Numerical Solution of Parabolic Optimal Control Problems

Ronald H.W. Hoppe[1,2]

[1] Department of Mathematics, University of Houston
[2] Institute of Mathematics, University of Augsburg

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- Parabolic Optimal Control Problems with Distributed Controls
- Parabolic Optimal Control Problems with Neumann Boundary Controls
- Parabolic Optimal Control Problems with Dirichlet Boundary Controls

Parabolic Optimal Control Problems

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Parabolic Optimal Control Problems

Distributed Control

Parabolic Problems: Distributed Control

Consider the distributed control problem

$$\inf_{y,u} \; J(y,u) \;:=\; \frac{1}{2}\int_0^T \|y-y^d\|^2_{L^2(\Omega)}\,dt + \frac{\alpha}{2}\int_0^T \|u\|^2_{L^2(\Omega)}\,dt,$$

$$\begin{aligned}
&(DBC)_1 \qquad & \frac{\partial y}{\partial t} + A(t)y &= u \quad \text{in } Q := \Omega\times(0,T),\\
&(DBC)_2 \qquad & y &= 0 \quad \text{on } \Sigma := \Gamma\times(0,T),\\
&(DBC)_3 \qquad & y(\cdot,0) &= y_0 \quad \text{in } \Omega,
\end{aligned}$$

where $u \in L^2(Q)$.

Approach: Weak formulation of $(DBC)_1 - (DBC)_3$.

Distributed Control: Weak Solution

Consider the parabolic initial-boundary value problem

$$
\begin{aligned}
(DBC)_1 \qquad \frac{\partial y}{\partial t} + A(t)y &= u \quad \text{in } Q := \Omega \times (0,T), \\
(DBC)_2 \qquad y &= 0 \quad \text{on } \Sigma := \Gamma \times (0,T), \\
(DBC)_3 \qquad y(\cdot,0) &= y_0 \quad \text{in } \Omega.
\end{aligned}
$$

A function $y \in H^1((0,T);H^{-1}(\Omega)) \cap L^2((0,T);H_0^1(\Omega))$ is called a weak solution of $(DBC)_1 - (DBC)_3$, if for all $v \in H_0^1(\Omega)$ there holds

$$
\begin{aligned}
\langle \frac{\partial y}{\partial t}, v \rangle + a(t;y,v) &= \int_\Omega uv dx, \\
(y(\cdot,0),v)_{L^2(\Omega)} &= (y_0,v)_{L^2(\Omega)}.
\end{aligned}
$$

where $a(t;y,v) := \int_\Omega \Big(a(\cdot,t)\nabla y \cdot \nabla v + b(\cdot,t)\cdot\nabla yv + c(\cdot,t)yv \Big) \, dx.$

Weak Solution: Necessary Optimality Conditions

Theorem. If $(y, u), y \in H^1((0,T); H^{-1}(\Omega)) \cap L^2((0,T); H_0^1(\Omega)), u \in L^2(Q)$ is the optimal solution of the optimal control problem, then there exists

$$p \in H^1((0,T); H^{-1}(\Omega)) \cap L^2((0,T); H_0^1(\Omega))$$

such that p satisfies the final time problem

$$\begin{aligned} -\frac{\partial p}{\partial t} + A^*(t)p &= -(y - y^d) && \text{in } Q, \\ p &= 0 && \text{on } \Sigma, \\ p(\cdot, T) &= 0 && \text{in } \Omega, \end{aligned}$$

and there holds

$$-p + \alpha u = 0 \quad \text{on } Q.$$

Parabolic Problems: Semidiscretization in Space

Let $V_h \subset H_0^1(\Omega)$ be the finite element space of P1 conforming finite elements with respect to a simplicial triangulation $\mathcal{T}_h(\Omega)$ of the computational domain Ω, and let $M_h \in \mathbb{R}^{N_h \times N_h}$ and $A_h(t) \in \mathbb{R}^{N_h \times N_h}, t \in [0,T]$, be the associated mass matrix and stiffness matrix. The semidiscretized parabolic optimal control problem reads as follows:

