Some modular varieties of low dimension II

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Introduction

Some years ago in [FH] have been studied some modular varieties related to the orthogonal group $\mathrm{O}(2,n)$.The most significant variety they studied was related to $\mathrm{O}(2,6)$, or —equivalently— to the symplectic group of degree two defined on the quaternions. Because of a mistake, they did not get a definite result regarding the structure of the graded ring of modular forms of degree two with respect to the Hurwitz integral quaternions. this has been obtained recently In particular they gave a finite map from this variety to a weighted 6-dimensional projective space. Recently Krieg determined the structure Krieg's result convinced the authors to reconsider [FH]. Here we found that in the computation of the covering degree of the mentioned finite map a factor three is missing in the denominator. The aim of this paper is to correct this mistake and to improve the result of [FH]. In this way we will get Krieg's structure theorem as a corollary of a more general result. In contrast to Krieg we work only in the orthogonal context. We think that this makes the underlying geometry much more visible.

This paper bases on many discussions with A. Krieg. We thank him for his support and for showing us unpublished manuscripts.

1. Orthogonal and elliptic modular forms

A lattice L is a free abelian group together with a real valued non degenerated bilinear form $(\cdot,\cdot)$. It is called even, if the quadratic form, the so-called norm, (x,x) is even for all x. We also will consider the quadratic space $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. Its orthogonal group is denoted by $\mathrm{O}(V)$. Recall that in the indefinite case there is a subgroup $\mathrm{O}^+(V)$ of index two, which is generated by all reflections along vectors a of negative norm (a,a). (The reflection along a is defined by the fact it changes the sign of a and fixes ist orthogonal complement.) The reflections along vectors of positive norm are not contained in this group. We consider the signature (p,q). In a suitable basis of V the quadratic form is given by $x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2$. It is easy to check that the transformation $x\mapsto -x$ is contained in $\mathrm{O}^+(V)$ when p is even. We also have to consider the integral orthogonal subgroups

$$\mathrm{O}(L):=\{\,g\in \mathrm{O}(V);\quad g(L)=L\,\}$$

and

$$\mathrm{O}^+(L)=\mathrm{O}(L)\cap \mathrm{O}^+(V).$$

Let V be a quadratic space of signature $(2,n)$. We denote by $\mathcal{H}(V)$ the set of all two dimensional positive definite subspaces. If L is a lattice of signature $(2,n)$ we write $\mathcal{H}(L)$ instead of $\mathcal{H}(V)$. We recall that $\mathcal{H}(L)$ is a complex manifold. For this one considers the projective space $P(V(\mathbb{C}))$ of the complexification $V(\mathbb{C})=V\otimes_{\mathbb{R}}\mathbb{C}$ of V. We extend $(\cdot,\cdot)$ to a $\mathbb{C}$-bilinear form. The zero quadric $(z,z)=0$ is a (smooth) complex submanifold. The set

$$\mathcal{K}(V):=\{\,z\in P(V(\mathbb{C}));\quad (z,z)=0,\ (z,\bar z)>0\,\}$$

is an open subset of $\mathcal{K}$ and hence a complex manifold. It has two connected components, which can be interchanged by the map $z\mapsto\bar z$. We choose one of the components and denote it by $\mathcal{K}^+$. The map

$$\mathcal{K}^+\xrightarrow{\sim}\mathcal{H}(V),\quad \mathbb{C}z\longmapsto \mathbb{R}x+\mathbb{R}y\qquad (z=x+\mathrm{i}y)$$

is bijective and defines the stated complex structure on $\mathcal{H}(V)$. The group $\mathrm{O}^+(V)$ is the subgroup of $\mathrm{O}(V)$ which preserves $\mathcal{K}^+$. Its action on $\mathcal{H}(V)$ is holomorphic. We recall the notion of a modular form. let $\Gamma\subset\mathrm{O}^+(L\otimes_{\mathbb{Z}}\mathbb{R})$ be a subgroup, which is commensurable with $\mathrm{O}^+(L)$. We consider the inverse image $\tilde{\mathcal{K}}^+$ of $\mathcal{K}^+$ under the natural map $V(\mathbb{C})-\{0\}\to P(V(\mathbb{C}))$. The group $\mathrm{O}^+(V)$ acts on $\tilde{\mathcal{K}}^+$ as well. A modular form of weight k and with respect to some character $v:\Gamma\to\mathbb{C}^\bullet$ is a holomorphic function $f:\tilde{\mathcal{K}}^+\to\mathbb{C}$ with the properties

a) $f(\gamma(z))=v(\gamma)f(z)$,
b) $f(tz)=t^{-k}f(z)$,
c) f is holomorphic at the cusps.

We denote by $[\Gamma,k,v]$ the space of all these forms or simply by $[\Gamma,k]$, when v is trivial. When k is odd and $-\mathrm{id}$ is contained in Γ then $[\Gamma,k]=0$. Sometimes we consider $\Gamma/\{\pm\mathrm{id}\}$ instead of Γ and write $[\Gamma/\{\pm\mathrm{id}\},k]$ instead of $[\Gamma,k]$ for even k.

