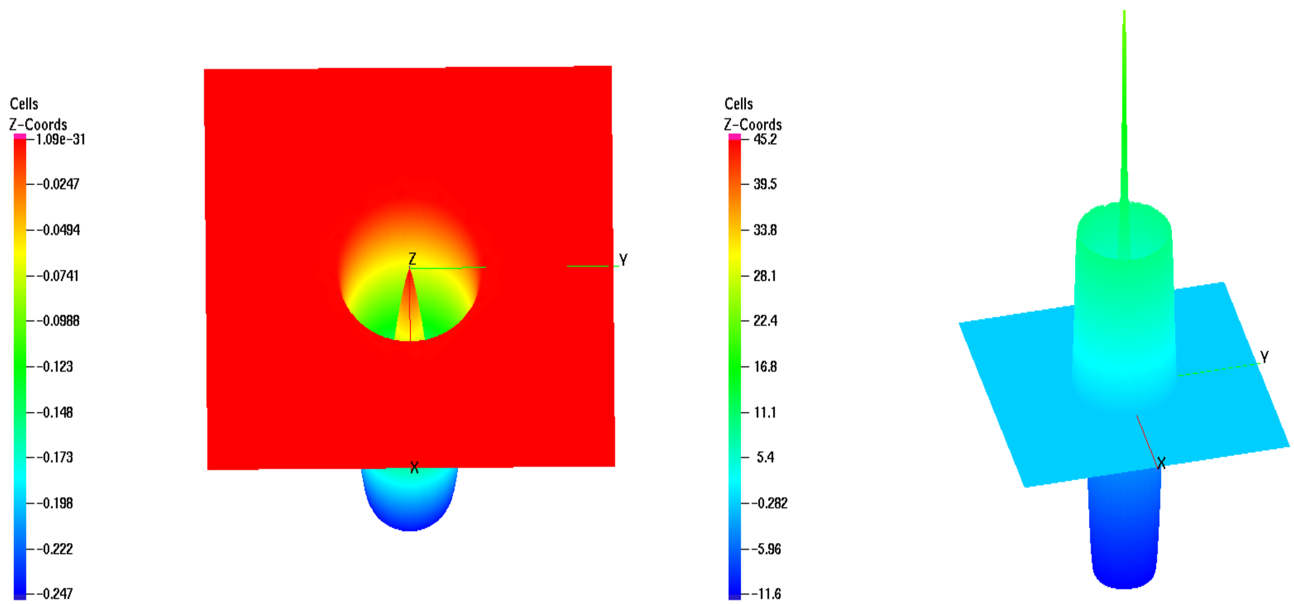
where $y(r), u(r), p(r), \sigma(r)$ is the solution of the problem:

$$y(r) := -r^{4/3} + \gamma_1(r) \ , \ u(r) = -\Delta y(r) \ , \ p(r) = \gamma_2(r) + r^4 - \frac{3}{2} r^3 + \frac{9}{16} r^2 \ , \ \sigma(r) := \begin{cases} 0.0 & , \quad r < 0.75 \\ 0.1 & , \quad \text{otherwise} \end{cases}$$

$$, \qquad \gamma_1 := \begin{cases} 1 & , \quad r < 0.25 \\ -192(r-0.25)^5 + 240(r-0.25)^4 - 80(r-0.25)^3 + 1 & , \quad 0.25 < r < 0.75 \\ 0 & , \quad \text{otherwise} \end{cases} ,$$

$$\gamma_2 := \begin{cases} 1 & , \quad r < 0.75 \\ 0 & , \quad \text{otherwise} \end{cases} .$$

Numerical Results: Distributed Control Problem with State Constraints I



Optimal state (left) and optimal control (right)

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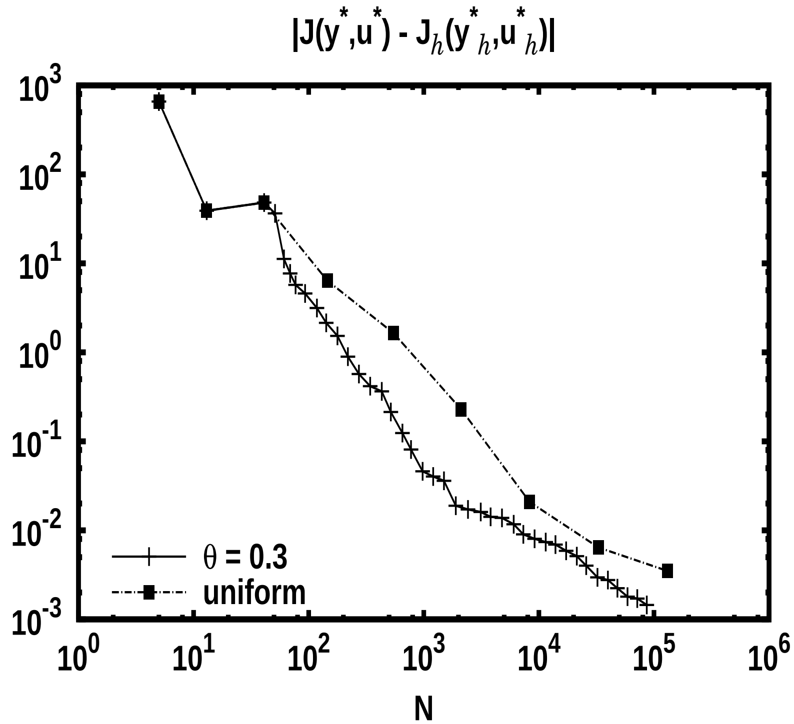
Numerical Results: Distributed Control Problem with State Constraints I



