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Marcel Berger

Introducing Dynamics in Elementary Geometry: Introduction to some Work of Richard Schwartz

We will consider three elementary geometric constructions. The first is the barycentric subdivision of a triangle, a tetrahedron, etc. The second is Pappus theorem, which starting with a pair of two sets of three points on a line in the plane, yields a third set of three points on a line. The third is, starting with a convex polygon (e.g. a pentagon), construct a new one by joining by lines the vertices from one to the next-next one.

Schwartz is studying what happens when one iterates up to infinity these constructions. In the case of barycentric subdivisons what are the possible shapes so obtained? In Pappus theorem iteration how do the figures made up by the infinite set of the so obtained pointed lines in the plane look? For polygons what are the shapes of the polygons so obtained?

In the first case the answer is known for triangles and for tetrahedrons, but still open starting dimension four or higher. For Pappus configuration precise information is obtained by introducing an action of the modular group into the structure. For polygons, things are known completely for pentagons, partially for hexagons, and mostly open for polygons with seven or more vertices. But numerical experiments suggest some conjectures.

Peter J. Cameron

Finite geometry and permutation groups: some polynomial links

A matroid is a combinatorial object which models (among other things) linear dependence in a vector space, and so can be used to describe configurations in finite projective spaces. Associated with any matroid is a 2-variable polynomial, the Tutte polynomial, which specialises to the weight enumerator of the corresponding code (for example).

Associated with any permutation group on n points is an n-variable polynomial, the cycle index, which has many applications in combinatorial enumeration.

There are several examples where one of these two polynomials is a specialisation of the other. Indeed, it is possible to define a polynomial which includes both the cycle index and (in some cases) the Tutte polynomial as specialisations.

In the talk I will cover these matters and speculate on possible future directions.

John H. Conway

Some Things You Can't Hear The Shape Of

The problem popularized by Mark Kac's famous lecture "Can you hear the shape of a drum?" was finally solved about ten years ago, buy Gordon, Wolpert and Webb. However, this is not the end of the problem, just as Kac's question wasn't the beginning. On the way to being solved, the problem led to several related notions of "isospectrality", each of which gives new problems of its own. I shall discuss these problems and their solutions, when known, for open and closed manifolds, lattices, codes, and groups. One most interesting particular result is the recent theorem of Juan-Pablo Rossetti that there is a unique non-trivial pair of isospectral "platycosms" (closed flat 3-manifolds). So I shall also discuss the 10 types of platycosm and their geometry.

Phillip Griffiths

Algebraic Cycles and Singularities of Normal Functions

(Report on joint work with Mark Green)

Given a smooth projective variety X of dimension $2n$, an ample line bundle $L \to X$ and a primitive Hodge class $\gamma \in H^n_g(X)_{\text{prim}}$, assuming the Hodge conjecture in lower dimensions we will give explicitly the equations of the locus

$$S(\gamma, k) = \left\{ \begin{array}{l} s \in H^0(X, L^k)\colon \text{there exists an} \\ \text{algebraic cycle } Z \text{ in } X_s \text{ with} \\ \langle \gamma, [Z] \rangle \neq 0 \end{array} \right\}$$

where $k >> 0$ and $X_s = \{s = 0\}$. By an inductive argument, the Hodge conjecture is equivalent to the statement

$$S(\delta, k) \neq \phi \quad \text{for} \quad k >> 0$$

asserting in effect that an explicit set of polynomial equations has non-trivial solutions. The definition of $S(\delta, k)$ is very geometric; showing that it is non-empty seems to require subtle arithmetic considerations.

Mikhael Gromov

Geometry of infinite Cartesian powers and related spaces

Many geometric invariants/theorems are (at least conjecturally) well behaved under taking finite Cartesian products. This suggests going to infinite products and passing to the infinite dimensional counterparts of such invariants/theorems. We indicate in our talk a few instances where such approach leads to a meaningful conclusion.

