On the varieties defined by Riemann-Mumford's relations

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1 Introduction

1.1. Let $\mathbb{H}_g$ be the Siegel upper half space, i.e. the set of complex symmetric matrices τ whose imaginary part is positive definite and $Sp(2g,\mathbb{R})$ be the real symplectic group. $Sp(2g,\mathbb{R})$ acts transitively on $\mathbb{H}_g$ via

$$\sigma \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$$

where $\sigma = \begin{pmatrix} AB \\ CD \end{pmatrix}$ is in $Sp(2g,\mathbb{R})$.

Let Γ be a subgroup of finite index of the integral symplectic group and k an half integer, thus a holomorphic function f defined on $\mathbb{H}_g$ is called a modular form of weight k and multiplier χ for Γ if

$$f(\sigma \cdot \tau) = f((A\tau + B)(C\tau + D)^{-1}) = \chi(\sigma) det(C\tau + D)^k f(\tau)$$

for all $\sigma \in \Gamma$. In the genus 1 case we require also the holomorphicity of f at the cusps. We denote by $[\Gamma, k, \chi]$ the vector space spanned by such forms.

Let q denote an even positive integer, m an element of $q^{-1}\mathbb{Z}^g/\mathbb{Z}^g$ for some $g \geq 1$, a Thetanullwert is defined by

$$\theta \begin{bmatrix} m \\ 0 \end{bmatrix} (q\tau) = \sum_{p \in \mathbb{Z}^g} exp(\pi i {}^t(p+m) q\tau (p+m)). \tag{1}$$

This is one of the simplest example of modular forms of weight $\frac{1}{2}$ for $\Gamma_g(q, 2q)$ and a suitable multiplier.

These Thetanullwerte induce well defined maps

$$\Theta_g(q) : \Gamma_g(q, 2q) \backslash \mathbb{H}_g \longrightarrow \mathbb{P}^{q^g - 1} \tag{2}$$

that are embeddings for all g if $q \geq 4$, cf.[**?**], [**?**] and generically injective if $q = 2$, cf.[**?**]

1.2. Really, we know more, in fact, let $R_g(q)$ be the ring generated by such Thetanullwerte. It is a well known fact that its integral closure $S_g(q)$ is the ring of modular forms $S(\Gamma(q, 2q), \chi)$, with χ equal to the multiplier relative to the Thetanullwerte,cf. [?] and[?]. The map $\Theta_g(q)$ extends to the boundary of the Satake's compactification of $\Gamma_g(q, 2q)\backslash\mathbb{H}_g$ that is $Proj(S_g(q))$.

In the ring $R_g(q)$ there are some standard relations. they are the quartic Riemann's relations and linear equations $X_{-m} = X_m$ with $m \in q^{-1}\mathbb{Z}/\mathbb{Z}$.

Let $(Q_g(q)$ be the ring defined by the above equations, in this note we proceed to compare the associate projective varieties $Proj(R_g(q))$ and $Proj(Q_g(q))$. The final result of [?], page 202, states that $\Theta_g(q)(Proj(R_g(q)))$ is an irreducible component of $Proj(Q_g(q))$ when $q \geq 6$.

We shall show that , if $q \neq 2^s$, $Proj(Q_g(q))$ is not irreducible and hence it cannot be isomorphic to $\Theta_g(q)(Proj(R_g(q)))$.

In the last section, from a detailed analysis of $Proj(Q_1(6)$, we shall show how to reconstruct the ring of modular forms.

Finally, in this case we shall exibit an explicit relation in the Thetanullwerte that is not a consequence of Riemann's relations.

2 Riemann-Mumford's relations

2.1. We fix representatives for the characteristics. We choose the entries in the set

$$\mathcal{F}(q) = \left[0, \frac{1}{q}, \dots, \frac{q-1}{q}\right]$$

and we set $1 - m$ for the only characteristic n such that $m + n \equiv 0 \, mod \, 1$.

