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Numerical Solution of Elliptic Optimal Control Problems

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Recent Books on Optimal Control of PDEs

R. Glowinski, J.L. Lions, and J. He; Exact and Approximate Controllability for Distributed Parameter Systems: A Numerical Approach. Cambridge University Press, Cambridge, 2008.

M. Hinze, R. Pinnau, and M. Ulbrich; Optimization with PDE Constraints. Springer, Berlin-Heidelberg-New York, 2008.

F. Tröltzsch; Optimal Control of Partial Differential Equations. Theory, Methods, and Applications. American Mathematical Society, Providence, 2010.

Numerical Solution of Elliptic Optimal Control Problems

- Unconstrained Elliptic Optimal Control Problems
- Control Constrained Elliptic Optimal Control Problems
- State Constrained Elliptic Optimal Control Problems

Elliptic Optimal Control Problems

Unconstrained Case

Elliptic Optimal Control Problems: Unconstrained Case

Consider the unconstrained distributed elliptic optimal control problem

$$\begin{aligned} \inf_{y,u} \; J(y,u) \; &:= \frac{1}{2} \int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx, \\ (EUC)_1 \qquad -\Delta y &= u \quad \text{in } \Omega, \\ (EUC)_2 \qquad y &= 0 \quad \text{on } \Gamma, \end{aligned}$$

where $u \in L^2(\Omega)$.

Existence and Uniqueness of a Solution

We introduce the control-to-state map

$$G : L^2(\Omega) \to H_0^1(\Omega) \quad , \quad y = Gu \quad \text{solves} \quad -\Delta y = u \ .$$

Substituting $y = Gu$ allows to consider the reduced control problem

$$\inf_u J_{red}(u) := \frac{1}{2}\|Gu - y^d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \ .$$

Theorem (Existence and uniqueness) The unconstrained elliptic control problem admits a unique solution $y \in H_0^1(\Omega)$.

Proof. Direct method of the calculus of variations (minimizing sequence).

Optimality Conditions for the Distributed Control Problem

There exists an adjoint state $p \in H_0^1(\Omega)$ such that the triple (y, p, u) satisfies

$$\begin{aligned} a(y, v) &= (u, v)_{L^2(\Omega)} \quad , \quad v \in H_0^1(\Omega) \ , \\ a(p, v) &= -\ (y - y^d, v)_{L^2(\Omega)} \quad , \quad v \in H_0^1(\Omega) \ , \\ p - \alpha u &= 0 \ , \end{aligned}$$

where $a(\cdot, \cdot)$ stands for the bilinear form

$$a(w, z) \ = \ \int_\Omega \nabla w \cdot \nabla z \, dx \quad , \quad w, z \in H_0^1(\Omega) \ .$$

Optimality Conditions for the FE Discretized Control Problem

There exists an adjoint state $p_h \in V_h$ such that the triple (y_h, p_h, u_h) satisfies

$$\begin{aligned} a(y_h, v_h) &= (u_h, v_h)_{L^2(\Omega)} \quad , \quad v_h \in V_h \ , \\ a(p_h, v_h) &= -\ (y_h - y^d, v_h)_{L^2(\Omega)} \quad , \quad v_h \in V_h \ , \\ p_h - \alpha u_h &= 0 \ . \end{aligned}$$

Finite Element Approximation of the Distributed Control Problem

Let $\mathcal{T}_h(\Omega)$ be a shape regular, simplicial triangulation of Ω and let

$$V_h := \{ v_h \in C(\Omega) \mid v_h|_T \in P_1(T) \ , \ T \in \mathcal{T}_h(\Omega) \ , \ v_h|_{\partial\Omega} = 0 \}$$

be the FE space of continuous, piecewise linear finite elements.

Consider the following FE Approximation of the distributed control problem

$$\begin{array}{lll} \text{Minimize} & J(y_h, u_h) \ := \ \frac{1}{2} \, \|y_h - y^d\|^2_{L^2(\Omega)} \ + \ \frac{\alpha}{2} \, \|u_h\|^2_{L^2(\Omega)} \ , \\ \text{over} & (y_h, u_h) \in V_h \times V_h \ , \\ \text{subject to} & a(y_h, v_h) \ = \ (u_h, v_h)_{L^2(\Omega)} \ , \ v_h \in V_h \ . \end{array}$$

Algebraic Formulation of the Discrete Optimality System

Let $V_\ell := \text{span}\{\varphi_\ell^{(1)}, \cdots, \varphi_\ell^{(N_\ell)}\}$, where $\varphi_\ell^{(i)}, 1 \le i \le N_\ell$, are the nodal basis functions. We refer to $M_\ell \in \mathbb{R}^{N_\ell \times N_\ell}$ and $A_\ell \in \mathbb{R}^{N_\ell \times N_\ell}$ as the associated mass matrix and stiffness matrix:

$$(M_\ell)_{ij} := (\varphi_\ell^{(i)}, \varphi_\ell^{(j)})_{L^2(\Omega)} \quad , \quad (A_\ell)_{ij} := a(\varphi_\ell^{(i)}, \varphi_\ell^{(j)}) \ , \ 1 \le i,j \le N_\ell.$$

The solution of the discrete optimality system requires the computation of $(y_\ell, p_\ell) \in \mathbb{R}^{N_\ell} \times \mathbb{R}^{N_\ell}$ as the solution of

$$\begin{pmatrix} A_\ell & -\alpha^{-1} M_\ell \\ M_\ell & A_\ell \end{pmatrix} \begin{pmatrix} y_\ell \\ p_\ell \end{pmatrix} = \begin{pmatrix} 0 \\ M_\ell y_\ell^d \end{pmatrix}.$$

Left and Right Transforming Iterations

For given $K \in \mathbb{R}^{N \times N}, d \in \mathbb{R}^N$, we want to solve the linear algebraic system

$$(*) \qquad Kx = d.$$

Let $K_L, K_R \in \mathbb{R}^{N \times N}$ be regular. Then, $(*)$ can be equivalently wriiten as

$$(**) \qquad K_L K K_R K_R^{-1} x = K_L d.$$

Assuming $\tilde{K}$ to be a suitable preconditioner for $K_L K K_R$, we consider the transforming iteration

$$K_R^{-1} x^{(\nu+1)} = K_R^{-1} x^{(\nu)} + \tilde{K}^{-1} (K_L d - K_L K x^{(\nu)}).$$

Backtransformation yields

$$x^{(\nu+1)} = x^{(\nu)} + \underbrace{K_R \tilde{K}^{-1} K_L}_{= (K_L^{-1} \tilde{K} K_R^{-1})^{-1}} (d - K x^{(\nu)}),$$

so that $\hat{K} := K_L^{-1} \tilde{K} K_R^{-1}$ is an appropriate preconditioner for the original system.

