Rho-classes, index theory and Stolz' positive scalar curvature sequence

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Abstract

In this paper, we study the space of metrics of positive scalar curvature using methods from coarse geometry.

Given a closed spin manifold M with fundamental group Γ, Stephan Stolz introduced the positive scalar curvature exact sequence.

Higson and Roe introduced a K-theory exact sequence $\to K_*(B\Gamma) \xrightarrow{\alpha} K_*(C_\Gamma^*) \xrightarrow{j} K_{*+1}(D_\Gamma^*) \to$ in coarse geometry. The K-theory groups in question are the home of interesting (secondary) invariants, in particular the rho-class $\rho_\Gamma(g) \in K_*(D_\Gamma^*)$ of a metric of positive scalar curvature.

One of our main results is the construction of a map from the Stolz exact sequence to the Higson–Roe exact sequence (commuting with all arrows), using coarse index theory throughout.

The main tool is an index theorem of Atiyah–Patodi–Singer (APS) type. Here, assume that Y is a compact spin manifold with boundary, with a Riemannian metric g which is of positive scalar curvature when restricted to the boundary (and $\pi_1(Y) = \Gamma$). One constructs an APS-index $\mathrm{Ind}_\Gamma(Y) \in K_*(C_\Gamma^*)$. This can be pushed forward to $j_*(\mathrm{Ind}_\Gamma(Y)) \in K_*(D_\Gamma^*)$ (corresponding to the 'delocalized part' of the index). The delocalized APS-index theorem then states that $j_*(\mathrm{Ind}_\Gamma(Z)) = \rho_\Gamma(g_{\partial Z}) \in K_*(D_\Gamma^*)$.

As a companion to this, we prove a secondary partitioned manifold index theorem for ρ-classes.

Contents

1. Introduction and main results

1.1. Summary of the results

Given a closed spin manifold M with fundamental group Γ, Stephan Stolz introduced the positive scalar curvature exact sequence, in analogy to the surgery exact sequence in topology. It calculates a structure group of metrics of positive scalar curvature on M (the object we want

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to understand) in terms of spin-bordism of $B\Gamma$ (the classifying space of Γ) and a further group $R^{\mathrm{spin}}(\Gamma)$.

Higson and Roe introduced a K-theory exact sequence $\to K_*(B\Gamma) \xrightarrow{\alpha} K_*(C_\Gamma^*) \xrightarrow{j} K_{*+1}(D_\Gamma^*) \to$ in coarse geometry which contains the Baum-Connes assembly map α, with $K_*(D_\Gamma^*)$ canonically associated to Γ. The K-theory groups in question are the home of interesting index invariants and secondary invariants, in particular the coarse index of the spin Dirac operator $\mathrm{Ind}(D) \in K_*(C_\Gamma^*)$ and the rho-class $\rho_\Gamma(g) \in K_*(D_\Gamma^*)$ of a metric of positive scalar curvature. This is a realm where calculations are feasible.

One of our main results is the construction of a canonical comparison map from the Stolz exact sequence to the Higson-Roe exact sequence (commuting with all arrows), using coarse index theory throughout. This theorem complements the results of Higson and Roe in [**11**–**13**] where they show that it is indeed possible to map the surgery exact sequence in topology to their sequence $\to K_*(B\Gamma) \xrightarrow{\alpha} K_*(C_\Gamma^*) \xrightarrow{j} K_{*+1}(D_\Gamma^*) \to$.

Our main tool is an index theorem of Atiyah-Patodi-Singer type, which we believe to be of independent interest. For this theorem, assume that Y is a compact spin manifold with boundary, with a Riemannian metric g which is a product near the boundary and whose restriction to the boundary has positive scalar curvature. Assume also that $\pi_1(Y) = \Gamma$. Because the Dirac operator on the boundary is invertible, one constructs an APS-index $\mathrm{Ind}_\Gamma(Y) \in K_*(C_\Gamma^*)$. This can be pushed forward to $j_*(\mathrm{Ind}_\Gamma(Y)) \in K_*(D_\Gamma^*)$ (corresponding to the "delocalized part" of the index). We then prove a "delocalized APS-index" theorem, equating this class to the rho-class of the boundary $j_*(\mathrm{Ind}_\Gamma(Z)) = \rho_\Gamma(g_{\partial Z}) \in K_*(D_\Gamma^*)$.

As a companion to this, we prove a secondary partitioned manifold index theorem. Given a (non-compact) spin manifold W with positive scalar curvature metric g, with a free and discrete isometric action by a group Γ and a Γ-invariant cocompact partitioning hypersurface M, one can use a "partitioned manifold construction" in order to obtain the partitioned manifold ρ-class $\rho_\Gamma^{\mathrm{pm}}(g) \in K(D_\Gamma^*)$. Assume in addition that M has a tubular neighborhood where the metric is a product $g = g_M + dt^2$. Then we prove the partitioned manifold ρ-class theorem $\rho_\Gamma(g_M) = \rho_\Gamma^{\mathrm{pm}}(g) \in K_*(D_\Gamma^*)$. We use this secondary partitioned manifold index theorem to distinguish isotopy classes of positive scalar curvature on W.

1.2. *Basics on coarse geometry and coarse index theory*

We start by recalling the basic constructions of coarse geometry, their associated C^*-algebras and K-theory as used in the paper. We shall freely use concepts and results from [**9**, **33**].

DEFINITION 1.1. Let X be a complete Riemannian manifold of positive dimension, let $C_c(X)$ be the compactly supported continuous functions with values in $\mathbb{C}$ and $C_0(X)$ its sup-norm closure, the continuous functions vanishing at infinity.

Let $E \to X$ be a Hermitean vector bundle. We consider $H := L^2(E)$ and $H' := L^2(E) \otimes l^2(\mathbb{N})$. These are so-called *adequate* X-modules, which means that H is a Hilbert space with a C^*-homomorphism $C_0(X) \to B(H)$, given here by pointwise multiplication, and if $0 \neq f \in C_0(X)$, then it does not act as compact operator, and that $C_0(X)H$ is dense in H. We have a canonical isometry $u \colon H \to H'$ mapping into the first direct summand of $l^2(\mathbb{N})$. Using this, we map an operator A on H to the operator uAu^* on H'. We will implicitly do this throughout the paper and this way consider the operators on H as operators on H', without explicitly mentioning it.

(1) We define $D_c^*(X, H)$ to be the algebra of bounded operators T on $L^2(E) \otimes l^2(\mathbb{N})$ with the following properties:

    (i) T has finite propagation, which means that there is an $R > 0$ such that for each $s \in L^2(E)$ and for each $x \in \mathrm{supp}(Ts)$, $d(x, \mathrm{supp}(s)) < R$;

(ii) T is *pseudo-local*: for each $\phi \in C_c(X)$, the commutator $[T, \phi]$ is compact.
Then define $D^*(X, H)$ as the norm closure of $D_c^*(X, H)$.

(2) We define $C_c^*(X, H)$ to be the subalgebra of $D_c^*(X, H)$ of operators which are, in addition, *locally compact*, that is, $T\phi$ and ϕT are compact for each $\phi \in C_c(X)$. The norm closure of $C_c^*(X, H)$ is called $C^*(X, H)$. This is the *Roe algebra* of X.

The definition generalizes to an arbitrary proper metric space X; $L^2(E)$ then has to be replaced by an abstract adequate $C_0(X)$-module.

There are natural functoriality properties that we recall from [**9**, **15**, **33**].

DEFINITION 1.2. A map $f : X \to Y$ between proper metric spaces is a *coarse map* if for each $R > 0$ there is $S > 0$ such that the image under f of every R-ball is contained in an S-ball and, moreover, the inverse image of every bounded set is bounded.

We will not prove the following functoriality results of Higson and Roe, but we recall, after the proposition, the relevant construction which we are going to use.

PROPOSITION 1.3. *If $f : X \to Y$ is a continuous coarse map and H_X, H_Y are adequate $C_0(X)$ or $C_0(Y)$-modules, respectively, then f induces a non-canonical, but with suitable choices functorial homomorphism $f_* : D^*(X, H_X) \to D^*(Y, H_Y)$ which maps $C^*(X, H_X)$ to $C^*(Y, H_Y)$. The induced map in K-theory is* canonical.

DEFINITION 1.4. Applying Proposition 1.3 to $\mathrm{id} : X \to X$, we observe that $C^*(X, H_X)$ and $D^*(X, H_X)$ depend only mildly on the adequate module H_X, and that their K-theory is independent of this choice. We follow the custom of [**33**] and drop H_X from the notation, writing simply $C^*(X)$ or $D^*(X)$ instead of $C^*(X, H_X)$, $D^*(X, H_X)$.

For the construction of f_* of Proposition 1.3, we need the following concepts:

DEFINITION 1.5. Let H_X and H_Y be two adequate modules and let $f : X \to Y$ be a coarse map. We say that an isometric embedding $W : H_X \to H_Y$ *covers f in the C^*-sense* if W is the norm-limit of linear maps V satisfying the following condition:

$$\exists R > 0 \text{ s.t. } \phi V \psi = 0 \ \forall \phi \in C_c(Y), \ \psi \in C_c(X) \text{ with } d(\mathrm{supp}(\phi), \mathrm{supp}(\psi)) > R. \quad (1.1)$$

Given a coarse map it is always possible to find such a W. Then the map $\mathrm{Ad}(W)(T) := WTW^*$, from the bounded operators on H_X to the bounded operators of H_Y, sends $C^*(X, H_X)$ to $C^*(Y, H_Y)$ and we define $f_* := \mathrm{Ad}(W) : C^*(X, H_X) \to C^*(Y, H_Y)$. The induced map in K-theory is independent of the choice of W, see [**15**, Lemma 3]. Moreover, by [**8**] the functor $K_*(C^*(X))$ is a coarse homotopy invariant.

Regarding $D^*(X, H)$ we have the following definition.

DEFINITION 1.6. Let $f : X \to Y$ be a *continuous* coarse map, let H_X, H_Y be two adequate modules. We shall say that an isometry $W : H_X \to H_Y$ *covers f in the D^*-sense*[†] if W is the

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[†] In [**39**, Definition 2.4], the same property is denoted 'W covers f topologically'.

norm-limit of bounded maps V satisfying the following two conditions:

(1) there is an $R > 0$ such that $\phi V \psi = 0$ if $d(\operatorname{supp}(\phi), f(\operatorname{supp}(\psi))) > R$, for $\phi \in C_c(Y)$ and $\psi \in C_c(X)$;

(2) $\phi V - V(\phi \circ f)$ is compact for each $\phi \in C_0(Y)$.

For such a W one proves that $\operatorname{Ad}(W)$ sends $D^*(X, H_X)$ into $D^*(Y, H_Y)$ and induces therefore a morphism $f_* := \operatorname{Ad}(W) \colon D^*(X, H_X) \to D^*(Y, H_Y)$. As for C^*, one proves that the induced map in K-theory does not depend on the choice of W, see [**15**, Lemma 3].

Up to tensoring with $\ell^2(\mathbb{N})$, see [**9**, Lemma 7.7], it is always possible to find an isometry W satisfying the required two properties, which is the reason why we included this tensor product with $\ell^2(\mathbb{N})$ in the definition of $D^*(X)$.

By [**9**, Lemma 7.8], $K_*(D^*(X))$ is invariant under *continuous* coarse homotopy.

To be able to use standard techniques from the K-theory of C^*-algebras, given a subspace $Z \subset X$ we replace C^*Z by an ideal $C^*(Z \subset X)$ of C^*X as follows:

DEFINITION 1.7.   Let X be a proper metric space and $Z \subset X$ a closed subset. Define $C^*(Z \subset X)$ as the closure of those operators $T \in C_c^*(X)$ such that there is an $R > 0$ satisfying $\phi T = 0 = T\phi$ whenever $\phi \in C_c(X)$ with $d(\operatorname{supp}(\phi), Z) > R$. Define $D^*(Z \subset X)^{\dagger}$ as the closure of those $T \in D_c^*(X)$ such that

(1) there is $R_T \geqslant 0$ satisfying $\phi T = 0 = T\phi$ whenever $\phi \in C_c(X)$ with $d(\operatorname{supp}(\phi), Z) > R_T$ and

(2) $\forall \phi \in C_0(X \setminus Z)$ ϕT and $T\phi$ are compact.

Then $D^*(Z \subset X)$ and $C^*(Z \subset X)$ are ideals in $D^*(X)$.

We now describe equivariant versions of the constructions made so far. Assume therefore in addition that a discrete group Γ acts freely and isometrically on the manifold X and the Hermitean bundle E. It then also acts by unitaries on $H = L^2(E)$.

(1) We define $D^*(X)^{\Gamma}$ to be the norm closure of the Γ-invariant part $D_c^*(X)^{\Gamma}$, and its ideal $C^*(X)^{\Gamma}$ as the norm closure of $C_c^*(X)^{\Gamma}$. If Z is a Γ-invariant subspace, then we define in the corresponding way the ideals $D^*(Z \subset X)^{\Gamma}$ and $C^*(Z \subset X)^{\Gamma}$.

(2) The construction generalizes to an arbitrary proper metric space X with proper isometric Γ-action, using a Γ-adequate‡ $C_c(X)$-module H with compatible unitary Γ-action.

(3) As indicated in the notation, one has suitable independence on E, along the way with the obvious generalization of functoriality to Γ-equivariant maps.

(4) If the quotient $V = X/\Gamma$ is a finite complex, then $K_*(D^*(X)^{\Gamma}/C^*(X)^{\Gamma}) \simeq K_{*-1}(V)$; see [**33**, Lemmas 5.14 and 5.15].

LEMMA 1.8 (cf. [**15**, Lemma 1; **39**, Proposition 3.8]).   *Given a closed Γ-subspace Z of a proper metric Γ-space X, the inclusion $Z \hookrightarrow X$ induces K-theory isomorphisms $K_*(C^*(Z)^{\Gamma}) \xrightarrow{\cong} K_*(C^*(Z \subset X)^{\Gamma})$, $K_*(D^*(Z)^{\Gamma}) \xrightarrow{\cong} K_*(D^*(Z \subset X)^{\Gamma})$.*

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†We deviate here from the notation employed by Roe, for example, $C_Z^*(X)$ for $C^*(Z \subset X)$ in [**33**, Definition 3.10]. Our notation and definitions agree with those used in [**39**].

‡Adequate requires a little bit of extra care, cf. [**33**, Definition 5.13]: replacing H by $H \otimes l^2(\Gamma) \otimes l^2(\mathbb{N})$ will do.

Directly from the above results and the short exact sequence

$$0 \longrightarrow C^*(X)^\Gamma \longrightarrow D^*(X)^\Gamma \longrightarrow D^*(X)^\Gamma/C^*(X)^\Gamma \longrightarrow 0,$$

we obtain the *Higson–Roe surgery sequence* for a Γ manifold X with quotient X/Γ compact:

$$\cdots \longrightarrow K_{n+1}(X/\Gamma) \longrightarrow K_{n+1}(C^*(X)^\Gamma) \longrightarrow K_{n+1}((D^*(X)^\Gamma)) \longrightarrow K_n(X/\Gamma) \longrightarrow \cdots. \quad (1.2)$$

We will also be interested in a universal version of this sequence. First, we give a definition:

DEFINITION 1.9.   Let Γ be a discrete group. Define

$$K_*(C^*_\Gamma) := \underset{X \subset E\Gamma \ \Gamma\text{-compact}}{\text{dirlim}} K_*(C^*(X)^\Gamma); \quad K_*(D^*_\Gamma) := \underset{X \subset E\Gamma \ \Gamma\text{-compact}}{\text{dirlim}} K_*(D^*(X)^\Gamma).$$

Here, $E\Gamma$ is any contractible CW-complex with free cellular Γ-action, a universal space for free Γ actions.

By coarse invariance of C^* and [**33**, Lemma 5.14], there is a canonical isomorphism $K_*(C^*(X)^\Gamma) \cong C^*_r\Gamma$ for any free cocompact Γ-space X. Therefore, the definition of $K_*(C^*_\Gamma)$ is along canonical isomorphisms and we get canonically $K_*(C^*_\Gamma) = K_*(C^*_r\Gamma)$. Once this definition is given, we obtain immediately the (universal) Higson–Roe surgery sequence

$$\cdots \longrightarrow K_{n+1}(B\Gamma) \longrightarrow K_{n+1}(C^*_\Gamma) \longrightarrow K_{n+1}(D^*_\Gamma) \longrightarrow K_n(B\Gamma) \longrightarrow \cdots, \quad (1.3)$$

which can be rewritten as

$$\cdots \longrightarrow K_{n+1}(B\Gamma) \longrightarrow K_{n+1}(C^*_r\Gamma) \longrightarrow K_{n+1}(D^*_\Gamma) \longrightarrow K_n(B\Gamma) \longrightarrow \cdots. \quad (1.4)$$

It is proved by Roe [**34**] that the homomorphism $K_{n+1}(B\Gamma) \to K_{n+1}(C^*_r\Gamma)$ appearing in (1.4) is precisely equal to the assembly map. This implies

if Γ is torsion free, then the Baum–Connes conjecture for Γ is equivalent to $K_{n+1}(D^*_\Gamma) = 0$. (1.5)

If M is a proper complete metric space with a free cocompact isometric Γ-action, then there is a universal Γ-map $u\colon M \to E\Gamma$ with range in a Γ-finite subcomplex (u is automatically coarse), and any two such maps are (coarsely continuously) Γ-homotopic. We therefore get canonical induced maps

$$u_*\colon K_*(C^*(M)^\Gamma) \longrightarrow K_*(C^*_\Gamma); \quad u_*\colon K_*(D^*(M)^\Gamma) \longrightarrow K_*(D^*_\Gamma).$$

Moreover, for $C^*(M)^\Gamma$ the map is a canonical isomorphism.