In this section we shall consider the projective variety $Proj(Q_g(q))$ defined in $\mathbb{P}^{q^g-1}$ by the equations

$$X_m = X_{1-m} \tag{1}$$

$$\Big(\sum_{c\in\mathcal{F}(2)^g} exp(4\pi i\,{}^t c' c)\, X_{a'+d+c}X_{b'+d+c}\Big)\Big(\sum_{c\in\mathcal{F}(2)^g} exp(4\pi i\,{}^t c' c)\, X_{a+c}X_{b+c}\Big) =$$

$$\Big(\sum_{c\in\mathcal{F}(2)^g} exp(4\pi i\,{}^t c' c)\, X_{a+d+c}X_{b+d+c}\Big)\Big(\sum_{c\in\mathcal{F}(2)^g} exp(4\pi i\,{}^t c' c)\, X_{a'+c}X_{b'+c}\Big) \tag{2}$$

with $c' \in \mathcal{F}(2)^g$ and $m, a, a', b, b', d \in \mathcal{F}(q)^g$ satisfying $a + b \equiv a' + b' \, mod \, 1$

The link between these varieties and $Proj(R_g(q))$ is consequence of the works of Mumford and Kempf, cf. [**?**] and [**?**]. In fact we have

Theorem 1. *a) For all $q \geq 3$, $\Theta_g(q)$ is an immersion of $\Gamma_g(q, 2q)\backslash\mathbb{H}_g$ in $\mathbb{P}^{q^g-1}$.*
b) if q is even and $q \geq 6$ then $Im(\Theta_g(q))$ is a Zariski open subset of $Proj(Q_g(q))$.

A proof of the above theorem can be found in [**?**] . Really in [**?**] the first statement is for $q \geq 4$

We recall that in the case $q = 3$ the injectivity of the map is proved in [**?**], then we proved in [**?**] the injectivity on the tangent spaces. Moreover the case $q = 2$ has been extensively studied in [**?**] and[**?**]. In [**?**] there are some inaccuracies, so at the moment we can say that the map $\Theta_g(2)$ is generically injective and it is injective when $g \leq 3$. Moreover we have to mention that when q is even the maps $\Theta_g(q)$ extend to the boundary of the Satake compactification

2.2. According to the above facts, when q is even, $\Theta_g(q)(ProjS_g(q)) = ProjR_g(q)$ is an irreducible reduced component of $Proj(Q_g(q))$.

Clearly we would like to show that equations **??** and **??** define $(ProjR_g(q))$. Unluckely we will get a negative answer.

For example, from this labyrinth of polynomial relations, when $g = 1$ and $q = 6$, identifying X_1 with X_5 and X_2 with X_4, we obtain 2 relations, namely

$$X_0^2X_1X_3 + X_0X_2X_3^2 = 2X_1^2X_2^2 \tag{3}$$

and

$$X_0^3X_2 + X_1X_3^3 = X_1^4 + X_2^4 \tag{4}$$

(We multiplied the indices by 6 to avoid heavier notations).

The projective line of equations $X_1 = X_2 = 0$ is contained in $ProjQ_6$ which is not irreducible.

We shall prove that this is a general fact, when $q \equiv 2\, mod.\, 4$. In fact, if we set $X_a = 0$ when $a \notin \mathcal{F}(2)^g$, the equations become trivial unless $\{a^{'}+d,\, b^{'}+d,\, a,\, b\}$ or $\{a^{'},\, b^{'},\, a+d,\, b+d\}$ are in $\mathcal{F}(2)^g$.

Each of these configurations implies that $2d \in \mathcal{F}(2)^g$, and, in these cases, we get $d \in \mathcal{F}(2)^g$.

We remark that these are exactly the quartic relations among Thetanullwerte with half integral characteristics and,it is a well known fact, that these relations do not exist .