Decrease in the quantity of interest versus total number of DOFs

Numerical Results: Distributed Control Problem with State Constraints II

$$\text{Minimize} \quad J(y,u) := \frac{1}{2}\,\|y - y^d\|_{0,\Omega}^2 + \frac{\alpha}{2}\,\|u - u^d\|_{0,\Omega}^2 \quad \text{over} \ (y,u) \in H^1(\Omega) \times L^2(\Omega)$$

$$\text{subject to} \quad -\Delta y + cy = u \ \text{ in } \Omega \quad , \quad y \in K := \{v \in H^1(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}$$

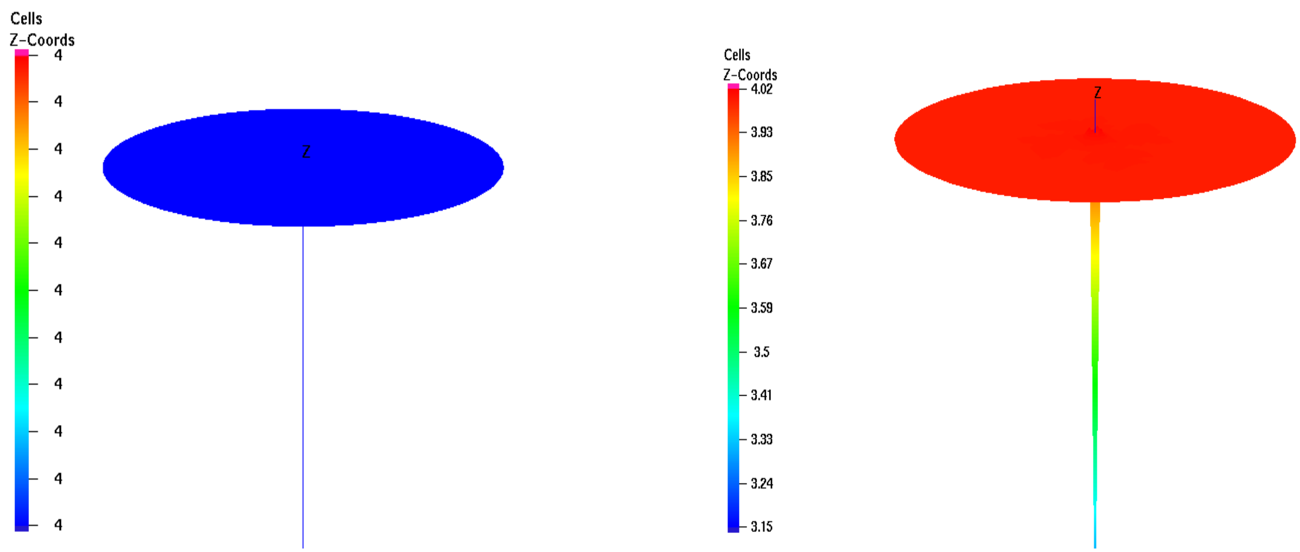
Data: $\Omega = B(0,1) := \{x = (x_1,x_2)^T \mid x_1^2 + x_2^2 < 1\}$, $y^d(r) := 4 + \frac{1}{\pi} - \frac{1}{4\pi}r^2 + \frac{1}{2\pi}\ln(r)$,

$u^d(r) := 4 + \frac{1}{4\pi}r^2 - \frac{1}{2\pi}\ln(r)$, $\psi := 4 + r$, $\alpha := 1$.

The solution $y(r), u(r), p(r), \sigma(r)$ of the problem is given by

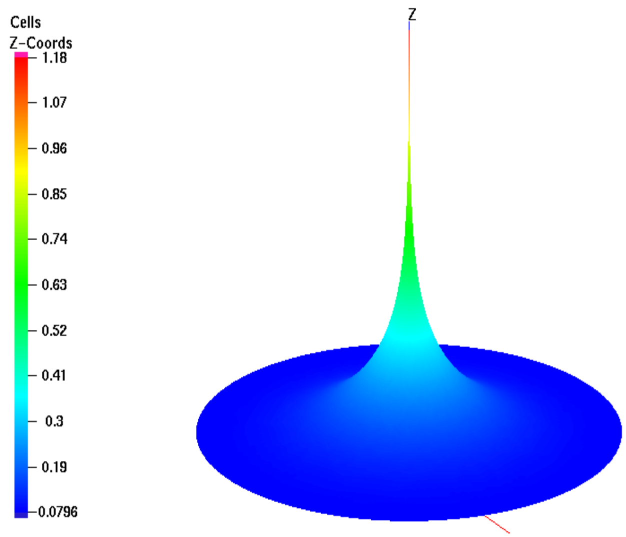
$$y(r) \equiv 4 \quad , \quad u(r) \equiv 4 \quad , \quad p(r) = \frac{1}{4\pi}r^2 - \frac{1}{2\pi}\ln(r) \quad , \quad \sigma(r) = \delta_0 \ .$$