We recall the standard realization of $\tilde{\mathcal{K}}^+$. For this purpose we decompose $V=\mathbb{R}^2\times\mathbb{R}^2\times V_0$, where the quadratic form is $x_1x_2+x_3x_4+(\mathfrak{x},\mathfrak{x})/2$ with a negative definite form $(\mathfrak{x},\mathfrak{x})$ on V_0. Then $\tilde{\mathcal{K}}^+$ can be taken as the set of all $t(1,*,z_0,z_2;\mathfrak{z})$ where $t\neq0$, $y_0>0$ and $y_0y_2+(\mathfrak{y},\mathfrak{y})/2>0$. A modular form f is determined by the function

$$F(z_0,z_2,\mathfrak{z}):=f(1,*,z_0,z_2,\mathfrak{z})$$

and this function satisfies the transformation formula

$$F(\gamma(z_0, z_2, \mathfrak{z})) = a(\gamma, (z_0, z_2, \mathfrak{z}))^k F(z_0, z_2, \mathfrak{z}).$$

Here $\gamma(z_0, z_2, \mathfrak{z})$ and $a(\gamma, (z_0, z_2, \mathfrak{z}))$ are defined as follows: Consider

$$\gamma((1, *, z_0, z_2, \mathfrak{z}) = t(1, *, w_0, w_2, \mathfrak{w})$$

and define

$$a(\gamma, (z_0, z_2, \mathfrak{z})) = t^{-1}, \quad \gamma(z_0, z_2, \mathfrak{z}) = (w_0, w_2, \mathfrak{w}).$$

1.1 Lemma. *The automorphy factor a is related to the Jacobian determinant* aF
J by the formula

$$J(\gamma, (z_0, z_2, \mathfrak{z})) = \det(\gamma) a(\gamma, (z_0, z_2, \mathfrak{z}))^n.$$

This formula can be verified directly for reflections, using some tedious but elementary calculation which we omit here.

Elliptic modular forms

Let $\mathrm{Mp}(2,\mathbb{Z})$ be the metaplectic cover of $\mathrm{SL}(2,\mathbb{Z})$. The elements of $\mathrm{Mp}(2,\mathbb{Z})$ are pairs $(M, \sqrt{c\tau+d})$, where $M = \binom{ab}{cd} \in \mathrm{SL}(2,\mathbb{Z})$, and $\sqrt{c\tau+d})$ denotes a holomorphic root of $c\tau+d$ on the upper half plane $\mathbf{H}$. It is well known that $\mathrm{Mp}(2,\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right),$$
$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad \mathrm{Re}\sqrt{\tau} > 0.$$

One has the relations $S^2 = (ST)^3 = Z$, where $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \mathrm{i}\right)$ is the standard generator of the center of $\mathrm{Mp}(2,\mathbb{Z})$.

Recall that there is a unitary representation ρ_L of $\mathrm{Mp}(2,\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$.

$$\varrho_L(T) = \left(e^{2\pi \mathrm{i} q_L(\alpha)}\right)_{\alpha \in L'/L} \quad \text{(diagonal matrix)}$$
$$\varrho_L(S) = \frac{\sqrt{\mathrm{i}}^{n-m}}{\sqrt{|L'/L|}} \left(e^{-2\pi \mathrm{i}(\alpha,\beta)}\right)_{\alpha,\beta \in L'/L}.$$

This representation is the Weil representation attached to the "finite quadratic form" $(L'/L, \bar{q}_L)$, where $q_L : L'/L \to \mathbb{Q}/\mathbb{Z}$ is induced by $(x,x)/2$.

We recall the notion of an elliptic modular form with respect to a finite dimensional representation $\varrho : \mathrm{Mp}(2,\mathbb{Z}) \to \mathrm{GL}(W)$. Let $k \in 1/2\mathbb{Z}$ and $f : \mathbf{H} \to W$ be a holomorphic function. Then f is called *modular form* of weight k with respect to ρ if

$$f(M\tau) = \sqrt{c\tau + d}^{2k} \varrho(M, \sqrt{c\tau + d}) f(\tau)$$

for all $(M, \sqrt{c\tau + d}) \in \mathrm{Mp}(2,\mathbb{Z})$ and if f is holomorphic at $\mathrm{i}\infty$. We denote the space of all these modular forms by $[\mathrm{SL}(2,\mathbb{Z}), k, \varrho]$. A form is called a cusp form if it vanishes at ∞. The space $[\mathrm{SL}(2,\mathbb{Z}), k, \varrho]$ decomposes into the direct sum of the subspace of cusp forms and into the space of Eisenstein series which can be defined as orthogonal complement with respect to the Petersson scalar product. An Eisenstein series is determined by its constant Fourier coefficient.

Borcherds' additive lift

We denote by

$$\Gamma_L := \mathrm{kernel}\big(\mathrm{O}^+(L) \longrightarrow \mathrm{Aut}(L'/L)\big)$$

the so called discriminant kernel. Borcherds defined for integral $k+n/2$ a linear map

$$[\mathrm{SL}(2,\mathbb{Z}), k, \varrho_L] \longrightarrow [\Gamma_L, k + n/2 - 1],$$

which generalizes constructions of Saito-Kurokawa, Shimura, Maaß, Gritsenko, Oda et.al. This map is equivariant with respect to the group $\mathrm{O}^+(L)$, which acts on both sides in a natural way. Borcherds defined in [Bo] this lift as a theta lift, but he also gives the Fourier expansion of the image of an elliptic modular form f at an arbitrary cusp in terms of the Fourier coefficients of f.

