SOME MODULAR VARIETIES IN LOW DIMENSION III

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1. INTRODUCTION

Some years ago in [1] the first named Author and F. Hermann studied some modular varieties related to the orthogonal group $\mathbb{O}(2, n)$. The most significant variety they studied was related to $\mathbb{O}(2, 6)$, or —equivalently— to the symplectic group of degree two defined on the quaternions. Because of a mistake, they did not get a definite result regarding the structure of the graded ring of modular forms of degree two with respect to the Hurwitz integral quaternions. This has been obtained recently by Krieg in [4]. Then in [2], we reconsidered [1] and in its spirit, we got the structure of the graded ring of modular forms of degree two with respect to a congruence subgroup of the Hurwitz integral quaternions. As a consequence we got also Krieg's results. Moreover in [1], it was explained a method for obtaining structure of the graded ring of modular forms related to $\mathbb{O}(2, n)$, $n < 6$, from the $\mathbb{O}(2, 6)$-case. In the $\mathbb{O}(2, 5)$- case they illustrate substantially two cases. One has been solved in [2]. In this note we will approach and solve the second case. Our results generalize some results which have been obtained by Klöcker [Kl] in his doctoral thesis.

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2. SOME BASIC FACTS ABOUT MODULAR FORMS

A lattice L is a free abelian group together with a real valued non degenerated bilinear form $(\cdot, \cdot)$. It is called even, if the quadratic form, the so-called norm, (x, x) is even for all x. We will consider lattices with signature $(2, n)$. We denote by $\mathbb{O}(V)$ the orthogonal group of the quadratic space $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by $\mathbb{O}^+(V)$ the subgroup of index two, which is generated by all reflections along vectors a of negative norm (a, a). We also have to consider the integral orthogonal subgroup

$$\mathbb{O}(L) := \{g \in \mathbb{O}(V); \quad g(L) = L\}$$

and its subgroup

$$\mathbb{O}^+(L) = \mathbb{O}(L) \cap \mathbb{O}^+(V).$$

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We consider the projective space $P(V(\mathbb{C}))$ of the complexification $V(\mathbb{C}) = V \otimes_{\mathbb{R}} \mathbb{C}$ of V. We extend $(\cdot,\cdot)$ to a $\mathbb{C}$-bilinear form. The zero quadric $(z,z) = 0$ is a (smooth) complex submanifold. The set

$$\mathcal{K}(V) := \{z \in P(V(\mathbb{C})); \quad (z,z) = 0,\ (z,\bar{z}) > 0\}$$

is an open subset of the quadric and hence a complex manifold. It has two connected components, which can be interchanged by the map $z \mapsto \bar{z}$. We choose one of the components and denote it by $\mathcal{K}^+$.

The group $\mathbb{O}^+(V)$ is the subgroup of $\mathbb{O}(V)$ which preserves $\mathcal{K}^+$. Its action on $\mathcal{K}^+$. is holomorphic.

We recall the notion of a modular form. Let $\Gamma \subset \mathbb{O}^+(V)$ be a subgroup, which is commensurable with $\mathbb{O}^+(L)$. We consider the inverse image $\tilde{\mathcal{K}^+}$ of $\mathcal{K}^+$ under the natural map $V(\mathbb{C}) - \{0\} \to P(V(\mathbb{C}))$. The group $\mathbb{O}^+(V)$ acts on $\tilde{\mathcal{K}^+}$ as well. A modular form of weight k and with respect to some character $v : \Gamma \to \mathbb{C}^\bullet$ is a holomorphic function $f : \tilde{\mathcal{K}^+} \to \mathbb{C}$ with the properties:

(1) $f(\gamma(z)) = v(\gamma) f(z)$,
(2) $f(tz) = t^{-k} f(z)$,
(3) f is holomorphic at the cusps.

We denote by $[\Gamma, k, v]$ the space of all these forms or simply by $[\Gamma, k]$, when v is trivial.