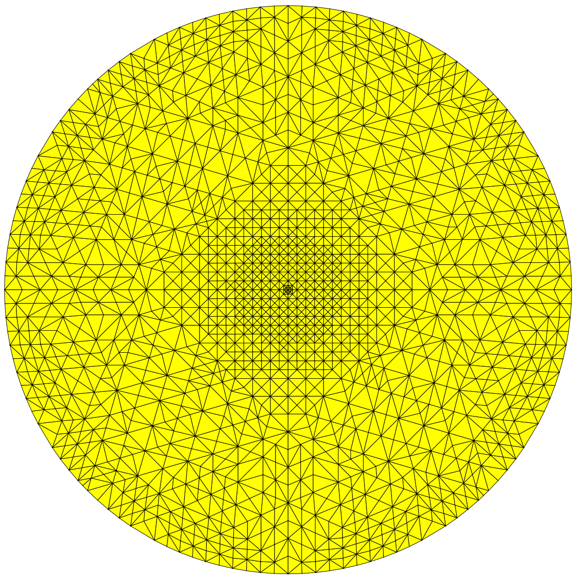
Numerical Results: Distributed Control Problem with State Constraints II



Optimal state (left) and optimal control (right)

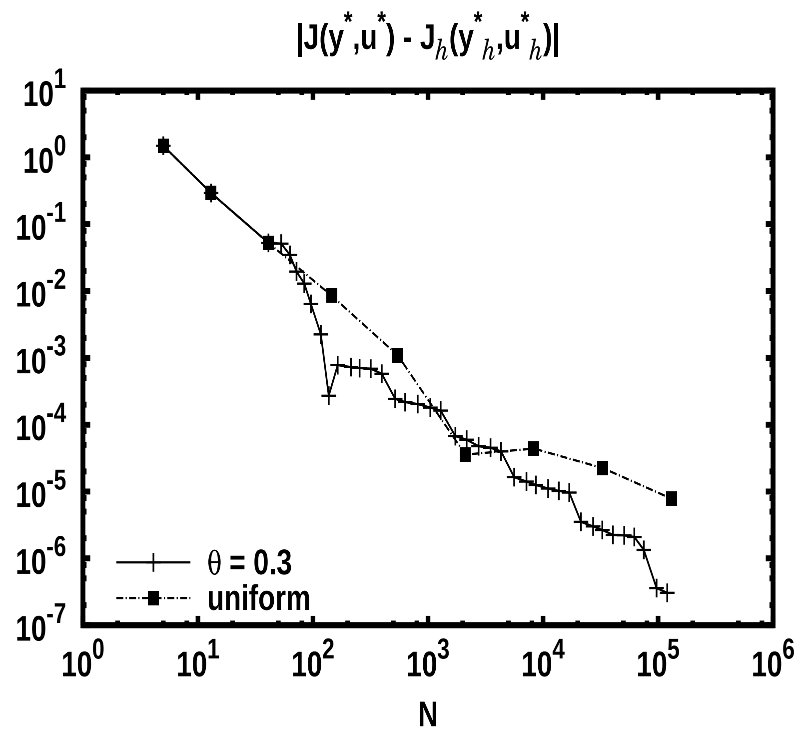
Numerical Results: Distributed Control Problem with State Constraints II





Optimal adjoint state (left) and mesh after 16 adaptive loops (right)

Numerical Results: Distributed Control Problem with State Constraints II



Decrease in the quantity of interest versus total number of DOFs

Control Constrained Elliptic Control Problems

A Posteriori Error Analysis of AFEM for Optimal Control Problems

(i) Unconstrained problems

R. Becker, H. Kapp, R. Rannacher (2000)   R. Becker, R. Rannacher (2001)

(ii) Control constrained problems

W. Liu and N. Yan (2000/01)   R. Li, W. Liu, H. Ma, and T. Tang (2002)

M. Hintermüller/H. et al. (2006)   A. Gaevskaya/H. et al. (2006/07)

A. Gaevskaya/H. and S. Repin (2006/07)   M. Hintermüller/H. (2007)

B. Vexler and W. Wollner (2007)

Model Problem (Distributed Elliptic Control Problem with Control Constraints)

Given a bounded domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\Gamma = \partial\Omega$, a function $y^d, \psi \in L^2(\Omega)$, and $\alpha > 0$, consider the distributed optimal control problem

$$
\begin{array}{ll}
\text{Minimize} & J(y,u) := \dfrac{1}{2} \|y - y^d\|_{0,\Omega}^2 + \dfrac{\alpha}{2} \|u\|_{0,\Omega}^2 , \\
\text{over} & (y,u) \in H_0^1(\Omega) \times L^2(\Omega) , \\
\text{subject to} & -\Delta y = u , \\
 & u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\} .
\end{array}
$$

Optimality Conditions for the Distributed Control Problem

There exists an adjoint state $p \in H_0^1(\Omega)$ and an adjoint control $\sigma \in L^2(\Omega)$ such that the quadruple (y, p, u, σ) satisfies

$$\begin{aligned} a(y, v) &= (u, v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \; , \\ a(p, v) &= (y^d - y, v)_{0,\Omega} \quad , \quad v \in H_0^1(\Omega) \; , \\ \alpha\, u &= p - \sigma \; , \\ \sigma \geq 0 \; , \; u \leq \psi \quad &, \quad (\sigma; u - \psi)_{0,\Omega} = 0 \; , \end{aligned}$$

where $a(\cdot,\cdot)$ stands for the bilinear form

$$a(w, z) = \int_\Omega \nabla w \cdot \nabla z \, dx \quad , \quad w, z \in H_0^1(\Omega) \; .$$