More generally, if W is a complete metric space with free Γ-action, $M \subset W$ is Γ-invariant and M/Γ is compact, then $D^*(M \subset W)^\Gamma$ is the limit of $D^*(U_R(M))^\Gamma$ as $R \to \infty$, where $U_R(M)$ is the closed R-neighborhood of M, again a Γ-compact metric space. We get a compatible system of universal maps to $E\Gamma$, all with image in finite subcomplexes, and an induced compatible system of maps in K-theory, giving rise to the maps

$$K_*(D^*(M)^\Gamma) \xrightarrow[\cong]{i_*} \lim_{R\to\infty} K_*(D^*(M \subset U_R(M))^\Gamma) \cong K_*(D^*(M \subset W)^\Gamma) \xrightarrow{u_*} K_*(D^*_\Gamma),$$

whose composition is the universal map for $D^*(M)^\Gamma$.

1.3.   *Index and ρ-classes*

We now recall Roe's method of applying C^*-techniques to the Dirac operator to efficiently define primary and secondary invariants for spin manifolds in the context of coarse geometry.

Let X be an arbitrary complete spin manifold with free isometric action by Γ of dimension $n > 0$. Fix an odd continuous chopping function $\chi\colon \mathbb{R} \to \mathbb{R}$, that is, $\chi(x) \xrightarrow{x \to +\infty} 1$. With the Dirac operator D_X we now consider $\chi(D_X)$. Roe proves, using finite propagation speed of the wave operator and ellipticity, that this is an element in $D^*(X)^\Gamma$, cf. [**31**, Proposition 2.3].

PROPOSITION 1.10. *Assume that $Y \subset X$ is a Γ-invariant closed subset and the scalar curvature is uniformly positive outside Y. Then $\chi(D_X)$ is an involution modulo $C^*(Y \subset X)^{\Gamma}$.*

In particular, if we have uniformly positive scalar curvature, then $\chi(D_X)$ is an involution in $D^*(X)^{\Gamma}$.

For the other extreme, without any further curvature assumption, $\chi(D_X)$ is an involution modulo $C^*(X)^{\Gamma}$.

This important proposition is at the heart of the method. It is stated by Roe [**33**, Proposition 3.11] but without a full proof. A complete proof is given independently in Pape's thesis [**27**, Theorem 1.4.28], compare [**5**, Theorem 1.7], or by Roe [**35**, Lemma 2.3].

Recall that, given an involution x in a C^*-algebra A, it defines in a canonical way the element $[\frac{1}{2}(x+1)] \in K_0(A)$. If $n := \dim(X)$ is odd, then, in the situation of Proposition 1.10, we obtain the corresponding class $[D_X] := [\frac{1}{2}(\chi(D_X) + 1)] \in K_0(D^*(X)^{\Gamma}/C^*(Y \subset X))$.

If n is even, then we have to use the additional Γ-invariant grading of the spinor bundle $L^2(S) = L^2(S_+) \oplus L^2(S_-)$. The operator D_X and, because χ is an odd function, $\chi(D_X)$ are odd with respect to this decomposition so that we obtain the positive part $\chi(D_X)_+ : L^2(S_+) \to L^2(S_-)$. We choose any isometry $U : L^2(S_-) \to L^2(S_+)$ covering id_X in the D^*-sense.[†]

Then $U\chi(D_X)_+$ is a unitary in $D^*(X)^{\Gamma}/C^*(Y \subset X)^{\Gamma}$ and represents $[D_X] \in K_1(D^*(X)^{\Gamma}/C^*(Y \subset X)^{\Gamma})$.

DEFINITION 1.11. Let (X, g) be a complete Riemannian spin manifold of dimension $n > 0$ with isometric free action of Γ. Define

$$\mathrm{Ind}^{\mathrm{coarse}}(D_X) := \partial([D_X]) \in K_n(C^*(X)^{\Gamma}).$$

Here, ∂ is the boundary map of the long exact sequence of the extension $0 \to C^*(X)^{\Gamma} \to D^*(X)^{\Gamma} \to D^*(X)^{\Gamma}/C^*(X)^{\Gamma} \to 0$.

Observe that, if we have uniformly positive scalar curvature outside of Y, then we have a canonical lift to

$$\mathrm{Ind}^{\mathrm{rel}}(D_X) := \partial([D_X]) \in K_n(C^*(Y \subset X)^{\Gamma}).$$

If we have uniformly positive scalar curvature throughout, then we define a secondary invariant, the ρ-class of the metric g, as

$$\rho(g) := [D_X] \in K_{n+1}(D^*(X)^{\Gamma}). \tag{1.6}$$

Finally, if X/Γ is compact, then there is the canonical map to $K_{n+1}(D^*_{\Gamma})$ of Definition 1.9 and we define $\rho_{\Gamma}(g) \in K_{n+1}(D^*_{\Gamma})$, the ρ_{Γ}-class of g, as the image of $\rho(g)$ under this map

$$\rho_{\Gamma}(g) := u_*(\rho(g)) \in K_{n+1}(D^*_{\Gamma}). \tag{1.7}$$

REMARK 1.12. It is important to point out that in contrast to the ρ-class $\rho(g) \in K_{n+1}(D^*(X)^{\Gamma})$, the ρ_{Γ}-class $\rho_{\Gamma}(g) \in K_{n+1}(D^*_{\Gamma})$ vanishes for groups without torsion, at least for those for which the Baum–Connes conjecture holds. See the fundamental remark appearing in (1.5). This means we expect $\rho_{\Gamma}(g)$ to be different from zero only for groups Γ *with* torsion.

Basic non-trivial examples of $\rho_{\Gamma}(g)$ for Γ with torsion are considered in [**14**].

[†]In [**33**], it is only required that U covers id_X. However, as pointed out by Ulrich Bunke, to make sure that $U\chi(D_X)_+ \in D^*(X)^{\Gamma}$ one needs the stronger assumption.

Note that the ρ-class is well-defined whenever the Dirac operator D_X is L^2-invertible; we denote it $\rho(D_X)$ in this more general case. In fact, we will sometime employ this notation also for the spin Dirac operator associated to a positive scalar curvature metric.

1.4. *Delocalized APS-index theorem*

GEOMETRIC SET-UP 1.13. Let now (W, g_W) be a n-dimensional Riemannian spin manifold with boundary, complete as metric space.[‡] We denote its boundary (M, g_M), and we assume always that we have product structures near the boundary. We assume that the scalar curvature of g_M is uniformly positive, and that Γ acts freely, isometrically and cocompactly on W and therefore also on M. We denote the quotient of (W, g_W) by the action of Γ as (Y, g_Y), a compact Riemannian manifold with boundary. Associated to these data is $W_\infty = W \cup_M M \times [0, \infty)$ with extended product structure on the cylinder. This defines a complete Riemannian metric g on W_∞ and we then have uniformly positive scalar curvature outside $W \subset W_\infty$.

The considerations of the previous subsection apply now to the pair $(W \subset W_\infty)$ and we obtain therefore a class $\mathrm{Ind}^{\mathrm{rel}}(D_{W_\infty}) \in K_n(C^*(W \subset W_\infty)^\Gamma)$ and thus a class

$$\mathrm{Ind}(D_W) := c_*^{-1} \mathrm{Ind}^{\mathrm{rel}}(D_{W_\infty}) \in K_n(C^*(W)^\Gamma). \tag{1.8}$$

Here, we use the canonical inclusion $c \colon C^*(W)^\Gamma \to C^*(W \subset W_\infty)^\Gamma$ which induces an isomorphism in K-theory by Lemma 1.8.

Let us remark here that, under the canonical isomorphism $K_n(C^*(W)^\Gamma) \cong K_n(C_r^*\Gamma)$, this index class corresponds to any of the other APS-indices for manifolds with boundary defined in this context, for example, using the Mishchenko–Fomenko approach and the b-calculus or using APS-boundary conditions, cf. Section 2.

The passage from C^*X to D^*X corresponds to the passage to the delocalized part of the index information (we will explain this later). This delocalized part we can compute by a K-theoretic version of the APS-index theorem.

THEOREM 1.14 (Delocalized APS-index theorem). *Let (W, g_W) be an even dimensional Riemannian spin-manifold with boundary ∂W such that $g_{\partial W}$ has positive scalar curvature. Assume that Γ acts freely isometrically and W/Γ is compact. Then*

$$\iota_*(\mathrm{Ind}(D_W)) = j_*(\rho(g_{\partial W})) \quad \text{in } K_0(D^*(W)^\Gamma). \tag{1.9}$$

Here, we use $j \colon D^*(\partial W)^\Gamma \to D^*(W)^\Gamma$ induced by the inclusion $\partial W \to W$ and $\iota \colon C^*(W)^\Gamma \to D^*(W)^\Gamma$ the inclusion.

COROLLARY 1.15. *By functoriality, using the canonical Γ-map $u \colon W \to E\Gamma$ of Definition 1.9, we have $\iota_* u_*(\mathrm{Ind}(D_W)) = \rho_\Gamma(g_{\partial W})$ in $K_0(D_\Gamma^*)$. If we define $\mathrm{Ind}_\Gamma(D_W) := u_*(\mathrm{Ind}(D_W))$ in $K_0(C_\Gamma^*)$, then the last equation reads*

$$\iota_*(\mathrm{Ind}_\Gamma(D_W)) = \rho_\Gamma(g_{\partial W}) \quad \text{in } K_0(D_\Gamma^*). \tag{1.10}$$

This gives immediately bordism invariance of the ρ-classes.

COROLLARY 1.16. *Let (M_1, g_1) and (M_2, g_2) be two odd-dimensional free cocompact spin Γ-manifolds of positive scalar curvature. Assume that they are bordant as manifolds with*

[‡]That is, every Cauchy sequence converges.

positive scalar curvature, that is, that there is a Riemannian spin manifold (W, g) *with free cocompact* Γ-*action such that* $\partial W = M_1 \amalg -M_2$, g *has positive scalar curvature and restricts to* g_j *on* M_j. *Then*

$$\rho_\Gamma(g_1) = \rho_\Gamma(g_2) \in K_0(D_\Gamma^*).$$

Proof.   The rho-class is additive for disjoint union and changes sign if one reverses the spin structure. Because W and W_∞ have uniformly positive scalar curvature, $\mathrm{Ind}(D_W) = 0$; thus $\mathrm{Ind}_\Gamma(D_W) = 0$. The assertion now follows directly from Corollary 1.15. $\square$

REMARK 1.17.   Note that bordism invariance holds only for ρ_Γ-classes; indeed, we need a common K-theory group where we can compare the two invariants. Precisely because of this last observation, the following variant of Corollary 1.16 holds:

Let (M_1, g_1) and (M_2, g_2) be two free cocompact spin Γ-manifolds of positive scalar curvature endowed with Γ-equivariant reference maps f_1, f_2 to a Hausdorff topological Γ-space $\tilde{X}$ with compact quotient $X := \tilde{X}/\Gamma$. Assume that there exists a Riemannian spin manifold (W, g) as in Corollary 1.16 endowed with a Γ-equivariant reference map $F: W \to \tilde{X}$ such that $F|_{M_j} = f_j$. Then, defining $\rho_{\tilde{X}}(g_j) := (f_j)_* \rho(g_j) \in K_0(D^*(\tilde{X})^\Gamma)$, we have the following identity:

$$\rho_{\tilde{X}}(g_1) = \rho_{\tilde{X}}(g_2) \in K_0(D^*(\tilde{X})^\Gamma). \tag{1.11}$$

Proof.   Denote by $\iota_{\tilde{X}} : C^*(\tilde{X})^\Gamma \to D^*(\tilde{X})^\Gamma$ the inclusion and similarly for ι_W. Let j_1 and j_2 be the natural inclusions $M_j \hookrightarrow W$. Then, from Theorem 1.14 we obtain

$$(\iota_W)_*(\mathrm{Ind}(D_W)) = (j_1)_* \rho(g_1) - (j_2)_* \rho(g_2) \quad \text{in } K_0(D^*(W)^\Gamma).$$

We now apply $F_*: K_0(D^*(W)^\Gamma) \to K_0(D^*(\tilde{X})^\Gamma)$. Since $F \circ j_1 = f_1$ and $F \circ j_2 = f_2$ and since $F_*(\iota_W)_* = (\iota_{\tilde{X}})_* F_*$, with the F_* on the right-hand side going from $K_0(C^*(W)^\Gamma)$ to $K_0(C^*(\tilde{X})^\Gamma)$, we see that

$$(\iota_X)_* F_*(\mathrm{Ind}(D_W)) = (f_1)_* \rho(g_1) - (f_2)_* \rho(g_2).$$

Since the left-hand side vanishes (recall that g on W is of positive scalar curvature), this is precisely what we wanted to prove. $\square$

REMARK 1.18.   We are convinced that the theorem also is correct if $\dim(W)$ is odd. In the present paper, we only deal with the even case. By using Cl_n-linear Dirac operators and an appropriate setup for Cl_n-linear (also called n-multigraded) cycles for K-theory, we expect that our method should generalize to all dimensions and also to the refined invariants in real K-theory one can get that way.

REMARK 1.19.   As we shall see, Theorem 1.14 has a surprisingly intricate proof. A different approach for proving it would be to develop a theory for the Calderon projector P associated to a Dirac-type operator on a Galois covering with boundary. In this direction, recall the classical formula for the APS numeric index in terms of the Calderon projection P and the APS projection $\Pi_\geqslant$: $\mathrm{ind}^{\mathrm{APS}} D^+ = \mathbf{i}(\Pi_\geqslant, P)$. If one were able to extend this formula to the APS-index class, then the theorem would follow provided one could establish, in addition, that the image of the class of the Calderon projector $[P]$ in $K_0(D^*(W)^\Gamma)$ vanishes. It would be very interesting to work out this alternative approach to Theorem 1.14, which seems to be, however, quite

an intricate question. A first step in this direction is carried out in [**1**], where the Calderon projector for C^*-module coefficients is constructed.

EXAMPLE 1.20. The morphisms $C^*(M)^\Gamma \to C^*(W)^\Gamma \to C^*(W \subset W_\infty)^\Gamma$ induce (canonical) isomorphisms in K-theory by Lemma 1.8 and because $M \to W$ is a coarse equivalence, as W/Γ is compact. Consequently, we can map $\mathrm{Ind}(D_W)$ also to $K_*(D^*(M)^\Gamma)$ and compare its image there to $\rho(g_M)$.

It turns out that in general these two objects are different, so that a corresponding sharpening of Theorem 1.14 is not possible. Indeed, an additional secondary term, a rho-class of a bordism, shows up. This secondary class appears naturally when one gives a proof of bordism invariance of the rho-index using suitable exact sequences of K-theory of Roe algebras and the principle that 'boundary of Dirac is Dirac'. We plan to work this out in a sequel publication.

Explicitly, take $W = D^{n+1}$ with $\partial W = S^n$, with the standard metrics (slightly modified to have product structure near the boundary, but clearly with positive scalar curvature as long as $n > 1$).

Because of overall positive scalar curvature, $\mathrm{Ind}(D_W) \in K_*(C^*(W)) = K_*(\mathbb{C})$ vanishes, and so does its image in $K_*(D^*(S^n))$.

On the other hand, the Dirac operator on S^n represents the fundamental class, a non-trivial element in $K_n(S^n)$. By the commutativity of the diagram (1.15), another main theorem of this paper, $\rho(g_{S^n}) \in K_n(D^*(S^n))$ has to be non-trivial, being mapped to a non-trivial element in $K_n(D^*(S^n)/C^*(S^n)) = K_n(S^n)$. Observe that this is a purely topological phenomenon, having nothing to do with analysis.

1.5. *Secondary index theorem for ρ-classes on partitioned manifolds*

In this section, we formulate a partitioned manifold secondary index theorem, for the ρ-class on a manifold of uniformly positive scalar curvature.

For this aim, let W be a (non-compact) Riemannian spin manifold of dimension $n + 1$ with isometric free Γ-action and assume that there is a Γ-invariant two-sided hypersurface $M \subset W$ such that M/Γ is compact. We get a decomposition $W = W_- \cup_M W_+$.

Let us quickly recall the primary partitioned manifold index theorem. The classical case is $\Gamma = \{1\}$, then we obtain $\mathrm{Ind}(D_W) \in K_{n+1}(C^*(W))$. The partition allows to construct a map (showing up in a corresponding Mayer–Vietoris sequence as in Section 3.3) to $K_n(C^*(M)) = K_n(\mathbb{C})$. The partitioned manifold theorem of Roe [**31**] then simply states that the image of $\mathrm{Ind}(D_W)$ under this map is $\mathrm{ind}(D_M)$. The corresponding statement for non-trivial Γ and even n is covered in [**46**].

We now treat the same question for the secondary rho class of manifolds with uniformly positive scalar curvature. Indeed, let us first give a direct definition of the partitioned manifold rho-class, similar to the definition of the partitioned manifold index as given by Higson [**6**].

DEFINITION 1.21. Assume, in the above situation, that W has dimension $n + 1$ and uniformly positive scalar curvature. Then we constructed $\rho(D_W) \in K_{n+2}(D^*(W)^\Gamma)$. Consider the image of $[D_W]$ under the D^*-Mayer–Vietoris boundary map for the decomposition of W into W_+ and W_- along W (discussed in Section 3.3): $\delta_{\mathrm{MV}}[D_W] \in K_{n+1}(D^*(M)^\Gamma)$. We set

$$\rho^{\mathrm{pm}}(g) := \delta_{\mathrm{MV}}[D_W] \in K_{n+1}(D^*(M)^\Gamma) \tag{1.12}$$

and we call it the *partitioned manifold ρ-class* associated to the partitioned manifold $W = W_- \cup_M W_+$. We shall be mainly concerned with a universal version of this class: we consider

the canonical map $u \colon M \to E\Gamma$ and we set

$$\rho_\Gamma^{\mathrm{pm}}(g) := u_*(\delta_{\mathrm{MV}}[D_W]) \in K_{n+1}(D_\Gamma^*). \tag{1.13}$$

We call this secondary invariant the *partitioned manifold ρ_Γ-class* associated to $W = W_- \cup_M W_+$.