We are interested in this construction especially in the case $k = 0$. Modular forms of weight zero are constant, hence we get for even n a map

$$\mathbb{C}[L'/L]^{\mathrm{SL}(2,\mathbb{Z})} \longrightarrow [\Gamma_L, n/2 - 1].$$

These orthogonal modular forms are the simplest examples of modular forms of several variables. The weight $n/2 - 1$ is the so-called singular weight. Every modular form of weight $0 < k < n/2 - 1$ vanishes.

We also have to recall the notion of a quadratic divisor. Let V be a quadratic space of signature $(2, n)$ and $W \subset V$ a subspace of signature $(2, n-1)$. Usually W is defined as orthogonal complement of a vector of negative norm. Then we obtain a natural holomorphic embedding $\mathcal{H}(W) \hookrightarrow \mathcal{H}(V)$. Assume that $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ with an even lattice L and that $W = M \otimes_{\mathbb{Z}} \mathbb{R}$ with $M = L \cap W$. Let $\Gamma \subset \mathrm{O}^+(V)$ a subgroup which is commensurable with $\mathrm{O}^+(L)$. The projected group Γ' consists of all elements of $\mathrm{O}^+(W)$, which are restrictions of an element of Γ. This group is commensurable with $\mathrm{O}^+(M)$. Then we get a natural map

$$\mathcal{H}(W)/\Gamma' \longrightarrow \mathcal{H}(V)/\Gamma.$$

from the theory of Baily-Borel [BB] we know that this is an algebraic map of quasiprojective varieties. Moreover this map is birational onto its image.

2. A distinguished point

A basic role will play the root lattice D_4. We take the realization given by the Hurwitz integers. This means the following: Denote by $1, \mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3$ the standard generators of the Hamilton quaternions. Then $\mathfrak{o}$ is the set of all $a_0 + a_1\mathrm{i}_1 + a_2\mathrm{i}_2 + a_2\mathrm{i}_3$ such that the a_i are all in $\mathbb{Z}$ or all in $1/2 + \mathbb{Z}$. The bilinear form is $(a, b) = 2\,\mathrm{Re}(a\bar{b})$, hence the norm is $(a, a) = 2a\bar{a} = 2(a_0^2 + a_1^2 + a_2^2 + a_3^2)$. We denote by H the lattice $\mathbb{Z}^2$ with bilinear form $(x, y) = x_1y_2 + x_2y_1$. This is an even lattice of signature $(1, 1)$. Finally we define the orthogonal sum

$$L(\mathfrak{o}) = H \times H \times D_4(-1).$$

Here $D_4(-1)$ means as usual D_4 equipped with the negative of the bilinear form of D_4. In the paper [FH] the six lattice points of $L(\mathfrak{o})$

$$\begin{aligned} &A_1 = (0, 0, 0, 0;\ \mathrm{i}_3), && A_2 = (0, 0, 0, 0;\ \omega), && A_3 = (1, 0, 0, 0;\ -\bar{\omega}),\\ &A_4 = (0, 1, 0, 0;\ -\bar{\omega}), && A_5 = (1, 1, 1, -1;\ -\bar{\omega}), && A_6 = (0, 0, 1, 0;\ -\mathrm{i}_2) \end{aligned}$$

have been considered. The determinant of their Gram matrix is 3. The sublattice generated by them is a copy of $E_6(-1)$, where E_6 denotes the well-known root lattice. We use the standard notations as in [CS]. It is a basic fact that the group $\mathrm{O}^+(L(\mathfrak{o}))$ acts transitively on the set of all sublattices of $L(\mathfrak{o})$ which are isomorphic to $E_6(-1)$ (proposition 8.1 in [FH]). Hence the vectors above generate a representative of this set. The orthogonal complement W of this $E_6(-1)$ in $L(\mathfrak{o})$ is generated by the two vectors

$$B_1 = (2, 2, 2, 0;\ -1 + \mathrm{i}_1), \quad B_2 = (0, 0, 2, 2;\ \mathrm{i}_1 - \mathrm{i}_2).$$

Their Gram matrix is

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

The determinant is 12. Hence the determinant of $W + E_6(-1)$ is 36. Since D_4 has determinant 4, it follows that the index of $W + E_6(-1)$ in $L(\mathfrak{o})$ is three. The vector

$$A := \frac{A_1 + A_2 + A_3 + A_4 + A_5 - B_1 - B_2}{3}$$

is contained in $L(\mathfrak{o})$. It follows that $L(\mathfrak{o})$ is generated by $W + E_6(-1)$ and A. We call A the *glue vector.*

We want to determine the subgroup G of $\mathrm{O}(L(\mathfrak{o}))$ which stabilizes W or what means the same, which stabilizes $E_6(-1)$. The automorphism group $\mathrm{O}(E_6)$ contains the Weyl group $W(E_6)$ as subgroup of index two. Recall that $W(E_6)$ is generated by the reflections along the 72 minimal vectors of E_6. The group $\mathrm{O}(E_6)$ is generated by this Weyl group and by the transformation $x \mapsto -x$. Since $W(E_6)$ is generated by reflections along vectors of norm 2 (roots) we obtain that the natural homomorphism $G \to \mathrm{O}(E_6)$ is surjective and moreover that it has a natural section

$$\mathrm{O}(E_6) \longrightarrow G.$$

An element $g \in \mathrm{O}(E_6)$ will act on W by $\pm\,\mathrm{id}$ depending on the fact whether it is contained in $W(E_6)$ or not.