Finite Element Approximation of the Distributed Control Problem

Let $\mathcal{T}_\ell(\Omega)$ be a shape regular, simplicial triangulation of Ω and let

$$V_\ell \;:=\; \{\; v_\ell \in C(\Omega) \mid v_\ell|_T \in P_{k_1}(T)\;,\; T \in \mathcal{T}_\ell(\Omega)\;,\; k_1 \in \mathbb{N}\;,\; v_H|_{\partial\Omega} = 0 \;\}$$

be the FE space of continuous, piecewise polynomial functions (of degree k_1) and

$$W_\ell \;:=\; \{\; w_\ell \in L^2(\Omega) \mid w_\ell|_T \in P_{k_2}(T)\;,\; T \in \mathcal{T}_\ell(\Omega)\;,\; k_2 \in \mathbb{N} \cup \{0\} \;\}$$

the linear space of elementwise polynomial functions (of degree k_2).
Consider the following FE Approximation of the distributed control problem

$$\begin{aligned}
&\text{Minimize} && J_\ell(y_\ell, u_\ell) \;:=\; \frac{1}{2}\,\|y_\ell - y_\ell^d\|_{0,\Omega}^2 \;+\; \frac{\alpha}{2}\,\|u_\ell\|_{0,\Omega}^2\;,\\
&\text{over} && (y_\ell, u_\ell) \in V_\ell \times W_\ell\;,\\
&\text{subject to} && a(y_\ell, v_\ell) \;=\; (u_\ell, v_\ell)_{0,\Omega}\;,\; v_\ell \in V_\ell\;,\\
& && u_\ell \in K_\ell \;:=\; \{w_\ell \in W_\ell \mid w_\ell|_T \le \psi_\ell|_T\;,\; T \in \mathcal{T}_\ell(\Omega)\}\;.
\end{aligned}$$

where $\psi_\ell \in W_\ell$ is the discrete control constraint.

Optimality Conditions for the FE Discretized Control Problem

There exists an adjoint state $p_\ell \in V_\ell$ and an adjoint control $\boldsymbol{\sigma}_\ell \in W_\ell$ such that the quadruple $(y_\ell, u_\ell, p_\ell, \boldsymbol{\sigma}_\ell)$ satisfies

$$\begin{aligned} a(y_\ell, v_\ell) &= (u_\ell, v_\ell)_{0,\Omega} \quad , \quad v_\ell \in V_\ell \, , \\ a(p_\ell, v_\ell) &= (y_\ell^d - y, v_\ell)_{0,\Omega} \quad , \quad v_\ell \in V_\ell \, , \\ \alpha \; u_\ell &= M_\ell \, p_\ell - \boldsymbol{\sigma}_\ell \, , \\ \boldsymbol{\sigma}_\ell \geq 0 \; , \; u_\ell \leq \psi_\ell \quad &, \quad (\boldsymbol{\sigma}_\ell, u_\ell - \psi_\ell)_{0,\Omega} = 0 \, , \end{aligned}$$

where $y_\ell^d \in V_\ell$ and $M_\ell : V_\ell \to W_\ell$, e.g., for $k_2 = 0$:

$$(M_\ell v_\ell)|_T := |T|^{-1} \int_T v_\ell \, dx \quad , \quad T \in \mathcal{T}_\ell(\Omega) \, .$$

Primal-Dual Weighted Error Representation (Control Constraints)

Theorem. Let $(\mathrm{x},\sigma) \in \mathrm{X} \times \mathrm{L}^2(\Omega)$ and $(\mathrm{x}_\ell,\sigma_\ell) \in \mathrm{X}_\ell \times \mathrm{W}_\ell$ be the solutions of the continuous and discrete optimality systems, respectively. Then, there holds

$$\mathrm{J}(\mathrm{y},\mathrm{u}) - \mathrm{J}_\ell(\mathrm{y}_\ell,\mathrm{u}_\ell) \;=\; -\frac{1}{2}\nabla_{\mathrm{xx}}\mathcal{L}(\mathrm{x}_\ell - \mathrm{x}, \mathrm{x}_\ell - \mathrm{x}) \;+\; (\sigma, \mathrm{u}_\ell - \psi)_{0,\Omega} \;+\; \mathrm{osc}_\ell^{(1)} \;.$$

Remark: In the unconstrained case, i.e., $\sigma = \sigma_\ell = 0$, the above result reduces to the error representation in [Becker, Kapp, and Rannacher (2000)].

Primal-Dual Weighted Error Representation (Control Constraints)

Theorem. Under the assumptions of the previous Theorem let $i_\ell^z, z \in \{y,u,p\}$, be the interpolation operators introduced before. Then, there holds

$$J(y,u) - J_\ell(y_\ell,u_\ell) \;=\; -\Big(r(i_\ell^y y - y) \;+\; r(i_\ell^p p - p) \;+\; r(i_\ell^u u - u)\Big) \;+\; \mu_\ell(x,\sigma) \;+\; \text{osc}_\ell^{(1)} \;+\; \text{osc}_\ell^{(2)} ,$$

where $r(i_\ell^y y - y)$, $r(i_\ell^p p - p)$ and $r(i_\ell^u u - u)$ stand for the primal-dual weighted residuals

$$r(i_\ell^y y - y) \;:=\; \frac{1}{2}\,\Big((y_\ell - y_\ell^d, i_\ell^y y - y)_{0,\Omega} \;+\; (\nabla(i_\ell^y y - y), \nabla p_\ell)_{0,\Omega}\Big) ,$$

$$r(i_\ell^p p - p) \;:=\; \frac{1}{2}\,\Big((\nabla(i_\ell^p p - p), \nabla y_\ell)_{0,\Omega} \;-\; (u_\ell, i_\ell^p p - p)_{0,\Omega}\Big) , \quad r(i_\ell^u u - u) \;:=\; \frac{1}{2}\,(M_\ell p_\ell - p_\ell, i_\ell^u u - u)_{0,\Omega} .$$

