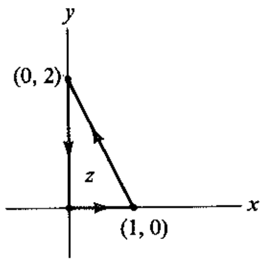
12 If α and β are closed differential forms, prove that $\alpha \wedge \beta$ is closed. If, in addition, β is exact, prove that $\alpha \wedge \beta$ is exact.

13 Consider the 1-form $\alpha = (x^2 + 7y)\,dx + (-x + y \sin y^2)\,dy$ on $\mathbb{R}^2$. Compute its integral over the following 1-cycle z.



14 Let $\alpha = (2x + y \cos xy)\,dx + (x \cos xy)\,dy$ on $\mathbb{R}^2$. Show that α is closed. Show that α is exact by finding a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with $\alpha = df$. What would the integral of α over the cycle of Exercise 13 be?

15 Let

$$\alpha = \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

Prove that α is a closed 1-form on $\mathbb{R}^2 - \{0\}$. Compute the integral of α over the unit circle S^1. How does this result show that α is not exact? How does this show that $\delta i(\alpha)$ is not exact, where $i\colon S^1 \to \mathbb{R}^2$ is the canonical imbedding?

16 (a) Prove that every closed 1-form on S^2 is exact.

(b) Let

$$\sigma = \frac{r_1\,dr_2 \wedge dr_3 - r_2\,dr_1 \wedge dr_3 + r_3\,dr_1 \wedge dr_2}{({r_1}^2 + {r_2}^2 + {r_3}^2)^{3/2}}$$

in $\mathbb{R}^3 - \{0\}$. Prove that σ is closed.

(c) Evaluate $\int_{S^2}\sigma$. How does this show that σ is not exact?

(d) Let

$$\alpha = \frac{r_1\,dr_1 + r_2\,dr_2 + \cdots + r_n\,dr_n}{({r_1}^2 + {r_2}^2 + \cdots + {r_n}^2)^{n/2}}$$

in $\mathbb{R}^n - \{0\}$. Find $*\alpha$, and prove that $*\alpha$ is closed.

(e) Evaluate $\int_{S^{n-1}} *\alpha$. Is $*\alpha$ exact?

17 Using de Rham cohomology, prove that the torus T^2 is not diffeomorphic with the 2-sphere S^2.

18 (a) Prove that every closed 1-form in the open shell

$$1 < \left(\sum_{i=1}^{3} r_i^2\right)^{1/2} < 2$$

in $\mathbb{R}^3$ is exact.

(b) Find a 2-form in the above shell that is closed but not exact.

(c) Prove that the above shell is not diffeomorphic with the open unit ball in $\mathbb{R}^3$.

19 Let f and g be C^∞ maps of M into N which are C^∞ homotopic; that is, there exists a C^∞ map F of $M \times (-\varepsilon, 1 + \varepsilon)$ into N, for some $\varepsilon > 0$, such that $F(m,0) = f(m)$ and $F(m,1) = g(m)$ for every $m \in M$. Prove that the induced homomorphisms f^* and g^* of $H^p_{\text{de R}}(N)$ into $H^p_{\text{de R}}(M)$ are equal for each integer p. (*Hint:* You will need to prove that the two injections $i_0(m) = (m,0)$ and $i_1(m) = (m,1)$ of M into $M \times (-\varepsilon, 1 + \varepsilon)$ induce the same homomorphisms on de Rham cohomology. To prove this, find suitable homotopy operators. The outline of the proof of the Poincaré lemma 4.18 should be helpful.)

20 (a) Let f: $M^n \to \mathbb{R}^{n+1}$ be an immersion, and let M^n be given the induced Riemannian structure; that is, for $m \in M$ and $u, v \in M_m$,

$$\langle u,v\rangle_m = \langle df(u), df(v)\rangle_{f(m)}.$$

Suppose that M is oriented, and that $\vec{n}$ is the oriented unit normal field along $f(M^n)$. (This means that $\vec{n}, df(v_1), \ldots, df(v_n)$ is to be an oriented orthonormal basis of the tangent space to Euclidean space at $f(m)$ whenever $v_1, \ldots, v_n$ is an oriented orthonormal basis of M_m.) Show that the volume form on M is given by

$$\omega = \delta f\big(i(\vec{n})(dr_1 \wedge \cdots \wedge dr_{n+1})\big).$$

(b) Let D be an open set in the xy plane, and let φ: $D \to \mathbb{R}^3$ be a smooth map of the form

$$\varphi(x,y) = \big(x,y,f(x,y)\big).$$

Thus φ determines an imbedded surface in $\mathbb{R}^3$. Give D and $\mathbb{R}^3$ the standard orientations. Use part (a) to prove that the induced volume form on D is given by

$$\omega = \left(\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\right) dx \wedge dy.$$