We have to determine the kernel of the homomorphism $G \to \mathrm{O}(E_6)$, i.e. the subgroup of G, which acts trivial on E_6. In [FH] it has been stated on p. 246 that the kernel of this homomorphism has order two. Unfortunately this is not true and we want to take the occasion to give the correct statement in some detail.

Let $g \in G$ an element of the kernel. We consider the elements $C_1 = g(B_1)$, $C_2 = g(B_2)$ which are elements of norm 4. A pair (C_1, C_2) of elements of W gives rise to an element of G, if and only is $(C_1, C_1) = (C_2, C_2) = 4$ and $(C_1, C_2) = 2$ and if the vector

$$\frac{A_1 + A_2 + A_3 + A_4 + A_5 - C_1 - C_2}{3}$$

is contained in $L(\mathfrak{o})$. This means that

$$\frac{B_1 + B_2 - C_1 - C_2}{3}$$

is contained in $L(\mathfrak{o})$. We call this the *glue condition.* Here is the list of all elements of norm 4 of W:

$$\pm B_1, \quad \pm B_2, \quad \pm(B_1 - B_2).$$

The set of all pairs (C_1, C_2) from this list with inner product $(C_1, C_2) = 2$ is

$$\begin{aligned}
&\pm\,(B_1, B_2), \quad \pm(B_2, B_1), \quad \pm(B_1, B_1 - B_2),\\
&\pm\,(B_2, B_2 - B_1), \quad \pm(B_1 - B_2, B_1), \quad \pm(B_1 - B_2, -B_2).
\end{aligned}$$

The glue condition exhibits the following pairs:

$$\begin{aligned}
&(B_1, B_2), \quad (B_2, B_1), \quad -(B_1, B_1 - B_2),\\
&-\,(B_2, B_2 - B_1), \quad -(B_1 - B_2, B_1), \quad -(B_2 - B_1, B_2).
\end{aligned}$$

This is a group order 6 which is isomorphic to S_3. It is the Weyl group of W. Hence we have the exact sequence

$$1 \longrightarrow S_3 \longrightarrow G \longrightarrow \mathrm{O}(E_6) \longrightarrow 1.$$

We have to intersect G with $\mathrm{O}^+(L(\mathfrak{o}))$. We denote this intersection by G^+. We mentioned that the transformation $x \mapsto -x$ is contained in $\mathrm{O}^+(L(\mathfrak{o}))$. Therefore the map $G^+ \to \mathrm{O}(E_6)$ remains surjective. The Weyl group of W is generated by reflections along vectors of positive norm. Hence they define elements in the kernel of G which are not contained in $\mathrm{O}^+(L(\mathfrak{o}))$. Only the products of two of them is in $\mathrm{O}^+(L(\mathfrak{o}))$. The subgroup of S_3 generated by products of two elements is the alternating group. Hence we get the exact sequence

$$1 \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow G^+ \longrightarrow \mathrm{O}(E_6) \longrightarrow 1.$$

Since this sequence splits, we have

2.1 Proposition. *The stabilizer of W in $\mathrm{O}^+(L(\mathfrak{o}))$ is isomorphic to* nSp

$$\mathbb{Z}/3\mathbb{Z} \times \mathrm{O}(E_6).$$

Sometimes it is better to consider the group

$$\Gamma := \mathrm{O}^+(L)/\{\pm\,\mathrm{id}\}$$

instead of $\mathrm{O}^+(L)$. This group also acts on the set of linear subspaces of $L \otimes_{\mathbb{Z}} \mathbb{R}$. The inclusion $W(E_6) \to \mathrm{O}(E_6)$ induces an ismorphism

$$W(E_6) \xrightarrow{\sim} \mathrm{O}(E_6)/\{\pm\,\mathrm{id}\}.$$

Hence we obtain:

2.2 Corollary of. *The stabilizer of W in $\Gamma(\mathfrak{o}) := \mathrm{O}^+(L(\mathfrak{o}))/\{\pm\,\mathrm{id}\}$ is isomorphic to* CnSp

$$\mathbb{Z}/3\mathbb{Z} \times W(E_6).$$

In terms of the pairs the group $\mathbb{Z}/3\mathbb{Z}$ corresponds to the three elements

$$(B_1, B_2), \quad -(B_1, B_1 - B_2), \quad -(B_2, B_2 - B_1).$$

There is an important congruence subgroup of $\mathrm{O}^+(L(\mathfrak{o}))$, namely the kernel of

$$\mathrm{O}^+(L(\mathfrak{o})) \longrightarrow \mathrm{Aut}(L(\mathfrak{o})/2L(\mathfrak{o})^*),$$

where $L(\mathfrak{o})^*$ denotes the dual lattice of $L(\mathfrak{o})$. We use the notation

$$\Gamma(\mathfrak{o})[\mathfrak{p}] := \mathrm{kernel}\big(\mathrm{O}^+(L(\mathfrak{o})) \longrightarrow \mathrm{Aut}(L(\mathfrak{o})/2L(\mathfrak{o})^*)\big)/\{\pm\,\mathrm{id}\}.$$

This notation comes from the fact that $2\mathfrak{o}^*$ is the two-sided ideal

$$\mathfrak{p} := (1 + \mathrm{i}_1)\mathfrak{o}.$$

As has been proved in [FH], the natural embedding $E_6 \to L(\mathfrak{o})$ induces an isomorphism

$$E_6/2E_6 \xrightarrow{\sim} L(\mathfrak{o})/2L(\mathfrak{o})^*.$$

It is known that the group $W(E_6)$ acts faithfully on $E_6/2E_6$. From we obtain:

2.3 Proposition. *There is a natural exact sequence* gZd

$$1 \longrightarrow \Gamma(\mathfrak{o})[\mathfrak{p}] \longrightarrow \Gamma(\mathfrak{o}) \longrightarrow W(E_6) \longrightarrow 1$$

which splits. The stabilizer $\Gamma(\mathfrak{o})[\mathfrak{p}]_W$ *of* W *inside* $\Gamma(\mathfrak{o})[\mathfrak{p}]$ *is isomorphic to* $\mathbb{Z}/3\mathbb{Z}$*.*

3. Examples of additive lifts

We return to the lattice $L(\mathfrak{o}) = H \times H \times D_4(-1)$. We replace this lattice by the (even) lattice $\sqrt{2}L(\mathfrak{o})^*$. Then the orthogonal group remains unchanged but the discrimant kernel changes. Since $(\sqrt{2}L^*)^* = L/\sqrt{2}$, we have a natural isomorphism

$$(\sqrt{2}L(\mathfrak{o})^*)^*/(\sqrt{2}L(\mathfrak{o})^*) = L(\mathfrak{o})/2L(\mathfrak{o})^*.$$

Hence the additive lift gives forms on the group $\Gamma_{\sqrt{2}L(\mathfrak{o})^*}$. Recall that we used the abbreviation

$$\Gamma(\mathfrak{o})[\mathfrak{p}] = \Gamma_{\sqrt{2}L(\mathfrak{o})^*}/\{\pm\,\mathrm{id}\}.$$

We see that we obtain an additive lift

$$\mathbb{C}[L(\mathfrak{o})/2L(\mathfrak{o})^*]^{\mathrm{SL}(2,\mathbb{Z})} \longrightarrow [\Gamma(\mathfrak{o})[\mathfrak{p}], 2].$$

Recall that the full group $\mathrm{O}^+(L(\mathfrak{o}))$ acts on both sides. More precisely it acts through its quotient $\Gamma(\mathfrak{o})/\Gamma(\mathfrak{o})[\mathfrak{p}] \cong W(E_6)$. A direct computation shows:

3.1 Proposition. *The dimension of* $\mathbb{C}[L(\mathfrak{o})/2L(\mathfrak{o})^*]^{\mathrm{SL}(2,\mathbb{Z})}$ *is* 7*. Under* AdL
$W(E_6)$ *this space splits into a one dimensional trivial and a* 6*-dimensional irreducible representation.*

The additive lift is general not injective. In theorem 14.3 of [Bo] the Fourier coefficients of an additive lifts are given as elementary expression in the Fourier coefficients of the input forms. By means of this formula it is easy to check:

3.2 Proposition. *The kernel of the additive lift is the one-dimensional trivial* Tod
subspace. Hence the image of the additive lift is a 6*-dimensional subspace of* $[\Gamma(\mathfrak{o})[\mathfrak{p}], 2]$.

This 6-dimensional space has been discovered erlier [FH]. But it has been described there in a different language using the isogeny between $\mathrm{O}(2,6)$ and the quaternionic symplectic group of degree two. In the symplectic context theta series are available and this 6-dimensional space has been obtained there as a space of theta series. Recently Krieg [Kr1] studied also this space from the symplectic of view and gave explicit simple Fourier expansions for a basis of this space. For sake of completeness we give the link between the two approaches. The vector space $V = L(\mathfrak{o}) \otimes_{\mathbb{Z}} \mathbb{R}$ appears in the form $V = \mathbb{R}^2 \times \mathbb{R}^2 \times V_0$, where $V_0 = D_4(-1) \otimes_{\mathbb{Z}} \mathbb{R}$. Hence we can use the coordinates $(z_0, z_2, \mathfrak{z})$ introduced in the previous section. We now simply write them as a matrix

$$Z = \begin{pmatrix} z_0 & \mathfrak{z} \\ \mathfrak{z}' & z_2 \end{pmatrix},$$

where $\mathfrak{z}'$ is defined as $\bar{\mathfrak{x}} + \mathrm{i}\bar{\mathfrak{y}}$. This matrix is an element of the quaternionic half-plane of degree two in the sense of Krieg.