THEOREM 1.22. *Let (W, g) be a connected spin manifold partitioned by a hypersurface M into $W_- \cup_M W_+$. Let Γ act freely on (W, M). Let $\dim(W) = n + 1$ be even. Assume that the metric g on W has uniformly positive scalar curvature and that the metric on a tubular neighborhood of the hypersurface M has product structure, so that the induced metric g_M also has positive scalar curvature. Assume, finally, that M/Γ is compact. Then*

$$\rho_\Gamma^{\mathrm{pm}}(g) = \rho_\Gamma(g_M) \in K_{n+1}(D_\Gamma^*).$$

REMARK 1.23. We are convinced that the assertion of Theorem 1.22 also holds if $\dim(W)$ is odd. As detailed in Remark 1.18, with appropriate new multigrading input our method might carry over. Again, we hope to work this out in the future.

COROLLARY 1.24. *Let W be as in Theorem 1.22 with two Γ-equivariant metrics g^0, g^1 of uniformly positive scalar curvature (and in the same coarse equivalence class) which are of product type near M. If g^0, g^1 are connected by a path of uniformly positive Γ-equivariant metrics g^t in the same coarse metric class (not necessarily product near M), then $\rho_\Gamma^{\mathrm{pm}}(g_M^0) = \rho_\Gamma^{\mathrm{pm}}(g_M^1) \in K_{n+1}(D_\Gamma^*)$.*

Proof. We simply have to observe that we get a homotopy $\rho_\Gamma^{\mathrm{pm}}(g^t)$ between $\rho_\Gamma^{\mathrm{pm}}(g^0)$ and $\rho_\Gamma^{\mathrm{pm}}(g^1)$ in D_Γ^* and then apply homotopy invariance of K-theory. □

As an application of this corollary, assume that M/Γ, which is assumed to be compact, has two metrics g_0, g_1 with Γ-invariant lifts $\tilde{g}_0, \tilde{g}_1$ that have the property that $\rho_\Gamma(\tilde{g}_0) \neq \rho_\Gamma(\tilde{g}_1) \in K_{n+1}(D_\Gamma^*)$. Of course, this implies that the two metrics are not concordant on M/Γ. Stabilize by taking the product with $\mathbb{R}$ (with the standard metric). We can now conclude that even with the extra room on $M/\Gamma \times \mathbb{R}$ we cannot deform $g_0 + dt^2$ to $g_1 + dt^2$ through metrics of uniformly positive scalar curvature. This follows directly from the corollary.

The strategy of proof for this ρ-version of the partitioned manifold index theorem is the same as the classical one:

(1) we prove it with an explicit calculation for the product case;
(2) we prove that the partitioned manifold rho-class depends only on a small neighborhood of the hypersurface.

REMARK 1.25. In the situation of Theorem 1.22, both $\rho^{\mathrm{pm}}(g)$ and $\rho(g_M)$ are defined in $K_*(D^*(M)^\Gamma)$. However, our method does not give any information about equality of these classes, only about their images in $K_*(D_\Gamma^*)$. This is in contrast to Theorem 1.14, where the equality is established in $K_*(D^*(W)^\Gamma)$.

On the other hand, we also do not have an example where the partitioned manifold ρ-class does not coincide with the ρ-class of the cross section. It is an interesting challenge to either find such examples, or to improve the partitioned manifold secondary index theorem. The latter would be important in particular in light of applications like the stabilization problem we just discussed: if g_1, g_2 on M are positive scalar curvature metrics which are not concordant, is the same true for $g_1 + dt^2$ and $g_2 + dt^2$ on $\mathbb{R} \times M$?

1.6. *Mapping the positive scalar curvature sequence to analysis*

The Stolz exact sequence is the companion for positive scalar curvature of the surgery exact sequence in the classification of high-dimensional manifolds. The latter connects the structure set, consisting of all manifold structures in a given homotopy type, with the generalized homology theory given by the L-theory spectrum and the algebraic L-groups of the fundamental group.

Similarly, Stolz' sequence connects the 'structure set' $\mathrm{Pos}^{\mathrm{spin}}$ (cf. Definition 1.26), which contains the equivalence classes of metrics of positive scalar curvature to the generalized homology group Ω^{spin} and to $R^{\mathrm{spin}}(X)$. The latter indeed is a group which only depends on the fundamental group of X. It is similar to the geometric definition of L-groups. Missing until now is an algebraic and computable description of these R-groups, in contrast to the L-groups of surgery.

In this subsection, we construct a map from the Stolz positive scalar curvature exact sequence to analysis. We give a picture which describes the transformation as directly as possible, using indices defined via coarse geometry.

DEFINITION 1.26. Fix a reference space X (often $X = B\Gamma$).

(1) Define $\mathrm{Pos}^{\mathrm{spin}}_n(X)$ as the set of singular bordism classes $(M, f \colon M \to X, g)$ of n-dimensional closed spin manifolds M together with a reference map f and a positive scalar curvature metric g on M. A bordism between $(M, f \colon M \to X, g)$ and $(M', f' \colon M' \to X, g')$ consists of a compact manifold with boundary W, with $\partial W = M \sqcup (-M')$, a reference map $F \colon W \to X$ restricting to f and f' on the boundary and a positive scalar curvature metric on W which has product structure near the boundary and restricts to g and g' on the boundary.

(2) We define $\Omega^{\mathrm{spin}}_{n+1}$ as the set of bordism classes (W, f, g) where W is a compact $(n+1)$-dimensional spin-manifold, possibly with boundary, with a reference map $f \colon W \to X$, and with a positive scalar curvature metric *on the boundary* when the latter is non-empty. Two triples $(W, f, g_{\partial W})$, $(W', f', g'_{\partial W})$ are bordant if there is a bordism with positive scalar curvature between the two boundaries, call it N, such that

$$Y := W \cup_{\partial W} N \cup_{-\partial W'} (-W')$$

is the boundary of a spin manifold Z. The reference maps to X have to extend over Z. By the surgery method for the construction of positive scalar curvature metrics, this set actually depends only on the fundamental group of X if X is connected, cf. [**36**, Section 5].

(3) Finally, $\Omega^{\mathrm{spin}}_n(X)$ is the usual singular spin bordism group of X.

PROPOSITION 1.27. *As a direct consequence of the definitions we get a long exact sequence, the* Stolz exact sequence

$$\longrightarrow \mathrm{Pos}^{\mathrm{spin}}_n(X) \longrightarrow \Omega^{\mathrm{spin}}_n(X) \longrightarrow R^{\mathrm{spin}}_n(X) \longrightarrow \mathrm{Pos}^{\mathrm{spin}}_{n-1}(X) \longrightarrow$$

with the obvious boundary or forgetful maps.

THEOREM 1.28. *For X a compact space with fundamental group Γ and universal covering $\tilde{X}$, there exists a well defined and commutative diagram, if n is odd,*

$$
\begin{array}{ccccccccc}
\longrightarrow & \Omega^{\mathrm{spin}}_{n+1}(X) & \longrightarrow & R^{\mathrm{spin}}_{n+1}(\Gamma) & \longrightarrow & \mathrm{Pos}^{\mathrm{spin}}_n(X) & \longrightarrow & \Omega^{\mathrm{spin}}_n(X) & \longrightarrow \\
& \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\mathrm{Ind}_\Gamma} & & \downarrow{\scriptstyle\rho_\Gamma} & & \downarrow{\scriptstyle\beta} & \\
\longrightarrow & K_{n+1}(X) & \longrightarrow & K_{n+1}(C^*_r\Gamma) & \longrightarrow & K_{n+1}(D^*(\tilde{X})^\Gamma) & \longrightarrow & K_n(X) & \longrightarrow.
\end{array}
$$
$$(1.14)$$

We also get a universal commutative diagram

$$\begin{array}{ccccccc}
\longrightarrow \Omega_{n+1}^{\mathrm{spin}}(B\Gamma) & \longrightarrow & R_{n+1}^{\mathrm{spin}}(B\Gamma) & \longrightarrow & \mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma) & \longrightarrow & \Omega_n^{\mathrm{spin}}(B\Gamma) \longrightarrow \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\mathrm{Ind}_\Gamma} & & \downarrow{\scriptstyle\rho_\Gamma} & & \downarrow{\scriptstyle\beta} \\
\longrightarrow K_{n+1}(B\Gamma) & \longrightarrow & K_{n+1}(C_r^*\Gamma) & \longrightarrow & K_{n+1}(D_\Gamma^*) & \longrightarrow & K_n(B\Gamma) \longrightarrow .
\end{array}$$

$$(1.15)$$

REMARK 1.29. As soon as the extension to arbitrary dimensions of our secondary index Theorem 1.14 has been carried out (as indicated in Remark 1.18), also Theorem 1.28 extends to arbitrary dimensions.

REMARK 1.30. In their seminal papers [**11**–**13**], Higson and Roe carry out a program similar to the one developed here: they construct a map from the surgery exact sequence in topology to exactly the same K-theory exact sequence showing up in (1.15) (with 2 inverted). Their construction is not quite as analytic as ours: it is not based on the index of the signature operator but rather on the manipulation of Poincaré duality complexes. In [**29**], we develop a direct analog of our index theoretic construction for the surgery exact sequence in topology. Note that the analysis is more difficult than the one developed here, given that the signature operator attached to an element of the structure set will not be invertible; similarly, the boundary-signature operator attached to an element of the L-groups of the fundamental group will not be invertible. One can use the homotopy equivalences built in the definition of the L-groups and the structure set in order to obtain a smoothing perturbation which makes the signature operator invertible, as in [**28**]. The issue is then to extend the constructions of index and rho-classes and the proofs of the secondary index theorems to this more general class of Dirac-type operators with smoothing perturbation (making the sum invertible).

We have completed this program in [**29**], reproving the main results of [**11**–**13**] with purely operator theoretic methods. The corresponding general index theorems should be useful in other contexts, as well.

REMARK 1.31. Beyond the extension of the method to the surgery exact sequence, a second goal for future work is to continue and map further from the K-theory exact sequence of (1.15) to a suitable exact sequence in cyclic (co)homology which should then allow one to obtain systematically numerical higher invariants. To achieve this, one has to overcome further analytic difficulties as the algebra D^*X is too large to allow for easy constructions of (higher) traces on it. Higson and Roe [**14**] carry out a small part of this program. They check that the pairing with the trace coming from a virtual representation of dimension zero (which gives rise to the APS rho invariant) is compatible with the K-theory exact sequence. It turns out that their construction of the relevant map on $K_*(D^*X)$ is very delicate. In [**42**], Wahl extends this to some more refined invariants, but working directly with the surgery exact sequences and its specific properties and, more importantly, mapping directly to cyclic homology, or rather non-commutative de-Rham homology, as was done in [**21**] for the Stolz sequence.

We now describe the structure of the rest of the paper. In Section 2, we review several alternative and previously used definitions of higher indices, in particular for manifolds with boundary, and check that they coincide with the approach via coarse C^*-algebras which we have described above (in the contexts where this makes sense). This puts 'coarse index theory'

in the context of usual index theory and allows us to use a few known properties of indices (like bordism invariance and gluing formulas) without having to prove them again in the coarse setting. In Section 3, we will work out basic properties of the K-theory of coarse C^*-algebras which we use in the course of the proofs. Section 4 finally is devoted to the proofs of the main Theorems 1.22 and 1.28, implementing the program set out above.

REMARK 1.32. After the first publication of the present paper in the arXiv, Xie and Yu, in the preprint [**45**], treated the problem with a different method. They use Yu's localization algebras and an exterior product structure between K-homology and the analytic structure group to reduce the proof of the main result of this paper to the known behavior of the K-homology fundamental class under the Mayer–Vietoris boundary map. Where for us the main difficulty lies in the explicit index calculation in the model situation, for them the main difficulty is the explicit calculation of certain exterior products which uses the full force of KK-theory. Their method does cover even and odd dimensions at the same time.

Moreover, also Paul Siegel announced a proof of the general case, along similar lines as Xie and Yu. In his PhD thesis, he develops a new model for K-homology and the structure set and develops an exterior product between those, and calculates the exterior product between a rho-class and a fundamental class. Paul Siegel has announced that he proved the compatibility between exterior product and Mayer–Vietoris. Again, this would lead to a proof of our main theorems in all dimensions, and would generalize to real K-theory.

2. Coarse, b- and APS index classes

The goal of this section is to give alternative descriptions of the relative coarse index class $\mathrm{Ind}^{\mathrm{rel}}(D_{W_\infty}) \in K_n(C^*(W \subset W_\infty)^\Gamma)$, $n = \dim W$, connecting it with classes that have already been defined in the literature. We also explain why we look at our index theorem as a delocalized APS index theorem.

2.1. Index classes in the closed case

First of all, we tackle the analogous problem in the boundaryless case. Thus, let V be a complete spin[†] Riemannian manifold with a free, isometric, cocompact spin structure preserving action of Γ. We denote the quotient V/Γ, a compact spin manifold without boundary, by Z; we thus get a Galois Γ-covering $V \xrightarrow{\pi} Z$. We denote the spinor bundles on V and Z by S_V and S_Z, respectively; S_V is Γ-equivariant and S_Z is obtained from S_V by passing to the quotient. There are five C^*-algebras which we consider:

(1) $C^*(V)^\Gamma$, the Roe algebra we have defined in Subsection 1.3;
(2) the C^*-algebra $C^*(G)$ defined by the groupoid G associated to the Γ-covering $\Gamma - V \to Z$; this is the groupoid with set of arrows $V \times_\Gamma V$, units Z and source and range maps defined by $s[v, v'] = \pi(v')$ and $r[v, v'] = \pi(v)$. We also consider the Morita equivalent C^*-algebra $C^*(G, S_V)$, which is defined by taking the closure of the algebra of smooth integral kernels $C_c^\infty(G, (s^* S_Z)^* \otimes r^* S_Z)$;
(3) the C^*-algebra of compact operators $\mathbb{K}(\mathcal{E})$ of the $C_r^*\Gamma$-Hilbert module $\mathcal{E}$ defined by taking the closure of the pre-Hilbert $\mathbb{C}\Gamma$-module $C_c^\infty(V, S_V)$;
(4) the C^*-algebra of compact operators $\mathbb{K}(\mathcal{E}_{\mathrm{MF}})$ of the Mishchenko–Fomenko $C_r^*\Gamma$-Hilbert module $\mathcal{E}_{\mathrm{MF}}$ which is, by definition, $L^2(Z, S_Z \otimes \mathcal{V}_{\mathrm{MF}})$; here $\mathcal{V}_{\mathrm{MF}}$ denotes the Mishchenko

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[†]Our arguments actually apply to any Dirac-type operator.

bundle: $\mathcal{V}_{\mathrm{MF}} := V \times_\Gamma C_r^*\Gamma$. We can also consider $\mathcal{E}_{\mathrm{MF}}^1 := H^1(Z, S_Z \otimes \mathcal{V}_{\mathrm{MF}})$, the first Sobolev $C_r^*\Gamma$-module.

(5) the reduced C^*-algebra of the group Γ: $C_r^*\Gamma$.

The relationships between these C^*-algebras are as follows:

$$C^*(V)^\Gamma = C^*(G, S) \simeq \mathbb{K}(\mathcal{E}) \simeq \mathbb{K}(\mathcal{E}_{\mathrm{MF}}) \simeq \mathcal{K} \otimes C_r^*\Gamma \tag{2.1}$$

where all the isomorphisms are canonical. The first equality, with $L^2(V, S_V)$ chosen as Hilbert $C_0(V)$-module for $C^*(V)^\Gamma$, follows from the inclusion

$$C_c^\infty(G, (s^*S_Z)^* \otimes r^*S_Z) \subset C_c(V)^\Gamma.$$

The second isomorphism is a special case of the corresponding result for foliated bundles in [**25**], and was well known before. The third isomorphism is induced by a canonical isomorphism of $\mathbb{C}\Gamma$-modules, $\psi \colon C_c^\infty(V, S_V) \to C_c^\infty(V, S_V \otimes \mathbb{C}\Gamma)^\Gamma \equiv C^\infty(Z, S_Z \otimes \mathcal{V}_{\mathrm{alg}})$ with $\mathcal{V}_{\mathrm{alg}} := V \times_\Gamma \mathbb{C}\Gamma$, see [**23**, Proposition 5] (it will suffice to replace $\mathcal{B}^\omega$ there with $\mathbb{C}\Gamma$). The last isomorphism is a consequence of the fact that $L^2(Z, S_Z \otimes \mathcal{V}_{\mathrm{MF}})$ is isomorphic to the standard $C_r^*\Gamma$-Hilbert module $\mathcal{H}_{C_r^*\Gamma}$.

From (2.1), we obtain

$$\begin{aligned} K_*(C^*(V)^\Gamma) = K_*(C^*(G, S)) &\simeq K_*(\mathbb{K}(\mathcal{E})) \simeq K_*(\mathbb{K}(\mathcal{E}_{\mathrm{MF}})) \\ &\simeq K_*(\mathcal{K} \otimes C_r^*\Gamma) \simeq K_*(C_r^*\Gamma), \end{aligned} \tag{2.2}$$

where all the isomorphisms are canonical.