We recall the standard realization of $\tilde{\mathcal{K}^+}$. For this purpose we decompose $V = \mathbb{R}^2 \times \mathbb{R}^2 \times V_0$, where the norm is $2(x_1x_2 + x_3x_4) + (\mathbf{x},\mathbf{x})$ with a negative definite form $(\mathbf{x},\mathbf{x})$ on V_0. Then $\tilde{\mathcal{K}^+}$ can be taken as the set of all $t(1, *, z_0, z_2; \mathbf{z})$ where $t \neq 0$, $y_0 > 0$ and $2y_0y_2 + (\mathbf{y},\mathbf{y}) > 0$. A modular form f is determined by the function

$$F(z_0, z_2, \mathbf{z}) := f(1, *, z_0, z_2, \mathbf{z}).$$

For details , we refer to [1] or [2].

3. Borcherds' additive lifting

We denote by L' the dual of the lattice L and by Γ_L the kernel of the map

$$\mathbb{O}^+(L) \longrightarrow \mathrm{Aut}(L'/L)$$

This is the so called discriminant kernel.

Borcherds defined for even n a linear map from the space of the elliptic modular forms with values in $\mathbb{C}[L'/L]$ to $[\Gamma_L, k+n/2-1]$, which generalizes constructions of Saito-Kurokawa, Shimura, Maass, Gritsenko, Oda et.al. We refer to [1] or [2] for details.

This map is equivariant with respect to the group $\mathbb{O}^+(L)$, which acts on both sides in a natural way.

We used this construction especially in the case $k = 0$. In this case the starting point are modular forms of weight zero, i.e. constant, and we get for even n a map

$$\mathbb{C}[L'/L]^{\mathrm{SL}(2,\mathbb{Z})} \longrightarrow [\Gamma_L, n/2 - 1].$$

We also have to recall the notion of a quadratic divisor. Let V be the quadratic space and $W \subset V$ a subspace of signature $(2, n-1)$. Usually W is defined as orthogonal complement of a vector of negative norm. Then we obtain a natural holomorphic embedding $\mathcal{K}^+(W) \to \mathcal{K}^+(V)$. Assume that $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ with an even lattice L and that $W = M \otimes_{\mathbb{Z}} \mathbb{R}$ with $M = L \cap W$. Let $\Gamma \subset \mathbb{O}^+(V)$ a subgroup which is commensurable with $\mathbb{O}^+(L)$. The projected group Γ' consists of all elements of $\mathbb{O}^+(W)$, which are restrictions of elements of Γ. This group is commensurable with $\mathbb{O}^+(M)$. Then we get a natural map

$$\mathcal{K}^+(W)/\Gamma' \to \mathcal{K}^+(V)/\Gamma.$$

From the theory of Baily-Borel [BB] we know that this is an algebraic map of quasiprojective varieties. Moreover this map is birational onto its image.

4. Quaternionic modular forms

In this paper, as in [2], we will be mainly interested to lattices related to the root lattice D_4.

We take the realization given by the Hurwitz integers. This means the following: Denote by 1, i_1, i_2, i_3 the standard generators of the Hamilton quaternions. Then $\mathcal{O}$ is the set of all $a_0 + a_1\mathrm{i}_1 + a_2\mathrm{i}_2 + a_3\mathrm{i}_3$ such that the a_i are all in $\mathbb{Z}$ or all in $1/2 + \mathbb{Z}$. The bilinear form is $(a, b) = 2\,\mathrm{Re}(a\bar{b})$, hence the norm is $(a, a) = 2a\bar{a} = 2(a_0^2 + a_1^2 + a_2^2 + a_3^2)$. We denote by H the lattice $\mathbb{Z}^2$ with bilinear form $(x, y) = x_1y_2 + x_2y_1$. This is an even lattice of signature $(1, 1)$. Finally we define the orthogonal sum

$$L(\mathcal{O}) = H \times H \times D_4(-1).$$

Here $D_4(-1)$ means as usual D_4 equipped with the negative of the bilinear form of D_4.