Left and Right Transforming Iterations II

We apply the idea of left/right transformations to

$$K = \begin{pmatrix} A_\ell & -\alpha^{-1}M_\ell \\ M_\ell & A_\ell \end{pmatrix} \quad , \quad d = \begin{pmatrix} 0 \\ M_\ell y_\ell^d \end{pmatrix} .$$

We choose the left and right transformations according to

$$K_L = \begin{pmatrix} \alpha^{1/2} I & 0 \\ 0 & -I \end{pmatrix} \quad , \quad K_R = \begin{pmatrix} \alpha^{-1/2} I & 0 \\ 0 & I \end{pmatrix} .$$

This leads to the transformed matrix

$$K_L K K_R = \begin{pmatrix} A_\ell & -\alpha^{-1/2} M_\ell \\ -\alpha^{-1/2} M_\ell & -A_\ell \end{pmatrix} .$$

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Left and Right Transforming Iterations III

Denoting by S_ℓ the Schur complement $S_\ell = A_\ell + \alpha^{-1} M_\ell A_\ell^{-1} M_\ell$, we have

$$\begin{pmatrix} A_\ell & -\alpha^{-1/2} M_\ell \\ -\alpha^{-1/2} M_\ell & -A_\ell \end{pmatrix} = \begin{pmatrix} A_\ell & -\alpha^{-1/2} M_\ell \\ -\alpha^{-1/2} M_\ell & -S_\ell + \alpha^{-1} M_\ell A_\ell^{-1} M_\ell \end{pmatrix}.$$

Choosing $\hat{A}_\ell$ as a preconditioner for A_ℓ and $\hat{S}_\ell := \tau_\ell^{-1} \operatorname{diag}(A_\ell + \alpha^{-1} M_\ell \hat{A}_\ell^{-1} M_\ell)$, we obtain the symmetric Uzawa preconditioner

$$\tilde{K}_\ell = \begin{pmatrix} \hat{A}_\ell & -\alpha^{-1/2} M_\ell \\ -\alpha^{-1/2} M_\ell & -\hat{S}_\ell + \alpha^{-1} M_\ell \hat{A}_\ell^{-1} M_\ell \end{pmatrix}.$$

Backtransformation results in the following preconditioner

$$\hat{K}_\ell = K_L^{-1} \tilde{K}_\ell K_R^{-1} = \begin{pmatrix} \hat{A}_\ell & -\alpha^{-1} M_\ell \\ M_\ell & \hat{S}_\ell - \alpha^{-1} M_\ell \hat{A}_\ell^{-1} M_\ell \end{pmatrix}.$$

Multigrid Solvers for Elliptic Optimal Control Problems

A. Borzi, K. Kunisch, and D. Y. Kwak; Accuracy and convergence properties of the finite difference multigrid solution of an optimal control optimality system.
SIAM J. Control Optimization 41, 1477-1497, 2003.

A. Borzi and V. Schulz; Multigrid methods for PDE optimization.
SIAM Rev. 51, 361-395, 2009.

J. Schöberl, R. Simon, and W. Zulehner; A robust multigrid method for elliptic optimal control problems.
Preprint, Inst. of Comput. Math., University of Linz, 2010.

Preconditioned Richardson Iteration I

Step 1: Given iterates $y_\ell^{(\nu)}, p_\ell^{(\nu)} \in \mathbb{R}^{N_\ell}$, compute the residuals

$$\begin{pmatrix} \mathrm{Res}_\ell^{(1)} \\ \mathrm{Res}_\ell^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ M_\ell y_\ell^d \end{pmatrix} - K_\ell \begin{pmatrix} y_\ell^{(\nu)} \\ p_\ell^{(\nu)} \end{pmatrix}.$$

Step 2: Solve the linear system

$$\begin{pmatrix} \hat{A}_\ell & -\alpha^{-1} M_\ell \\ M_\ell & \hat{S}_\ell - \alpha^{-1} M_\ell \hat{A}_\ell^{-1} M_\ell \end{pmatrix} \begin{pmatrix} \Delta y_\ell^{(\nu)} \\ \Delta p_\ell^{(\nu)} \end{pmatrix} = \begin{pmatrix} \mathrm{Res}_\ell^{(1)} \\ \mathrm{Res}_\ell^{(2)} \end{pmatrix}.$$

Step 3: Compute the new iterates

$$\begin{pmatrix} y_\ell^{(\nu+1)} \\ p_\ell^{(\nu+1)} \end{pmatrix} = \begin{pmatrix} y_\ell^{(\nu)} \\ p_\ell^{(\nu)} \end{pmatrix} + \begin{pmatrix} \Delta y_\ell^{(\nu)} \\ \Delta p_\ell^{(\nu)} \end{pmatrix}.$$

Preconditioned Richardson Iteration II

Consider the linear system

$$(*) \qquad \begin{pmatrix} \hat{A}_\ell & -\alpha^{-1}M_\ell \\ M_\ell & \hat{S}_\ell - \alpha^{-1}M_\ell\hat{A}_\ell^{-1}M_\ell \end{pmatrix} \begin{pmatrix} \Delta y_\ell^{(\nu)} \\ \Delta p_\ell^{(\nu)} \end{pmatrix} = \begin{pmatrix} Res_\ell^{(1)} \\ Res_\ell^{(2)} \end{pmatrix}.$$

Elimination of $\Delta y_\ell^{(\nu)}$ results in the Schur complement system

$$\hat{S}_\ell \Delta p_\ell^{(\nu)} = Res_\ell^{(2)} - M_\ell \hat{A}_\ell^{-1} Res_\ell^{(1)}.$$

Hence, the solution of $(*)$ can be reduced to the successive solution of the three linear systems

$$\begin{aligned} \hat{A}_\ell \tilde{\Delta} y_\ell^{(\nu)} &= Res_\ell^{(1)}, \\ \hat{S}_\ell \Delta p_\ell^{(\nu)} &= Res_\ell^{(2)} - M_\ell \tilde{\Delta} y_\ell^{(\nu)}, \\ \hat{A}_\ell \Delta y_\ell^{(\nu)} &= Res_\ell^{(1)} + \alpha^{-1} M_\ell \Delta p_\ell^{(\nu)}. \end{aligned}$$

Preconditioned Richardson Iteration III

Theorem [Schöberl/Simon/Zulehner] If the preconditioners $\hat{A}_\ell$ and $\hat{S}_\ell$ are chosen according to

$$\hat{A}_\ell = \sigma_\ell^{-1} \operatorname{diag}(A_\ell) \quad , \quad \hat{S}_\ell = \tau_\ell^{-1} \operatorname{diag}(S_\ell),$$
$$\sigma_\ell \le \lambda_{\max}((\operatorname{diag}(A_\ell))^{-1} A_\ell)^{-1} \quad , \quad \tau_\ell \le \lambda_{\max}((\operatorname{diag}(S_\ell))^{-1} S_\ell)^{-1},$$

then there holds

$$\|(y_\ell^{(\nu)}, p_\ell^{(\nu)}) - (y_\ell, p_\ell)\| \le C\ (1 + \alpha^{-1/2}\ h_\ell^2)\ \eta(\nu)\ \|(y_\ell^{(0)}, p_\ell^{(0)}) - (y_\ell, p_\ell)\|,$$

where

$$\eta(\nu) \le \begin{cases} \sqrt{\frac{2}{\pi(\nu-1)}} \ , \text{ if } \nu \text{ is even} \\ \sqrt{\frac{2}{\pi\nu}} \ , \text{ if } \nu \text{ is odd} \end{cases},$$

and C is a positive constant independent of h_ℓ and α.

Multigrid Solution of the Optimality System

The previous theorem suggests to use the preconditioned iterative solver as a smoother within a multigrid solution of the optimality system.

Theorem. In case of two nested grids $\mathcal{T}_{\ell-1}(\Omega)$ and $\mathcal{T}_{\ell}(\Omega)$ let $(y_\ell^{(0)}, p_\ell^{(0)})$ be a level ℓ startiterate and let $(y_\ell^{(1)}, p_\ell^{(1)})$ be the result of a two-grid cycle with $\nu > 0$ smoothing steps. Then there holds

$$\|(y_\ell^{(1)}, p_\ell^{(1)}) - (y_\ell, p_\ell)\| \leq C \, \frac{1 + \alpha^{-1/2} h_\ell^2}{\sqrt{\nu}} \, \|(y_\ell^{(0)}, p_\ell^{(0)}) - (y_\ell, p_\ell)\|,$$

where $C > 0$ is a constant independent of h_ℓ and α.