$$\inf_{y_h, u_h} \; J_h(y_h, u_h) \; := \; \frac{1}{2} \int_0^T |M_h y_h - y_h^d|^2 \, dt + \frac{\alpha}{2} \int_0^T |M_h u_h|^2 \, dt,$$

$$(DBC)_{h,1} \qquad M_h \frac{dy_h}{dt} + A_h(t) y_h \; = \; M_h u_h \quad t \in (0,T],$$

$$(DBC)_{h,2} \qquad M_h y_h(0) \; = \; y_h^0.$$

Semidiscretized Problem: Necessary Optimality Conditions

Theorem. If $(y_h, u_h), y_h \in C^1([0,T];V_h), u_h \in L^2((0,T);V_h)$ is the optimal solution of the semidiscretized parabolic optimal control problem, then there exists $p_h \in C^1([0,T];V_h)$ such that p_h satisfies the final time problem

$$\begin{aligned} -M_h \frac{dp_h}{dt} + A_h^*(t) p_h &= -M_h(y_h - y_h^d) \quad t \in [0,T), \\ M_h p_h(T) &= 0, \end{aligned}$$

and there holds

$$p_h(t) - \alpha u_h(t) = 0 \quad t \in [0,T].$$

Parabolic Problems: Fully Discrete Optimality System

Discretization in time by the implicit Euler method with respect to a partition $0 =: t_0 < t_1 < \cdots < t_M := T$ of the time interval $[0, T]$ with step size $\Delta t := T/M, M \in \mathbb{N}$, results in the fully discrete optimality system

$$\begin{aligned} (M_h + \Delta t \; A_h(t_m))y_h^{(m)} - \alpha^{-1} M_h p_h^{(m)} &= M_h y_h^{(m-1)} \;, \; m = 1, \cdots, M, \\ M_h y_h^{(m)} + (M_h + \Delta t \; A_h^*(t_m))p_h^{(m)} &= M_h p_h^{(m+1)} + M_h y_h^d(t_m) \;, \; m = M-1, \cdots, 0, \end{aligned}$$

where $M_h y_h^{(0)} = y_h^0$ and $M_h p_h^{(M)} = 0$.

Parabolic Optimal Control Problems

Neumann Boundary Control

Parabolic Problems: Neumann Boundary Control

Consider the Neumann boundary control problem

$$\inf_{y,u} \; J(y,u) \; := \; \frac{1}{2}\int_0^T \|y - y^d\|^2_{L^2(\Omega)} \, dt + \frac{\alpha}{2} \int_0^T \|u\|^2_{L^2(\Gamma)} \, dt,$$

$$\begin{array}{llcll}
(DBC)_1 & \dfrac{\partial y}{\partial t} + A(t)y & = & 0 & \text{in } Q := \Omega \times (0,T), \\
(DBC)_2 & \nu_\Gamma \cdot a\nabla y & = & u & \text{on } \Sigma := \Gamma \times (0,T), \\
(DBC)_3 & y(\cdot,0) & = & y_0 & \text{in } \Omega,
\end{array}$$

where $u \in L^2(Q)$.

Approach: Weak formulation of $(DBC)_1 - (DBC)_3$.

Neumann Boundary Control: Weak Solution

Consider the parabolic initial-boundary value problem

$$
\begin{aligned}
(\mathrm{DBC})_1 \qquad \frac{\partial y}{\partial t} + A(t)y &= 0 \quad \text{in } Q := \Omega \times (0,T),\\
(\mathrm{DBC})_2 \qquad \nu_\Gamma \cdot a\nabla y &= u \quad \text{on } \Sigma := \Gamma \times (0,T),\\
(\mathrm{DBC})_3 \qquad y(\cdot,0) &= y_0 \quad \text{in } \Omega.
\end{aligned}
$$