The K-theory of these C^*-algebras are the home of different equivalent definition of the index class associated to the Dirac operator D_V. Let us recall these definitions in the even dimensional case:

(1) the coarse index class $\mathrm{Ind}^{\mathrm{coarse}}(D_V) \in K_0(C^*(V)^\Gamma)$ of Definition 1.11;

(2) the Connes–Skandalis index class $\mathrm{Ind}^{\mathrm{CS}}(D_V) \in K_0(C^*(G, S_V))$ defined via the Connes–Skandalis projector associated to a parametrix Q for D_V. Thus, we first choose Q, a Γ-compactly supported pseudodifferential operator of order (-1) so that

$$QD_V^+ = \mathrm{Id} - S_+, \quad D_V^+ Q = \mathrm{Id} - S_- \tag{2.3}$$

with remainders S_- and S_+ that are in $C_c^\infty(G, (s^*S_Z^\pm)^* \otimes r^*S_Z^\pm)$; then we consider $\mathrm{Ind}^{\mathrm{CS}}(D_V) := [P_Q] - [e_1] \in K_0(C^*(G, S_V))$, with

$$P_Q := \begin{pmatrix} S_+^2 & S_+(I + S_+)Q \\ S_- D_V^+ & I - S_-^2 \end{pmatrix}; \quad e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.4}$$

Here, we have in fact defined the compactly supported index class $\mathrm{Ind}_c^{\mathrm{CS}}(D_V) \in K_0(C_c^\infty(G, (s^*S_Z)^* \otimes r^*S_Z))$; the C^*-index class is the image of this class under the K-theory homomorphism induced by the inclusion $C_c^\infty(G, (s^*S_Z)^* \otimes r^*S_Z) \hookrightarrow C^*(G, S_V)$. There are other equivalent descriptions of this C^*-index class, such as the one defined by the Wassermann projector [**4**, p. 356] or the graph projector [**25**, Section 8]; these are obtained from parametrices that are of Sobolev order (-1) but are not compactly supported.

(3) the analogous index class $\mathrm{Ind}_{\mathrm{MF}}^{\mathrm{CS}}(D_V) \in K_0(\mathbb{K}(\mathcal{E}_{\mathrm{MF}}))$, defined via a Mishchenko–Fomenko parametrix $\mathcal{Q}$ for $\mathcal{D}^+$, with $\mathcal{D}^+$ equal to the Dirac operator on Z twisted by $\mathcal{V}_{\mathrm{MF}}$;

(4) the Mishchenko–Fomenko index class $\mathrm{Ind}_{\mathrm{MF}}(D_V) := [L_+] - [L_-] \in K_0(C_r^*\Gamma)$, defined via a Mishchenko–Fomenko decomposition induced by $\mathcal{D}^+$ on $\mathcal{E}_{\mathrm{MF}}^1$ and $\mathcal{E}_{\mathrm{MF}}$.

PROPOSITION 2.1. *Up to canonical K-theory isomorphisms we have*

$$\text{Ind}^{\text{coarse}}(D_V) = \text{Ind}^{\text{CS}}(D_V) = \text{Ind}^{\text{CS}}_{\text{MF}}(D_V) = \text{Ind}_{\text{MF}}(D_V). \tag{2.5}$$

Proof. The equality $\text{Ind}^{\text{CS}}(D_V) = \text{Ind}^{\text{CS}}_{\text{MF}}(D_V)$ follows from the second and third isomorphism in (2.1). A detailed proof appears in [**4**, Lemma 6.1], where it is actually proved that $\text{Ind}^{\text{CS}}_{\text{MF}}(D_V)$ is equal to the image of the compactly supported index class $\text{Ind}^{\text{CS}}_c(D_V)$ under the K-theory homomorphism induced by the inclusion $C^\infty_c(G, (s^*S_Z)^* \otimes r^*S_Z) \hookrightarrow C^*(G, S_V)$.

For the other equalities we make a preliminary remark. It is clear that the Connes–Skandalis index class of D_V is equal to the Connes–Skandalis index class of the bounded transform of D_V. This is true by general principles but can also be checked directly: consider $A = (1 + D_V^2)^{-1/2}D_V$ and $B = (1 + D_V^- D_V^+)^{1/2}Q$, which will be written shortly as $(1 + D_V^2)^{1/2}Q$. Then $A^+B = \text{Id} - R_-$ and $BA^+ = \text{Id} - R_+$, with $R_- = S_-$ and $R_+ = (1 + D_V^2)^{1/2}S_+(1 + D_V^2)^{-1/2}$. We can now write the Connes–Skandalis projector associated to A^+, B and $R_\pm$ and we call it P_B. This is homotopic to the Connes–Skandalis projector (2.4) (just consider $(1 + sD_V^2)^{\pm 1/2}$, with $s \in [0,1]$, throughout). Moreover, the K-theory class defined by P_B is nothing but $\partial[A^+]$, the index class associated to A^+ via the short exact sequence $0 \to \mathbb{K}(\mathcal{E}) \to \mathbb{B}(\mathcal{E}) \to \mathbb{B}(\mathcal{E})/\mathbb{K}(\mathcal{E}) \to 0$. Now, the same remark applies to $\text{Ind}^{\text{CS}}_{\text{MF}}(D_V)$ but for the isomorphic short exact sequence $0 \to \mathbb{K}(\mathcal{E}_{\text{MF}}) \to \mathbb{B}(\mathcal{E}_{\text{MF}}) \to \mathbb{B}(\mathcal{E}_{\text{MF}})/\mathbb{K}(\mathcal{E}_{\text{MF}}) \to 0$. We can now invoke the results in [**43**, Section 17], stating the equality of the latter index with the Mishchenko–Fomenko index of $(1 + \mathcal{D}^2)^{-1/2}\mathcal{D}^+$. Using the Mishchenko–Fomenko calculus the latter is easily seen to be the same as the Mishchenko–Fomenko index of $\mathcal{D}^+$, see for example [**37**, Theorem 6.22] for the details.

Summarizing: we have also proved that $\text{Ind}^{\text{CS}}_{\text{MF}}(D_V) = \text{Ind}_{\text{MF}}(D_V)$. It remains to show that $\text{Ind}^{\text{coarse}}(D_V) = \text{Ind}^{\text{CS}}(D_V)$. To this end we start with the expression of the Connes–Skandalis index class in terms of A^+, B, $R_\pm$

$$\text{Ind}^{\text{CS}}(D_V) = [P_B] - [e_1] \quad \text{with} \quad P_B = \begin{pmatrix} R_+^2 & R_+(I + R_+)B \\ R_-A^+ & I - R_-^2 \end{pmatrix}.$$

Consider now the function $\chi(x) := x/\sqrt{1 + x^2}$; this is a chopping function (so, it can be used to define the coarse index) and $\chi(D_V) = A$. The inverse of U^*A^+ in $D^*(V)^\Gamma/C^*(V)^\Gamma$ can be represented by $BU \in D^*(V)^\Gamma$: we observe that

$$(BU)(U^*A^+) = \text{Id} - R_+; \quad (U^*A^+)(BU) = U^*(\text{Id} - R_-)U.$$

We now write the expression of $\partial(U^*A^+)$ in terms of these choices and we obtain

$$\text{Ind}^{\text{coarse}}(D_V) = [\Pi_B] - [e_1] \quad \text{with} \quad \Pi_B = \begin{pmatrix} R_+^2 & R_+(I + R_+)BU \\ U^*R_-A^+ & U^*(I - R_-^2)U \end{pmatrix}.$$

Observing now that

$$\Pi_B = \begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} P_B \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix},$$

we conclude that $\text{Ind}^{\text{coarse}}(D_V) = \text{Ind}^{\text{CS}}(D_V)$, as required. $\qquad\square$

2.2. *Index classes on manifolds with boundary*

We now pass to manifolds with boundary and we adopt the notation explained in the Geometric set-up 1.13. We remark that the complete Riemannian spin manifold (W_∞, g) is in a natural way a Γ-covering of Y_∞, the manifold with cylindrical end associated to Y. We consider the groupoid $G_\infty := W_\infty \times_\Gamma W_\infty$ and the associated C^*-algebra $C^*(G_\infty)$, obtained by taking the closure of $C^\infty_c(G_\infty)$. Similarly, we can consider $C^*(G_\infty, S)$ with S the spinor bundle of W_∞.

Then, as in (2.1)

$$C^*(W \subset W_\infty)^\Gamma = C^*(G_\infty, S). \tag{2.6}$$

PROPOSITION 2.2. *Assume that W has dimension $n = 2\ell$ and that the boundary has positive scalar curvature. Then, using a b-parametrix, there is a well-defined b-index class associated to the spin Dirac operator on W_∞*

$$\mathrm{Ind}^b(D_{W_\infty}) \in K_0(C^*(W \subset W_\infty)^\Gamma).$$

Proof. This class is, by definition, the Connes–Skandalis projector associated to a suitable parametrix. We take the parametrix construction explained in [**26**] (which is directly inspired by the b-parametrix construction of Melrose [**24**]). The parametrix in [**26**] is explicitly proved to have remainders in $C^*(G_\infty, S)$, thus, by (2.6), in $C^*(W \subset W_\infty)^\Gamma$. □

Together with this b-index class we consider other K-theory classes:

(1) the Mishchenko–Fomenko b-index class $\mathrm{Ind}^b_{\mathrm{MF}}(D_{W_\infty}) \in K_0(C^*_r\Gamma)$, obtained from a Mishchenko–Fomenko decomposition theorem induced by a b-parametrix in the b-Mishchenko-Fomenko calculus on Y_∞, see [**19**] and the Appendix of [**20**];
(2) the index class $\mathrm{Ind}^{\mathrm{APS}}_{\mathrm{MF}}(D_W) \in K_0(C^*_r\Gamma)$ defined by a Mishchenko–Fomenko boundary value problem on Y à la APS, see [**44**];
(3) the conic index class $\mathrm{Ind}^{\mathrm{conic}}_{\mathrm{MF}}(D_W) \in K_0(C^*_r\Gamma)$ defined by a Mishchenko–Fomenko conic parametrix, see [**20**].

REMARK 2.3. Each one of these index classes has interesting features: $\mathrm{Ind}^b_{\mathrm{MF}}(D_{W_\infty})$ is the most suitable for proving higher index formulas (see the next subsection); the conic index class displays the most interesting stability properties (see [**18**], where these two properties are used in order to define higher signatures on manifolds with boundary and proving, under additional assumptions on Γ, their homotopy invariance). The APS index class, on the other hand, makes the study of gluing and cut-and-paste problems particularly easy.

PROPOSITION 2.4. *Up to natural K-theory isomorphisms the following equalities hold*:

$$\mathrm{Ind}^{\mathrm{rel}}(D_{W_\infty}) = \mathrm{Ind}^b(D_{W_\infty}) = \mathrm{Ind}^b_{\mathrm{MF}}(D_{W_\infty}) = \mathrm{Ind}^{\mathrm{APS}}_{\mathrm{MF}}(D_W) = \mathrm{Ind}^{\mathrm{conic}}_{\mathrm{MF}}(D_W).$$

Proof. The equalities $\mathrm{Ind}^b_{\mathrm{MF}}(D_{W_\infty}) = \mathrm{Ind}^{\mathrm{APS}}_{\mathrm{MF}}(D_W) = \mathrm{Ind}^{\mathrm{conic}}_{\mathrm{MF}}(D_W)$ are proved in [**18**] (the case treated here, with full invertibility of the boundary operator, is actually simpler than the one discussed in [**18**]). The proof that $\mathrm{Ind}^b(D_{W_\infty}) = \mathrm{Ind}^b_{\mathrm{MF}}(D_{W_\infty})$ is as in the closed case (hence, using the results in [**43**]). Thus, we only need to prove the first equality: $\mathrm{Ind}^{\mathrm{rel}}(D_{W_\infty}) = \mathrm{Ind}^b(D_{W_\infty})$. However, this follows once again from the reasoning given in the proof of Proposition 2.1. Indeed, the relative-coarse index class is defined in terms of $U^*\chi(D_V)_+$, with χ a chopping function equal to ± 1 on the spectrum of the boundary operator. We have claimed that $\chi(D)_- U$ is an inverse of $U^*\chi(D_V)_+$ modulo $C^*(W \subset W_\infty)^\Gamma$; now, in order to define $\partial[U^*\chi(D_V)_+]$ we can choose an arbitrary inverse in the quotient. Consider BU, with $B = (1 + D_V^2)^{1/2}Q$ and Q equal to a b-parametrix for D_V^+ as in [**26**]. It is easy to see, from the expression of Q, that BU is indeed an element in $D^*(W_\infty)^\Gamma$; moreover, since $S_\pm$, the remainders given by Q, are residual terms in the b-calculus, it follows that $R_\pm$ are also residual, so that BU is indeed an inverse of $U^*\chi(D_V)_+$ mod $C^*(G_\infty, S)$, that is, mod $C^*(W \subset W_\infty)^\Gamma$. The proof now proceeds as in the closed case. □

2.3. *Delocalizing*

The goal of this subsection is to explain why we look at our index formula (1.10)

$$\iota_*(\mathrm{Ind}_\Gamma(D_W)) = \rho_\Gamma(g_{\partial W}) \in K_0(D_\Gamma^*)$$

as an equality between a *delocalized* part of the index class and the rho-class of the boundary operator. This is not needed in the sequel, but meant to put our considerations in the context of previously established index theorems.

To this end we recall the higher APS-index formula proved in [**19**, Theorem 14.1]. This is a formula for the Karoubi Chern-character of the index class $\mathrm{Ind}^b(D_{W_\infty})$, an element in the non-commutative de Rham homology $\bar{H}_*(\mathcal{B}^\infty)$ ($\mathcal{B}^\infty$ is a suitable dense holomorphically closed subalgebra of $C_r^*\Gamma$, for example the Connes–Moscovici algebra). The formula reads

$$\mathrm{Ch}(\mathrm{Ind}^b_{\mathrm{MF}}(D_{W_\infty})) = \left[\int_Y \hat{A}(Y, \nabla^Y) \wedge \omega - \frac{1}{2}\tilde{\eta}(D_{\partial W}) \right] \quad \text{in } \bar{H}_*(\mathcal{B}^\infty), \qquad (2.7)$$

where $Y = W/\Gamma$, ω is a certain bi-form in $\Omega^*(Y) \otimes \Omega_*(\mathbb{C}\Gamma)$ and where

$$\tilde{\eta}(D_{\partial W}) \in \bar{\Omega}_*(\mathcal{B}^\infty) := \Omega_*(\mathcal{B}^\infty)/\overline{[\Omega_*(\mathcal{B}^\infty), \Omega_*(\mathcal{B}^\infty)]}$$

is Lott's higher eta invariant of the boundary operator [**22**], an invariant which is well defined for any L^2-invertible Dirac operator D_V on the total space of a boundaryless Galois Γ-covering V with base Z. Assume now, for simplicity, that Γ is *virtually nilpotent*. Under this additional assumption, the complex $\bar{\Omega}_*(\mathcal{B}^\infty)$ splits as the direct sum of sub-complexes labeled by the conjugacy classes of Γ. Write $\langle\Gamma\rangle$ for the set of conjugacy classes. Thus $\tilde{\eta}(D_V)$ splits as a direct sum

$$\tilde{\eta}(D_V) = \bigoplus_{\langle x \rangle \in \langle \Gamma \rangle} \tilde{\eta}_{\langle x \rangle}(D_V).$$

The higher ρ-invariant of Lott [**22**] associated to D_V is, by definition,

$$\tilde{\rho}(D_V) := \bigoplus_{\langle x \rangle \neq \langle e \rangle} \tilde{\eta}_{\langle x \rangle}(D_V).$$

As pointed out in [**22**, p. 222], the higher ρ-invariant lies in fact in $\bar{H}_*(\mathcal{B}^\infty)$. Note that $\bar{H}_*(\mathcal{B}^\infty)$ splits as the direct sum $\bar{H}_*(\mathcal{B}^\infty) = \bar{H}_{\langle e \rangle, *}(\mathcal{B}^\infty) \oplus \bar{H}_*^{\mathrm{deloc}}(\mathcal{B}^\infty)$ with the first group on the right-hand side associated to the subcomplex of $\bar{\Omega}_*(\mathcal{B}^\infty)$ labeled by the trivial conjugacy class $\langle e \rangle$ and the second group associated to the subcomplexes labeled by the non-trivial conjugacy classes; $\bar{H}_*^{\mathrm{deloc}}(\mathcal{B}^\infty)$ is thus the *delocalized* part of $\bar{H}_*(\mathcal{B}^\infty)$. Then $\tilde{\rho}(D_V)$ lies in $\bar{H}_*^{\mathrm{deloc}}(\mathcal{B}^\infty)$. We let π^{deloc} be the natural projection map. Then on the basis of the higher APS index formula (2.7) one proves easily that

$$\pi^{\mathrm{deloc}}(\mathrm{Ch} \circ \mathrm{Ind}^b_{MF}(D_{W_\infty})) = -\tfrac{1}{2}\tilde{\rho}(D_{\partial W}). \qquad (2.8)$$

Thus we see that Lott's higher rho invariant corresponds to the delocalized part of the Chern character of the index class. We regard our equation $\iota_*(\mathrm{Ind}_\Gamma(D_W)) = \rho_\Gamma(g_{\partial W})$ as a sharpening in K-theory of (2.8).