Moreover, $\mu_\ell(x,\sigma)$ represents the primal-dual mismatch in complementarity

$$\mu_\ell(x,\sigma) \;:=\; \frac{1}{2}\,\Big((\sigma, u_\ell - \psi)_{0,\Omega} \;+\; (\sigma_\ell, \psi_\ell - u)_{0,\Omega}\Big) ,$$

and $\text{osc}_\ell^{(2)}$ is a further oscillation term

$$\text{osc}_\ell^{(2)} \;:=\; \sum_{T \in \mathcal{T}_\ell(\Omega)} \text{osc}_T^{(2)} \quad , \quad \text{osc}_T^{(2)} \;:=\; \frac{1}{2}\,(y^d - y_\ell^d, y_\ell - y)_{0,T} .$$

Primal-Dual Weighted Residuals (Control Constraints)

Theorem. The primal-dual residuals can be estimated according to

$$|r(i_\ell^y y - y)| \ \leq \ C \sum_{T \in \mathcal{T}_\ell(\Omega)} \omega_T^y \rho_T^y \quad , \quad |r(i_\ell^p p - p)| \ \leq \ C \sum_{T \in \mathcal{T}_\ell(\Omega)} \left(\omega_T^p \rho_T^{p,1} \ + \ \omega_T^u \rho_T^{p,2} \right) \ .$$

Here, ρ_T^y and $\rho_T^{p,1}$ are L^2-norms of the residuals associated with the state and the adjoint state

$$\rho_T^y \ := \ \Big(\ \|u_\ell\|_{0,T}^2 \ + \ h_T^{-1} \ \|\frac{1}{2}\nu \cdot [\nabla y_\ell]\|_{0,\partial T}^2 \ \Big)^{1/2} \ ,$$

$$\rho_T^{p,1} \ := \ \Big(\ \|y_\ell - y_\ell^d\|_{0,T}^2 \ + \ h_T^{-1} \ \|\frac{1}{2}\nu \cdot [\nabla p_\ell]\|_{0,\partial T}^2 \ \Big)^{1/2} \ .$$

The corresponding dual weights ω_T^u and ω_T^p are given by

$$\omega_T^y \ := \ \Big(\ \|i_\ell^p p - p\|_{0,T}^2 \ + \ h_T \ \|i_\ell^p p - p\|_{0,\partial T}^2 \ \Big)^{1/2} \ ,$$

$$\omega_T^p \ := \ \Big(\ \|i_\ell^y y - y\|_{0,T}^2 \ + \ h_T \ \|i_\ell^y y - y\|_{0,\partial T}^2 \ \Big)^{1/2} \ .$$

The residual $\rho_T^{p,2}$ and its dual weight ω_T^u are given by

$$\rho_T^{p,2} \ := \ \|M_\ell p_\ell - p_\ell\|_{0,T} \quad , \quad \omega_T^u \ := \ \|i_\ell^u u - u\|_{0,T} \ .$$

Primal-Dual Mismatch in Complementarity (Control Constraints)

Using the complementarity conditions

$$u \le \psi \quad , \quad \sigma \ge 0 \quad , \quad (\sigma, u - \psi)_{0,\Omega} = 0 \quad , \quad \alpha u - p + \sigma = 0 \, ,$$

$$u_\ell \le \psi_\ell \quad , \quad \sigma_\ell \ge 0 \quad , \quad (\sigma_\ell, u_\ell - \psi_\ell)_{0,\Omega} = 0 \quad , \quad \alpha u_\ell - M_\ell \, p_\ell + \sigma_\ell = 0 \, ,$$

the primal-dual mismatch $\mu_\ell := \mu_\ell(x, \sigma)$ can be further assessed according to

$$\begin{aligned}
\mu_\ell(\mathcal{I} \cap \mathcal{I}_\ell) &= 0 \, , \\
\mu_\ell(\mathcal{A} \cap \mathcal{A}_\ell) &= \frac{1}{2} \, (\sigma + \sigma_\ell, \psi_\ell - \psi)_{0,\mathcal{A} \cap \mathcal{A}_\ell} \, , \\
\mu_\ell(\mathcal{I} \cap \mathcal{A}_\ell) &= \frac{1}{2} \, (\sigma_\ell, \psi_\ell - \alpha^{-1} p)_{0,\mathcal{I} \cap \mathcal{A}_\ell}, \\
\mu_\ell(\mathcal{A} \cap \mathcal{I}_\ell) &= \frac{\alpha}{2} \, \|u - u_\ell\|^2_{0,\mathcal{I} \cap \mathcal{A}_\ell} \; + \; \frac{1}{2} \, (p - M_\ell \, p_\ell, u_\ell - u)_{0,\mathcal{I} \cap \mathcal{A}_\ell}.
\end{aligned}$$

and we finally obtain

$$|\mu_\ell(\mathcal{I} \cap \mathcal{A}_\ell) \; + \; \mu_\ell(\mathcal{A} \cap \mathcal{I}_\ell)| \; \le \; \nu_\ell$$

with a fully computable a posteriori term ν_ℓ (consistency error).