To be clearer , it can be easily verified that the equations (**??**) become

$$\Big(\sum_{c\in\mathcal{F}(2)^g} exp(4\pi i\, {}^t c^{'} c)\, X_{a^{'}+c}X_{b^{'}+c}\Big)\Big(\sum_{c\in\mathcal{F}(2)^g} exp(4\pi i\, {}^t c^{'} c)\, X_{a+c}X_{b+c}\Big) =$$

$$\Big(\sum_{c\in\mathcal{F}(2)^g} exp(4\pi i\,{}^tc^{'}c)\,X_{a+c}X_{b+c}\Big)\Big(\sum_{c\in\mathcal{F}(2)^g} exp(4\pi i\,{}^tc^{'}c)\,X_{a^{'}+c}X_{b^{'}+c}\Big). \quad (5)$$

These are obviously tautological, so we proved the following

Theorem 2. *Let us assume $q\equiv 2\,mod\,4$, then , for any g the projective variety $Proj(Q_g(q))$ defined by the equations **??** and **??** has more than an irreducible component. In particular it contains a linear variety of dimension at least 2^g-1.*

Really, with some modifications we can prove more, in fact we have

Theorem 3. *Let us assume $q\neq 2^s$ for some positive integer $s>1$, then , for any g the projective variety $Proj(Q_g(q))$ defined by the equations **??** and **??** has more than an irreducible component.*

Proof. Let us assume $q=2^s o$, with o an odd number bigger than 1 and $Proj(Q_g(q))$ irreducible.

If we set $X_a=0$ when $a\notin\mathcal{F}(2^s)^g$, also in these cases the equations (23) become trivial unless $\{a^{'}+d,\,b^{'}+d,\,a,\,b\}$ or $\{a^{'},\,b^{'},\,a+d,\,b+d\}$ are in $\mathcal{F}2^s)^g$.

Each of these configurations implies that $2d\in\mathcal{F}(2^s)^g$, and we get $d\in\mathcal{F}(2^s)^g$.These are exactly the equations defining $Proj(Q_g(2^s)$;

Thus, if L_q is the linear space defined by $X_a=0$ when $a\notin\mathcal{F}(2^s)^g$, we have

$$L_q\cap Proj(Q_g(q))=Proj(Q_g(2^s) \quad (6)$$

Now if $Proj(Q_g(q))$ is irreducible we know that its dimension is exactly $(1/2)g(g+1)$ and in any case $Proj(Q_g(2^s)$ has dimension bigger or equal to $(1/2)g(g+1)$, since $Proj R_g(q)$ is an irreducible component, thus we get a contradiction once we prove that the inclusion

$$L_q\cap Proj(Q_g(q))\subset Proj(Q_g(q))$$

is proper.

Let $\tau\in\mathbb{H}_g$ be purely imaginary, i.e. $\tau=iy$, we verify that $\Theta_g(q)(iy)$ in $\mathbb{P}^{q^g-1}$ has all entries different from 0. This is an immediate consequence of the definition of Thetanullwerte, since

$$\theta\begin{bmatrix} m\\ 0\end{bmatrix}(iqy)=\sum_{p\in\mathbb{Z}^g} exp(-\pi qy[p+m])$$

is the convergent sum of positive terms.

This shows that

$$\Theta_g(q)(iy)\notin L_q.$$

These results are useful for a better understanding of the variety $Proj S_g(q)$ at least for small values of g. This will be discuss in the next section

3 An example

3.1. Using the results of the last section we will obtain a good description of $Proj S_1(6)$. We know that it has exactly 24 cusps. It is a Riemann surface of genus 13, since it is a Galois covering of degree 12 of $Proj S_1(2) \cong \mathbb{P}^1$ and ramifies only on the 6 cusps of $Proj S_1(2)$.