Let $a \in L(\mathfrak{o})$ a vector of norm $(a, a) = -2$. Its orthogonal complement defines an irreducible subvariety of codimension one (divisor) of $\mathcal{H}(L(\mathfrak{o}))/\Gamma(\mathfrak{o})[\mathfrak{p}]$. Since there are 36 orbits of pairs $\pm a$ with respect to $\Gamma(\mathfrak{o})[\mathfrak{p}]$ we get 36 such divisors. The reflection along a defines an automorphism of $\mathcal{H}(L(\mathfrak{o}))/\Gamma(\mathfrak{o})[\mathfrak{p}]$ which fixes the corresponding divisor. If f is a modular form which changes its sign under this reflection, it will vanish along this divisor. Now we need some information about the six-dimensional irreducible representations of $W(E_6)$. Actually $W(E_6)$ admits two isomorphy classes of irreducible representations. The first one is represented by the obvious representation on $E_6 \otimes_{\mathbb{Z}} \mathbb{C}$. The second one is its twist with the non-trivial character of $W(E_6)$ (determinant). In both cases there exist elements which change their sign under the reflection. Hence we obtain a non-vanishing modular form f in $[\Gamma(\mathfrak{o})[\mathfrak{p}], 2]$ (in the additive lift sub-space) which vanishes along the divisor. From the theory of Borcherds products follows [FH] that there exists a form of weight 2 with precisely this divisor. Hence f must be this Borcherds product. This also shows that the subspace of $[\Gamma(\mathfrak{o})[\mathfrak{p}], 2]$, which changes its sign under the reflection is one-dimensional. Hence we obtain:

3.3 Remark. *The representation of $W(E_6)$ on the additive lift space in $[\Gamma(\mathfrak{o})[\mathfrak{p}], 2]$ is isomorphic to $E_6 \otimes_{\mathbb{Z}} \mathbb{C}$.* Ass

In [FH] it also has been shown:

3.4 Proposition. *The intersection of the 36 divisors belonging to the norm-two vectors of L consists in the Baily-Borel compactification [BB] of $\mathcal{H}(L(\mathfrak{o}))/\Gamma(\mathfrak{o})[\mathfrak{p}]$ of precisely one point, namely the distinguished point from section one.* Dpd

4. The structure theorem

In section 2 we introduced a certain two-dimensional positive definite subspace $W \subset V$, which defines a point $\varrho \in \mathcal{H}(L)$. We consider a basis $f_1, \dots, f_6$ of the six-dimensional additive lift space. Assume that $f \in \Gamma(\mathfrak{o})[\mathfrak{p}]$ is a modular form of even weight, which does not vanish at the distinguished point. Because ϱ is a fixed point of order 3 of $\Gamma(\mathfrak{o})[\mathfrak{p}]$ its weight has to divisible by 3. We write $6k$ for the weight. Now we consider a neighbourhood $U(\varrho) \subset \mathcal{H}(L)$, which is invariant under the stabilizer $\Gamma(\mathfrak{o})[\mathfrak{p}]_\varrho \cong \mathbb{Z}/3\mathbb{Z}$. and such that $f = g^{3k}$ with some invertible holomorphic function g on $U(\varrho)$. We consider the map

$$\phi : U(\varrho) \longrightarrow \mathbb{C}^6, \quad z \longmapsto (f_1(z)/g(z), \dots, f_6(z)/g(z)).$$

Now we use that the space $\mathbb{C}f_1 + \cdots + \mathbb{C}f_6$ can be generated by Borcherds products with the divisors described in . This implies that this space admits a basis of forms whose zero divisors have normal crossings in ϱ. Hence for sufficiently small $U(\varrho)$ the map ϕ is biholomorphic from $U(\varrho)$ onto an open subset $V(0)$ of $\mathbb{C}^n$.

We also consider the map

$$\mathcal{H}(L(\mathfrak{o}))/\Gamma(\mathfrak{o})[\mathfrak{p}] \longrightarrow P^N(\mathbb{C}),$$

which is defined by the modular forms of weight $6k$

$$f,\ f_1^{i_1} \cdots f_6^{i_6}, \quad i_1 + \cdots + i_6 = 3k.$$

This an algebraic map onto an algebraic variety Y. We want to compute its covering degree d. We denote by $Y_0 \subset Y$ the small open subset which corresponds to $U(\varrho)$. Because of the uniqueness of the distinguished point, d is the same as the covering degree of $U(\varrho)/\Gamma(\mathfrak{o})[\mathfrak{p}]_\varrho \longrightarrow Y_0$. This can be computed from the commutative diagram

$$\begin{array}{ccc} U(\varrho) & \longrightarrow & V(0) \\ \downarrow & & \downarrow \\ U(\varrho)/\Gamma(\mathfrak{o})[\mathfrak{p}]_\varrho & \longrightarrow & Y_0 \end{array}$$

The fist vertical arrow has covering degree 3, the second has covering degeree $3k$. The first row has covering degree 1. Hence the second row has covering k.