J.W.P. Hirschfeld

The number of points on a curve, and applications

Curves defined over a finite field have various applications, such as

(a) the construction of good error-correcting codes,

(b) the correspondence with arcs in a finite Desarguesian plane,

(c) the Main Conjecture for maximum-distance-separable (MDS) codes.

Bounds for the number of points of such a curve imply results in these cases.

For plane curves, there is a variety of bounds that can be considered, such as the Hasse–Weil bound (1934/1948), the Stöhr–Voloch bound (1986), as well as bounds that depend on the plane embedding. Curves that achieve these bounds can sometimes be characterized.

Segre applied bounds for the number of points on a curve to obtain bounds on the sizes of complete arcs. He also considered plane Fermat curves that achieve the Hasse–Weil bound.

Various of these results and their applications are surveyed.

Dieter Jungnickel

Some geometric aspects of abelian groups

Consider a finite projective plane admitting a large abelian collineation group. It is well-known that this situation may be studied by algebraic means (via a representation by suitable types of difference sets), namely using group rings and algebraic number theory and leading to rather strong non-existence results. What is less well-known is the fact that the abelian group (and sometimes its group ring) can also be used in a much more geometric way, which will be the topic of this lecture. In one direction, abelian collineation groups may be applied for the construction of interesting geometric objects such as unitals, arcs and (hyper-)ovals, (Baer) subplanes and projective triangles. On the other hand, this approach makes it sometimes possible to provide simple geometric proofs for non-trivial structural restrictions on the given collineation group, avoiding algebraic machinery.

Gábor Korchmáros

Segre-type theorems in finite geometry

(Research supported by the Italian Ministry MURST, Strutture geometriche, combinatoria e loro applicazioni.)

The concept of an oval in finite geometry arises from two combinatorial properties of a closed convex curve Ω in the real planes; namely

(1) no three points in Ω are collinear;

(2) there is exactly one line at every point $P \in \Omega$ that meets Ω only in P.

In a finite projective plane π, a k-*arc* is a set Ω of k points that has property (1). An *oval* is a k-arc which also has property (2). If π has order n, then every oval consists of $n + 1$ points. An example of an oval is the irreducible conic in $PG(2, q)$, the projective plane coordinatised by the Galois field $GF(q)$ of order q.

Segre's famous theorem states that every oval in $PG(2, q)$, with q odd, is an irreducible conic. Segre noted that his theorem does not hold true in $PG(2, q)$ when q is even and $q \geq 8$. The classification project of ovals in $PG(2, q)$ with q even is still in progress.

Segre published his theorem in 1954. Ever since, many results have been found concerning ovals and their generalisations in Galois geometry, that is, in higher-dimensional projective spaces over a Galois field.

In a joint paper with the author, Segre gave a purely combinatorial characterisation of external lines to an irreducible conic $\mathcal{C}$ in $PG(2, q)$. If every chord and tangent of an irreducible conic meets a set $\mathcal{L}$ in exactly one point, then $\mathcal{L}$ consists of all points of an external line to the conic.

In the abstract of the paper, the following remark is made: "While the result admits no analogue in the real field, a number of similar properties can be established or investigated in any Galois geometry." In this spirit, combinatorial characterisations of geometric objects related to ovals are *Segre-type theorems*.

The proof of the above, and some other Segre-type theorems dating back to the late seventies and early eighties, uses the powerful idea in Segre's original proof to connect combinatorics to number theory via finite geometry.

Ideas depending on lacunary polynomials, originally developed to investigate blocking sets, produced several, interesting Segre-type theorems in the nineties.

A different approach to Segre-type theorems is based on a surprising result on algebraic curves in positive characteristic: the linear system of algebraic curves of minimum degree, which is $q-1$, passing through every internal point of an irreducible conic in $PG(2,q)$, q odd, has dimension $q-1$. So, such points impose independent conditions on the algebraic curves of degree $q-1$ which pass through them. This result, together with the classification of all subgroups of $PGL(2,q)$, is the main ingredient in the current investigation of a new generation of Segre-type theorems. A prototype is the following result.

