Esercizio: Una successione esatta corta

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

spacca se esiste un'applicazione $\gamma : B \to A$ t.c. $\gamma\alpha \cong \mathrm{id}_A$ oppure se esiste $\delta : C \to B$ t.c. $\beta\delta = \mathrm{id}_C$. Verificare che le due affermazioni sono equivalenti

Dim $\gamma : B \to A$ suriettiva. $\mathrm{Ker}\,\gamma \subseteq B$.

$\alpha(A), \mathrm{Ker}\,\gamma \subseteq B$ $\quad \alpha(A) \cap \mathrm{Ker}\,\gamma = \left\{ \begin{matrix} b \,|\, \gamma(b) = 0 \\ b = \alpha(a) \end{matrix} \right\} = 0$

$\alpha(A) \oplus \mathrm{Ker}\,\gamma \subseteq B$.

$b \in B \quad \gamma(b) = a. \quad \alpha(a) \in B \quad x = b - \alpha(a)$

$\gamma(x) = \gamma(b) - a = a - a \iff x \in \mathrm{Ker}\,\gamma$.

$b \in \alpha(A) + \mathrm{Ker}\,\gamma. \Rightarrow B = \alpha(A) \oplus \mathrm{Ker}\,\gamma$

In modo simile $B = \mathrm{Ker}\,\beta \oplus \delta(C)$.

Adesso $B = \alpha(A) \oplus \mathrm{Ker}\,\gamma \qquad \overline{\beta} : \mathrm{Ker}\,\gamma \to C$ isomorfismo.

$$\begin{aligned} \delta : C &\longrightarrow B \\ c &\longrightarrow \overline{\beta}^{-1}(c) \in \mathrm{Ker}\,\gamma. \end{aligned}$$

Teorema di Mayer-Vietoris useremo i risultati di Algebra Omologica.

X spazio topologico $U, V \subseteq X$ aperti tali che $X = U \cup V$. Vogliamo calcolare l'omologia di X conoscendo quelle di U, di V e di $U \cap V$.

Dato c un ciclo in $W \overset{\text{aperto}}{\subseteq} X$. Ovviamente c è anche un ciclo in X. $[c]_W$ e $[c]_X$ le rispettive classi di omologia.

<u>Teorema</u> (Mayer-Vietoris) Sia X uno spazio topologico U, V aperti di X / $X = U \cup V$. Allora dato un c in X esistono catene c_U in U, c_V in V t.c. $[c]_X = [c_U + c_V]_X$ e vi è una successione esatta lunga in omologia

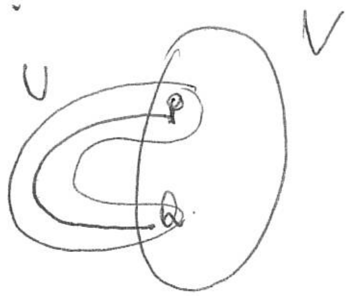
$$\dashrightarrow H_k(U\cap V) \xrightarrow{\alpha_*} H_k(U) \oplus H_k(V) \xrightarrow{\beta_*} H_k(X) \xrightarrow{\delta_*} H_{k-1}(U \cap V) \to \cdots$$

dove $\alpha_*([C]_{U\cap V}) = ([C]_U, -[C]_V)$

$$\beta_*([c]_U, [c']_V) = [c]_X + [c']_X$$

$$\partial_*([c]_X) = [\partial c_U]_{U \cap V}.$$

Oss. $[\partial c_U]_{U\cap V}$ è un ciclo, ma non è necessariamente un bordo.



$$\partial c_U = P - Q \notin B_0(U \cap V)$$

Dimostrazione del teorema

$$C'_K(X) = C_K(U) + C_K(V) \subseteq C_K(X)$$

Abbiamo una successione esatta di complessi

$$0 \to C_K(U\cap V) \xrightarrow{\alpha} C_K(U) \oplus C_K(V) \xrightarrow{\beta} C_K(U) + C_K(V) \to 0$$

$$\alpha(c) = (c, -c) \quad \beta(c, c') = c + c'$$

Abbiamo una successione esatta corta perchè

$$\alpha(\partial c) = (\partial c, -\partial c) = \partial(c, -c) = \partial\alpha(c)$$

$$\beta(\partial(c, c')) = \beta(\partial c, \partial c') = \partial c + \partial c' = \partial\beta(c, c').$$

Possiamo applicare il teorema dell'algebra omologica ④

$$\dashrightarrow H_k(U\cap V) \to H_k(U)\oplus H_k(V) \to H'_k(X) \xrightarrow{\delta_*} H_{k-1}(U\cap V) \dashrightarrow$$

$H'_k(X)$ è l'omologia di $C'_\bullet(X)$

Tutto compatibile con le definizioni date.