A function $y \in H^1((0,T);H^1(\Omega)^*) \cap L^2((0,T);H^1(\Omega))$ is called a weak solution of $(\mathrm{DBC})_1 - (\mathrm{DBC})_3$, if for all $v \in H^1(\Omega)$ there holds

$$
\begin{aligned}
\langle \frac{\partial y}{\partial t}, v\rangle + a(t;y,v) &= \int_\Gamma uv\,d\sigma,\\
(y(\cdot,0),v)_{L^2(\Omega)} &= (y_0,v)_{L^2(\Omega)}.
\end{aligned}
$$

where $a(t;y,v) := \int_\Omega \Big(a(\cdot,t)\nabla y \cdot \nabla v + b(\cdot,t)\cdot\nabla y v + c(\cdot,t) y v \Big)\, dx.$

Weak Solution: Necessary Optimality Conditions

Theorem. If $(y,u), y \in H^1((0,T);H^1(\Omega)^*) \cap L^2((0,T);H^1(\Omega)), u \in L^2(\Sigma)$ is the optimal solution of the optimal control problem, then there exists

$$p \in H^1((0,T);H^{-1}(\Omega)) \cap L^2((0,T);H_0^1(\Omega))$$

such that p satisfies the final time problem

$$\begin{aligned} -\frac{\partial p}{\partial t} + A^*(t)p &= -(y - y^d) \quad &&\text{in } Q, \\ p &= 0 &&\text{on } \Sigma, \\ p(\cdot,T) &= 0 &&\text{in } \Omega, \end{aligned}$$

and there holds

$$-p + \alpha u = 0 \quad \text{on } \Sigma.$$

Parabolic Optimal Control Problems

Dirichlet Boundary Control

Parabolic Problems: Dirichlet Boundary Control I

Consider the Dirichlet boundary control problem

$$\inf_{y,u} J(y,u) := \frac{1}{2}\int_0^T \|y - y^d\|^2_{L^2(\Omega)}\, dt + \frac{\alpha}{2}\int_0^T \|u\|^2_{L^2(\Gamma)}\, dt,$$

$$(\mathrm{DBC})_1 \qquad \frac{\partial y}{\partial t} + A(t)y = 0 \quad \text{in } Q := \Omega \times (0,T),$$

$$(\mathrm{DBC})_2 \qquad y = u \quad \text{on } \Sigma := \Gamma \times (0,T),$$

$$(\mathrm{DBC})_3 \qquad y(\cdot,0) = y_0 \quad \text{in } \Omega,$$

where $u \in L^2((0,T); H^{1/2}(\Gamma))$.

Approach: Weak formulation of $(\mathrm{DBC})_1 - (\mathrm{DBC})_3$.

Dirichlet Boundary Control: Weak Solution

Consider the parabolic initial-boundary value problem

$$
\begin{aligned}
(DBC)_1 \qquad \frac{\partial y}{\partial t} + A(t)y &= 0 \quad \text{in } Q := \Omega \times (0,T),\\
(DBC)_2 \qquad y &= u \quad \text{on } \Sigma := \Gamma \times (0,T),\\
(DBC)_3 \qquad y(\cdot,0) &= y_0 \quad \text{in } \Omega.
\end{aligned}
$$

A function $y \in H^1((0,T);H^1(\Omega)^*) \cap L^2((0,T);H^1(\Omega))$ is called a weak solution of $(DBC)_1 - (DBC)_3$, if for all $v \in H^1_0(\Omega)$ there holds

$$
\begin{aligned}
\langle \frac{\partial y}{\partial t}, v\rangle + a(t;y,v) &= 0,\\
y|_\Sigma &= u,\\
(y(\cdot,0),v)_{L^2(\Omega)} &= (y_0,v)_{L^2(\Omega)}.
\end{aligned}
$$

where $a(t;y,v) := \int_\Omega \Big(a(\cdot,t)\nabla y \cdot \nabla v + b(\cdot,t)\cdot\nabla y v + c(\cdot,t) y v \Big)\, dx$.