Finally, it is proved in [**21**] that the higher rho-invariant induces a well-defined group homomorphism

$$\hat{\rho} : \mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma) \longrightarrow \bar{H}_*^{\mathrm{deloc}}(\mathcal{B}^\infty) \qquad (2.9)$$

with $\hat{\rho}[Z, u\colon Z \to B\Gamma, g_Z] = -\frac{1}{2}\tilde{\rho}(D_V)$, $V = u^*E\Gamma$, and $* = $ odd if $n = 2\ell$ and $* = $ even is $n = 2\ell + 1$. Together with (2.8) this shows that the following diagram is commutative:

$$
\begin{array}{ccc}
R_{n+1}^{\mathrm{spin}}(B\Gamma) & \longrightarrow & \mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma) \\
\downarrow{\scriptstyle\mathrm{ChoInd}_\Gamma} & & \downarrow{\scriptstyle\hat{\rho}} \\
\bar{H}_*(\mathcal{B}^\infty) & \longrightarrow & \bar{H}_*^{\mathrm{deloc}}(\mathcal{B}^\infty).
\end{array}
\tag{2.10}
$$

In future work, we plan to tackle the problem of defining a group homomorphism $\mathrm{Ch}_{\mathrm{deloc}}\colon K_*(D_\Gamma^*) \to \bar{H}_*^{\mathrm{deloc}}(\mathcal{B}^\infty)$ so that the following double diagram is commutative and the composition of the two vertical arrows on the right-hand side is precisely $\hat{\rho}$ in (2.9):

$$
\begin{array}{ccc}
R_{n+1}^{\mathrm{spin}}(B\Gamma) & \longrightarrow & \mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma) \\
\downarrow{\scriptstyle\mathrm{Ind}_\Gamma} & & \downarrow{\scriptstyle\rho} \\
K_{n+1}(C_r^*\Gamma) & \longrightarrow & K_{n+1}((D_\Gamma^*)) \\
\downarrow{\scriptstyle\mathrm{Ch}} & & \downarrow{\scriptstyle\mathrm{Ch}_{\mathrm{deloc}}} \\
\bar{H}_*(\mathcal{B}^\infty) & \longrightarrow & \bar{H}_*^{\mathrm{deloc}}(\mathcal{B}^\infty).
\end{array}
$$

In fact, we hope to map the whole Higson–Roe surgery sequence to a sequence in non-commutative de Rham homology.

3. K-theory homomorphisms

3.1. Geometrically induced homomorphisms

Let W, W_∞, $M = \partial W$ be as in the Geometric set-up 1.13.

In the following, with slight abuse of notation we will denote all inclusions $C^*(X) \to D^*(X)$ (and their equivariant and relative versions) by ι and the induced map in K-theory by ι_*.

PROPOSITION 3.1. *We have commutative diagrams*

$$
\begin{array}{ccc}
K_*(C^*(\partial W)^\Gamma) & \xrightarrow{\ \iota_*\ } & K_*(D^*(\partial W)^\Gamma) \\
\downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\
K_*(C^*(W)^\Gamma) & \xrightarrow{\ \iota_*\ } & K_*(D^*(W)^\Gamma) \\
{\scriptstyle c}\downarrow{\scriptstyle\cong} & & {\scriptstyle c}\downarrow{\scriptstyle\cong} \\
K_*(C^*(W \subset W_\infty)^\Gamma) & \xrightarrow{\ \iota_*\ } & K_*(D^*(W \subset W_\infty)^\Gamma),
\end{array}
\tag{3.1}
$$

$$
\begin{array}{ccc}
K_*(D^*(\partial W)^\Gamma) & \longrightarrow & K_*(D^*(W)^\Gamma) \\
\cong\downarrow{\scriptstyle j_\partial} & & \cong\downarrow{\scriptstyle c} \\
K_*(D^*(\partial W \subset [0,\infty) \times \partial W)^\Gamma) & \xrightarrow{\ j_+\ } & K_*(D^*(W \subset W_\infty)^\Gamma)
\end{array}
\tag{3.2}
$$

with j_+ and j_∂ induced by the natural inclusions.

Proof. To construct the maps $C^*(\partial W)^\Gamma \to C^*(W)^\Gamma$ and $D^*(\partial W)^\Gamma \to D^*(W)^\Gamma$, we can and will use the *same* isometry covering the inclusion $j\colon \partial W \to W$. Then all the maps are induced

by inclusions of algebras and therefore the commutativity follows from naturality of K-theory. The isomorphism claims in the statement have already been discussed. $\qquad\square$

3.2. *Kasparov lemma*

The following lemma, stated for the first time in [**17**, Proposition 3.4], see also [**7**, Lemma 7.2], is a useful tool in proving pseudolocality.

LEMMA 3.2 (Kasparov lemma). *Let H be an adequate X-module. A bounded operator $A\colon H \to H$ is pseudo-local if and only if $\psi A\phi$ is compact whenever ψ and ϕ are bounded continuous functions on X such that the supports $\mathrm{supp}(\psi)$ and $\mathrm{supp}(\phi)$ are disjoint and at least one of them is compact.*

If A is a norm limit of operators of bounded propagation, then it is sufficient to consider only functions of compact support.

Proof. A proof of the first statement for the case that X is compact is given in [**10**, 5.4.7]. It directly covers the general case, as well.

The second statement about finite propagation operators is already remarked in [**7**, footnote 6]. To prove it, it suffices (by a limit argument) to assume that A has bounded propagation R. Then, given ϕ, ψ with ϕ of compact support and such that $\phi\psi = 0$, write $\psi = \psi_1 + \psi_2$ such that ψ_1 has compact support and ψ_2 has support of distance R from ϕ. Then by the bounded propagation property, $\phi A\psi_2 = 0$, and by assumption $\phi A\psi_1$ is compact, so also $\phi A\psi$ is compact and the assumptions of the usual form of the Kasparov lemma are fulfilled. $\qquad\square$

3.3. *The Mayer–Vietoris sequence*

Assume that $X = X_1 \cup X_2$ is a Riemannian manifold (typically non-compact), decomposed into two closed subsets X_1, X_2, with $X_0 := X_1 \cap X_2$. (The more general case of metric spaces is treated in exactly the same way.) We make the following *excision assumption*: $X_0 := X_1 \cap X_2$ is big enough in the following sense (of Higson, Roe and Yu [**15**]): for each $R > 0$ there is $S > 0$ such that $U_R(X_1) \cap U_R(X_2) \subset U_S(X_0)$.

Using along the way the relative Roe-algebras for $X_i \subset X$ and Lemma 1.8, one finally gets the expected commuting diagram of 6-terms exact Mayer–Vietoris sequences [**39**, Section 3]

$$
\begin{array}{ccccccc}
\ldots \longrightarrow K_0(C^*(X_1)) \oplus K_0(C^*(X_2)) & \longrightarrow & K_0(C^*(X)) & \xrightarrow{\delta_{\mathrm{MV}}} & K_1(C^*(X_1 \cap X_2)) \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
\ldots \longrightarrow K_0(D^*(X_1)) \oplus K_0(D^*(X_2)) & \longrightarrow & K_0(D^*(X)) & \xrightarrow{\delta_{\mathrm{MV}}} & K_1(D^*(X_1 \cap X_2)) \longrightarrow.
\end{array}
\tag{3.3}
$$

Exactly the same works for the Γ-equivariant versions.

3.4. *Mayer–Vietoris for the cylinder*

Let M be a Galois Γ-cover of a compact manifold Z and consider the cylinder $X := \mathbb{R} \times M$, with Γ acting in a trivial way on $\mathbb{R}$. We set $X_1 = (-\infty, 0] \times M$ and $X_2 = [0, \infty) \times M$ so that $X_0 = \{0\} \times M = M$. This decomposition of the cylinder clearly satisfies the excision axiom; thus, we have the commuting diagram of long exact sequence for C^* and D^*

$$\ldots \longrightarrow K_0(C^*((-\infty,0] \times M)^\Gamma) \longrightarrow K_0(C^*(\mathbb{R} \times M)^\Gamma) \xrightarrow{\delta_{\mathrm{MV}}} K_1(C^*(M)^\Gamma) \longrightarrow \ldots$$

$$\oplus$$

$$K_0(C^*([0,\infty) \times M)^\Gamma)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\ldots \longrightarrow K_0(D^*((-\infty,0] \times M)^\Gamma)$$

$$\oplus$$

$$K_0(D^*([0,\infty) \times M)^\Gamma) \longrightarrow K_0(D^*(\mathbb{R} \times M)^\Gamma) \xrightarrow{\delta_{\mathrm{MV}}} K_1(D^*(M)^\Gamma) \longrightarrow \ldots$$

$$\tag{3.4}$$

LEMMA 3.3. *For any metric space M with isometric Γ-action*
$$K_*(C^*([0,\infty) \times M)^\Gamma) = 0; \quad K_*(D^*([0,\infty) \times M)^\Gamma) = 0.$$

Proof.   This is a special instance of a general principle introduced by Roe: any space of the form $[0,\infty) \times M$ is *flasque* in the sense of [**33**, Definition 9.3], and then by [**33**, Proposition 9.4] the K-theory of $C^*([0,\infty) \times M)$ vanishes.

The argument given there, based on an Eilenberg swindle, word by word applies also to the Γ-invariant subalgebras and to D^*. $\square$

In particular, we conclude that the boundary maps in the Mayer–Vietoris sequence induce compatible isomorphisms

$$\begin{array}{ccc} K_*(C^*(\mathbb{R} \times M)^\Gamma) & \xrightarrow{\ \delta_{\mathrm{MV}}\ }_{\simeq} & K_{*+1}(C^*(M)^\Gamma) \\ \downarrow & & \downarrow \\ K_*(D^*(\mathbb{R} \times M)^\Gamma) & \xrightarrow{\ \delta_{\mathrm{MV}}\ }_{\simeq} & K_{*+1}(D^*(M)^\Gamma). \end{array} \tag{3.5}$$

REMARK 3.4.   The maps δ_{MV} of (3.5) are given as follows: take a representative $\alpha \in C^*(\mathbb{R} \times M)^\Gamma$ of the K-theory class, that is, either a projector or an invertible (using that the algebra is stable). One now has to trace the definition of the splicing argument which gives rise to (3.3): map α to $C^*(\mathbb{R} \times M)^\Gamma / C^*((-\infty,0] \times M \subset \mathbb{R} \times M)^\Gamma$. Then we lift it (through the inverse of the natural isomorphism) to $C^*([0,\infty) \times M)^\Gamma / C^*(\{0\} \times M \subset [0,\infty) \times M)^\Gamma$. For this, let ψ_+ be the characteristic function of $[0,\infty) \times M$, then the compression $\psi_+ \alpha \psi_+$ (where ψ_+ acts as multiplication operator) is such a lift. That it has the relevant properties follows, as proved in Lemmas 4.6 and 4.4. Finally, $\delta_{MV}(\alpha) = c^{-1}\delta([\psi_+\alpha\psi_+])$ where δ is the boundary homomorphism of the K-theory long exact sequence for $C^*(\{0\} \times M \subset [0,\infty) \times M)^\Gamma \hookrightarrow C^*([0,\infty) \times M)^\Gamma$ and c is the isomorphism of Lemma 1.8.

Exactly the same construction works for D^*.

4. *Proofs of the main theorems*

The goal of this section is to provide a proof of our two main theorems. We shall begin by stating a key result, the 'cylinder delocalized index theorem'. This result is the cornerstone for the proof of both theorems. We state the cylinder delocalized index theorem in Subsection 4.1, but we defer the (quite technical) proof to Subsection 4.4. Next we explain how the

cylinder delocalized index theorem can be employed in order to prove both theorems. We do this in Subsections 4.2 and 4.3. Finally, as anticipated, we give a detailed proof of the cylinder delocalized index theorem in Subsection 4.4.

4.1. *The cylinder delocalized index theorem*

NOTATION 4.1. We let M be a boundaryless manifold with a free, isometric and cocompact action of Γ. We assume that M is endowed with a Γ-invariant metric of positive scalar curvature. We assume M to be of dimension n, with n odd. We shall consider $\mathbb{R} \times M$, $\mathbb{R}_{\geqslant} \times M$, $\mathbb{R}_{\leqslant} \times M$. We consider the Dirac operators

$$D_M \text{ on } M \text{ and } D_{\text{cyl}} \text{ on } \mathbb{R} \times M.$$

We shall also employ the notation $D_{\mathbb{R} \times M}$ for D_{cyl}.

The positive scalar curvature assumption on M implies that D_M is L^2-invertible; thus there is a well-defined ρ-class $\rho(D_M) \in K_0(D^*(M)^\Gamma)$. Also D_{cyl} is L^2-invertible; hence $\chi(D_{\text{cyl}})$, with χ a suitable chopping function, is an involution. This means that there is a well-defined ρ-class on the cylinder: $\rho(D_{\mathbb{R} \times M}) \in K_1(D^*(\mathbb{R} \times M)^\Gamma)$. We know by (3.5) that for the cylinder $\mathbb{R} \times M = (\mathbb{R}_{\leqslant} \times M) \cup_M (\mathbb{R}_{\geqslant} \times M)$ there is a well-defined Mayer–Vietoris isomorphism

$$\delta_{\text{MV}} : K_1(D^*(\mathbb{R} \times M)^\Gamma) \longrightarrow K_0(D^*(M)^\Gamma).$$

Thus, it makes sense to consider $\delta_{\text{MV}}(\rho(D_{\mathbb{R} \times M})) \in K_0(D^*(M)^\Gamma)$. The following result will be crucial:

THEOREM 4.2 (Cylinder delocalized index theorem).

$$\delta_{\text{MV}}(\rho(D_{\mathbb{R} \times M})) = \rho(D_M) \quad \text{in } K_0(D^*(M)^\Gamma). \tag{4.1}$$

4.2. *Proof of the delocalized APS index theorem assuming Theorem 4.2*

In this subsection, we make use of the fundamental identity on the cylinder, (4.1), in order to give a proof of Theorem 1.14, the delocalized APS index theorem.

NOTATION 4.3. We consider W, ∂W and W_∞ as in the Geometric set-up 1.13. We also consider

$$\mathbb{R} \times \partial W, \quad \mathbb{R}_{\geqslant} \times \partial W, \quad \mathbb{R}_{\leqslant} \times \partial W.$$

We consider the Dirac operators

$$D \text{ on } W_\infty, \quad D_\partial \text{ on } \partial W \quad \text{and} \quad D_{\text{cyl}} \text{ on } \mathbb{R} \times \partial W.$$

Recall that the boundary ∂W is endowed with a metric of positive scalar curvature, so that the coarse index class of D is well defined as an element $\text{Ind}^{\text{rel}}(D) \in K_0(C^*(W \subset W_\infty)^\Gamma)$. The positive scalar curvature assumption implies that also D_{cyl} is L^2-invertible; hence $\chi(D_{\text{cyl}})$ is an involution, too.

We denote by ψ the characteristic function of $[0, \infty) \times \partial W$ on W_∞ and by ψ_+ the corresponding characteristic function on $\mathbb{R} \times \partial W$.

Consider the operator $\psi_+ \chi(D_{\text{cyl}}) \psi_+$ on $\mathbb{R}_{\geqslant} \times \partial W$; obviously, this is not an involution any more. Similarly, consider the operator $\psi \chi(D_{\text{cyl}}) \psi$ on W_∞, which also fails to be an involution.

We start with a basic lemma about commutators with ψ_+.

LEMMA 4.4. *If $T \in D^*(\mathbb{R} \times \partial W)^\Gamma$, then $[T, \psi_+]$ is in $D^*(\partial W \subset \mathbb{R} \times \partial W)^\Gamma$, and correspondingly if $T \in D^*(W_\infty)^\Gamma$, then $[T, \psi] \in D^*(W \subset W_\infty)^\Gamma$.*

Proof. We follow the proof of [**33**, Lemma 4.3]. We already know that $[T, \psi_+]$ belongs to $D^*(\mathbb{R} \times \partial W)^\Gamma$ and we only need to show that it lies actually in the ideal $D^*(\partial W \subset \mathbb{R} \times \partial W)^\Gamma$. We can assume that T has finite propagation R. Then, outside a sufficiently large neighborhood of the support of ψ_+, $[T, \psi_+]$ is zero, because there ψ_+ acts as the identity. It follows that it is compactly supported in the $\mathbb{R}$ direction, as desired.

Finally, given $\phi \in C_c((0, \infty) \times \partial W)$, we have to show that $[T, \psi_+]\phi$ is compact. But

$$[T, \psi_+]\phi = T\phi - \psi_+ T\phi = (1 - \psi_+)T\phi.$$

Because of finite propagation of T we can replace $(1 - \psi_+)$ by $(1 - \psi_+)\alpha$ where α has compact support. Then, as $(1 - \psi_+)\alpha\phi = 0$, by the pseudolocality of T this operator $(1 - \psi_+)T\phi = (1 - \psi_+)\alpha T\phi$ indeed is compact. □

REMARK 4.5. The first part of Lemma 4.6 holds unchanged if we consider, more generally, a boundaryless manifold M with a free, isometric and cocompact action of Γ and endowed with a Γ-invariant metric of positive scalar curvature. In this case $\psi_+\chi(D_{\mathrm{cyl}})\psi_+$ is an involution in $D^*(\mathbb{R}_\geqslant \times M)^\Gamma / D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma$.

LEMMA 4.6. $\psi_+\chi(D_{\mathrm{cyl}})\psi_+$ *is an involution in* $D^*(\mathbb{R}_\geqslant \times \partial W)^\Gamma / D^*(\partial W \subset \mathbb{R}_\geqslant \times \partial W)^\Gamma$, *where we write briefly $D^*(\partial W \subset \mathbb{R}_\geqslant \times \partial W)^\Gamma$ instead of $D^*(\{0\} \times \partial W \subset \mathbb{R}_\geqslant \times \partial W)^\Gamma$. Similarly, $\psi\chi(D_{\mathrm{cyl}})\psi$ is an involution in $D^*(W_\infty)^\Gamma / D^*(W \subset W_\infty)^\Gamma$.*

Proof. We choose χ such that $\chi(D_{\mathrm{cyl}})^2 = 1$, this is possible because 0 is not in the spectrum of D_{cyl} by the positive scalar curvature assumption. Then, using that $\psi_+^2 = \psi_+$

$$(\psi_+\chi(D_{\mathrm{cyl}})\psi_+)^2 = \psi_+\chi(D_{\mathrm{cyl}})^2\psi_+ + \psi_+\chi(D_{\mathrm{cyl}})[\psi_+, \chi(D_{\mathrm{cyl}})]\psi_+$$
$$= 1 + (\psi_+ - 1) + \psi_+\chi(D_{\mathrm{cyl}})[\psi_+, \chi(D_{\mathrm{cyl}})]\psi_+.$$

Observe that the second and the third operator are both in $D^*(W_\infty)^\Gamma$. Note that, on $[0, \infty) \times \partial W$, $\psi_+ - 1 = 0$. On W_∞, the corresponding $\psi - 1$ is the negative of the characteristic function of W, so has propagation 0 and vanishes identically on $[0, \infty) \times \partial W$, therefore $(\psi - 1) \in D^*(W \subset W_\infty)^\Gamma$. Using Lemma 4.4, we see that also the third summand belongs to $D^*(W \subset W_\infty)^\Gamma$ or $D^*(\partial W \subset [0, \infty) \times \partial W)^\Gamma$, so the statement follows. □

DEFINITION 4.7. Let $n + 1$, the dimension of W, be even.