$\boxed{\text{Basta}}$ verificare che l'applicazione

$$J : H'_k(X) \longrightarrow H_k(X)$$
$$[c_U + c_V]' \longrightarrow [c_U + c_V]$$

indotta da $C'_k(X) \hookrightarrow C_k(X)$

è un isomorfismo a livello di omologia.

Basta costruire $S : C_k(X) \longrightarrow C_k(X)$

suddivisione baricentrica

t.c. 1) $\forall c \in C_k(X)\ \exists n_c$ t.c. $S^{n_c} c \sim c'_U + c'_V \in C'_k(X)$

2) $\forall x \in Z_k(X)\ \forall n\quad S^n x = x + \partial H_n x$

$H_n : C_k(X) \longrightarrow C_{k+1}(X)$ è operatore lineare

con

$$H_n(C_k(U)) \subseteq C_{k+1}(U)$$
$$H_n(C_k(V)) \subseteq C_{k+1}(V).$$

3) $S\partial = \partial S$.

Supponiamo che valgano 1), 2), 3) $\Rightarrow$ I è suriettiva

$$I_* : H_k'(X) \longrightarrow H_k(X)$$

$[x] \in H_k(X)$, $S^n x = x + \partial H_n x \Rightarrow [x] = [S^n x] =$
$[x_U' + x_V']$.

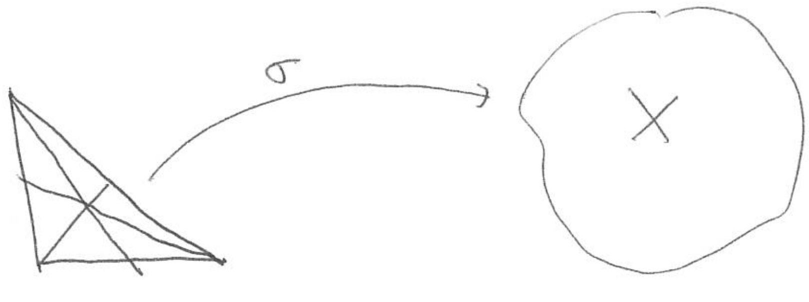
I è iniettiva: Supponiamo che $[c_U + c_V] = 0$

$c_U + c_V = \partial c$. $= S^n \partial c - \partial H_n \partial c =$ usiamo $S^n_c \sim c_U' + c_V'$
$\partial S^n c - \partial H_n(c_U) - \partial H_n(c_V) \sim$
$(\partial c_U' - \partial H_n(c_U)) + (\partial c_V' - \partial H_n(c_V))$

quindi $[c_U + c_V]' = 0$.

<u>Costruzione di S.</u>

Idea prendiamo un k-simplesso e lo scriviamo come somma di k-catene



Come già fatto, $\forall X$, costruiremo 2 operatori

$$S : C_k(X) \longrightarrow C_k(X)$$
$$T : C_k(X) \longrightarrow C_{k+1}(X)$$

Con le seguenti proprietà

$$Sf_* = f_* S \qquad \forall f : X \to Y.$$
$$Tf_* = f_* T \qquad \forall f : X \to Y.$$

$\cdot \quad \partial S = S \partial.$

$\cdot\cdot \quad \partial T + T \partial = 1 - S.$

Posto $1 = 1_{\Delta_k} : \Delta_k \longrightarrow \Delta_k.$

e definiti $S(1)$, $S(\partial 1)$, $T(1)$ e $T(\partial 1)$

In modo che valgano $(\cdot)$ e $(\cdot\cdot)$

vediamo che $\forall X$ sono univocamente definiti

S, T

Invece è sempre che $\sigma : \Delta_k \longrightarrow X$
è $\sigma = \sigma_*(1)$

Facciamo un caso (il più difficile!!)

$$(\partial T + T\partial)\sigma = (\partial T + T\partial)\sigma_*(1) \overset{\sigma_* \text{ commuta}}{=} \sigma_*\big(\partial T(1) + T\partial(1)\big) =$$
$$= \sigma_*\big(1 - S(1)\big) = \sigma - S\sigma_*(1) = \sigma - S\sigma = (1-S)\sigma.$$

Quindi basta definire $S(1)$, $T(1)$ $S\partial(1)$ $T\partial(1)$

Simplesso standard e il suo bordo sono catene particolari dette catene affini. Definiremo S, T sulle catene affini

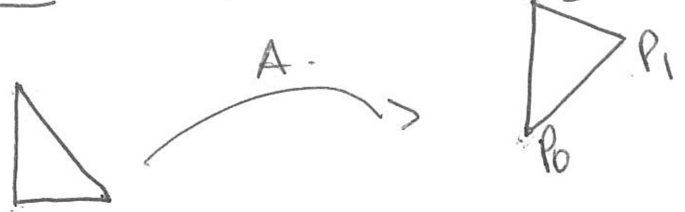
Denotiamo con E_0 l'origine e $E_1 \dots E_k$ i punti unità di $\mathbb{R}^k$.