Numerical Results: Distributed Control Problem with Control Constraints I

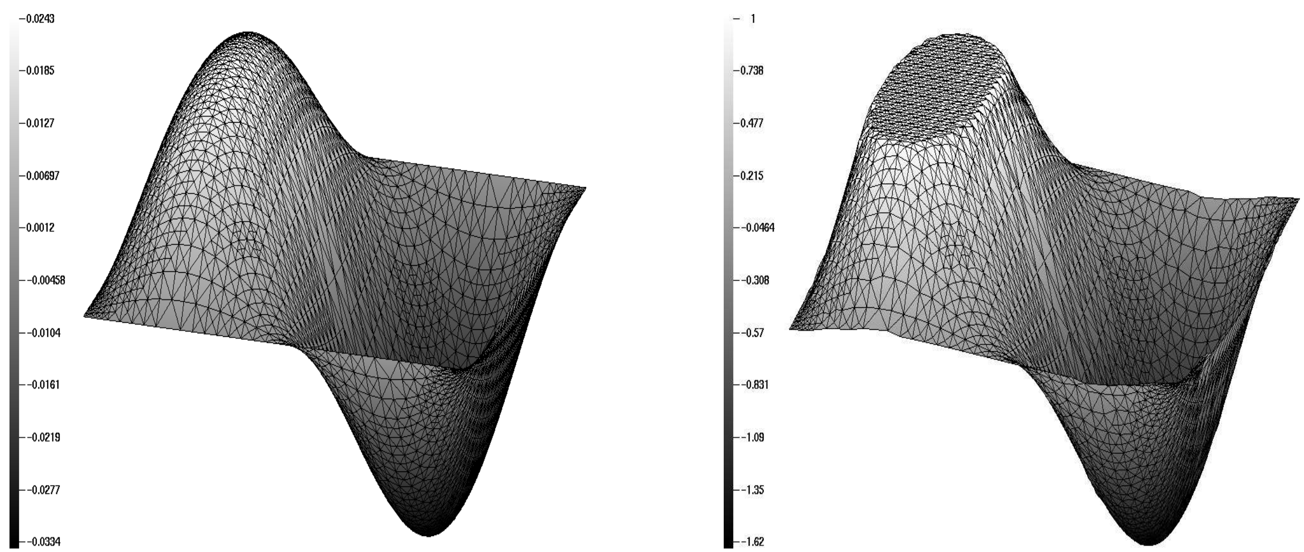
$$
\begin{aligned}
&\text{Minimize} \qquad && J(y,u) \ := \ \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} \ + \ \frac{\alpha}{2}\,\|u\|^2_{0,\Omega} \\
&\text{over} && (y,u) \in H^1_0(\Omega) \times L^2(\Omega) \\
&\text{subject to} && -\ \Delta y \ = \ u \quad \text{in } \Omega \ , \\
& && u \in K \ := \ \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}
\end{aligned}
$$

Data: $\Omega \ := (0,1)^2$,

$$
y^d \ := \begin{cases} 200\ x_1\ x_2\ (x_1 - 0.5)^2\ (1 - x_2)\ , & 0 \le x_1 \le 0.5 \\ 200\ (x_1 - 1)\ (x_2\ (x_1 - 0.5)^2\ (1 - x_2)\ , & 0.5 < x_1 \le 1 \end{cases} ,
$$

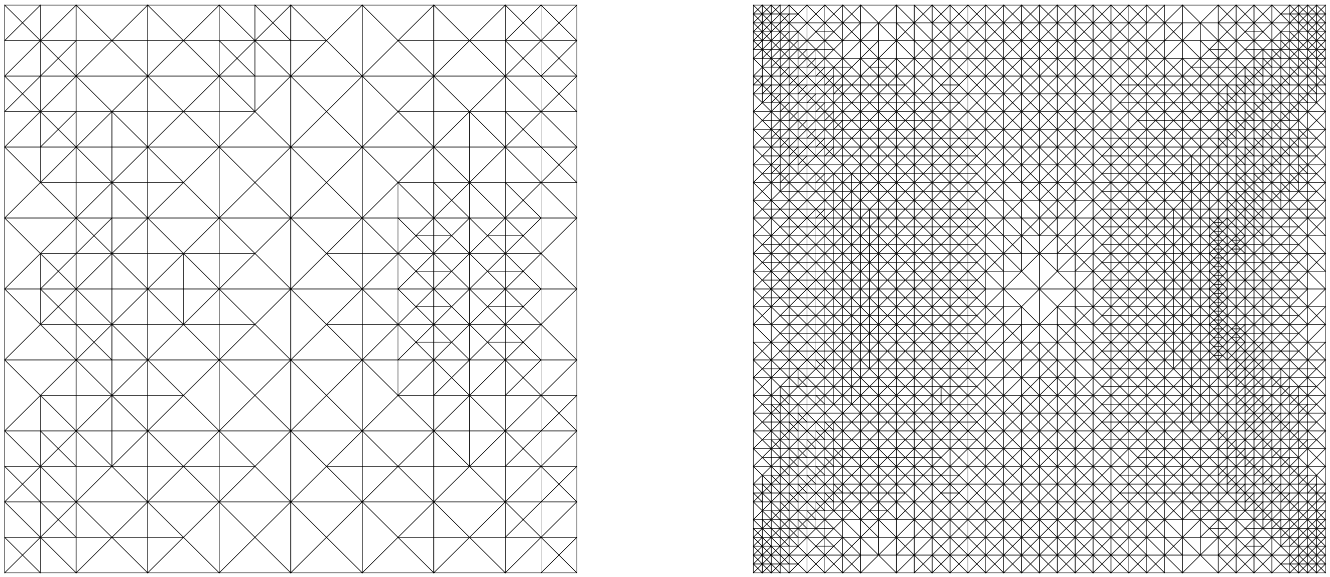
$\alpha \ = \ 0.01 \quad , \quad \psi \ = \ 1$.