Elliptic Optimal Control Problems

Control Constraints

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Elliptic Optimal Control Problems: Control Constraints

Consider the control constrained distributed elliptic optimal control problem

$$\inf_{y,u} J(y,u) := \frac{1}{2}\int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx,$$

$$(\mathrm{ECC})_1 \qquad -\Delta y = u \quad \text{in } \Omega,$$

$$(\mathrm{ECC})_2 \qquad y = 0 \quad \text{on } \Gamma,$$

and

$$u \in K_C := \{v \in L^2(\Omega) \mid v(x) \le \psi(x) \text{ f.a.a. } x \in \Omega\},$$

where $u \in L^2(\Omega)$.

Control Constraints: Necessary Optimality Conditions

Theorem. If $(y, u) \in H_0^1(\Omega) \times K_C$ is the optimal solution of the optimal control problem, then there exists

$$(p, \lambda) \in H_0^1(\Omega)) \times L^2(\Omega)$$

such that p satisfies the adjoint state equation

$$\begin{aligned} -\Delta p &= -(y - y^d) && \text{in } \Omega, \\ p &= 0 && \text{on } \Gamma, \end{aligned}$$

and there holds

$$\begin{aligned} p - \lambda &= \alpha u && \text{in } \Omega, \\ \lambda &\in \partial I_{K_c}(u). \end{aligned}$$

Elliptic Optimal Control Problems

Primal-Dual Active Set Strategy

Moreau-Yosida Approximation of Multivalued Maps I

Weighted Duality Mapping: Assume that V is a Banach space with dual V^* and let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and non-decreasing function such that $h(0) = 0$ and $h(t) \to \infty$ as $t \to \infty$. Then the mapping $J_h : V \to 2^{V^*}$

$$J_h(u) := \{u^* \in V^* \mid \langle u^*, u\rangle = \|u\|\|u^*\| \ , \ \|u^*\| = h(\|u\|)\}$$

is called the duality mapping with weight h.

Example: For $V = L^p(\Omega), V^* = L^q(\Omega), 1 < p, q < \infty, 1/p + 1/q = 1$, and $h(t) = t^{p-1}$ we have

$$J_h(u)(x) = \begin{cases} |u(x)|^{p-1} \operatorname{sgn}(u(x)) \ , \ u(x) \neq 0 \\ 0 \ , \ u(x) = 0 \end{cases} .$$

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Moreau-Yosida Approximation of Multivalued Maps II

Moreau-Yosida proximal map: Let $f : V \to \bar{\mathbb{R}}$ be a lower semi-continuous proper convex function with subdifferential ∂f. For $c > 0$, the Moreau-Yosida proximal map $P_c^{\partial f} : V \to 2^V$ is defined such that $P_c^{\partial f}(w)$, $w \in V$, is the set of minimizers of

$$\inf_{v \in V} f(v) + c j_h\left(\frac{v - w}{c}\right),$$

where $\partial j_h = J_h$.

Moreau-Yosida approximation: If J_h is single-valued, then for $c > 0$ the Moreau-Yosida approximation $(\partial f)_c$ of ∂f is given by

$$(\partial f)_c(w) := J_h(c^{-1}w - c^{-1}P_c^{\partial f}(w)).$$

Moreau-Yosida Approximation of ∂I_{K_C}

Idea: Approximate ∂I_{K_C} by its Moreau-Yosida approximation $(\partial I_{K_C})_c$.

Theorem. For any $c > 0$, we have

$$\lambda \in (\partial I_{K_C})_c,$$

if and only if there holds

$$\lambda = c\Big(u + c^{-1}\lambda - \Pi_{K_C}(u + c^{-1}\lambda)\Big) = c\ \max(0, u + c^{-1}\lambda - \psi),$$

and this is equivalent to

$$u = \Pi_{K_C}(u + c^{-1}\lambda),$$

where Π_{K_C} denotes the L^2-projection onto K_C.

Primal-Dual Active Set Strategy I

Step 1 (Initialization):

Choose $c > 0$, start-iterates $y_h^{(0)}, u_h^{(0)}, \lambda_h^{(0)}$ and set $n = 1$.

Step 2 (Specification of active/inactive sets):

Compute the active/inactive sets $\mathcal{A}_n$ and $\mathcal{I}_n$ according to

$$\mathcal{A}_n := \{1 \le i \le N \mid (u_h^{(n-1)} + c^{-1}\lambda_h^{(n-1)})_i > (\psi_h)_i\} \quad , \quad \mathcal{I}_n := \{1, \cdots, N\} \setminus \mathcal{A}_n.$$

Step 3 (Termination criterion):

If $n \ge 2$ and $\mathcal{A}_n = \mathcal{A}_{n-1}$, stop the algorithm. Otherwise, go to Step 4.

Primal-Dual Active Set Strategy II

Step 4 (Update of the state, adjoint state, and control):

Compute $y_h^{(n)}, p_h^{(n)}$ as the solution of

$$(A_h y_h^{(n)})_i = \begin{cases} (\psi_h)_i \ , & \text{if } i \in \mathcal{A}_n \\ \alpha^{-1}(M_h^{-1} p_h)_i \ , & \text{if } i \in \mathcal{I}_n \end{cases} \quad , \quad A_h p_h^{(n)} = -M_h(y_h - y_h^d),$$

and set

$$(u_h^{(n)})_i := \begin{cases} (\psi_h)_i \ , & \text{if } i \in \mathcal{A}_n \\ \alpha^{-1}(M_h^{-1} p_h^{(n)})_i \ , & \text{if } i \in \mathcal{I}_n \end{cases} .$$

Step 5 (Update of the multiplier):

Set $\lambda_h^{(n)} := p_h^{(n)} - \alpha M_h u_h^{(n)}$, $n := n + 1$, and go to Step 2.

PDAS As A Semi-Smooth Newton Method I

Newton differentiability: A function $f : S \subset X \to Y$ (X, Y Banach spaces) is called Newton differentiable at $x \in S$, if there exist a neighborhood $\mathcal{U}(x) \subset S$ and a family of mappings $g : \mathcal{U}(x) \to \mathcal{L}(X, Y)$ such that

$$\|f(x+h) - f(x) - g(x+h)(h)\|_Y = o(\|h\|_X) \quad (\text{as } \|h\|_X \to \infty).$$

$g(x)$ is said to be the generalized derivative of f at x.

Example: The max-function $\max : L^p(\Omega) \to L^q(\Omega)$ is Newton differentiable for $1 \leq q < p < \infty$. If $f : L^r(\Omega) \to L^p(\Omega)$ is Fréchet-differentiable for some $1 < r \leq \infty$, then the function

$$x \longmapsto \chi_A(x)\ f(u(x))$$

is the generalized derivative of $\max(0, f(\cdot))$. Here, χ_A denotes the characteristic function of the set A where $f(u(\cdot))$ is nonnegative.

PDAS As A Semi-Smooth Newton Method II

Setting $z_h := (y_h, u_h, \lambda_h)$, the discrete optimality conditions require the solution of the nonlinear system

$$F_h(z_h) := \begin{pmatrix} A_h y_h - M_h u_h \\ \alpha A_h u_h + M_h y_h - y_h^d + \lambda_h \\ -\lambda_h + c \max(0, u_h + c^{-1}\lambda_h - \psi_h) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The primal-dual active set strategy (PDAS) is equivalent to the semi-smooth Newton method

$$\begin{aligned} F_h'(z_h^{(n)})\Delta z_h^{(n)} &= -F_h(z_h^{(n)}), \\ z_h^{(n+1)} &= z_h^{(n)} + \Delta z_h^{(n)}, \end{aligned}$$

where $F_h'(z_h^{(n)})$ is the generalized derivative of F_h at $z_h^{(n)}$.