Dati $P_0 \dots P_k \in \mathbb{R}^n$ $\exists!$ Trasformazione affine $A : \mathbb{R}^k \longrightarrow \mathbb{R}^n$.

$$A(E_i) = P_i$$

$A : \Delta_k \longrightarrow \mathbb{R}^n$ k- simplesso affine di vertici $P_0 \dots P_k$

<u>Notazione</u> $\langle P_0 \dots P_K \rangle$.



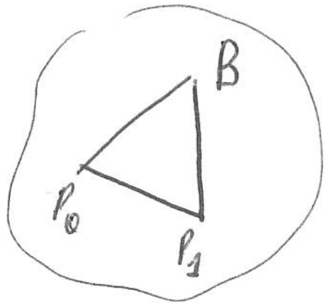
$$\partial \langle P_0 \dots P_K \rangle = \sum_{i=0}^{k} (-1)^i \langle P_0 \dots \hat{P_i} \dots P_K \rangle.$$

Sia $B \in \mathbb{R}^n$.

$B \langle P_0 \dots P_K \rangle = \langle B, P_0 \dots P_K \rangle$

$k+1$ simplesso

$\langle P_0, P_1 \rangle$



B lo possiamo estendere per linearità a tutti i k- ~~simplessi~~ catene affini

$C = \sum_i a_i A_i$.

Verifichiamo manualmente che

$\partial B(C) + B \partial C = C$ $\quad k > 0$

$\partial B(C) + \sum_i a_i \langle B \rangle = C$. $\quad k = 0$

Definiamo induttivamente S, T.

$k=0$ $\quad S\langle P \rangle = P$

$\quad\quad\quad T\langle P \rangle = 0$.

Supponiamo di averli definiti su $(k-1)$ catene affini.

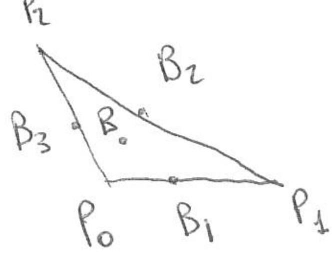
Dato $A = \langle P_0 \dots P_k \rangle$ sia B l'immagine del baricentro di Δ_k via A.

$$SA = BS\partial A$$
$$TA = B(A - SA - T\partial A)$$

$A = \langle P_0, P_1 \rangle$ $\quad SA = BS\partial A = BS(P_1 - P_0) = \langle BP_1 \rangle - \langle B, P_0 \rangle$

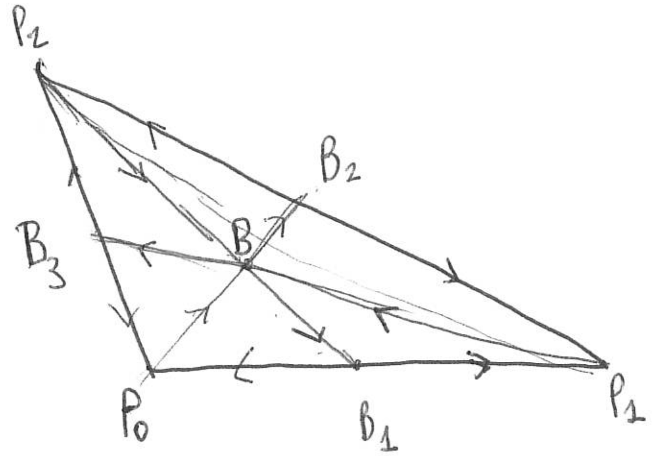
$$S\langle P_0, P_1, P_2 \rangle = BS(\langle P_1, P_2 \rangle - \langle P_0, P_2 \rangle + \langle P_0, P_1 \rangle)$$

$$= B(\langle B_1, P_1 \rangle - \langle B_1, P_0 \rangle + \langle B_3, P_0 \rangle - \langle B_3, P_2 \rangle + \langle B_2, P_2 \rangle - \langle B_2, P_1 \rangle)$$



$$= \langle B, B_1, P_1\rangle - \langle B, B_1, P_0\rangle + \langle B, B_3, P_0\rangle - \langle B, B_3, P_2\rangle$$

$$+ \langle B, B_2, P_2\rangle - \langle B, B_2, P_1\rangle$$



- Verifichiamo che $\partial S = S\partial$
  $\partial T + T\partial = C - S$.

Induzione su k. Facciamolo su simplessi affini

$\partial S(A) = \partial B S(\partial A) = S\partial A - B\partial S\partial A$ proprietà di B

$= S\partial A - BS\partial^2 A = S\partial A.$

$\partial T(A) + T\partial(A) = \partial BA - \partial BSA - \partial BT\partial A + T\partial A.$

$= A - B\partial A - SA + B\partial SA - \cancel{T\partial A} + B\partial T\partial A + \cancel{T\partial A}$

$A - SA + B(-\partial A + S\partial A + \partial T\partial A) =$

$A - SA + B(-\partial A + S\partial A - \cancel{T\partial^2 A} + \partial A - S\partial A) =$

$A - SA + B($

$\boxed{A - SA}$

Adesso abbiamo costruito S e T

$\sigma : \Delta_k \longrightarrow X$ $\quad \Delta_k$ compatto

per $n >> 0$ $\quad S^n \sigma = c_U + c_V \quad c_U \in C_k(U)$
$c_V \in C_k(V)$

e $S \partial = \partial S$.