Theorem 1 *Let $\mathcal{C}$ be an irreducible conic in $PG(2,q)$, q odd. Let $\mathcal{B}$ be a point set in $PG(2,q)$ which meets every external line to $\mathcal{C}$. Then $|\mathcal{B}| \geq q-1$ with equality occurring for $q=3$ and $q \geq 9$ in the linear case only, that is, when $\mathcal{B}$ consists of all points of a chord r of $\mathcal{C}$ minus the two common points of r and $\mathcal{C}$. For $q=5,7$ there exists just one more example, up to* *projectivities.*

Ovals are known to exist in almost all known finite projective planes. They have been found by various different methods depending on polarities, collineations, quasifield properties, ad hoc constructions as well as on extensive computer search. As a part of a classification project of ovals in finite projective planes, collineation groups have been intensively studied. In this context, the following result plays an important role.

Theorem 2 *Let G be a simple group acting on a projective plane of odd order as a collineation group preserving an oval. Then $G \cong PSL(2, q)$ with $q \geq 5$ odd.*

Yuri I. Manin

Manifolds with multiplication in tangent bundle

The talk will be a review of the theory of manifolds endowed with associative and commutative multiplication in the tangent bundle, which satisfies a certain integrability condition: F-manifolds introduced by Hertling and Manin. A somewhat stronger structure of Frobenius manifolds axiomatized and studied by B. Dubrovin play a central role in the theory of quantum cohomology and Mirror Symmetry. I will discuss relationships with motives, tensor categories, and theory of unfolding singularities.

Edoardo Sernesi

Segre's works on curves and their moduli

I will overview the contributions of B. Segre about the geometry of algebraic curves and their moduli from the perspective of today's algebraic geometry.

Nicholas Shepherd-Barron

Cubic surfaces and rationality

This talk describes two examples of the intervention of cubic surfaces in questions of number theory and algebraic geometry since Segre's time.

Joseph A. Thas

Finite Geometries: Classical Problems and Recent Developments

In recent years there has been an increasing interest in finite projective spaces, and important applications to practical topics such as coding theory, cryptography and design of experiments have made the field even more attractive. Pioneering work has been done by B. Segre and each of the four topics of my talk is related to his work. It is my intention to speak about two classical problems and two recent developments. First I will mention a purely combinatorial characterization of Hermitian curves in $\mathrm{PG}(2, q^2)$; here, from the beginning, the considered point set is contained in $\mathrm{PG}(2, q^2)$. It is a characterzation in the spirit of Segre's famous characterization of conics in $\mathrm{PG}(2, q), q$ odd. A second approach is where the object is described as an incidence structure satisfying certain properties; here the geometry is not a priori embedded in a projective space. This will be illustrated by a characterization of the classical inverse plane in the odd case. A recent beautiful result in Galois geometry is the discovery of an infinite class of hemisystems of the Hermitian variety in $\mathrm{PG}(3, q^2)$, leading to new interesting classes of incidence structures, graphs and codes; before this result, just one example for GF(9), due to Segre, was known. An exemplary example of research combining combinatorics, incidence geometry, Galois geometry and group theory is the determination of embeddings of generalized polygons in finite projective spaces. As an illustration we will speak about the embeddings of the flag geometry of a projective plane and about the embedding of the generalized quadrangle of order (4,2), that is, the Hermitian variety $H(3, 4)$, in PG(3,$\mathbf{K}$) with $\mathbf{K}$ any field.

Gudlaugur Thorbergsson

Transformation groups and submanifold geometry

Beniamino Segre classified in 1938 isoparametric hypersurfaces in Euclidean spaces. As a consequence of his classification it turns out that these hypersurfaces are precisely the homogeneous ones (or pieces of such). In the last 20 years the theory of isoparametric hypersurfaces has been generalized in various directions. First one studied isoparametric submanifolds in Euclidean spaces with arbitrary codimension. Then more general ambient spaces were also considered. The theory is by now well developed in symmetric spaces and one has started to generalize it to general Riemannian manifolds. Transformation groups, or more precisely polar actions (isometric actions admitting canonical forms), have always played a central role, since the models for all of the generalizations are principal orbits of such actions. There is a recurrent question that brings us back to Segre's paper: When are the generalized isoparametric submanifolds homogeneous?