Elliptic Optimal Control Problems

Interior Point Methods

Interior-Point Methods (General)

Gonzaga [1992] , Wright [1992] , Zhang/Tapia/Dennis [1992]

Conn/Gould/Toint [1996] , Heinkenschloss [2001]

Forsgren/Gill/Wright [2002] , M. Ulbrich [2002] , S. Ulbrich [2003]

Interior-Point Methods (PDE Constrained Optimization)

Ulbrich/Ulbrich/Heinkenschloss [1999] , Ulbrich/Ulbrich [2000]

H./Petrova/Schulz [2002] , H./Petrova [2004] , Weiser [2005]

Ulbrich/Ulbrich [2007] , Antil/H./Linsenmann [2007]

Interior Point Method I

Idea: Coupling of control constraints $u_h \leq \psi_h$ by, e.g., logarithmic barrier functions with barrier parameter $\beta > 0$:

$$\inf_{y_h, u_h} B_h^{(\beta)}(y_h, u_h) := J_h(y_h, u_h) + \beta \sum_{i=1}^{N} \ln((\psi_h - u_h)_i),$$

which represents a parameter-dependent family of minimization subproblems.

Coupling of the discrete state equation by a Lagrange multiplier:

$$\inf_{y_h, u_h} \sup_{p_h} \mathcal{L}_h^{(\beta)}(y_h, u_h, p_h) := B_h^{(\beta)}(y_h, u_h) + \langle p_h, A_h y_h - M_h u_h \rangle.$$

Interior Point Method II

The barrier path $\beta \longmapsto x_h(\beta) := (y_h(\beta), u_h(\beta), p_h(\beta))$ is the solution of the parameter-dependent nonlinear system:

$$F_h(x_h(\beta), \beta) = \begin{pmatrix} A_h p_h + M_h y_h - y_h^d \\ A_h y_h - M_h u_h \\ \alpha (M_h u_h)_1 + \frac{\beta}{(\psi_h - u_h)_1} \\ \cdots \\ \alpha (M_h u_h)_N + \frac{\beta}{(\psi_h - u_h)_N} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix} .$$

Solution by path-following predictor-corrector continuation methods in the barrier parameter.

Continuation Strategies in Interior-Point Methods

(i) Static Strategies

Fiacco/McCormick [1990] , Byrd/Hribar/Nocedal [1999]

Wächter/Biegler [2005]

(ii) Dynamic Update Strategies

El-Bakry/Tapia/Tsuchiya/Zhang [1996] , Gay/Overton/Wright [1998]

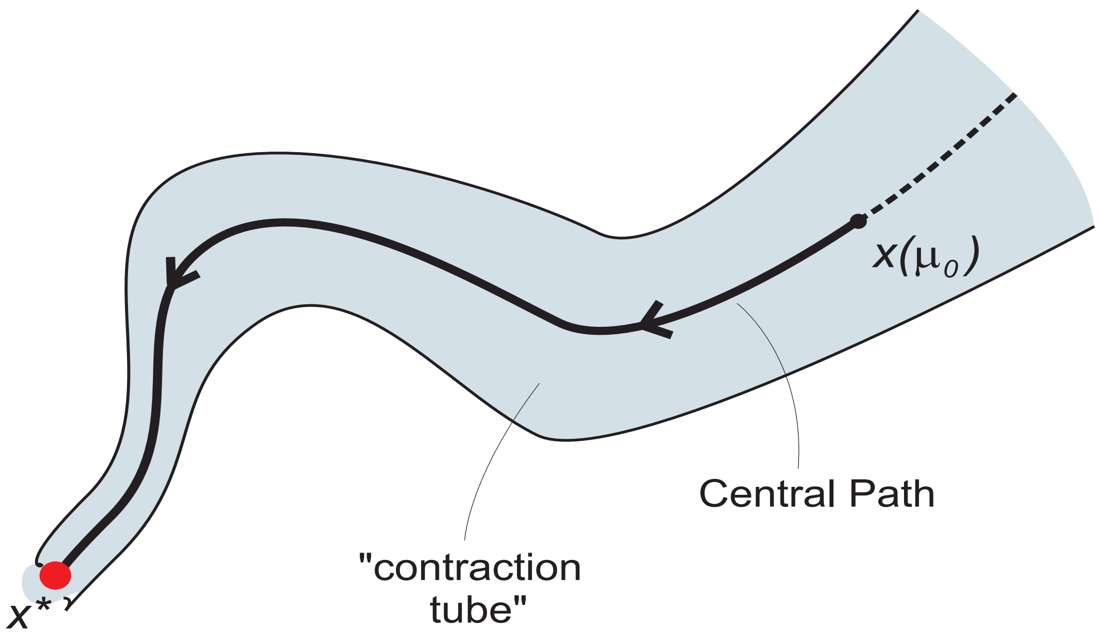
Vanderbei/Shanno [1999] , H./Petrova/Schulz [2002]

Tits/Wächter/Bakhtiari/Urban/Lawrence [2003] , H./Petrova [2004]

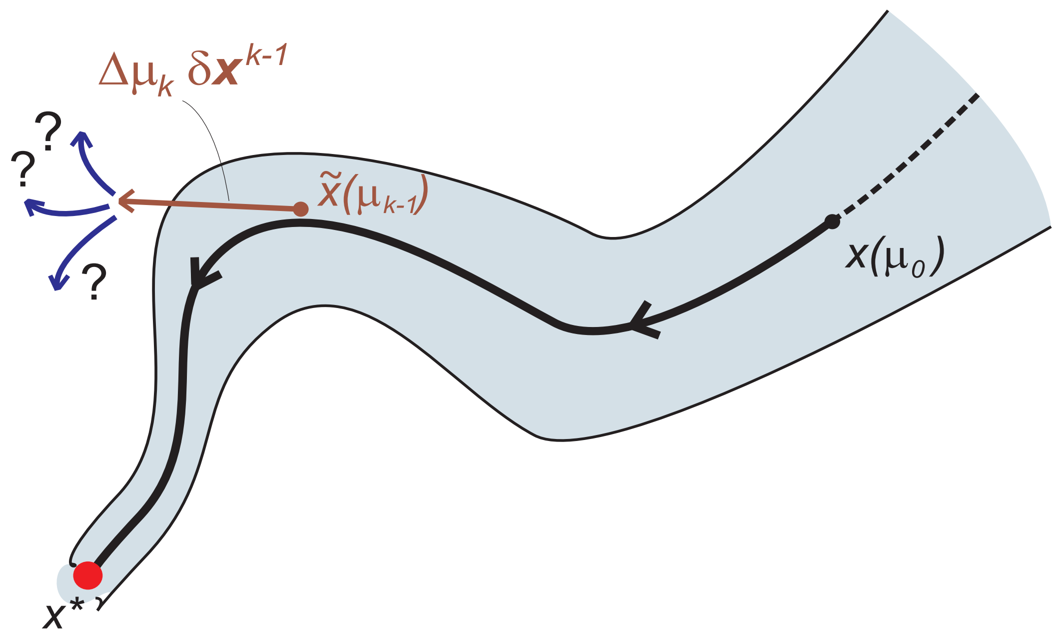
Ulbrich/Ulbrich/Vicente [2004] , H./Linsenmann/Petrova [2006]

Nocedal/Wächter/Waltz [2006] , Armand/Benoist/Orban [2007]