An important congruence subgroup of $\mathbb{O}^+(L(\mathcal{O}))$ is the group $\Gamma_{\sqrt{2}L(\mathcal{O})^*}$ that is the kernel of

$$\mathbb{O}^+(L(\mathcal{O})) \longrightarrow \mathrm{Aut}(L(\mathcal{O})/2L(\mathcal{O})^*),$$

where $L(\mathcal{O})^*$ denotes the dual lattice of $L(\mathcal{O})$.

We obtain this subgroup replacing the lattice $L(\mathcal{O})$ by the (even) lattice $\sqrt{2}L(\mathcal{O})^*$. Then the orthogonal group remains unchanged, but the discriminant kernel changes. Since $(\sqrt{2}L^*)^* = L/\sqrt{2}$, we have a natural isomorphism

$$(\sqrt{2}L(\mathcal{O})^*)^*/\sqrt{2}L(\mathcal{O})^* = L(\mathcal{O})/2L(\mathcal{O})^*.$$

Hence the additive lift gives forms on the group $\Gamma_{\sqrt{2}L(\mathcal{O})^*}$.

Really we are interested to the congruence subgroup

$$\Gamma(\mathcal{O})[\mathcal{P}] := \Gamma_{\sqrt{2}L(\mathcal{O})^*}/\pm 1.$$

This notation comes from the fact that $2\mathcal{O}^*$ is the two-sided ideal

$$\mathcal{P} := (1+\mathrm{i})\mathcal{O}.$$

As it has been proved in [1], the natural embedding $E_6 \to L(\mathcal{O})$ induces an isomorphism between $E_6/2E_6$ and $L(\mathcal{O})/2L(\mathcal{O})^*$. It is known that the group $W(E_6)$ acts faithfully on $E_6/2E_6$.

In [2] we determined the ring of modular forms with respect to $\Gamma(\mathcal{O})[\mathcal{P}]$, in fact we have

Theorem 4.1. *The graded algebra*

$$A(\Gamma(\mathcal{O})[\mathcal{P}]) := \sum_{k=0}^{\infty} [\Gamma(\mathcal{O})[\mathcal{P}], 2k]$$

is a weighted polynomial ring generated by six forms $f_1, \dots, f_6$ of weight two and a form G_6 of weight six.

The modular forms $f_1, \dots, f_6$ are additive lifting of the constants and the form G_6 is also an additive lifting of an elliptic modular form of weight 4.

As we said in the introduction , it is possible to derive structure theorems for several known and unknown modular varieties which can be embedded as Heegner divisors into $\mathcal{K}^+(V)/\Gamma(\mathcal{O})[\mathcal{P}]$.

In [1] are listed several sublattices of $L(\mathcal{O})$ of the form $H \times H \times \mathcal{O}'$ with $\mathcal{O}' := \mathcal{O} \cap \mathbb{H}'$, here $\mathbb{H}'$ is a subspace of the real quaternions $\mathbb{H}$. In particular we shall consider the subspace $\mathbb{H}'$ defined by $a_0 = a_1$. This defines an irreducible divisor in $\mathcal{K}^+(V)/\Gamma(\mathcal{O})[\mathcal{P}]$ and it is defined by a modular form Ψ^2 of weight 48, that has been described in [1] and [2]. The exponent 2 is due to the fact that the form Ψ has a not trivial character with respect to $\Gamma(\mathcal{O})[\mathcal{P}]$.

Moreover in [4], we have

$$\Psi^2 = H_{24}^2 - 4H_{16}^3$$

with H_{24} and H_{16} modular forms relative to $\Gamma(\mathcal{O})$ and it is given an explicit expression as polynomial in $f_1, \dots, f_6$ and G_6.

Really, we got this expression for Ψ^2 using a different method that we shall explain, since it is very fruitful, once one knows the weight of the expected relation.

Let us consider the ring of formal power series $R[[X_0, X_2]]$, where

$$R = \mathbb{Z}[X_{11}, X_{11}^{-1}, \dots, X_{14}, X_{14}^{-1}]$$

means the ring of Laurent polynomials in four variables. We denote by A the ring of quaternionic modular forms with integral Fourier coefficients, there is an obvious homomorphism $A \to R[[X_0, X_2]]$.