In the paper [FH] we consider the case of a form of weight 48. (In the present paper the weights are doubled compared to [FH]). Hence in this case the covering degree is $d = 8$ and not $d = 24$ as stated in 12.2 of [FH].

Krieg constructed in [Kr2] a form of weight 6 with respect to the full modular group. His structure theorem shows that this form cannot vanish at ϱ. Because this result is basic for our approach, we want to give a new construction of this form and include a direct proof for the non-vanishing at the point ϱ:

For this we need another example of an additive lift, lifting an elliptic modular form of weight 4. We use now $L(\mathfrak{o})$. The discriminant $L(\mathfrak{o})'/\mathfrak{o}$ is isomorphic to $L(\mathfrak{o})/2L(\mathfrak{o})'$. The latter group is represented by the elements

$$0,\ 1,\ \omega,\ \bar{\omega}, \quad \omega = \frac{1 + \mathrm{i}_1 + \mathrm{i}_2 + \mathrm{i}_3}{2}.$$

we need solutions of

$$h_1(\tau+1)=h_1(\tau),\qquad h_i(\tau+1)=-h_i(\tau)\quad (2\le i\le 4)$$

and

$$\begin{pmatrix}h_1\\h_2\\h_3\\h_4\end{pmatrix}\Bigl(-\frac{1}{\tau}\Bigr)=-\frac{\tau^4}{2}\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}h_1\\h_2\\h_3\\h_4\end{pmatrix}(\tau)$$

Upto a constant factor the only solution is (compare [FH], p. 260).

$$h_1(\tau)=\vartheta^8(\tau)-4\vartheta^4(\tau)\vartheta^4(\tau+1)+\vartheta^8(\tau+1)$$
$$h_2(\tau)=h_3(\tau)=h_4(\tau)=\vartheta^8(\tau)-\vartheta^8(\tau+1)$$

Here

$$\vartheta(\tau)=\sum_{n=-\infty}^{\infty}e^{\pi i n^2\tau}$$

is the classical theta function.

Since $h_2=h_3=h_4$ the additive lift will be a modular from with respect to the full modular group $\Gamma(\mathfrak{o})$ Computing some Fourier coefficients by means of Borcherds formula 14.3 of [Bo] it is possible to verify, that it does not vanish identically.

4.1 Proposition. *The additive lift G_6 of the above elliptic modular form is a non-vanishing element of* $[\Gamma(\mathfrak{o}),6]$. Pas

It is possible to compute some Fourier coefficients of the determinant

$$f=\det\begin{pmatrix}f_1&\dots&f_6&3G_6\\\nabla f_1&\dots&\nabla f_6&\nabla G_6\end{pmatrix},$$

where ∇g denotes the gradient of a function g written as column. It turns out that this determinant is not zero. Using it can be checked that the determinant above is a modular form of weight 24 with respect to the character $\det(\gamma)$. Since the permutations of the coordinates of the variable $\mathfrak{z}$ are contained in $\Gamma(\mathfrak{o})[\mathfrak{p}]$ the determinant vanishes along $\mathfrak{z}_1=\mathfrak{z}_2$. The mentioned form of weight 48 is a square of a form of weight 24 that has precisely this divisor (see [FH], 12.1). Hence it is the determinant f up to a constant factor. In [FH] it has been proved that the point ϱ is not contained in this divisor. Hence the determinant does not vanish at this point and we obtain:

4.2 Proposition (Krieg). *There exists a a modular form $G_6\in[\Gamma(\mathfrak{o}),6]$, which does not vanish at the distinguished point.* Nvd

Moreover we obtain that the covering degree of the map

$$\mathcal{H}(L(\mathfrak{o}))/\Gamma(\mathfrak{o})[\mathfrak{p}]\longrightarrow P^N(\mathbb{C}),$$
$$Z\longmapsto[G_6(Z),\ f_1^{i_1}(Z)\cdots f_6^{i_6}(Z)]\quad(i_1+\cdots+i_6=3)$$

is one. This implies:

4.3 Theorem. *The graded algebra* GRM

$$A(\Gamma(\mathfrak{o})[\mathfrak{p}]) := \sum_{k=0}^{\infty} [\Gamma(\mathfrak{o})[\mathfrak{p}], 2k]$$

is a weighted polynomial ring generated by six forms $f_1, \dots, f_6$ *of weight two and a form* G_6 *of weight six.*

Proof. The above geometric result in connection with a criterion of Hilbert (compare [Fr], p. 123f) implies that the ring of all modular forms is the normalization of $\mathbb{C}[G_6, f_1, \dots, f_6]$. Since the seven forms are algebraically independent, this is a polynomial ring and hence normal.

Since we know the action of $\Gamma(\mathfrak{o})/\Gamma(\mathfrak{o})[\mathfrak{p}] = W(E_6)$ on the generators, we can compute the ring of modular forms for the full modular group and reproduce Krieg's theorem [Kr1]. Using well-known results due to Burkhardt [Bu] and Coble [Co] , see also [Hu], we have:

4.4 Theorem (Krieg). *The ring of modular forms with respect to the group* $\Gamma(\mathfrak{o})$ *is a weighted polynomial ring generated by seven forms of weight* 4, 6, 10, 12, 16, 18, 24. TK

It is possible to derive structure theorems for several known and unknown modular varieties which can be embedded as Heegner divisors into $\mathcal{H}(V)/\Gamma(\mathfrak{o})[\mathfrak{p}]$.