The Barrier Path

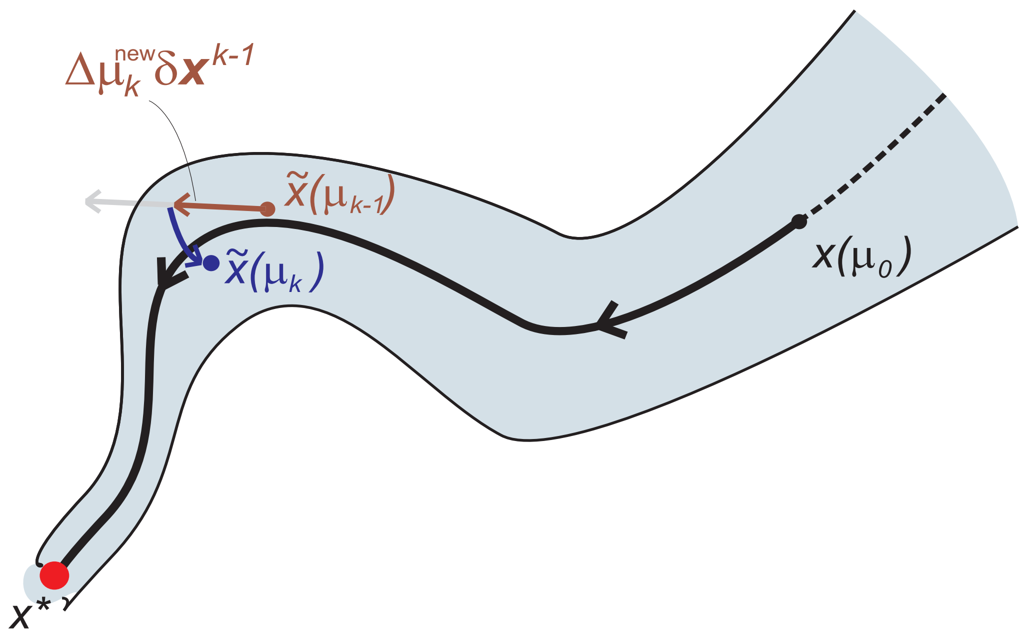


Barrier Path: The Predicted Continuation Step



$\cdots$ may leave the Kantorovich neighborhood.

Barrier Path: The Newton Correction



$\cdots$ reduces the continuation steplength adaptively.

Numerical Results: Distributed Control Problem with Control Constraints I

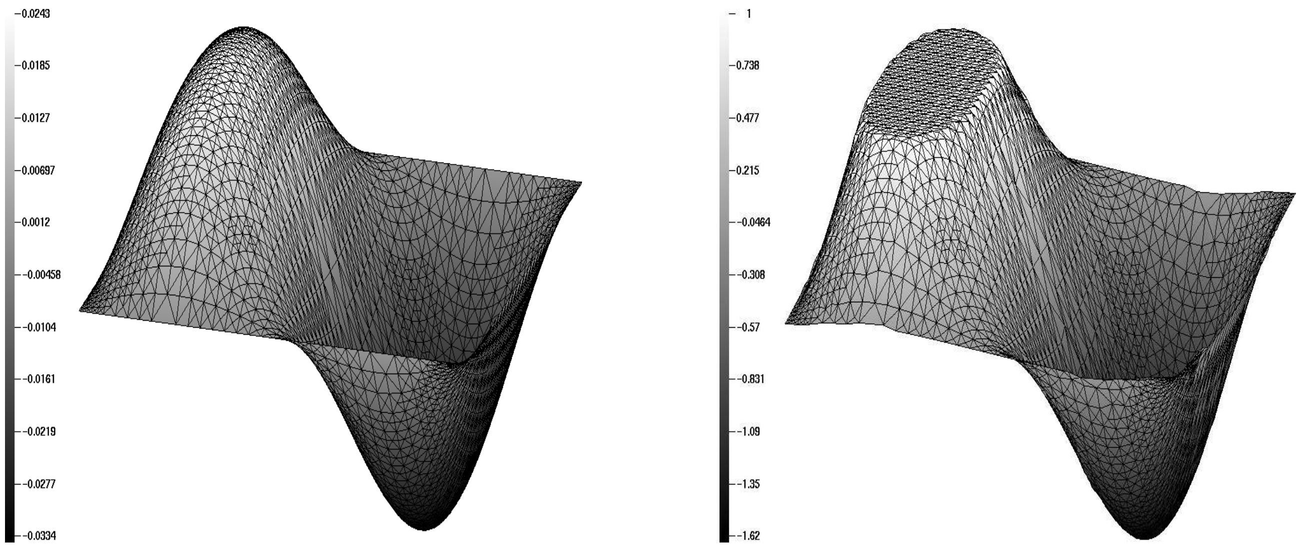
$$
\begin{array}{ll}
\text{Minimize} & J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u\|^2_{0,\Omega} \\
\text{over} & (y,u) \in H^1_0(\Omega) \times L^2(\Omega) \\
\text{subject to} & -\Delta y = u \quad \text{in } \Omega\,, \\
 & u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}
\end{array}
$$

Data: $\Omega := (0,1)^2$,

$$
y^d := \begin{cases} 200\; x_1\; x_2\; (x_1 - 0.5)^2\; (1 - x_2)\,, & 0 \le x_1 \le 0.5 \\ 200\; (x_1 - 1)\; (x_2\; (x_1 - 0.5)^2\; (1 - x_2)\,, & 0.5 < x_1 \le 1 \end{cases} ,
$$

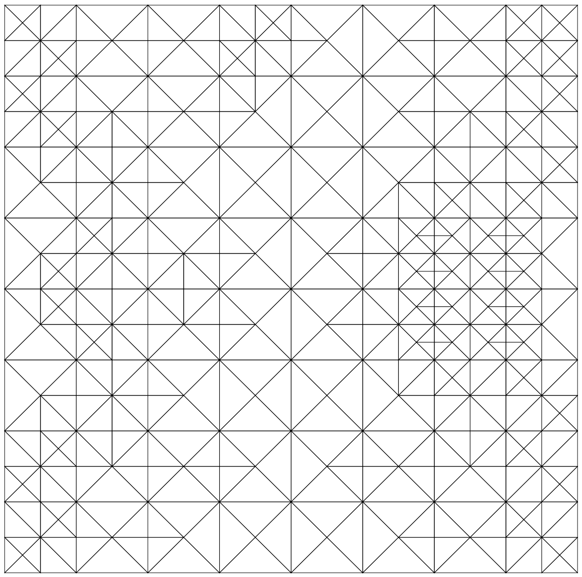
$$\alpha = 0.01 \quad , \quad \psi = 1 \, .$$

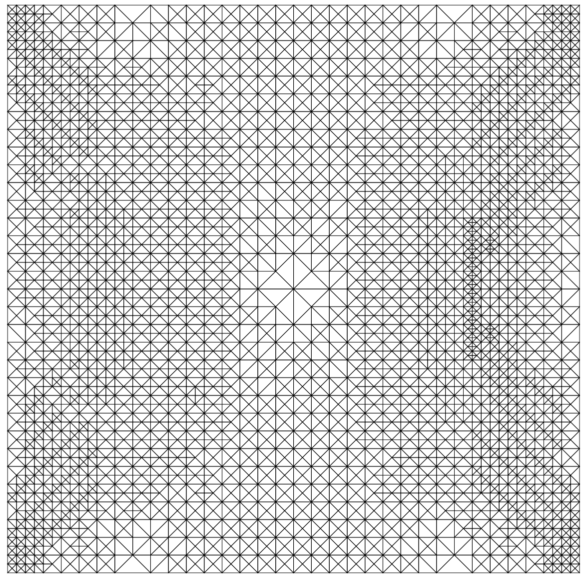
Numerical Results: Distributed Control Problem with Control Constraints I



Optimal state (left) and optimal control (right)

Numerical Results: Distributed Control Problem with Control Constraints I





Grid after 6 (left) and 10 (right) refinement steps

Numerical Results: Distributed Control Problem with Control Constraints II

$$\begin{array}{ll} \text{Minimize} & J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u - u^d\|^2_{0,\Omega} \\ \text{over} & (y,u) \in H^1_0(\Omega) \times L^2(\Omega) \\ \text{subject to} & -\Delta y = f + u \quad \text{in } \Omega\,, \\ & u \in K := \{v \in L^2(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\} \end{array}$$