Consider the half cylinder $\mathbb{R}_\geqslant \times \partial W$; observe that the spinor bundle is in this case the pullback of the direct sum of two copies of the spinor bundle on ∂W. Thus, in this case, we could choose U to be the identity. Using Lemma 4.6, we can define the class

$$[U^*(\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+)] \in K_1(D^*(\mathbb{R}_\geqslant \times \partial W)^\Gamma / D^*(\partial W \subset \mathbb{R}_\geqslant \times \partial W)^\Gamma) \qquad (4.2)$$

and thus, applying the boundary map for the obvious 6-terms long exact sequence

$$\partial\colon K_1(D^*(\mathbb{R}_\geqslant \times \partial W)^\Gamma / D^*(\partial W \subset \mathbb{R}_\geqslant \times \partial W)^\Gamma) \longrightarrow K_0(D^*(\partial W \subset \mathbb{R}_\geqslant \times \partial W)^\Gamma), \qquad (4.3)$$

we obtain a class

$$\partial[U^*(\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+)] \in K_0(D^*(\partial W \subset \mathbb{R}_\geqslant \times \partial W)^\Gamma). \qquad (4.4)$$

Similarly, if $n + 1$ is odd, then we have a well-defined class

$$\partial(\tfrac{1}{2}[1 + \psi_+\chi(D_{\mathrm{cyl}})\psi_+]) \in K_1(D^*(\partial W \subset \mathbb{R}_\geqslant \times \partial W)^\Gamma). \tag{4.5}$$

REMARK 4.8.   There is a corresponding statement, obtained by replacing ∂W by a general M with positive scalar curvature. In particular, if M is odd dimensional and if we choose U to be the identity, then this gives classes

$$[\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+] \in K_1(D^*(\mathbb{R}_\geqslant \times M)^\Gamma/D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma) \tag{4.6}$$

and

$$\partial[\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+] \in K_0(D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma). \tag{4.7}$$

REMARK 4.9.   We can restate Lemma 4.6, and its obvious extensions in Remarks 4.5 and 4.8 in a more conceptual way. Indeed, exactly the same proof establishes the following statements:
   If $n + 1$ is even, then *compression by χ_+ gives a well-defined homomorphism*

$$K_1(D^*(\mathbb{R} \times M)^\Gamma) \longrightarrow K_1(D^*(\mathbb{R}_\geqslant \times M)^\Gamma/D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma),$$

which sends the ρ-class defined by $D_{\mathbb{R}\times M}$ to the class (4.6).
Therefore, composition of this homomorphism with the boundary map

$$\partial\colon K_1(D^*(\mathbb{R}_\geqslant \times M)^\Gamma/D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma) \longrightarrow K_0(D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma)$$

gives a homomorphism

$$H\colon K_1(D^*(\mathbb{R} \times M)^\Gamma) \longrightarrow K_0(D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma).$$

Further composing with the inverse of the isomorphism

$$j_M \colon K_0(D^*(M)^\Gamma) \longrightarrow K_0(D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma)$$

induced by the inclusion (it is the analog of (3.2)) gives finally a well-defined homomorphism:

$$K_1(D^*(\mathbb{R} \times M)^\Gamma) \longrightarrow K_0(D^*(M)^\Gamma) \tag{4.8}$$

and this homomorphism sends $\rho(D_{\mathbb{R}\times M})$ into $j_M^{-1}\partial[\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+]$. It is not difficult to show, proceeding exactly as in Remark 3.4, that the homomorphism (4.8) is precisely the Mayer–Vietoris homomorphism δ_{MV} we have described in Subsection 3.4; in particular, by the argument given in Subsection 3.4, the homomorphism (4.8) is an isomorphism. Moreover, by the above remarks, the following identity holds in $K_0(D^*(M)^\Gamma)$:

$$\delta_{\mathrm{MV}}(\rho(D_{\mathbb{R}\times M})) = j_M^{-1}\partial[\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+]. \tag{4.9}$$

The corresponding statement holds in the odd dimensional case.

We now go back to the manifold with cylindrical ends W_∞. Then, by the second part of Lemma 4.6 we have, in the even dimensional case,

$$[U^*(\psi\chi(D_{\mathrm{cyl}})_+\psi)] \in K_1(D^*(W_\infty)^\Gamma/D^*(W \subset W_\infty)^\Gamma)$$

and thus applying the boundary map

$$\partial\colon K_1(D^*(W_\infty)^\Gamma/D^*(W \subset W_\infty)^\Gamma) \longrightarrow K_0(D^*(W \subset W_\infty)^\Gamma),$$

we obtain an element

$$\partial[U^*(\psi\chi(D_{\mathrm{cyl}})_+\psi)] \in K_0(D^*(W \subset W_\infty)^\Gamma). \tag{4.10}$$

In the odd-dimensional case we have a corresponding class

$$\partial(\tfrac{1}{2}[1 + \psi\chi(D_{\mathrm{cyl}})\psi]) \in K_1(D^*(W \subset W_\infty)^\Gamma). \tag{4.11}$$

LEMMA 4.10. $\chi(D) - \psi\chi(D_{\mathrm{cyl}})\psi \in D^*(W \subset W_\infty)^\Gamma$.

Proof.   This is a consequence of the proof of [**32**, Proposition 1.5]. We have to show two things:

   (i) if $\phi \in C_0(W_\infty)$ with $d(W, \mathrm{supp}(\phi)) > \epsilon > 0$, then $(\chi(D) - \psi\chi(D_{\mathrm{cyl}}\psi))\phi$ is compact;
   (ii) for each $\epsilon > 0$ there is $R > 0$ such that $\|(\chi(D) - \psi\chi(D_{\mathrm{cyl}}\psi))\phi\| < \epsilon$ whenever $\phi \in C_0(W_\infty)$ with $d(W, \mathrm{supp}(\phi)) > R$.

Write the distribution $\hat\chi(t) = \alpha(t) + \beta(t)$ such that α is smooth and rapidly decreasing and β is a distribution supported in $(-\epsilon, \epsilon)$. Then $\chi(D) = \int_{\mathbb{R}} \alpha(t)e^{itD} + \int_{-\epsilon}^{\epsilon} \beta(t)e^{itD}$. By unit propagation and isometry invariance of the wave operator, $e^{itD}\phi = e^{itD_{\mathrm{cyl}}}\phi$ for $t \leqslant R$ if $d(W, \mathrm{supp}(\phi)) > R$. Therefore, in this situation

$$(\chi(D) - \psi\chi(D_{\mathrm{cyl}}))\phi = \int_{|t| \geqslant R} \alpha(t)(e^{itD} - \psi e^{itD_{\mathrm{cyl}}})\phi$$
$$\implies \|(\chi(D) - \psi\chi(D_{\mathrm{cyl}}))\phi\| \leqslant 2 \sup_{|t| \geqslant R} |\alpha(t)|.$$

As α is smooth and rapidly decreasing, the operator belongs to $C^*(W_\infty)^\Gamma$ and therefore is compact for each $R > 0$, and the norm converges to 0 as $R \to \infty$. This establishes the two properties. $\square$

LEMMA 4.11.   *Let $\iota_*\colon K_*(C^*(W \subset W_\infty)^\Gamma) \longrightarrow K_*(D^*(W \subset W_\infty)^\Gamma)$ be the homomorphism induced by the inclusion (we considered ι_* in Lemma 3.1). If $n + 1$ is even, then*

$$\iota_*(\mathrm{Ind}^{\mathrm{rel}}(D)) = \partial[U^*(\psi\chi(D_{\mathrm{cyl}})_+\psi)] \quad \text{in } K_0(D^*(W \subset W_\infty)^\Gamma). \tag{4.12}$$

If $n + 1$ is odd, then

$$\iota_*(\mathrm{Ind}^{\mathrm{rel}}(D)) = \partial[\tfrac{1}{2}(1 + \psi\chi(D_{\mathrm{cyl}})\psi)] \quad \text{in } K_1(D^*(W \subset W_\infty)^\Gamma). \tag{4.13}$$

Proof.   Let us prove the case in which $n + 1$ is even. Recall the inclusion of the ideal $C^*(W \subset W_\infty)^\Gamma \subset D^*(W \subset W_\infty)^\Gamma$. Using an obvious commutative diagram we see that the left-hand side of (4.12) is nothing but the boundary map applied to the involution

$$[U^*\chi(D)_+] \in K_1(D^*(W_\infty)^\Gamma/D^*(W \subset W_\infty)^\Gamma).$$

The lemma follows immediately from Lemma 4.10. The odd case is similar. $\square$

For the next lemma and the following proposition recall the homomorphisms j_+ and j_∂ appearing in Proposition 3.1.

LEMMA 4.12.   *If $n + 1$ is even, then*

$$\partial[U^*(\psi\chi(D_{\mathrm{cyl}})_+\psi)] = j_+(\partial[U^*(\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+)]) \quad \text{in } K_{n+1}(D^*(W \subset W_\infty)^\Gamma). \tag{4.14}$$

If $n + 1$ is odd

$$\partial[\tfrac{1}{2}(1 + \psi\chi(D_{\mathrm{cyl}})\psi)] = j_+(\partial[\tfrac{1}{2}(1 + \psi_+\chi(D_{\mathrm{cyl}})\psi_+)]) \quad \text{in } K_{n+1}(D^*(W \subset W_\infty)^\Gamma. \tag{4.15}$$

Proof. We only prove (4.14), the other statements can be derived similarly. We choose as adequate modules on $\mathbb{R}_{\geqslant} \times \partial W$ and on W_∞ the L^2-sections of the corresponding spinor bundles. Observe now that the adequate module for $\mathbb{R}_{\geqslant} \times \partial W$ is a direct summand of the one for W_∞. Thus, we can choose the isometry V covering the inclusion $\mathbb{R}_{\geqslant} \times \partial W \hookrightarrow W_\infty$ to be simply given by the inclusion of the first module as a direct summand into the second. Then, by definition, $j_+([U^*(\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+)]) = [U^*(\psi\chi(D_{\mathrm{cyl}})_+\psi)]$ and since j_+ commutes with the boundary map, we are done. □

Note that we can choose U to induce the identity on the cylindrical end, where the positive and the negative spinor bundles are both the pullback of the spinor bundle on ∂W. Then the above identity reads

$$\partial[U^*(\psi\chi(D_{\mathrm{cyl}})_+\psi)] = j_+(\partial[\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+]). \tag{4.16}$$

Finally, we have the following proposition which is of crucial importance.

PROPOSITION 4.13. *If $n + 1$ is even, then*

$$\partial[\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+] = j_\partial\rho(D_\partial) \quad \text{in } K_{n+1}(D^*(\partial W \subset \mathbb{R}_{\geqslant} \times \partial W)^\Gamma). \tag{4.17}$$

Proof. We apply j_∂^{-1} to both sides. Using (4.9) for $M = \partial W$ we see that (4.17) is equivalent to

$$\delta_{\mathrm{MV}}(\rho(D_{\mathbb{R}\times\partial W})) = \rho(D_{\partial W}) \quad \text{in } K_{n+1}(D^*(\partial W)^\Gamma) \tag{4.18}$$

which is precisely the content of Theorem 4.2 (this is the cylinder delocalized index theorem). □

REMARK 4.14. Of course, we expect that the corresponding formula to Equation (4.17) holds if $n + 1$ is odd, namely

$$\partial[\tfrac{1}{2}(1 + \psi_+\chi(D_{\mathrm{cyl}})\psi_+)] = j_\partial\rho(D_\partial) \quad \text{in } K_{n+1}(D^*(\partial W \subset \mathbb{R}_{\geqslant} \times \partial W)^\Gamma). \tag{4.19}$$

Proof of Theorem 1.14. We can finally give the proof of Theorem 1.14. Indeed, if $n + 1$ is even, then from (4.12), (4.16), (4.17) we obtain at once

$$\iota_*(\mathrm{Ind}^{\mathrm{rel}}(D)) = \partial[U^*(\psi\chi(D_{\mathrm{cyl}})_+\psi) = j_+(\partial[\psi_+\chi(D_{\mathrm{cyl}})_+\psi_+]) = j_+(j_\partial\rho(D_\partial)).$$

Applying c^{-1} and using the commutativity of (3.2) we get precisely what we have to show. □

4.3. *Proof of the partitioned manifold theorem for ρ-classes assuming Theorem 4.2*

In this subsection, we show how to prove Theorem 1.22 assuming the cylinder delocalized index Theorem 4.2, namely that

$$\delta_{\mathrm{MV}}(\rho(D_{\mathbb{R}\times M})) = \rho(D_M) \quad \text{in } K_0(D^*(M)^\Gamma).$$

Consider (W, g), an $(n + 1)$-dimensional Riemannian manifold with uniformly positive scalar curvature metric g, partitioned by a two-sided hypersurface M, $W = W_- \cup_M W_+$, with product structure near M and with signed distance function $f\colon W \to \mathbb{R}$. We also assume an isometric action of Γ, preserving M and with the property that M/Γ is compact. There is then a resulting Γ-map $u\colon M \to E\Gamma$. We have defined in Subsection 1.5 the partitioned manifold ρ-class $\rho^{\mathrm{pm}}(g) \in K_n(D^*(M)^\Gamma)$ and the partitioned manifold ρ_Γ-class $\rho_\Gamma^{\mathrm{pm}}(g) \in K_n(D_\Gamma^*)$. We shall also employ the notation $\rho_\Gamma^{\mathrm{pm}}(D_W)$ for this class.

Recall that our goal is to show that

$$\rho_\Gamma^{\mathrm{pm}}(g) = \rho_\Gamma(g_M) \quad \text{in } K_n(D_\Gamma^*) \tag{4.20}$$

We first show that the left-hand side is unchanged if we replace W by $\mathbb{R} \times M$. Note that our proof only applies to $\rho_\Gamma^{\mathrm{pm}}(g) \in K_n(D_\Gamma^*)$; it does not apply to $\rho^{\mathrm{pm}}(g) \in K_n(D^*(M)^\Gamma)$.

First of all, we extend our discussion in Remark 3.4 and give a more detailed description of $\delta_{\mathrm{MV}}[D_W]$, with δ_{MV} the Mayer–Vietoris boundary homomorphism associated to the partition $W = W_- \cup_M W_+$. To this end, we recall that $\delta_{\mathrm{MV}} : K_n(D^*(W)^\Gamma) \to K_{n+1}(D^*(M)^\Gamma)$ is obtained by composing

$$K_n(D^*(W)^\Gamma) \longrightarrow K_n(D^*(W)^\Gamma / D^*(W_- \subset W)^\Gamma) \simeq K_n(D^*(W_+ \subset W)^\Gamma) / D^*(M \subset W)^\Gamma$$

$$\xrightarrow{\partial} K_{n+1}(D^*(M \subset W)^\Gamma) \xrightarrow[\cong]{c^{-1}} K_{n+1}(D^*(M)^\Gamma).$$

Let $\chi_\pm$ be the characteristic functions of $W_\pm$. By writing an element x in $K_n(D^*(W)^\Gamma)$ as $\chi_+ x \chi_+ + \chi_- x \chi_- + R$, with $R \in D^*(M \subset W)^\Gamma$, we see first of all that $\delta_{\mathrm{MV}}(\rho(D_W))$, with $\rho(D_W) \in K_n(D^*(W)^\Gamma)$, is equal to $c^{-1} \partial [\chi_+ \rho(D_W) \chi_+]$ where ∂ is equal to the connecting homomorphism for the ideal $D^*(M \subset W)^\Gamma$ in $D^*(W_+ \subset W)^\Gamma$. Observe now that there is an isomorphism of algebras

$$D^*(W_+ \subset W)^\Gamma / D^*(M \subset W)^\Gamma \xleftarrow{\alpha} D^*(W_+)^\Gamma / D^*(M \subset W_+)^\Gamma \tag{4.21}$$

and that the lift of $[\chi_+ \rho(D_W) \chi_+] \in D^*(W_+ \subset W)^\Gamma / D^*(M \subset W)^\Gamma$ through α is the class $[\chi_+ \rho(D_W) \chi_+] \in D^*(W_+)^\Gamma / D^*(M \subset W_+)^\Gamma$, that is, the same element seen in a different algebra. Of course this correspondence will hold also for the associated K-theory elements. Consider now the manifold $W_{\mathrm{cyl},+} := (\mathbb{R}_{\leqslant} \times M) \cup_M W_+$; thus, we cut out W_- and we glue at its place $\mathbb{R}_{\leqslant} \times M$. We can also consider the class $[\chi_+ (D_{\mathrm{cyl},+}) \chi_+] \in K_n(D^*(W_+ \subset W_{\mathrm{cyl},+})^\Gamma / D^*(M \subset W_{\mathrm{cyl},+})^\Gamma)$ and its lift to $K_n(D^*(W_+)^\Gamma / D^*(M \subset W_+)^\Gamma)$ under the K-theory isomorphism induced by the analog to (4.21) but for $W_{\mathrm{cyl},+}$. The two lifts can now be compared, as they live in the K-theory of the same algebra, which is $D^*(W_+)^\Gamma / D^*(M \subset W_+)^\Gamma$.