Numerical Results: Distributed Control Problem with Control Constraints I



Optimal state (left) and optimal control (right)

Numerical Results: Distributed Control Problem with Control Constraints I



Grid after 6 (left) and 10 (right) refinement steps

Numerical Results: Distributed Control Problem with Control Constraints I

l	N_{dof}	δ_h	η_h	osc_h	ν_h
0	12	2.73E-03	1.47E-02	1.17E-01	0.00E+00
1	25	8.57E-04	2.03E-02	6.23E-02	2.04E-03
2	42	5.09E-04	1.42E-02	3.44E-02	4.86E-03
4	138	1.52E-04	4.61E-03	1.27E-02	1.66E-04
6	478	4.24E-05	1.35E-03	4.20E-03	3.67E-05
8	1706	9.91E-06	3.67E-04	2.08E-03	4.27E-06
10	6237	2.52E-06	9.95E-05	6.60E-04	3.82E-07
12	22639	5.92E-07	2.74E-05	1.63E-04	1.63E-07
14	81325	1.57E-07	7.57E-06	5.05E-05	7.60E-09
16	299028	4.65E-08	2.05E-06	1.58E-05	1.32E-09

Error (quantity of interest), estimator, oscillations, and consistency error

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Numerical Results: Distributed Control Problem with Control Constraints II

$$\begin{array}{ll} \text{Minimize} & J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u - u^d\|^2_{0,\Omega} \\ \text{over} & (y,u) \in H^1_0(\Omega) \times L^2(\Omega) \\ \text{subject to} & -\Delta y = f + u \quad \text{in } \Omega\,, \\ & u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\} \end{array}$$

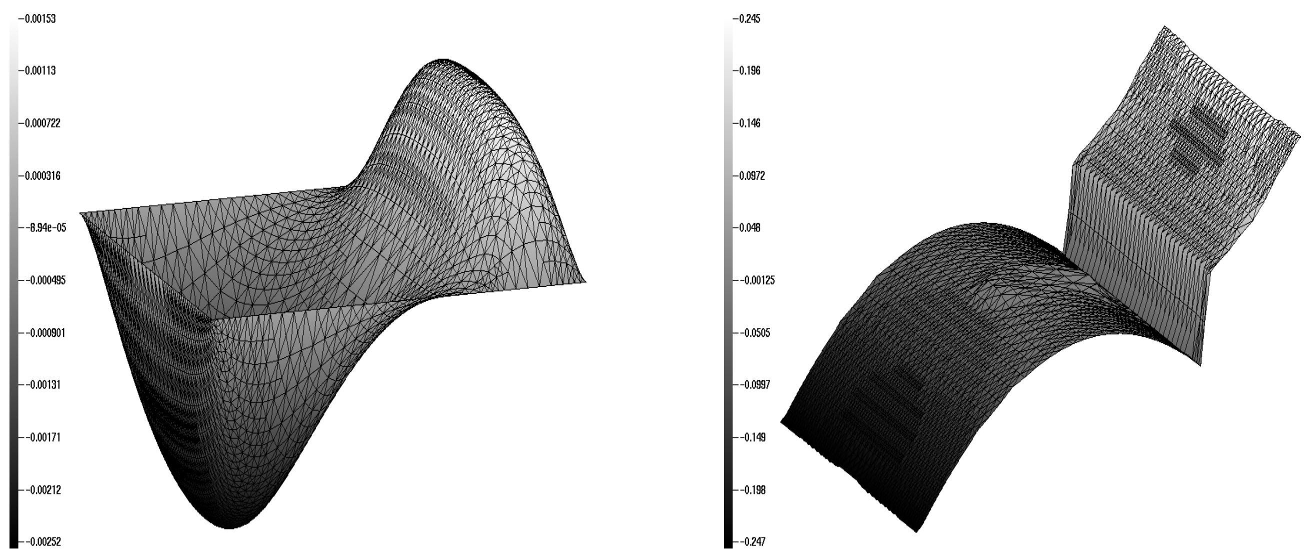
Data: $\Omega := (0,1)^2 \quad , \quad y^d := 0 \quad , \quad u^d := \hat{u} + \alpha^{-1}(\hat{\sigma} - \Delta^{-2}\hat{u})\,,$

$$\psi := \begin{cases} (x_1 - 0.5)^8 & , \quad (x_1,x_2) \in \Omega_1, \\ (x_1 - 0.5)^2 & , \quad \text{otherwise} \end{cases} \quad , \quad \alpha := 0.1 \quad , \quad f := 0$$

$$\hat{u} := \begin{cases} \psi & , \quad (x_1,x_2) \in \Omega_1 \cup \Omega_2, \\ -1.01\,\psi & , \quad \text{otherwise} \end{cases} \quad , \quad \hat{\sigma} := \begin{cases} 2.25\,(x_1 - 0.75)\cdot 10^{-4} & , \quad (x_1,x_2) \in \Omega_2, \\ 0 & , \quad \text{otherwise} \end{cases} \quad ,$$