Quindi $x \in Z_k(X)$ abbiamo

$$S(x) = x - \partial T x - \cancel{T \partial x} = x + \partial H_1(x)$$

$$S^2(x) = S(x) - S(\partial T_x) = x + \partial H_2(X)$$

$$S^n(x) = x + \partial H_n(X)$$

H_n è un polinomio in S, T e ∂ e delle definizioni

$$H_n : C_k(V) \longrightarrow C_k(V)$$

$$H_n : C_k(U) \longrightarrow C_k(U)$$

Teorema è dimostrato.

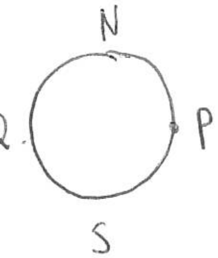
OSSERVAZIONE Assumiamo $f : X \longrightarrow Y$. continue

$X = U \cup V$ e $Y = U' \cup V'$ $\quad f(U) \subseteq U'$, $f(V) \subseteq V'$

Allora c'è un diagramma commutativo

$$\begin{tikzcd}
\arrow[r] & H_k(U\cap V) \arrow[r] \arrow[d, "f_*"] & H_k(U)\oplus H_k(V) \arrow[r] \arrow[d, "f_*"'] \arrow[d, "f_*"] & H_k(X) \arrow[r] \arrow[d, "f_*"] & H_{k-1}(U\cap V) \arrow[r] \arrow[d, "f_*"] & \cdots \\
 & H_k(U'\cap V') & H_k(U')\oplus H_k(V') \arrow[r] & H_k(Y) \arrow[r] & H_{k-1}(U'\cap V') \arrow[r] & \cdots
\end{tikzcd}$$

Esempio : La sfera S^n — $\quad H_k(S^n) = \begin{cases} \mathbb{R} & k=0,n \\ 0 & \text{altrimenti} \end{cases}$

$S^1 =$ 

$U = S^1 \setminus \{N\}$ $\qquad U \sim \mathbb{R} \sim$ pto

$V = S^1 \setminus \{S\}$ $\qquad V \sim \mathbb{R} \sim$ pto

$U\cap V \sim \{p, q\}$

$$H_k(U\cap V) \to H_k(U) \to \underset{0}{\underset{\|}{H_k(V)}} \to H_k(S^1) \to \underset{0}{\underset{\|}{H_{k-1}(U\cap V)}}$$

$k \geq 2 \Rightarrow H_k(S^1) = 0$

$$0 \to H_1(S^1) \xrightarrow{\delta} H_0(U\cap V) \xrightarrow{\alpha_*} H_0(U)\oplus H_0(V) \xrightarrow{\beta_*} H_0(S^1) \to 0$$

$$\mathbb{R}\cdot p \oplus \mathbb{R}\cdot q \qquad \mathbb{R}\cdot s \oplus \mathbb{R}\cdot N \longrightarrow \mathbb{R} \longrightarrow 0$$

$$\alpha_*[ap+bq] = [ap+bq, -ap-bq] = [(a+b)p, -(a+b)q)$$

$\ker \alpha_* = [a(p-q)]$ $\qquad H_1(S^1) \sim \mathbb{R}(\underset{\substack{\| \\ \partial\gamma_U}}{p-q})$

generatore $\gamma : \Delta_1 \longrightarrow S^1$

$t \longrightarrow e^{2\pi i t}$

$\gamma = \gamma_U + \gamma_V \qquad [\partial \gamma_U]_{U \cap V} = p - q.$

$S^n \quad n \geq 2$ per induzione.

$S^n \setminus \{pto\} \sim \mathbb{R}^n \qquad U = S^n \setminus \{N\} \sim pto$

$V = S^n \setminus \{S\} \sim pto$

$U \cap V \simeq S^n \setminus \{N, S\} \sim S^{n-1}$

$$0 \longrightarrow H_k(S^n) \xrightarrow{\delta_*} H_{k-1}(S^n) \longrightarrow 0 \qquad k \geq 2.$$

$H_n(S^n) \cong R \qquad H_k(S^n) = 0 \quad k \geq 2 \quad k \neq n.$

$$\begin{array}{ccccccccc} & & & R & & R \quad R & & R \\ 0 \longrightarrow & H_1(S^n) & \xrightarrow{\delta_*} & H_0(S^{n-1}) & \xrightarrow{\alpha_*} & H_0(pto) \oplus H_0(pto) & \rightarrow & H_0(S^n) \\ & & & p & & [p \ , -p] & & \downarrow \\ & & & & & & & 0 \end{array}$$

α_* è iniettivo $\Rightarrow \quad \operatorname{Im} \delta_* = \ker \alpha_* = 0 \Rightarrow H_1(S^n) = 0$

OSSERVAZIONE: Se $U \cap V$ è connesso per archi

$\delta_* : H_1(X) \longrightarrow H_0(U \cap V)$ è sempre!!