4.5 Remark. *Let* $W \subset V = L(\mathfrak{o}) \otimes_{\mathbb{Z}} \mathbb{R}$ *be a subspace of signature* $(2, m)$. *For any subgroup* $\Gamma \subset \Gamma(\mathfrak{o})$ *of finite index we consider the projected group* Γ_W, *which consists of all elements* $g \in \mathrm{O}^+(W)/\{\pm\,\mathrm{id}\}$ *that extend to* Γ. *Restricting the modular forms (even weight) of the ring* $A(\Gamma)$ *to* $\mathcal{H}(W)$ *one obtains a ring* B, *whose normalization is* RF

$$A(\Gamma_W) = \sum_{k=0}^{\infty} [\Gamma_W, 2k] = \bar{B}.$$

This follows from the fact that the map $\mathcal{H}/\Gamma_W \to \mathcal{H}(V)/\Gamma$ is generically injective in connection with the mentioned criterion of Hilbert.

Hence the determination of $A(\Gamma_W)$ means to find the ideal of relations $\mathfrak{a}_W$ and to normalize $A(\Gamma)/\mathfrak{a}_W$, which is a purely algebraic problem.

We give just one very simple example, namely the 5-dimensional variety defined as orthogonal complement of a vector of norm -2 from $L(\mathfrak{o})$. It can be described as follows. Take as W the subspace of V which is defined be the equation $\mathfrak{x}_0 = 0$. Recall that the elements of V can be written in the form $(x_1, x_2, x_3, x_4, \mathfrak{x})$, where the x_i are reals and $\mathfrak{x}$ is a Hamilton quaternion, which we can write as $\mathfrak{x} = \mathfrak{x}_0 + \mathfrak{x}_1 \mathrm{i}_1 + \mathfrak{x}_2 \mathrm{i}_2 + \mathfrak{x}_3 \mathrm{i}_3$. The half plane $\mathcal{H}(W)$ can be identified with the set of all matrices

$$Z = \begin{pmatrix} z_0 & \mathfrak{z} \\ \mathfrak{z}' & z_2 \end{pmatrix}, \quad \mathfrak{z}_0 = 0.$$

We know that there is an element f in the space generated by $f_1, \dots, f_6$ which vanishes along $\mathcal{H}(W)$. We may assume that $f = f_1 + \cdots + f_6$ is it. Since $A(\Gamma(\mathfrak{o})[\mathfrak{p}])/(f)$ is a polynomial ring, especially an integral domain, a dimension argument shows that the kernel of $A(\Gamma(\mathfrak{o})[\mathfrak{p}]) \to A(\Gamma(\mathfrak{o})[\mathfrak{p}]_W)$ is generated by f. We obtain from :

4.6 Theorem. *The graded algebra* GRK

$$A(\Gamma(\mathfrak{o})[\mathfrak{p}]_W) := \sum_{k=0}^{\infty} [\Gamma(\mathfrak{o})[\mathfrak{p}]_W, 2k]$$

is a polynomial ring generated by five forms of weight two (the restrictions of $f_1, \dots, f_5$) and a form of weight six (the restriction of G_6).

The determination of the group $K = \Gamma(\mathfrak{o})_W / \Gamma(\mathfrak{o})[\mathfrak{p}]_W$ and its action on the f_i can be taken from [FH]. On page 278 a certain copy of S_6 inside $W(E_6)$ has been defined. This group is an image of the Siegel modular group $\mathrm{Sp}(2, \mathbb{Z})$. From this and the fact that S_6 is a maximal subgroup of the quotient of $W(E_6)$ by the involution $Z \mapsto Z'$ it follows that

$$K = \Gamma(\mathfrak{o})_W / \Gamma(\mathfrak{o})[\mathfrak{p}]_W \cong S_6.$$

The action of this S_6 on the six-dimensional representation space of $W(E_6)$ has been described in [FH] in a very explicit manner on p. 278. There have been defined generators $F_1, \dots, F_6, U$ of this space with the defining relation $F_1 + \cdots + F_6 = 0$. The form U vanishes along the five dimensional subvariety in consideration. The group S_6 leaves U invariant up to the sign-character and acts on the $F_1, \dots, F_6$ as standard permutation group. Choosing the $f_i := F_i$, we obtain:

4.7 Corollary. *The ring $A(\Gamma(\mathfrak{o})_W)$ is a polynomial ring in the restrictions of* CK

$$G_6, \quad f_1^k + \cdots + f_6^k \ (2 \le k \le 6).$$

In a similar way the structures for rings of modular forms corresponding to subgroups of S_6 can be obtained. For example the ring of modular forms for the subgroup of index two in $\Gamma(\mathfrak{o})_W$ which corresponds to A_6 needs as additional generator $\prod_{i<j}(f_i - f_j)$.

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