LEMMA 4.15.   In $K_n(D^*(W_+)^\Gamma / D^*(M \subset W_+)^\Gamma)$ the following equality holds:

$$[\chi_+ D_W \chi_+] = [\chi_+ (D_{\mathrm{cyl},+}) \chi_+]. \tag{4.22}$$

Assuming the lemma we now conclude the proof of the partitioned manifold theorem for ρ-classes.

Theorem 4.2, the cylinder delocalized index theorem, states that for any n-dimensional Γ-manifold without boundary, n odd, with isometric, free, cocompact action and positive scalar curvature one has

$$\delta_{\mathrm{MV}} \rho(D_{\mathbb{R} \times M}) = \rho(D_M) \quad \text{in } K_{n+1}(D^*(M)^\Gamma).$$

In particular,

$$\delta_{\mathrm{MV}} \rho_\Gamma(D_{\mathbb{R} \times M}) = \rho_\Gamma(D_M) \quad \text{in } K_{n+1}(D_\Gamma^*).$$

From this equation and the very definition of $\rho_\Gamma^{\mathrm{pm}}$ class, we obtain at once that

$$\rho_\Gamma^{\mathrm{pm}}(D_{\mathbb{R} \times M}) = \rho_\Gamma(D_M) \quad \text{in } K_{n+1}(D_\Gamma^*).$$

Thus, it suffices to prove that

$$\rho_\Gamma^{\mathrm{pm}}(D_W) = \rho_\Gamma^{\mathrm{pm}}(D_{\mathbb{R} \times M}). \tag{4.23}$$

In order to show this equality we observe, first of all, that it suffices to prove that

$$\rho_\Gamma^{\mathrm{pm}}(D_W) = \rho_\Gamma^{\mathrm{pm}}(D_{\mathrm{cyl},+}) \quad \text{in } K_{n+1}(D_\Gamma^*).$$

Indeed, if this equality holds we can further modify $W_{\mathrm{cyl},+}$ by cutting out M_+ and gluing in at its place $\mathbb{R}_{\geqslant} \times M$, obtaining from $W_{\mathrm{cyl},+}$ the manifold $W_{\mathrm{cyl,cyl}}$, which is nothing but $\mathbb{R} \times M$. By the same argument above we obtain the equality

$$\rho_\Gamma^{\mathrm{pm}}(D_{\mathrm{cyl},+}) = \rho_\Gamma^{\mathrm{pm}}(D_{\mathrm{cyl,cyl}}) \equiv \rho^{\mathrm{pm}}(D_{\mathbb{R} \times M}) \quad \text{in } K_{n+1}(D_\Gamma^*).$$

This proves that

$$\rho_\Gamma^{\mathrm{pm}}(D_W) = \rho_\Gamma^{\mathrm{pm}}(D_{\mathbb{R} \times M}) \quad \text{in } K_{n+1}(D_\Gamma^*),$$

which is precisely (4.23). In order to show that $\rho_\Gamma^{\mathrm{pm}}(D_W) = \rho_\Gamma^{\mathrm{pm}}(D_{\mathrm{cyl},+})$ we use Lemma 4.15 and the following commutative diagram. For the sake of brevity we set

$$A := D^*(W_+ \subset W_{\mathrm{cyl},+})^\Gamma / D^*(M \subset W_{\mathrm{cyl},+})^\Gamma, \quad B := D^*(W_+)^\Gamma / D^*(M \subset W_+)^\Gamma,$$
$$C = D^*(W_+ \subset W)^\Gamma / D^*(M \subset W)^\Gamma,$$
$$A_\Gamma := D^*(\mathbb{R} \times E\Gamma \subset \mathbb{R}_{\geqslant} \times E\Gamma)^\Gamma / D^*(\{0\} \times E\Gamma \subset \mathbb{R} \times E\Gamma)^\Gamma,$$
$$B_\Gamma := D^*(\mathbb{R}_{\geqslant} \times E\Gamma)^\Gamma / D^*(\{0\} \times E\Gamma \subset \mathbb{R}_{\geqslant} \times E\Gamma)^\Gamma,$$
$$\mathfrak{A}_\Gamma := D^*(\{0\} \times E\Gamma \subset \mathbb{R} \times E\Gamma)^\Gamma, \quad \mathfrak{B}_\Gamma := D^*(\{0\} \times E\Gamma \subset \mathbb{R}_{\geqslant} \times E\Gamma)^\Gamma.$$

We note furthermore that $n \equiv 1 \bmod 2$.

$$
\begin{array}{ccccccc}
K_1(A) & \longrightarrow & K_1(A_\Gamma) & \xrightarrow{\ \partial\ } & K_0(\mathfrak{A}_\Gamma) & \xrightarrow{\ \cong\ } & K_0(D_\Gamma^*) \\
\Big\uparrow{\scriptstyle\cong} & & \Big\uparrow{\scriptstyle\cong} & & \Big\uparrow & & \Big\uparrow{\scriptstyle=} \\
K_1(B) & \longrightarrow & K_1(B_\Gamma) & \xrightarrow{\ \partial\ } & K_0(\mathfrak{B}_\Gamma) & \xrightarrow{\ \cong\ } & K_0(D_\Gamma^*) \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow & & \Big\downarrow{\scriptstyle=} \\
K_1(C) & \longrightarrow & K_1(A_\Gamma) & \xrightarrow{\ \partial\ } & K_0(\mathfrak{A}_\Gamma) & \xrightarrow{\ \cong\ } & K_0(D_\Gamma^*).
\end{array}
$$

The class $\rho_\Gamma^{\mathrm{pm}}(D_W)$ can be obtained by mapping the class $[\chi_+ D_W \chi_+] \in K_1(C)$ all the way to $K_0(D_\Gamma^*)$ via the homomorphisms of the bottom horizontal line. Here the naturality of the boundary map in K-theory has been used. This same class can also be computed, always applying commutativity and naturality, by lifting $[\chi_+ D_W \chi_+] \in K_1(C)$ to $K_1(B)$ and then traveling on the central horizontal line. The same argument applies to $\rho_\Gamma^{\mathrm{pm}}(D_{\mathrm{cyl},+})$, which is originally defined by considering $[\chi_+(D_{\mathrm{cyl},+})\chi_+]$ in $K_1(A)$ and then traveling on the top horizontal line; the resulting class in $K_0(D_\Gamma^*)$ can also be obtained by lifting $[\chi_+(D_{\mathrm{cyl},+})\chi_+]$ to $K_1(B)$ and then traveling on the central horizontal line. Since, by the lemma, the two lifts of $[\chi_+ D_W \chi_+]$ and $[\chi_+(D_{\mathrm{cyl},+})\chi_+]$ are equal in $K_1(B)$, we see that $\rho_\Gamma^{\mathrm{pm}}(D_W) = \rho_\Gamma^{\mathrm{pm}}(D_{\mathrm{cyl},+})$, as required.

Summarizing, assuming Lemma 4.15 and the cylinder delocalized index Theorem 4.2, we have proved that

$$\rho_\Gamma^{\mathrm{pm}}(D_W) = \rho_\Gamma^{\mathrm{pm}}(D_{\mathbb{R} \times M}) \equiv \delta_{\mathrm{MV}}(\rho_\Gamma(D_{\mathbb{R} \times M})) = \rho_\Gamma(D_M)$$

and this is precisely what we need to show in order to establish Theorem 1.22

We shall now prove Lemma 4.15.

Consider more generally the following situation: we have two complete Γ-manifolds as above, W and Z, both endowed with metrics of positive scalar curvature and with partitions

$$W = W_1 \cup_M W_2, \quad Z = W_1 \cup_M Z_2.$$

In other words, the two partitions have one component equal, W_1, they (necessarily) involve the same hypersurface, M, but have the other component of the partition different. Choose a chopping function χ equal to ± 1 on both the spectrum of D_W and D_Z. We want to show that

$$\chi_{W_1}(\chi(D_W))\chi_{W_1} = \chi_{W_1}(\chi(D_Z))\chi_{W_1} \quad \text{in } D^*(W_1)^\Gamma / D^*(M \subset W_1)^\Gamma, \qquad (4.24)$$

with χ_{W_1} denoting the characteristic function of W_1. Consider the left-hand side of the above equation, $\chi_{W_1}(\chi(D_W))\chi_{W_1}$. The Fourier transform of χ is a smooth rapidly decreasing function away from 0, see [**31**, p. 121]. Thus, $\chi_{W_1}(\chi(D_W))\chi_{W_1}$ is (up to multiplication with $\sqrt{2\pi}$) approximated in norm for $R \in \mathbb{R}$ large by

$$\chi_{W_1}\left(\int_{-R}^{R} \hat{\chi}(\xi)\, e^{i\xi D_W}\, d\xi\right) \chi_{W_1}.$$

We rewrite this latter term as

$$\chi_{(W_1\setminus U_R(M))}\left(\int_{-R}^{R} \hat{\chi}(\xi)\, e^{i\xi D_W}\, d\xi\right) \chi_{(W_1\setminus U_R(M_1))} + \chi_{W_1}\left(\int_{-R}^{R} \hat{\chi}(\xi)\, e^{i\xi D_W}\, d\xi\right) \chi_{U_R(M)}$$

$$+ \chi_{U_R(M)}\left(\int_{-R}^{R} \hat{\chi}(\xi)\, e^{i\xi D_W}\, d\xi\right) \chi_{(W_1\setminus U_R(M))}. \tag{4.25}$$

Because of the unit propagation property, the first summand is unchanged if we replace D_W by D_Z, that is, is equal to

$$\chi_{(W_1\setminus U_R(M))}\left(\int_{-R}^{R} \hat{\chi}(\xi)\, e^{i\xi D_Z}\, d\xi\right) \chi_{(W_1\setminus U_R(M))}.$$

We then rewrite (4.25) as

$$\chi_{W_1}\left(\int_{-R}^{R} \hat{\chi}(\xi)\, e^{i\xi D_Z}\, d\xi\right) \chi_{W_1} + \chi_{W_1}\left(\int_{-R}^{R} \hat{\chi}(\xi)\, e^{i\xi D_W}\, d\xi\right) \chi_{U_R(M)}$$

$$+ \chi_{U_R(M)}\left(\int_{-R}^{R} \hat{\chi}(\xi)\, e^{i\xi D_W}\, d\xi\right) \chi_{(W_1\setminus U_R(M))} - \chi_{W_1}\left(\int_{-R}^{R} \hat{\chi}(\xi)\, e^{i\xi D_Z}\, d\xi\right) \chi_{U_R(M)}$$

$$- \chi_{U_R(M)}\left(\int_{-R}^{R} \hat{\chi}(\xi)\, e^{i\xi D_Z}\, d\xi\right) \chi_{(W_1\setminus U_R(M))}.$$

The first summand in this sum approximates $\chi_{W_1}(\chi(D_Z))\chi_{W_1}$; moreover, by unit propagation the remaining four summands are elements in the ideal $D^*(M \subset W_1)^\Gamma$. Therefore, the difference

$$\chi_{W_1}(\chi(D_W))\chi_{W_1} - \chi_{W_1}(\chi(D_Z))\chi_{W_1}$$

is approximated by a sequence of elements in the ideal $D^*(M \subset W_1)^\Gamma$; since this ideal is closed we have proved that $\chi_{W_1}(\chi(D_W))\chi_{W_1} - \chi_{W_1}(\chi(D_Z))\chi_{W_1} \in D^*(M \subset W_1)^\Gamma$, that is, that

$$\chi_{W_1}(\chi(D_W))\chi_{W_1} = \chi_{W_1}(\chi(D_Z))\chi_{W_1} \bmod D^*(M \subset W_1)^\Gamma.$$

The lemma is proved.

The proof of the partitioned manifold theorem for ρ-classes, Theorem 1.22, is now complete.

REMARK 4.16. In [**38**, **39**], the classical partitioned manifold index theorem is extended to a multi-partitioned situation, that is, to a manifold partitioned by k suitably transversal hypersurfaces. It would be interesting to generalize also our ρ-index theorem to the multi-partitioned situation. This does not seem straightforward if one only assumes a product structure near the (codimension k) intersection of the k hypersurfaces.

4.4. Proof of Theorem 4.2

In this subsection, we finally prove Theorem 4.2, the cylinder delocalized index theorem.

We will only treat the case in which $n + 1$, the dimension of the cylinder $\mathbb{R} \times M$, is even. We want to show the equality

$$\delta_{\mathrm{MV}}(\rho(D_{\mathbb{R} \times M})) = \rho(D_M) \quad \text{in } K_0(D^*(M)^\Gamma).$$

Equivalently, see Remark 4.9, we want to show that

$$\partial[\psi_+ \chi(D_{\mathrm{cyl}})_+ \psi_+] = j_M \rho(D_M) \quad \text{in } K_0(D^*(M \subset [0, \infty) \times M)^\Gamma),$$

where $j_M \colon K_*(D^*(M)^\Gamma) \to K_*(D^*(M \subset [0, \infty) \times M)^\Gamma)$ is the map induced by the inclusion $M \hookrightarrow [0, \infty) \times M$ and ∂ is the boundary map $K_1(D^*([0, \infty) \times M)^\Gamma / D^*(M \subset [0, \infty) \times M)^\Gamma \to K_0(D^*(M \subset [0, \infty) \times M)^\Gamma)$.

NOTATION 4.17. In order to lighten the notation we shall always write $\mathcal{L}^2(M)$ for the covariant M-module given by the L^2-section of the spinor bundle of M; the latter is denoted S_M. We shall consider $\mathbb{R} \times M$ and write $\mathcal{L}^2_\oplus(\mathbb{R} \times M)$ for the L^2-sections of the bundle obtained by pulling back $S_M \oplus S_M$ from M to $\mathbb{R} \times M$. We keep the notation $\mathcal{L}^2(\mathbb{R} \times M)$ for the L^2-sections of S_M. Similar notations are adopted for $[0, \infty) \times M \equiv \mathbb{R}_\geqslant \times M$, the half cylinder. Departing from the notation adopted so far, and only for this subsection, we denote by D the (Γ-equivariant) Dirac operator on M and by D_{cyl} the (Γ-equivariant) operator on $\mathbb{R} \times M$ (these being the only Dirac operators we will be concerned with).

We thus tackle the proof of the identity

$$\partial[\psi_+ \chi(D_{\mathrm{cyl}})_+ \psi_+] = j_M \rho(D) \tag{4.26}$$

in $K_{n+1}(D^*(M \subset [0, \infty \times M))^\Gamma)$, with ψ_+ the characteristic function of $[0, \infty) \times M$ in $\mathbb{R} \times M$.

In order to establish (4.26) we need an explicit representative for the right-hand side. To define j_M we must find an isometry $V \colon \mathcal{L}^2(M) \to \mathcal{L}^2(\mathbb{R}_\geqslant \times M)$ covering in the D^*-sense the inclusion $M \hookrightarrow \mathbb{R}_\geqslant \times M$, $m \mapsto (0, m)$; see Section 1. Of course, we know that one can always find covariant modules H_1 for M, H_2 for $\mathbb{R}_\geqslant \times M$ and an isometry $V \colon H_1 \to H_2$ covering the inclusion in the D^*-sense; here we want to show that we can choose $H_1 = \mathcal{L}^2(M)$, $H_2 = \mathcal{L}^2(\mathbb{R}_\geqslant \times M)$ and then describe explicitly the isometry V.

Consider $\mathcal{L}^2(\mathbb{R}_\geqslant \times M)$; this can be identified with $L^2([0, \infty), \mathcal{L}^2(M))$. Define V as follows:

$$\mathcal{L}^2(M) \ni s \longmapsto Vs \in L^2([0, \infty), \mathcal{L}^2(M)), \quad (Vs)(t) := \sqrt{2|D|} e^{-t|D|}(s). \tag{4.27}$$

PROPOSITION 4.18. The bounded linear operator $V \colon \mathcal{L}^2(M) \to \mathcal{L}^2(\mathbb{R}_\geqslant \times M)$ of (4.27) covers in the D^*-sense the inclusion $i \colon M \hookrightarrow \mathbb{R}_\geqslant \times M$, $m \to (0, m)$.

Proof. We prove this in Subsection 4.5. $\square$

We now observe that by its very definition the ρ-class of the operator on M, $\rho(D) = [\chi_{[0,\infty)}(D)]$ in $K_0(D^*(M)^\Gamma)$.

We have, by definition of j_M,

$$j_M[\chi_{[0,\infty)}(D)] = [V \chi_{[0,\infty)}(D) V^*] \in K_0(D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma).$$

The operator $P := V \chi_{[0,\infty)}(D) V^*$, which is a projector, acts as follows on $L^2([0, \infty), \mathcal{L}^2(M))$:

$$(V \chi_{[0,\infty)}(D) V^* g)(t) = \int_0^\infty \sqrt{2|D|} e^{-t|D|} \chi_{[0,\infty)}(D) \sqrt{2|D|} e^{-\tau|D|} g \, d\tau. \tag{4.28}$$

Thus, we need to show that the K-theory class of the projector P given by (4.28) coincides with the index class $\partial[\psi_+ \chi(D_{\mathrm{cyl}})_+ \psi_+]$. In order to achieve this it suffices to show that there

exists an $L \in D^*(\mathbb{R}_{\geqslant} \times M)$ such that

$$(\psi_+ \chi(D_{\mathrm{cyl}})_+ \psi_+) \circ L = \mathrm{Id}; \quad L \circ (\psi_+ \chi(D_{\mathrm{cyl}})_+ \psi_+) = \mathrm{Id} - P. \tag{4.29}$$

Indeed, from the very definition of the boundary map, see [2], we would then have that $\partial[\psi_+ \chi(D_{\mathrm{cyl}})_+ \psi_+] = [P]$ which is what we wish to prove.