Weak Solution: Necessary Optimality Conditions

Theorem. If $(y,u), y \in H^1((0,T);H^1(\Omega)^*) \cap L^2((0,T);H^1(\Omega)), y|_\Sigma = u \in L^2((0,T);H^{1/2}(\Gamma))$ is the optimal solution of the optimal control problem, then there exists

$$p \in H^1((0,T);H^{-1}(\Omega)) \cap L^2((0,T);H_0^1(\Omega))$$

such that p satisfies the final time problem

$$\begin{aligned} -\frac{\partial p}{\partial t} + A^*(t)p &= -(y - y^d) && \text{in } Q, \\ p &= 0 && \text{on } \Sigma, \\ p(\cdot, T) &= 0 && \text{in } \Omega, \end{aligned}$$

and there holds

$$\nu_\Gamma \cdot a\nabla p + \alpha u = 0 \quad \text{on } \Sigma.$$

Parabolic Problems: Dirichlet Boundary Control II

Consider the Dirichlet boundary control problem

$$\inf_{y,u} J(y,u) := \frac{1}{2}\int_0^T \|y - y^d\|^2_{L^2(\Omega)}\, dt + \frac{\alpha}{2}\int_0^T \|u\|^2_{L^2(\Gamma)}\, dt,$$

$$\begin{aligned}
&(DBC)_1 \qquad & \frac{\partial y}{\partial t} + A(t)y &= 0 \quad \text{in } Q := \Omega \times (0,T),\\
&(DBC)_2 \qquad & y &= u \quad \text{on } \Sigma := \Gamma \times (0,T),\\
&(DBC)_3 \qquad & y(\cdot,0) &= y_0 \quad \text{in } \Omega,
\end{aligned}$$

where $u \in L^2(\Sigma)$.

Problem: The weak formulation of $(DBC)_1 - (DBC)_3$ is not appropriate.

Department of Mathematics, University of Houston
Institut für Mathematik, Universität Augsburg

Dirichlet Boundary Control: Very Weak Solution

Consider the parabolic initial-boundary value problem

$$
\begin{aligned}
(DBC)_1 \qquad & \frac{\partial y}{\partial t} + A(t)y = 0 \quad \text{in } Q := \Omega \times (0,T),\\
(DBC)_2 \qquad & y = u \quad \text{on } \Sigma := \Gamma \times (0,T),\\
(DBC)_3 \qquad & y(\cdot,0) = y_0 \quad \text{in } \Omega,
\end{aligned}
$$

A function $y \in H^1((0,T);H^{-2}(\Omega)) \cap L^2(Q)$ is called a very weak solution of $(DBC)_1 - (DBC)_3$, if for all $v \in L^2((0,T);H^2(\Omega) \cap H_0^1(\Omega))$ there holds

$$\int_0^T \Big(\langle \frac{\partial y}{\partial t}, v \rangle_{H^{-2},H^2} + (y, A^*(t)v)_{L^2(\Omega)} \Big) \, dt = \int_0^T (u, \nu_\Gamma \cdot a \nabla v)_{L^2(\Gamma)} \, dt.$$

where $A^*(t)v := -\nabla \cdot (a \nabla v) - b \cdot \nabla v + (c - \nabla \cdot b)v$.

Department of Mathematics, University of Houston
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Very Weak Solution: Necessary Optimality Conditions

Theorem. If (y,u) , $y \in H^1((0,T);H^{-2}(\Omega)) \cap L^2(Q)$, $u \in L^2(\Sigma)$ is the optimal solution of the optimal control problem, then there exists

$$p \in H^1((0,T);L^2(\Omega)) \cap L^2((0,T);H^2(\Omega) \cap H_0^1(\Omega))$$

such that p satisfies the final time problem

$$\begin{aligned} -\frac{\partial p}{\partial t} + A^*(t)p &= -(y - y^d) \quad && \text{in } Q, \\ p &= 0 && \text{on } \Sigma, \\ p(\cdot,T) &= 0 && \text{in } \Omega, \end{aligned}$$

and there holds

$$\nu_\Gamma \cdot a\nabla p + \alpha u = 0 \quad \text{on } \Sigma.$$