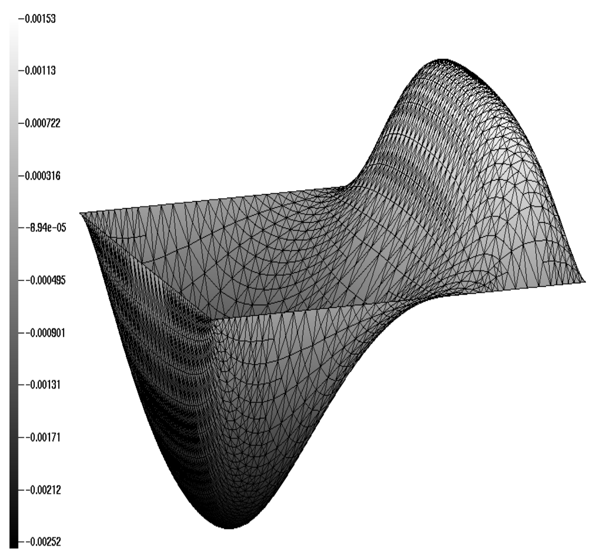
Data: $\Omega := (0,1)^2 \quad , \quad y^d := 0 \quad , \quad u^d := \hat{u} + \alpha^{-1}(\hat{\sigma} - \Delta^{-2}\hat{u})\,,$

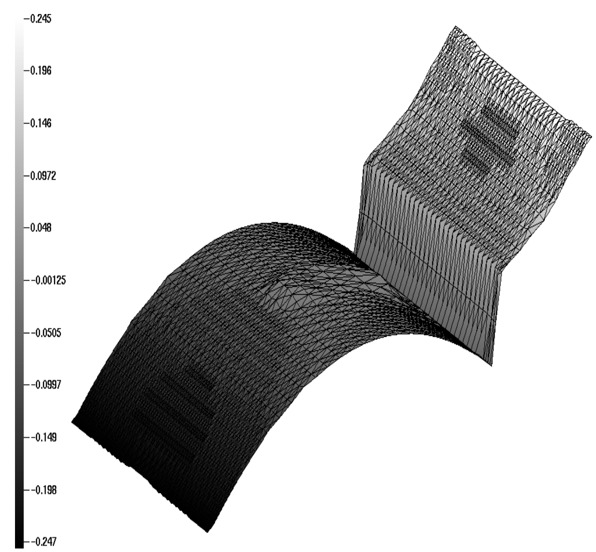
$$\psi := \begin{cases} (x_1 - 0.5)^8 & , \quad (x_1,x_2) \in \Omega_1, \\ (x_1 - 0.5)^2 & , \quad \text{otherwise} \end{cases} \quad , \quad \alpha := 0.1 \quad , \quad f := 0$$

$$\hat{u} := \begin{cases} \psi & , \quad (x_1,x_2) \in \Omega_1 \cup \Omega_2, \\ -1.01\,\psi & , \quad \text{otherwise} \end{cases} \quad , \quad \hat{\sigma} := \begin{cases} 2.25\,(x_1 - 0.75) \cdot 10^{-4} & , \quad (x_1,x_2) \in \Omega_2, \\ 0 & , \quad \text{otherwise} \end{cases} \quad ,$$

$$\Omega_1 := \{(x_1,x_2) \in \Omega \mid \Big((x_1 - 0.5)^2 + (x_2 - 0.5)^2\Big)^{1/2} \le 0.15\} \quad , \quad \Omega_2 := \{(x_1,x_2) \in \Omega \mid x_1 \ge 0.75\}\,.$$

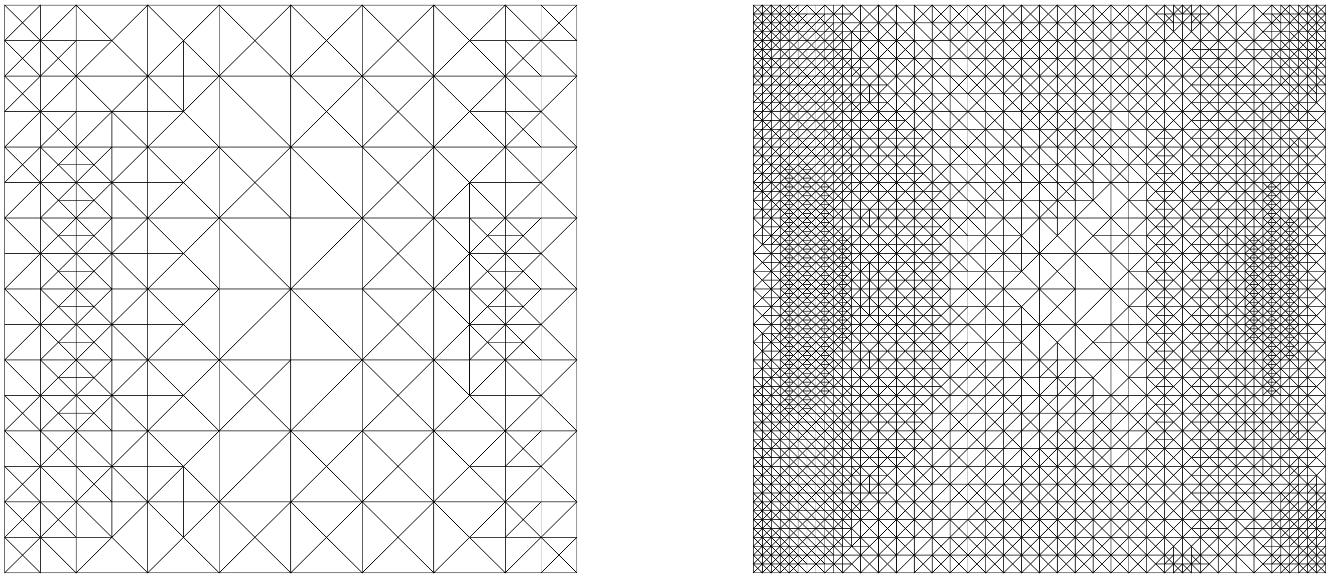
Numerical Results: Distributed Control Problem with Control Constraints II





Optimal state (left) and optimal control (right)

Numerical Results: Distributed Control Problem with Control Constraints II



Grid after 6 (left) and 10 (right) refinement steps

Elliptic Optimal Control Problems

State Constraints

Literature on State-Constrained Optimal Control Problems

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Elliptic Optimal Control Problems: State Constraints

Consider the state constrained distributed elliptic optimal control problem

$$\inf_{y,u} J(y,u) := \frac{1}{2}\int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2}\int_\Omega |u|^2 \, dx,$$

$$(ESC)_1 \qquad -\Delta y = u \quad \text{in } \Omega,$$

$$(ESC)_2 \qquad y = 0 \quad \text{on } \Gamma,$$

and

$$y \in K_S := \{y \in H_0^1(\Omega) \mid y(x) \le \psi(x) \text{ f.a.a. } x \in \Omega\},$$

where $u \in L^2(\Omega)$.

State Constraints: Necessary Optimality Conditions

Theorem. The state constrained optimal control problem admits a unique solution (y, u) with $y \in W_0^{1,r}(\Omega), r > 2$, and $u \in L^2(\Omega)$.