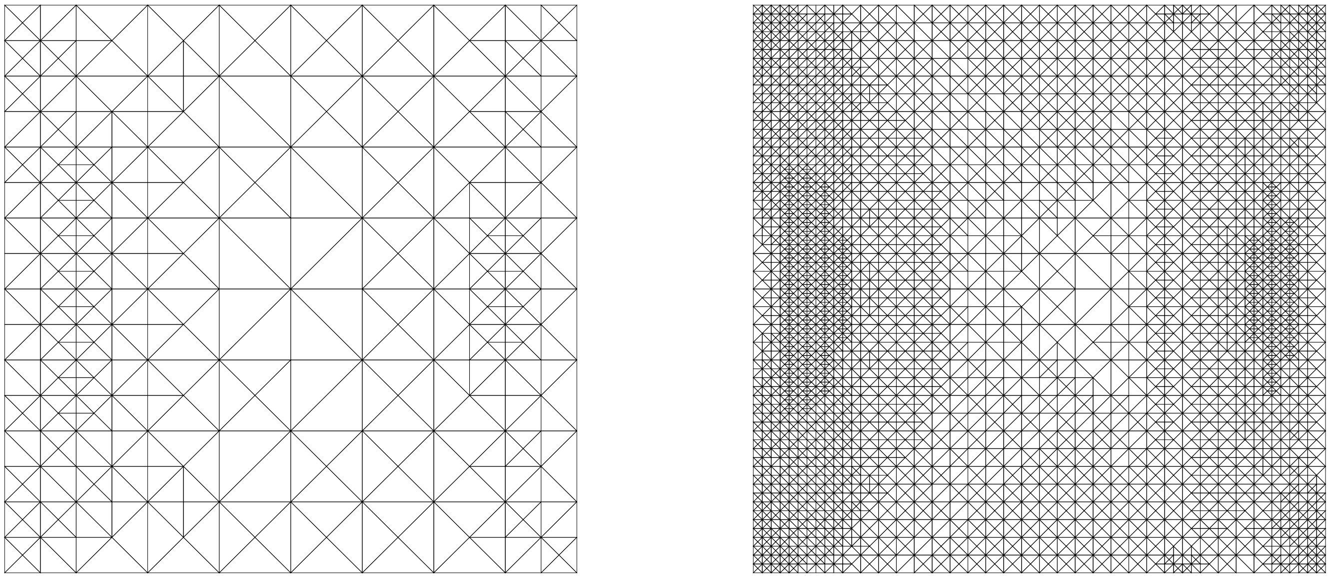
$$\Omega_1 := \{(x_1,x_2) \in \Omega \mid \Big((x_1 - 0.5)^2 + (x_2 - 0.5)^2\Big)^{1/2} \le 0.15\} \quad , \quad \Omega_2 := \{(x_1,x_2) \in \Omega \mid x_1 \ge 0.75\}\,.$$

Numerical Results: Distributed Control Problem with Control Constraints II



Optimal state (left) and optimal control (right)

Numerical Results: Distributed Control Problem with Control Constraints II



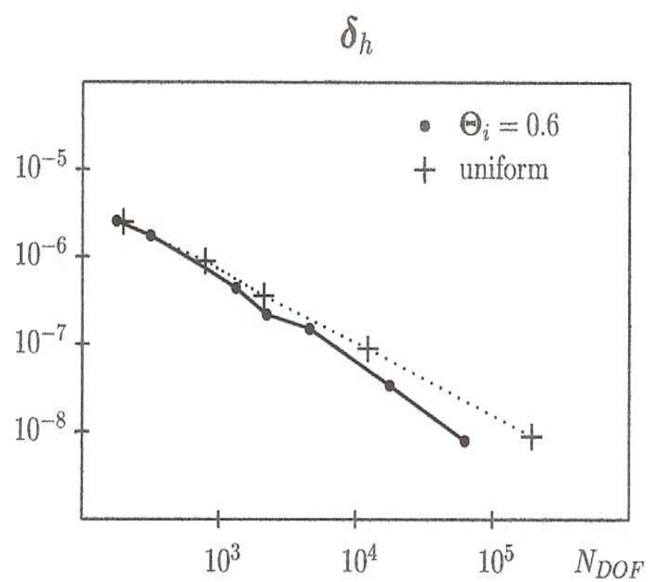
Grid after 6 (left) and 10 (right) refinement steps

Numerical Results: Distributed Control Problem with Control Constraints I

l	N_{dof}	δ_h	η_h	osc_h	ν_h
0	5	2.41E-04	2.58E-06	1.07E-01	0.00E+00
1	12	1.61E-04	5.26E-06	8.11E-02	2.71E-07
2	26	7.62E-05	4.78E-06	5.25E-02	4.19E-07
4	73	1.54E-05	2.08E-06	2.89E-02	0.00E+00
6	253	4.09E-06	6.45E-07	1.59E-02	0.00E+00
8	953	1.16E-06	1.79E-07	8.39E-03	9.86E-12
10	3507	3.41E-07	4.87E-08	4.70E-03	2.66E-13
12	12684	1.03E-07	1.33E-08	2.59E-03	3.08E-14
14	45486	2.99E-08	3.71E-09	1.52E-03	2.23E-15
16	165366	8.12E-09	1.05E-09	9.06E-04	2.65E-16

Error (quantity of interest), estimator, oscillations, and consistency error

Numerical Results: Distributed Control Problem with Control Constraints II



Decrease in the quantity of interest versus total number of DOFs