$\delta_*[c] = \partial c_U = \underset{\shortparallel \atop 0}{\sum a_i (P_i - Q_i)} \; \forall P.$ <u>nullo</u>...!

$X = \mathbb{R}^n \setminus \mathbb{R}^m \quad m \leq n-2$ altrimenti sconnesso.

$\mathbb{R}^n \setminus \mathbb{R}^m \sim \mathbb{R}^{n-m} \setminus \{(0 \dots 0)\}$.

$$\mathbb{R}^n \setminus \mathbb{R}^m \xleftarrow{i} \mathbb{R}^{n-m} \setminus \{0 \dots 0\}$$
$$(0 \dots 0\ x_{m+1} \dots x_n) \longleftarrow (x_{m+1} \dots x_n)$$

$$\pi : \mathbb{R}^n \setminus \mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^{n-m} \setminus \{0 \dots 0\}$$
$$(x_1 \dots x_n) \longrightarrow (x_{m+1} \dots x_n)$$

$\pi \circ i = id_{\mathbb{R}^{n-m} \setminus \{(0 \dots 0)\}}$.

$$i \circ \pi : \mathbb{R}^n \setminus \mathbb{R}^m \longrightarrow \mathbb{R}^n \setminus \mathbb{R}^m$$
$$(x_1 \dots x_n) \longrightarrow (0 \dots 0\ x_{m+1} \dots x_n)$$

Omotopia $F((x_1 \dots x_n), t) = (tx_1 \dots tx_m, x_{m+1} \dots x_n)$

$\mathbb{R}^{n-m} \setminus \{(0 \dots 0)\} \sim S^{n-m-1}$.

$$H_k(\mathbb{R}^{n-m}) \cong \begin{cases} R & k=0,\ k=n-m-1 \\ 0 & \text{altrimenti} \end{cases}$$

$\mathbb{P}^n\mathbb{C}$ $\qquad H_k(\mathbb{P}^n\mathbb{C}) = \begin{cases} R & \text{se } k=2i \le 2n \\ 0, & \text{altrimenti} \end{cases}$

Sugli appunti modo diverso $\mathbb{P}^n\mathbb{C}$ come fibrazione.

Induzione $\mathbb{P}^1(\mathbb{C}) \cong S^2$ quindi o.k.

Supponiamolo vero fino a $\mathbb{P}^{n-1}$

$\mathbb{P}^n \setminus \text{pto} \sim \mathbb{P}^{n-1}$

$$\mathbb{P}^n = \mathbb{P}^n \setminus \{[0 \dots 0\,1]\} \cup \mathbb{C}^n = \{[x_0 \dots x_n] \,/\, x_n \neq 0\}$$

$U \sim \mathbb{P}^{n-1}$, $\quad V \sim \text{pto}$.

$U \cap V = \mathbb{C}^n \setminus \{\text{pto}\} = \mathbb{R}^{2n} \setminus \{\text{pto}\}$. $\qquad H_k(U \cap V) = \begin{cases} R & k = 0, 2n-1 \\ 0 & \text{altrim} \end{cases}$

connesso per archi

$$H_k(U\cap V) \to H_k(U) \oplus H_k(V) \to H_k(\mathbb{P}^n) \to H_{k-1}(U \cap V)$$

$k = 2 \dots 2n-2$ $\qquad H_k(\mathbb{P}^n) = H_k(\mathbb{P}^{n-1})$

$k=1$ $\qquad 0 \to \underset{\overset{\shortparallel}{0}}{H_1(\mathbb{P}^{n-1})} \to H_1(\mathbb{P}^n) \to 0 \quad \Rightarrow H_1(\mathbb{P}^n) = 0$

$k = 2n-1$ $\qquad 0 \to H_{2n-1}(\mathbb{P}^n) \to 0 \qquad H_{2n-1}(\mathbb{P}^n) = 0$

$k = 2n$

$$0 \longrightarrow H_{2n}(\mathbb{P}^n) \longrightarrow H_{2n-1}(U \cap V) \longrightarrow 0$$

$$\Rightarrow \quad H_{2n}(\mathbb{P}^n) = R.$$

$k \geq 2n+1$ sempre nullo

$R = k$ campo.

$$X = T_n \simeq \underbrace{S^1 \times S^1 \times \dots S^1}_{n \text{ volte}}$$

$$H_k(T_n) = \begin{cases} k^{\oplus \binom{n}{k}} & k \leq n \\ 0 & n > n \end{cases}$$

Per induzione su n

$n = 1$ o.k.