Let Y_1 and Y_2 the quartics defined by **??** and **??**, thus $Proj Q_1(6) = Y_1 \cap Y_2$ contains 4 lines L_1, L_2, L_3, L_4 of equations

$$X_1 = 0,\ X_2 = 0; \quad X_2 - X_0 = 0,\ X_3 - X_1 = 0;$$

$$X_2 - \phi^4 X_0 = 0,\ X_3 - \phi^8 X_1 = 0; \quad X_2 - \phi^8 X_0 = 0,\ X_3 - \phi^4 X_1 = 0$$

with $\phi = exp\left(\frac{2\pi i}{12}\right)$. Thus we can write

$$Proj Q_1(6) = L_1 \cup L_2 \cup L_3 \cup L_4 \cup \mathcal{C}$$

It has exactly 24 singular points. They are

$$[1,0,0,1] \ ,[1,0,0,-1] \ ,[1,0,0,i] \ ,[1,0,0,-i] \ ,[0,0,0,1] \ ,[1,0,0,0]$$

$$[1,0,1,0] \ ,[1,i,1,i] \ ,[1,-i,1,-i] \ ,[1,1,1,1] \ ,[1,-1,1,-1] \ ,[0,1,0,1]$$

$$[1,0,\phi^4,0] \ ,[1,\phi,\phi^4,-i] \ ,[1,\phi^7,\phi^4,i] \ ,[1,\phi^{-2},\phi^4,-1] \ ,[1,\phi^4,\phi^4,1] \ ,[0,\phi^4,0,1]$$

$$[1,0,\phi^8,0] \ ,[1,\phi^{-1},\phi^8,-i] \ ,[1,\phi^5,\phi^8,i] \ ,[1,\phi^2,\phi^8,-1] \ ,[1,\phi^8,\phi^8,1] \ ,[0,\phi^8,0,1]$$

The first set of six points are contained in L_1; the second, the third and the fourth set in L_2, L_3, L_4 respectively.

Since $\Theta_g(q)$ is $\Gamma_g/\Gamma_g(q,2q)$-equivariant and it can be easily verified that the above points have non trivial stabilizer for the action of $\Gamma_1(2,4)/\Gamma_1(6,12)$, they are the image of the cusps, and have the same singularity.

Hence to prove that $Proj S_1(6) \cong Proj R_1(6)$ it is enough to check that a singularity is nodal.

This can be easily verified at the point $[1,0,0,0]$. In fact passing to affine coordinates, we have the following equations

$$xz + yz^2 = 2x^2y^2 \tag{7}$$

and

$$y + xz^3 = x^4 + y^4. \tag{8}$$

Then obtaining y in the second equation and substituting in the first we get that the principal tangent have equation $xz = 0$.

Thus $ProjS_1(6) \cong ProjR_1(6)$ is a curve in $\mathbb{P}^3$ whose automorphism group has order divisible for

$$|\Gamma_1 / \pm \Gamma_1(6, 12)| = 288.$$

3.2. A priori we cannot say that $ProjS_1(6) \cong \mathcal{C}$, in fact we have not shown that $\mathcal{C}$ is irreducible.

Since $\mathcal{C}$ is smooth, it is enough to show that it is connected.

We observe that the quartic Y_2 is smooth and the effective divisor $\mathcal{C}$ induces a linear system on the quartic that is equivalent to

$$(4H - L_1 - L_2 - L_3 - L_4).$$

Here with H we denote the hyperplane section of $\mathbb{P}^3$ restricted to Y_2.

Let us consider the linear system $H - L_i$, $i = 1 \dots 4$. These induce maps $f_i : Y_2 \to \mathbb{P}^1$ that describe one dimensional families of cubics curves in the quartic surface.

Each of these systems is without base points ; in fact this can be easily proved for points of the quartic that are not on the line and for the points on the line we remark that if one is a base point then it should be a singular point of Y_2, but this is impossible.

Consequently the linear system $(4H - L_1 - L_2 - L_3 - L_4)$ is without base points and thus it induces a morphism

$$f : Y_2 \to \mathbb{P}^n.$$

Moreover we have that $dim\,(f(Y_2)) = 2$, in fact each of the maps f_i the generic fiber is a cubic and the generic fiber of the map

$$f_1 \times f_2 : Y_2 \to \mathbb{P}^1 \times \mathbb{P}^1$$

is finite, since is the intersection of two cubics contained in the quartic surface that are not in the same plane.