Now we specialize the four variables $X_{10}, X_{11}, X_{12} X_{13}$ to four non-zero integers, namely $k_0,\ k_1,\ k_2,\ k_3$. In this way we get an homomorphism

$$A \longrightarrow \mathbb{Z}[X_0, X_2].$$

Since $k_i = exp(\pi i z_{1i})$ means a transcendental value for z_{1i}, $(i = 0, \dots 3)$, it can be expected that this homomorphism is injective, if the intergers k_i are taken generic. We did not write this statement as a theorem, because there is no need to prove it.

If we take not generic k_i, for example $k_0 = k_1$, the homomorphism factors through the ring of restrictions to a certain subvariety. In this way, every relation induces a relation in $\mathbb{Z}[[X_0, X_2]]$. The point is that calculations in this ring are very fast compared to calculations in A itself. In our specific case, we know the existence of a relation of the weight 48. One actually computes the coefficients of this relation by means of a calculation inside $\mathbb{Z}[[X_0, X_2]]$.

To be correct, it is not optimal to do the calculations directly in $\mathbb{Z}[[X_0, X_2]]$ but better in $(\mathbb{Z}/p\mathbb{Z})[[X_0, X_2]]$ for some primes p.

Thus to reconstruct the characteristic 0-case, one needs some rough a priori estimate for the Fourier coefficients. This method already has been used in [1] to prove theorem 15.3.

Let us denote by $\Gamma(\mathcal{O}')$ and $\Gamma(\mathcal{O}')[\mathcal{P}]$ the respective projected subgroups, then we have generically injective maps:

$$\mathcal{K}^+(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{H}')/\Gamma(\mathcal{O}') \longrightarrow \mathcal{K}^+(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{H})/\Gamma(\mathcal{O})$$

and

$$\mathcal{K}^+(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{H}')/\Gamma(\mathcal{O}')[\mathcal{P}] \longrightarrow \mathcal{K}^+(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{H})/\Gamma(\mathcal{O})[\mathcal{P}]$$

that extend along the boundaries.

In both cases we have that the defining equation for these divisors is $\Psi^2 = 0$. Unfortunately the quotient rings are not normal, as we will show in a moment. So they are not the ring of modular forms.

In fact in their quotient ring a further modular form appears , cf. [3] and [4]

$$\psi_8 = H_{24}/H_{16}$$

that satisfies the obvious integral equation

$$\Psi_8^2 = 4H_{16},$$

then we have

Theorem 4.2. *The graded algebra*

$$\sum_{k=0}^{\infty} [\Gamma(\mathcal{O}')[\mathcal{P}], 2k] = \mathbb{C}[f_1, \dots, f_6, G_6, \Psi_8]/ < G_8^2 - 4H_{16}, G_8^3 - 4H_{24} >$$

is the ring of modular forms $A(\Gamma(\mathcal{O}')[\mathcal{P}])$ Here the six forms $f_1, \dots, f_6$ have weight two and the form G_6 and Ψ_8 have weights six and eight respectively. The Hilbert function is

$$\sum_{k=0}^{\infty} \dim[\Gamma(\mathcal{O})'[\mathcal{P}], 2k] X^k = \frac{1 - X^{24}}{(1 - X)^6(1 + X + X^2)}$$

Proof. The above described ring is obviously Cohen Macaulay, since it is complete intersection, moreover with the aid of a computer we checked that its spectrum is regular in codimension one. These fact and the irreducibility imply by a well-known criterion of Serre that the ring is integrally closed. Hence it is the ring of modular forms.

Similarly, using the structure theorem for the ring of modular forms with respect to the group $\Gamma(\mathcal{O})$, or taking the invariants with respect to the quotient group, we have

Theorem 4.3. (Klöcker) *The ring of modular forms with respect to the group $\Gamma(\mathcal{O}')$ is a weighted polynomial ring generated by six forms of weight* 4, 6, 8, 10, 12, 18.

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