$T_n = U \cup V$

$$U = \underbrace{S^1 \times \dots \times S^1}_{n-1} \times (S^1 \setminus \{i\})$$

$$V = S^1 \times \dots \times S^1 \times (S^1 \setminus \{-i\}).$$

$U \sim V \sim T_{n-1}$ $\quad$ $U \cap V \sim T_{n-1} \cup T_{n-1}$

$$H_k(T_{n-1}) \oplus H_k(T_{n-1}) \xrightarrow{\alpha} H_k(T_{n-1}) \oplus H_k(T_{n-1}) \xrightarrow{\beta} H_k(T^n_n))$$

$$(x, y) \longrightarrow (x+y, -x-y)$$

$\operatorname{Im} \alpha_K \cong H_K(T_{n-1})$

$$0 \longrightarrow \underset{}{\overset{\|}{K^{\oplus\binom{n-1}{K}}}} \xrightarrow{\alpha_K} K^{\oplus 2\binom{n-1}{K}} \xrightarrow{\beta_K} H_K(T_n) \xrightarrow{\delta_K} \underset{\underset{\operatorname{Ker} \alpha_{K-1}}{\|}}{K^{\oplus\binom{n-1}{K-1}}} \longrightarrow 0$$

$$\operatorname{rg} H_K(T_n) = \binom{n-1}{K-1} + \binom{n-1}{K} = \binom{n}{K}.$$

Omologia singolare di $\mathbb{P}^n(\mathbb{R})$

La strategia è ~~dive~~ uguale, ma risultato diverso $R = \mathbb{Z}$

$\mathbb{P}^1(\mathbb{R}) \cong S^1 \qquad H_0(\mathbb{P}^1(\mathbb{R}) = H_1(\mathbb{P}^1(\mathbb{R})) = \mathbb{Z}$

$\mathbb{P}^2 = U \cup V \qquad U = \mathbb{P}^2 \setminus \{pto\} \sim \mathbb{P}^1 \qquad V \simeq \mathbb{R}^2$

$U \cap V \simeq \mathbb{R}^2 \setminus \{(0,0)\}$

$$0 \xrightarrow{K \geq 3} H_K(\mathbb{P}^2) \to H_{K-1}\underset{\underset{0}{\|}}{(\mathbb{R}^2 \setminus \{(0,0)\}} \Rightarrow H_K(\mathbb{P}^2) = 0$$

$$0 \to H_2(\mathbb{P}^2) \to H_1(\mathbb{R}^2 \setminus \{0,0\}) \to H_1(\mathbb{P}^1) \to H_1(\mathbb{P}^2) \to 0$$

$$\to H_0(\mathbb{R}^2 \setminus \{0,0\} \to H_0(pto) \oplus H_0(\mathbb{P}^1) \to H_0(\mathbb{P}^2) \to 0$$

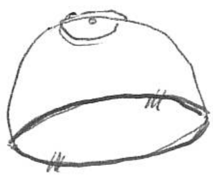
$$0 \longrightarrow H_2(\mathbb{P}^2) \xrightarrow{\delta_*} \mathbb{Z} \xrightarrow{\alpha_*} \mathbb{Z} \xrightarrow{\beta_*} H_1(\mathbb{P}^2) \longrightarrow 0$$

$$\mathbb{Z} = H_1(S^1) \longrightarrow H_1(\mathbb{P}^1)$$

mappa

porta $1 \longrightarrow 2$

$\mathbb{P}^2(\mathbb{R}$



quindi $H_2(\mathbb{P}^2) = 0 \qquad H_1(\mathbb{P}^2) = \mathbb{Z}/2\mathbb{Z}$.

$\mathbb{P}^3(\mathbb{R}) \qquad k \geq 4 \quad H_k(\mathbb{P}^3) = 0$

$$0 \to H_3(\mathbb{P}^3) \to H_2(S^2) \to \underset{\overset{\parallel}{0}}{H_2(\mathbb{P}^2)} \to H_2(\mathbb{P}^3)$$

$$\downarrow$$

$$0 \longleftarrow H_1(\mathbb{P}^3) \longleftarrow \underset{\overset{\parallel}{\mathbb{Z}/2\mathbb{Z}}}{H_1(\mathbb{P}^2)} \longleftarrow \underset{\overset{\parallel}{0}}{H_1(S^2)}$$

Quindi $H_3(\mathbb{P}^3) \cong \mathbb{Z} \qquad H_1(\mathbb{P}^3) = \mathbb{Z}/2\mathbb{Z}$