Theorem. Let us assume that there exists $u_0 \in L^2(\Omega)$ such that the associated solution $y_0 \in W_0^{1,r}(\Omega)$ of the state equation satisfies $y_0 \in \text{int}(K_S)$ (Slater condition).
If (y, u) is the unique solution of the state constrained optimal control problem, there exist $p \in W_0^{1,s}(\Omega), 1/r + 1/s = 1$, and $\lambda \in \mathcal{M}_+(\Omega)$ such that

$$
\begin{aligned}
(\nabla p, \nabla v)_{L^2(\Omega)} &= -(y - y^d, v)_{L^2(\Omega)} + \langle \lambda, v \rangle \quad , \quad v \in W_0^{1,r}(\Omega), \\
p &= \alpha u, \\
\langle \lambda, y - \psi \rangle &= 0.
\end{aligned}
$$

Finite Element Approximation

Let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation of Ω and let

$$V_h := \{ v_h \in C(\overline{\Omega}) \mid v_h|_T \in P_1(T) , \ T \in \mathcal{T}_h(\Omega) , \ v_h|_\Gamma = 0 \}$$

be the FE space of continuous, piecewise linear functions. Let ψ_h be the V_h-interpoland of ψ. Consider the following FE Approximation of the state constrained control problem

$$\begin{aligned} &\text{Minimize} && J_h(y_h, u_h) := \frac{1}{2} \|y_h - y^d\|^2_{0,\Omega} + \frac{\alpha}{2} \|u_h\|^2_{0,\Omega} , \\ &\text{over} && (y_h, u_h) \in V_h \times V_h , \\ &\text{subject to} && (\nabla y_h, \nabla v_h)_{0,\Omega} = (u_h, v_h)_{0,\Omega} , \ v_h \in V_h , \\ & && y_h \in K_{h,S} := \{ v_h \in V_h \mid v_h(x) \le \psi_h(x) , \ x \in \overline{\Omega} \} . \end{aligned}$$

Since the constraints are point constraints associated with the nodal points, the discrete multipliers are chosen from

$$\mathcal{M}_h := \{ \lambda_h \in \mathcal{M}(\Omega) \mid \lambda_h = \sum_{a \in \mathcal{N}_h(\Omega)} \kappa_a \delta_a , \ \kappa_a \in \mathbb{R} \} .$$

Primal-Dual Active Set Strategy I

Step 1 (Initialization):

Choose $\mathbf{c} > \mathbf{0}$, start-iterates $\mathbf{y}_h^{(0)}, \mathbf{u}_h^{(0)}, \boldsymbol{\lambda}_h^{(0)}$ and set $\mathbf{n} = \mathbf{1}$.

Step 2 (Specification of active/inactive sets):

Compute the active/inactive sets $\mathcal{A}_n$ and $\mathcal{I}_n$ according to

$$\mathcal{A}_n := \{\mathbf{1} \leq \mathbf{i} \leq \mathrm{N} \mid (\mathbf{y}_h^{(n-1)} + \mathbf{c}^{-1}\boldsymbol{\lambda}_h^{(n-1)})_i > (\psi_h)_i\} \quad , \quad \mathcal{I}_n := \{\mathbf{1}, \cdots, \mathrm{N}\} \setminus \mathcal{A}_n.$$

Step 3 (Termination criterion):

If $\mathbf{n} \geq \mathbf{2}$ and $\mathcal{A}_n = \mathcal{A}_{n-1}$, stop the algorithm. Otherwise, go to Step 4.

Primal-Dual Active Set Strategy II

Step 4 (Update of the state, adjoint state, control, and multiplier):

Compute $(y_h^{(n)}, u_h^{(n)}, p_h^{(n)}, \lambda_h^{(n)})$ as the solution of

$$\begin{aligned} A_h y_h^{(n)} - M_h u_h^{(n)} &= 0, \\ A_h p_h^{(n)} - (M_h y_h^{(n)} - y_h^d) - \lambda_h^{(n)} &= 0, \\ p_h^{(n)} - \alpha M_h u_h^{(n)} &= 0, \\ (y_h^{(n)})_i &= (\psi_h)_i \quad \text{for } i \in \mathcal{A}_n, \\ (\lambda_h^{(n)})_i &= 0 \quad \text{for } i \in \mathcal{A}_n. \end{aligned}$$

Set $n := n + 1$ and go to Step 2.

Lavrentiev Regularization: Mixed Control-State Constraints

Introduce a regularization parameter $\varepsilon > 0$ and consider the mixed control-state constrained optimal control problem

$$\text{Minimize} \quad J(y^\varepsilon, u^\varepsilon) := \frac{1}{2} \|y^\varepsilon - y^d\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u^\varepsilon\|^2_{L^2(\Omega)} ,$$

$$\text{subject to} \quad -\Delta y^\varepsilon = u^\varepsilon \text{ in } \Omega \quad , \quad y^\varepsilon = 0 \text{ on } \Gamma_D,$$

$$\varepsilon u^\varepsilon + y^\varepsilon \in K := \{v \in L^2(\Omega) \mid v(x) \leq \psi(x) \ , \ \text{f.a.a. } x \in \Omega\} \ .$$

Theorem (Optimality conditions). The optimal solution $(y^\varepsilon, u^\varepsilon) \in V \times L^2(\Omega)$ is characterized by the existence of an adjoint state $p^\varepsilon \in V$ and a multiplier $\lambda^\varepsilon \in L^2_+(\Omega)$ such that

$$(\nabla y^\varepsilon, \nabla v)_{L^2(\Omega)} = (u^\varepsilon, v)_{L^2(\Omega)} \quad , \quad v \in V \ ,$$

$$(\nabla p^\varepsilon, \nabla v)_{L^2(\Omega)} = -(y^\varepsilon - y^d, v)_{L^2(\Omega)} + (\lambda^\varepsilon, v)_{L^2(\Omega)} \quad , \quad v \in V \ ,$$

$$p^\varepsilon - \alpha u^\varepsilon + \varepsilon \lambda^\varepsilon = 0 \quad , \quad (\lambda^\varepsilon, \varepsilon u^\varepsilon + y^\varepsilon - \psi)_{L^2(\Omega)} = 0 \ .$$

Numerical Results: Distributed Control Problem with State Constraints I

$$\text{Minimize} \quad J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u - u^d\|^2_{0,\Omega} \quad \text{over} \quad (y,u) \in H^1_0(\Omega) \times L^2(\Omega)$$

$$\text{subject to} \quad -\Delta y = u \quad \text{in } \Omega \quad , \quad y \in K := \{v \in H^1_0(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}$$

Data: $\Omega := (-2,+2)^2 \quad , \quad y^d(r) := y(r) + \Delta p(r) + \sigma(r) \quad , \quad u^d(r) := u(r) + \alpha^{-1} p(r) ,$
$\psi := 0 \quad , \quad \alpha := 0.1 ,$

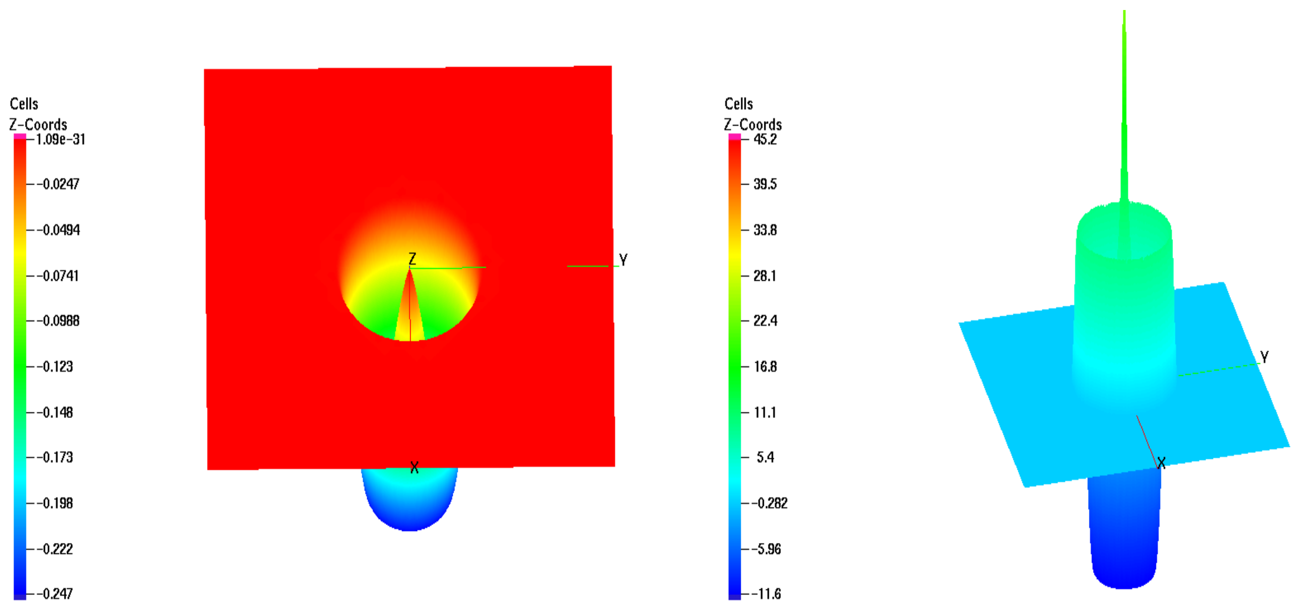
where $y(r), u(r), p(r), \sigma(r)$ is the solution of the problem:

$$y(r) := -r^{4/3} + \gamma_1(r) \ , \ u(r) = -\Delta y(r) \ , \ p(r) = \gamma_2(r) + r^4 - \frac{3}{2}r^3 + \frac{9}{16}r^2 \ , \ \sigma(r) := \begin{cases} 0.0 & , \quad r < 0.75 \\ 0.1 & , \quad \text{otherwise} \end{cases}$$

$$, \qquad \gamma_1 := \begin{cases} 1 & , \quad r < 0.25 \\ -192(r-0.25)^5 + 240(r-0.25)^4 - 80(r-0.25)^3 + 1 & , \quad 0.25 < r < 0.75 \\ 0 & , \quad \text{otherwise} \end{cases} ,$$

$$\gamma_2 := \begin{cases} 1 & , \quad r < 0.75 \\ 0 & , \quad \text{otherwise} \end{cases} .$$

Numerical Results: Distributed Control Problem with State Constraints I



Optimal state (left) and optimal control (right)

Numerical Results: Distributed Control Problem with State Constraints II

$$\text{Minimize} \quad J(y,u) := \frac{1}{2}\,\|y - y^d\|^2_{0,\Omega} + \frac{\alpha}{2}\,\|u - u^d\|^2_{0,\Omega} \quad \text{over} \;\; (y,u) \in H^1(\Omega) \times L^2(\Omega)$$

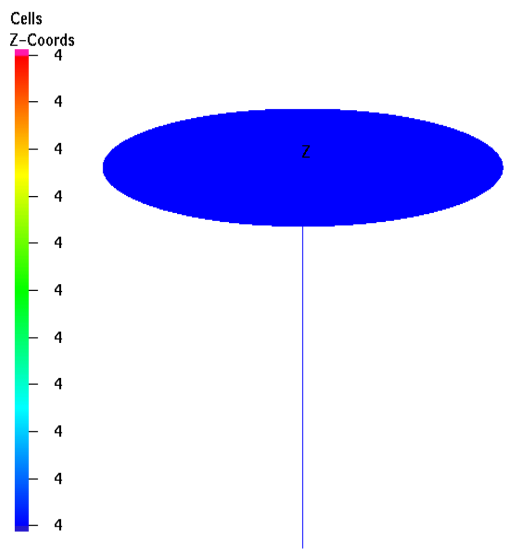
$$\text{subject to} \quad -\,\Delta y + cy = u \;\; \text{in } \Omega \;\; , \quad y \in K := \{v \in H^1(\Omega) \mid v \le \psi \text{ a.e. in } \Omega\}$$

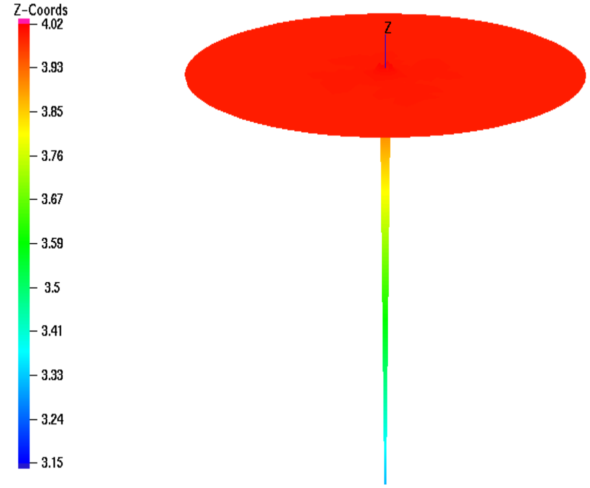
Data: $\Omega = B(0,1) := \{x = (x_1,x_2)^T \mid x_1^2 + x_2^2 < 1\}$, $y^d(r) := 4 + \frac{1}{\pi} - \frac{1}{4\pi} r^2 + \frac{1}{2\pi} \ln(r)$,
$u^d(r) := 4 + \frac{1}{4\pi} r^2 - \frac{1}{2\pi} \ln(r)$, $\psi := 4 + r$, $\alpha := 1$.

The solution $y(r), u(r), p(r), \sigma(r)$ of the problem is given by

$$y(r) \equiv 4 \quad , \quad u(r) \equiv 4 \quad , \quad p(r) = \frac{1}{4\pi} r^2 - \frac{1}{2\pi} \ln(r) \quad , \quad \sigma(r) = \delta_0 \; .$$

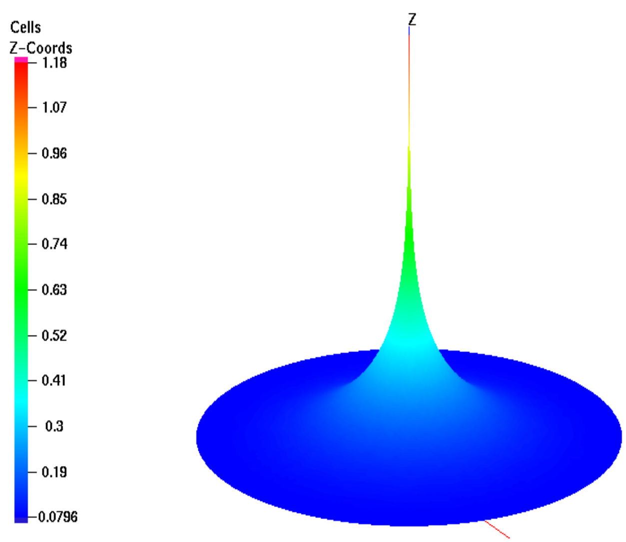
Numerical Results: Distributed Control Problem with State Constraints II

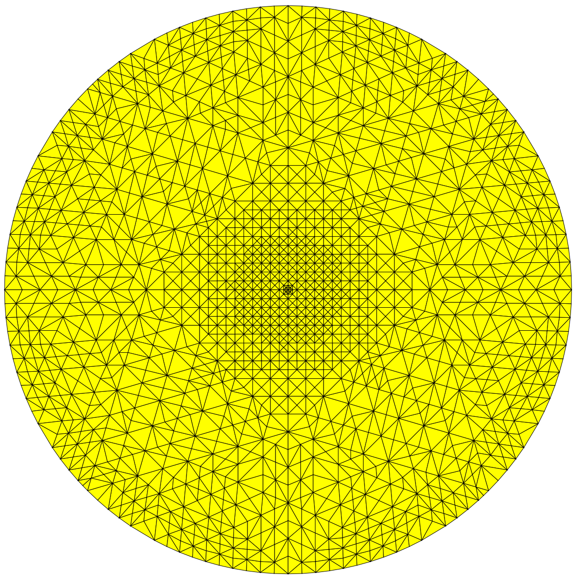




Optimal state (left) and optimal control (right)

Numerical Results: Distributed Control Problem with State Constraints II





Optimal adjoint state (left) and mesh after 16 adaptive loops (right)