Giuseppe Tomassini

Extension problems in complex geometry

One of the recurring problems in Complex Analysis is the one of extending “analytic objects”. The Hartogs theorem (holomorphic functions “fill compact holes” in $\mathbb{C}^n$, $n \geq 2$) is the prototype of all extension theorems. In this talk we will treat two aspects of the problem, namely the extension of analytic subsets and the extension by Levi-flat hypersurfaces (i.e. foliated by complex hypersurfaces).

The theme on the extension analytic subsets is very classical and a vast literature is available. We will sketch some new results on the extension of an analytic subset given on the complement of a not necessarely compact subset of a complex space.

The interest for Levi-flat hypersurfaces is reletively recent and is related to the construction of global hulls of holomorphy. In $'83$, using Bishop's method of the “analytic discs”, Bedford and Gaveau proved the following fundamental result: let M be a generic graph of a smooth function g on the sphere $S^2 \subset \mathbb{C} \times \mathbb{R}$; assume that M is embedded in the boundary of a strictly pseudoconvex domain in $\mathbb{C}^2$; then M is *extendable* by a Levi-flat graph $\widetilde{M}$. The analytic counterpart of this result is the existence of solutions to the Dirichlet problem for a quasi-linear second order degenerate elliptic equation (the so called *Levi equation*). The Bedford-Gaveau theorem has been meanwhile generalized to generic smooth spheres embedded in the boundary of a strictly pseudoconvex domain in $\mathbb{C}^2$.

Concerning the extension by Levi-flat hypersurfaces, we will discuss some new results for $\mathbb{C}^2$ and $\mathbb{C}^3$. Precisely, we will state a theorem on Levi-flat extension from a part of the boundary in $\mathbb{C}^2$. This can be seen as the first step for a general theory of the "domains of existence" for Levi-flat hypersurfaces.

The situation in $\mathbb{C}^3$ is quite different. Generically a 4-manifold M is not even locally extendable by a Levi-flat hypersurface $\widetilde{M}$. We first find local necessary conditions for the extension and then, under suitable hypothesis, using a Harvey-Lawson Theorem with C^∞ parameters, we prove the existence of a "Levi-flat" chain $\widetilde{M}$ such that $d\widetilde{M} = M$.

Joseph Zaks

Geometric graphs and the Beckman-Quarles Theorem

A graph G is called geometric if, for some d and r, its vertex set is of the form F^d, where F is a subfield of the reals $\mathbb{R}$, and xy is an edge if, and only if, $||x-y|| = r$. In this case we denote G by $F(d,r)$.

The Beckman-Quarles Theorem states that every unit-distance preserving mapping $f : E^d \to E^d$ is an isometry, provided $d \geq 2$.

One may try to extend this theorem by treating other fields, like the field of the rationals $\mathbb{Q}$, or treat mappings of F^d to F^d which preserve other distances. A major open problem asks for the analogue for mappings $f : E^n \to E^m$, $m > n$; there is an example of a unit-distance preserving mapping of E^2 to E^6 which is not an isometry.

One can easily show that if the graph $F(d,r)$ is not connected, then there exist mappings $f : F^d \to F^d$ which preserve the distance r yet f is not an isometry.

M. Perles raised the following question: Suppose a graph G is given, by means of its vertices and edges, and it is also given that G is of the form $F(d,r)$; can the Euclidean distance $||x-y||$ be found, in terms of some multiple of r? Is it true for $F = \mathbb{Q}$, or for $F = \mathbb{R}$ in case $||x-y||$ is an algebraic number?

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