In generale

$$H_n(\mathbb{P}^n) = \begin{cases} 0 & n \text{ pari} \\ \mathbb{Z} & n \text{ dispari} \end{cases}$$

$$H_q(\mathbb{P}^n) = \mathbb{Z}/2\mathbb{Z} \quad q = 2k+1 < n.$$

$$0 \to H_n(\mathbb{P}^n) \to H_{n-1}(\mathbb{R}^n \setminus \{pto\}) \xrightarrow{\alpha_*} H_{n-1}(\mathbb{P}^{n-1}) \to H_{n-1}(\mathbb{P}^n)$$

$$1 \longrightarrow 2$$

$$H_{n-1}(\mathbb{P}^{n-1}) = 0 \Rightarrow H_n(\mathbb{P}^n) = \mathbb{Z}$$

$H_{n-1}(\mathbb{P}^{n-1}) = \mathbb{Z}$ l'applicazione α_* è iniettiva

$$\Rightarrow H_n(\mathbb{P}^n) = 0$$

$n-1$ pari

$$0 \to H_{n-1}(\mathbb{P}^n) \to \underset{\parallel\, 0}{H_{n-2}(\mathbb{R}^n \setminus \{pto\})} \to H_{n-2}(\mathbb{P}^{n-1}) \to H_{n-2}(\mathbb{P}^n)$$

$$H_{n-2}(\mathbb{P}^n) \to H_{n-3}(\mathbb{R}^n \setminus \{pto\}) = 0$$

$$H_{n-1}(\mathbb{P}^n) = 0$$

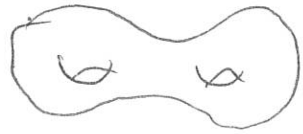
$H_{n-2}(\mathbb{P}^{n-1}) = H_{n-2}(\mathbb{P}^n)$ e poi ✓ $H_k(\mathbb{P}^{n-1}) = H_k(\mathbb{P}^n)$

$n-1$ dispari

$$\underset{\substack{\parallel \\ H_n(\mathbb{P}^n)}}{0} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \underset{\substack{\parallel \\ \mathbb{Z}/2\mathbb{Z}}}{H_{n-1}(\mathbb{P}^n)} \longrightarrow 0$$

$$1 \longrightarrow 2$$

Calcolo dell'omologia delle superfici compatte

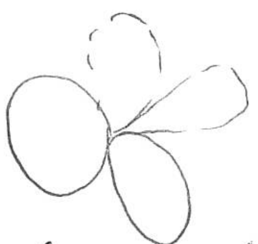
con g buchi  ----  ---

Toro con un buco $X \simeq S^1 \times S^1$

quindi $H_k(X) = \begin{cases} R & k=2 \\ R \oplus R & k=1 \\ R & k=0 \end{cases}$

In generale sia β_k il bouchet di k-circonferenze $\underbrace{S_1 \vee S_1 \cdots\cdots \vee S_1}_{k \text{ volte}}$ tutte.

passanti per lo stesso punto



$$H_k(\beta_n) = \begin{cases} 0 & k \geq 2 \\ R^n & k=1 \\ R & k=0 \end{cases}$$

Mayer-Vietoris per induzione $\beta_1 = S_1$ o.k.

$\beta_n = U \cup V$ con $U = \beta_{n-1} \vee (S_1 \setminus p)$ $V = (S_1 \setminus q)$

$U \sim \mathcal{B}_{n-1}$, $\quad V \sim$ pto $\quad U \cap V \sim \{p_1, p_2\}$

per $k \geq 2$

$$0 \longrightarrow \underset{\overset{\shortparallel}{0}}{H_k(\mathcal{B}_{n-1})} \longrightarrow H_k(\mathcal{B}_n) \longrightarrow 0$$

$$\Rightarrow H_k(\mathcal{B}_n) = 0$$

$$0 \longrightarrow \underset{R^{n-1}}{H_1(\mathcal{B}_{n-1})} \xrightarrow{\beta_*} H_1(\mathcal{B}_n) \xrightarrow{\delta_*} H_0(U \cap V) \quad R_{p_1} \oplus R_{p_2}$$

$$\big\downarrow \alpha_*$$

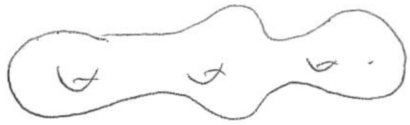
$$0 \longleftarrow \underset{R}{H_0(\mathcal{B}_n)} \xleftarrow{\beta_*} \underset{R \oplus R}{H_0(U) \oplus H_0(V)}$$

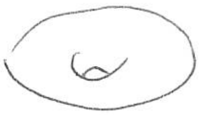
$\ker \alpha_* = a(p_1 - p_2)$ perchè i punti p_1 e p_2 sono nella stessa classe di omologia in $H_0(U)$ e $H_0(V)$, i.e. p_1, p_2 sono connessi x archi

$$0 \longrightarrow R^{n-1} \longrightarrow H_1(\mathcal{B}_n) \longrightarrow R \longrightarrow 0$$