As a consequence of Zariski's Main Theorem , cf [**?**] p.280 Ex.11.3, we get that all divisors in $|4H - L_1 - L_2 - L_3 - L_4|$,and in particular $\mathcal{C}$, are connected. Hence we get

$$ProjS_1(6) \cong \mathcal{C}.$$

We are grateful to Marco Manetti that suggested us this proof.

Really one could prove more, in fact with some computation it is possible to show that the divisor $(4H - L_1 - L_2 - L_3 - L_4)$ satisfies the condition of a criterion (Nakai-Moishezon) of ampleness.

3.3. Now we shall treat the relations in $R_1(6)$; in particular we look for relations that are not induced by Riemann's relations.

For this reason we analyze the graded rings $Q_1(6)$ and $S_1(6)$.

About the first graded ring we have that its Poincaré serie is

$$P(t) = \sum_{k=0}^{\infty} dimQ_1(6)_k t^k = \frac{(1-t^4)^2}{(1-t)^4}. \tag{9}$$

Thus we have

$$dimQ_1(6)_4 = 33, \quad dimQ_1(6)_6 = 65. \tag{10}$$

Moreover from [**?**] p.61 , we get

$$P'(s) = \sum_{k=0}^{\infty} dimS_1(6)_{2k} s^k = \frac{1+10s+13s^2}{(1-s)^2}. \tag{11}$$

Consequently we get

$$dimS_1(6)_4 = dim\left[\Gamma_1(6,12), 2, id\right] = 36, \;\; dimS_1(6)_6 = dim\left[\Gamma_1(6,12), 3, id\right] = 60.$$

Hence we have that the Thetanullwerte satisfy some relations in degree 6, in fact, using the decomposition of these spaces with respect to some characters of $\Gamma_1(2,4)$.

Theorem 4. *The following relation holds*

$$2\left(\theta\begin{bmatrix}1\\0\end{bmatrix}^4(6\tau) - \theta\begin{bmatrix}2\\0\end{bmatrix}^4(6\tau)\right)\theta\begin{bmatrix}1\\0\end{bmatrix}(6\tau)\theta\begin{bmatrix}2\\0\end{bmatrix}(6\tau) -$$

$$\left(\theta\begin{bmatrix}0\\0\end{bmatrix}^4(6\tau) - \theta\begin{bmatrix}3\\0\end{bmatrix}^4(6\tau)\right)\theta\begin{bmatrix}0\\0\end{bmatrix}(6\tau)\theta\begin{bmatrix}3\\0\end{bmatrix}(6\tau). \tag{12}$$

This relation is not induced from Riemann's relations.

Proof. To avoid problem induced by the multiplier we shall consider the modular form

$$g(\tau) =$$

$$2\left(\theta\begin{bmatrix}1\\0\end{bmatrix}^4 - \theta\begin{bmatrix}2\\0\end{bmatrix}^4\right)\theta\begin{bmatrix}1\\0\end{bmatrix}^2\theta\begin{bmatrix}2\\0\end{bmatrix}^2 -$$

$$\left(\theta\begin{bmatrix}0\\0\end{bmatrix}^4 - \theta\begin{bmatrix}3\\0\end{bmatrix}^4\right)\theta\begin{bmatrix}0\\0\end{bmatrix})\theta\begin{bmatrix}3\\0\end{bmatrix}\theta\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}2\\0\end{bmatrix} \tag{13}$$

$g(\tau)$ is a cusp form, since it vanishes at all 24 cusps of $Proj S_1(6)$. An easy , but rather tedious computation, involving the Fourier coefficients of $g(\tau)$, shows that its vanishing at the cusps is so high that $g(\tau) \equiv 0$,

Elementary computations show that the projective lines L_1, L_2, L_3, L_4 are not contained in the surface Y_3 of degree 6 defined by the equation **??**, consequently the above relation is not induced from Riemann's relations.

We remark that the curve $Y_1 \cap Y_2 \cap Y_3$ is isomorphic to $Proj(S_1(6))$.

However we have numerical evidence that we did not give a complete description of all relations among the Thetanullwerte, however we can find all other relations using the action of $\Gamma_1/\Gamma_1(6, 12)$.

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