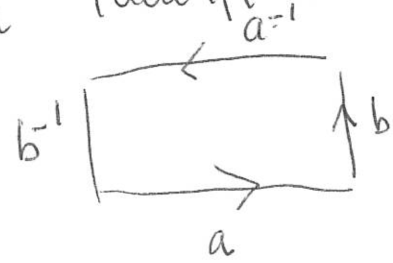
spacca

Consideriamo le "ciambelle" con g buchi

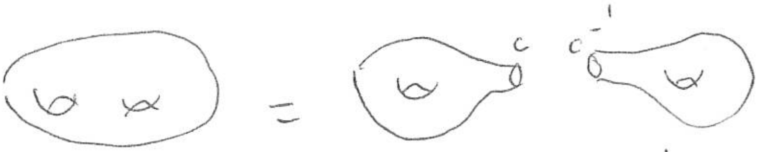


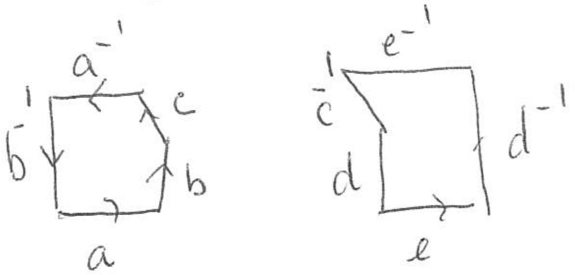


lo possiamo ottenere come un poligono con identificazione

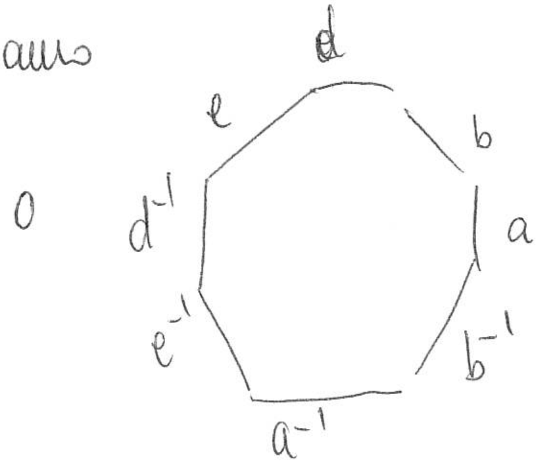


identificando il bordo.

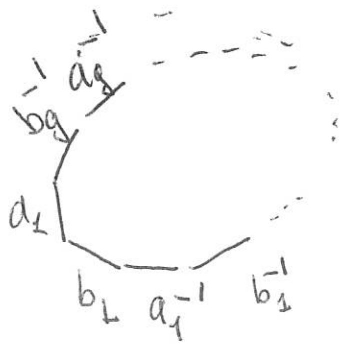




Attacchiamo

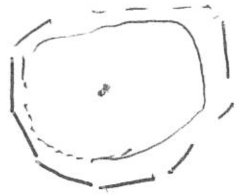


In generale



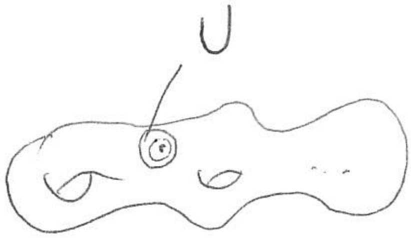
$= \Pi_{4g}$

Prendiamo come $U = \mathring{\Pi}_{4g} =$ 

$V =$ 

Procedendo alle identificazioni $U \sim$ pto p.

$V \sim \beta_{2g}$.



$V \simeq X \setminus$ pto.

Inoltre $U \cap V \sim S^1$.

Usando M-V abbiamo

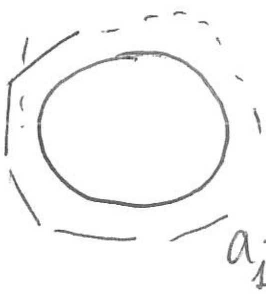
$k \geq 3$

$$0 = H_k(\beta_{2g}) \longrightarrow H_k(X) \longrightarrow \underset{\overset{\shortparallel}{0}}{H_{k-1}(S^1)}$$

$$\Rightarrow H_k(X) = 0$$

$$0 \longrightarrow H_2(X) \longrightarrow H_1(S^1) \xrightarrow{\alpha_*} H_1(\beta_{2g}) \longrightarrow H_1(X) \longrightarrow 0$$

$$0 \longrightarrow H_2(X) \xrightarrow{\delta_*} R \cdot \gamma \xrightarrow{\alpha^*} R^{2g} \longrightarrow H_1(X) \longrightarrow 0$$

γ è il ciclo  schiacciandolo nel bordo otteniamo

$a_1 + b_1 - a_1 - b_1 + a_2 + b_2 - a_2 - b_2 + \dots = 0$

quindi $\alpha_* = 0$

e abbiamo $H_2(X) \cong R$ e $H_1(X) \cong R^{2g}$.