In order to find such an L we first perform a deformation of the representatives of the class $[\psi_+ \chi(D_{\mathrm{cyl}})_+ \psi_+]$ in $K_1(D^*(\mathbb{R}_{\geqslant} \times M)^\Gamma / D^*(M \subset \mathbb{R}_{\geqslant} \times M)^\Gamma)$. Let us denote by t the variable on the line $\mathbb{R}$ appearing in $\mathbb{R} \times M$; let us denote by ∂_t the derivative with respect to t. We first concentrate our analysis on $\mathbb{R} \times M$. The operator D_{cyl} is given as

$$\begin{pmatrix} 0 & D - \partial_t \\ D + \partial_t & 0 \end{pmatrix}, \tag{4.30}$$

where we recall that D denotes the Γ-equivariant Dirac operator on M. We have already observed that since D is assumed to be L^2-invertible (the scalar curvature on M is positive), also D_{cyl} is L^2-invertible. We choose as a chopping function χ the one given by $\chi(t) = 1$ for $t \geqslant 0$, $\chi(t) = -1$ for $t < 0$, which is continuous on the spectrum of D_{cyl}. Thus, $\chi(D_{\mathrm{cyl}})_+$ is the bounded operator on $\mathcal{L}^2(\mathbb{R} \times M)$ given by the left bottom corner of

$$\begin{pmatrix} 0 & \dfrac{D - \partial_t}{\sqrt{D^2 - \partial_t^2}} \\ \dfrac{D + \partial_t}{\sqrt{D^2 - \partial_t^2}} & 0 \end{pmatrix}.$$

The operator $\chi(D_{\mathrm{cyl}})_+$ will be written as $(D + \partial_t)/\sqrt{D^2 - \partial_t^2}$. We shall connect it to

$$\frac{|D| + \partial_t}{D - \partial_t},$$

which is also an invertible operator on $\mathcal{L}^2(\mathbb{R} \times M)$. We observe here a few useful identities: $[D, \partial_t] = 0 = [|D|, \partial_t]$; $(|D| - \partial_t)(|D| + \partial_t) = D^2 - \partial_t^2 = (D - \partial_t)(D + \partial_t)$. The latter equality gives

$$\frac{|D| + \partial_t}{D - \partial_t} = \frac{D + \partial_t}{|D| - \partial_t}.$$

Note, in particular, that

$$\left(\frac{|D| + \partial_t}{D - \partial_t}\right)^{-1} = \frac{|D| - \partial_t}{D + \partial_t}. \tag{4.31}$$

We claim that the line segment joining the two operators,

$$s \longmapsto s\frac{D + \partial_t}{\sqrt{D^2 - \partial_t^2}} + (1 - s)\frac{|D| + \partial_t}{D - \partial_t}$$

is through L^2-invertible operators in $D^*(\mathbb{R} \times M)^\Gamma$.

The fact that for each $s \in [0, 1]$ the above operator is invertible can be seen by rewriting it as $A_s/((D - \partial_t)\sqrt{D^2 - \partial_t^2})$ with $A_s = s(D^2 - \partial_t^2) + (1 - s)(|D| + \partial_t)\sqrt{D^2 - \partial_t^2}$; it suffices to show that A_s is invertible for each $s \in [0, 1]$, which in turn is proved with an elementary computation by showing that $A_s^* A_s > 0$ (here the L^2-invertibility of D and D_{cyl} is used).

Next we address the fact that $s((D + \partial_t)/\sqrt{D^2 - \partial_t^2}) + (1 - s)((|D| + \partial_t)/(D - \partial_t)) \in D^*(\mathbb{R} \times M)^\Gamma$ for each s. Since D^* is a C^*-subalgebra of the bounded operators of $\mathcal{L}^2(\mathbb{R} \times M)$, it suffices to show that the end points of the convex combination are in $D^*(\mathbb{R} \times M)^\Gamma$. We already know that $(D + \partial_t)/\sqrt{D^2 - \partial_t^2}$ is in $D^*(\mathbb{R} \times M)^\Gamma$ (given that is the left bottom corner of $D_{\mathrm{cyl}}/|D_{\mathrm{cyl}}|$). Thus, we only need to establish the following proposition.

PROPOSITION 4.19.   *The operator $(|D| + \partial_t)/(D - \partial_t)$ belongs to $D^*(\mathbb{R} \times M)^\Gamma$.*

Proof. We prove this in Subsection 4.5. □

Using the above convex combination we see easily that

$$\left[\psi_+ \frac{D + \partial_t}{\sqrt{D^2 - \partial_t^2}} \psi_+ \right] = \left[\psi_+ \frac{|D| + \partial_t}{D - \partial_t} \psi_+ \right] \quad \text{in } K_1(D^*(\mathbb{R}_\geqslant \times M)^\Gamma / D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma).$$

In particular, $\partial[\psi_+((D + \partial_t)/\sqrt{D^2 - \partial_t^2})\psi_+] = \partial[\psi_+((|D| + \partial_t)/(D - \partial_t))\psi_+]$ in $K_0(D^*(M \subset \mathbb{R}_\geqslant \times M)^\Gamma)$ and so we are reduced to the problem of finding $Q \in D^*(\mathbb{R}_\geqslant \times M)^\Gamma$ such that

$$\left(\psi_+ \frac{|D| + \partial_t}{D - \partial_t} \psi_+ \right) \circ Q = \mathrm{Id}_{\mathcal{L}^2(\mathbb{R}_\geqslant \times M)}, \tag{4.32}$$

$$Q \circ \left(\psi_+ \frac{|D| + \partial_t}{D - \partial_t} \psi_+ \right) = \mathrm{Id}_{\mathcal{L}^2(\mathbb{R}_\geqslant \times M)} - P. \tag{4.33}$$

We claim that we can take $Q := \psi_+((|D| - \partial_t)/(D + \partial_t))\psi_+$. First, by Proposition 4.19, $Q \in D^*(\mathbb{R}_\geqslant \times M)^\Gamma$. Next we need to show that with this choice (4.32) holds. Using that $(\psi_+)^2 = \psi_+$, together with (4.31) we see that it suffices to show that $\psi_+((|D| - \partial_t)/(D + \partial_t))\psi_+ = ((|D| - \partial_t)/(D + \partial_t))\psi_+$ on $\mathcal{L}^2(\mathbb{R} \times M)$. We decompose $\mathcal{L}^2(\mathbb{R} \times M)$ using the spectral transform induced by the Browder–Garding decomposition associated to the self-adjoint operator D, see [30]. Thus, there is an isometry

$$T \colon \mathcal{L}^2(\mathbb{R} \times M) \longrightarrow L^2(\mathbb{R}, \oplus_j L^2(\mathbb{R}, d\mu_j)) \tag{4.34}$$

such that $\psi_+((|D| - \partial_t)/(D + \partial_t))\psi_+ = ((|D| - \partial_t)/(D + \partial_t))\psi_+$ if and only if

$$\chi_{[0,\infty)} \frac{|\lambda| - \partial_t}{\lambda + \partial_t} \chi_{[0,\infty)} = \frac{|\lambda| - \partial_t}{\lambda + \partial_t} \chi_{[0,\infty)}$$

on each $L^2(\mathbb{R}, L^2(\mathbb{R}, d\mu_j))$. Conjugate both sides of the last equation by the Fourier transformation $\mathcal{F}$ for the t-variable, then write $\mathcal{F}^{-1}((|\lambda| - \partial_t)/(\lambda + \partial_t))\chi_{[0,\infty)}\mathcal{F}$ as $(\mathcal{F}^{-1}((|\lambda| - \partial_t)/(\lambda + \partial_t))\mathcal{F})(\mathcal{F}^{-1}\chi_{[0,\infty)}\mathcal{F})$. Then the right-hand side is equal to the projection onto the Hardy space, see [40], followed by the multiplication operator by $(|\lambda| + i\tau)/(\lambda - i\tau) = (\lambda + i\tau)/(|\lambda| - i\tau)$. The latter function is holomorphic on the upper half plane, so multiplication by it does preserve the Hardy space. It follows that projecting once again onto the Hardy space leaves it unchanged. This proves that $\chi_{[0,\infty)}((|\lambda| - \partial_t)/(\lambda + \partial_t))\chi_{[0,\infty)} = (|\lambda| - \partial_t)/(\lambda + \partial_t))\chi_{[0,\infty)}$ and thus that (4.32) holds. Next we tackle (4.33), that is, the equation

$$\psi_+ \left(\frac{|D| - \partial_t}{D + \partial_t} \right) \psi_+ \circ \left(\psi_+ \frac{|D| + \partial_t}{D - \partial_t} \psi_+ \right) = \mathrm{Id}_{\mathcal{L}^2(\mathbb{R}_\geqslant \times M)} - P \tag{4.35}$$

with $P \colon L^2([0,\infty), \mathcal{L}^2(M)) \to L^2([0,\infty), \mathcal{L}^2(M))$ defined by

$$(Pg)(t) = \int_0^\infty \sqrt{2|D|} \, e^{-t|D|} \chi_{[0,\infty)}(D_\partial) \sqrt{2|D|} \, e^{-u|D|} g \, du. \tag{4.36}$$

LEMMA 4.20. *Under the spectral transform*

$$T \colon \mathcal{L}^2([0,\infty) \times M) \longrightarrow L^2([0,\infty), \oplus_j L^2(\mathbb{R}, d\mu_j)) \tag{4.37}$$

the projector P diagonalizes. Let P_j be the restriction of P to $L^2([0,\infty), L^2(\mathbb{R}_\lambda, d\mu_j))$ and consider the decomposition

$$L^2([0,\infty), L^2(\mathbb{R}, d\mu)) = L^2([0,\infty), L^2([0,\infty)_\lambda, d\mu)) \oplus L^2([0,\infty), L^2((-\infty, 0)_\lambda, d\mu)).$$

Then P_j is diagonal and equal to

$$\begin{pmatrix} P_j^+ & 0 \\ 0 & 0 \end{pmatrix}$$

with P_j^+ described in the following way: view $L^2([0,\infty), L^2([0,\infty)_\lambda), d\mu_j)$ as $L^2_{d\mu_j}([0,\infty)_\lambda,$ $L^2[0,\infty)_t)$. With respect to this decomposition, P_j^+ is the direct integral $\int_{[0,\infty)} P_\lambda\, d\mu_j(\lambda)$ where $P_\lambda \colon L^2([0,\infty)) \to L^2([0,\infty))$ is the projector onto the subspace spanned by $f_\lambda(t) = \sqrt{2\lambda}\,e^{-\lambda t}$.

Observe that, because 0 is not in the spectrum of the operator D, 0 is not in the support of any of the measures μ_j so that f_λ indeed is in $L^2([0,\infty))$ and depends continuously on λ for all λ relevant to us.

Proof. With respect to the decomposition of our Hilbert space, as D acts as multiplication with λ and $|D|$ acts as multiplication with $|\lambda|$ under the spectral transform T, the formula (4.36) becomes the direct integral over the operators

$$g \longmapsto \int_0^\infty \sqrt{2|\lambda|}\, e^{-t|\lambda|} \chi_{[0,\infty)}(\lambda) \sqrt{2|\lambda|}\, e^{-u|\lambda|} g(u)\, du = \begin{cases} f_\lambda\langle g, f_\lambda\rangle_{L^2([0,\infty))}; & \lambda > 0, \\ 0; & \lambda < 0. \end{cases} \qquad \square$$

We are now in the position to prove (4.35). We use the spectral transform and we reduce to a computation on each single $L^2([0,\infty), L^2(\mathbb{R}, d\mu))$. First, we remark that the operator induced by the left-hand side of (4.35) diagonalizes with respect to the decomposition

$$L^2([0,\infty), L^2([0,\infty)_\lambda, d\mu)) \oplus L^2([0,\infty), L^2((-\infty,0)_\lambda, d\mu),$$

even better, as before it becomes a direct integral over $\lambda \in \mathbb{R}$ with measure $\mu_j(\lambda)$. Moreover, the restriction to the second summand is equal to the identity, given that the restriction of the operator induced by $(|D| + \partial_t)/(D - \partial_t)$ on $L^2(\mathbb{R}, L^2(\mathbb{R}, d\mu))$ is equal to $(|\lambda| + \partial_t)(\lambda - \partial_t)^{-1}$ and it is therefore equal to $-\mathrm{Id}$ on $L^2(\mathbb{R}, L^2(-\infty,0)_\lambda, d\mu)$, and the same holds with the same argument for $(|D| - \partial_t)/(D + \partial_t)$. The conclusion is that the restriction of the two sides of (4.35) to the second summand of $L^2([0,\infty), L^2([0,\infty)_\lambda, d\mu)) \oplus L^2([0,\infty), L^2((-\infty,0)_\lambda, d\mu)$ agree.

Thus, we are left with the task of showing that the operator induced by the left-hand side of (4.35) on $L^2([0,\infty), L^2([0,\infty)_\lambda, d\mu))$ in its direct integral decomposition for each $\lambda \in (0,\infty)$ has a one-dimensional null space, generated by $f_\lambda(t) = \sqrt{2\lambda}\,e^{-t\lambda}$, $t \geqslant 0$ and it is equal to the identity on the orthogonal complement of this null space. Using the direct integral decomposition, we treat $\lambda > 0$ as a constant. Let us then check that the function $f_\lambda(t)$ is in the null space of the operator induced by the left-hand side of (4.35). In order to check this property we conjugate by Fourier transform. The inverse Fourier transform of $\sqrt{2\lambda}\,e^{-\lambda t}$, $t \geqslant 0$, is up to a constant equal to $1/(\lambda - i\tau)$ (cf. [**16**, Appendix]) which is holomorphic outside $\tau = -i\lambda$; in particular, it is holomorphic on the upper half plane which means that it is left unchanged by the projection onto the Hardy space (as it should, given that $\chi_{[0,\infty)} f_\lambda = f_\lambda$). Now we apply the operator of multiplication by $(\lambda - i\tau)(\lambda + i\tau)^{-1}$, getting the function $1/(\lambda + i\tau)$. This is holomorphic on the lower half plane and therefore its boundary value is in the orthogonal complement of the Hardy space, so it is mapped to 0 by projecting onto the Hardy space. The conclusion is that f_λ is indeed in the null space of the left-hand side of (4.35).

Consider now the orthogonal complement of f_λ, that is,

$$\left\{ g \in L^2([0,\infty)); \int_0^\infty g(t) f_\lambda(t)\, dt = \int_\mathbb{R} g(t)\, e^{it(i\lambda)}\, dt = 0 \right\}.$$

This is the space of functions g such that $(\mathcal{F}^{-1}g)(i\lambda) = 0$. This means that

$$\tau \longmapsto \frac{(\lambda - i\tau)}{(\lambda + i\tau)}\left(\mathcal{F}^{-1}g\right)(\tau)$$

is still holomorphic on the upper half plane and thus projection onto the Hardy space leaves it unchanged. Composing with the multiplication operator by $(\lambda + i\tau)/(\lambda - i\tau)$ gives back $\mathcal{F}^{-1}g$. Thus the left-hand side of (4.35) acts as the identity on the orthogonal complement of f_λ and the conclusion is that the left-hand side of (4.35) is precisely equal to $\mathrm{Id} - P$, thanks to Lemma 4.20.

4.5. *Proof of Propositions* 4.18 *and* 4.19

We begin by proving Proposition 4.18.

We wish to prove that the bounded linear operator $V \colon \mathcal{L}^2(M) \to \mathcal{L}^2(\mathbb{R}_\geqslant \times M)$ defined by $(Vs)(t) := \sqrt{2|D|}e^{-t|D|}(s)$ is an isometry that is the norm-limit of bounded linear operators U satisfying the propagation condition appearing in Definition 1.6 such that $\phi U - U(\phi \circ i)$ is compact for each $\phi \in C_0(\mathbb{R}_\geqslant \times M)$.

The fact that V is an isometry is proved by direct computation, using the fact that the spectrum of D does not contain zero, so that $e^{-t|D|}$ converges (exponentially) to zero for $t \to +\infty$. Consider next the propagation condition which we recall here: *there exists an $R > 0$ such that $\phi U \psi = 0$ if $d(\mathrm{supp}\phi, i(\mathrm{supp}\psi)) > R$, with $\phi \in C_0(\mathbb{R}_\geqslant \times M)$ and $\psi \in C_0(M)$.* We must find an approximating sequence of bounded linear operators U with this property. Consider the function $h_t(x) := \sqrt{|x|}\, e^{-t|x|}$. Our operator V is obtained from h_t by

$$(Vs)(t) = \frac{\sqrt{2}}{\sqrt{2\pi}} \int_\mathbb{R} \hat{h}_t(\xi)\, e^{i\xi D}(s)\, d\xi.$$

We consider the function

$$f_t(x) := \begin{cases} 0; & x < 0, \\ \sqrt{x}\, e^{-tx}; & x \geqslant 0. \end{cases}$$

We write $h_t = f_t + g_t$, with $g_t(x) := f_t(-x)$. Its Fourier–Laplace transform is

$$\hat{f}_t(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \sqrt{x}e^{-x(t+i\zeta)}\, dx,$$

and we observe that this is a holomorphic function in the region $\mathrm{Im}(\zeta) < t$. For $s < t$ the integral can easily be evaluated, giving

$$\hat{f}_t(is) = C\frac{1}{(t-s)^{3/2}}, \quad \text{with } C = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \sqrt{x}\, e^{-x}\, dx.$$

Thus, by the identity principle for holomorphic functions, we deduce that

$$\hat{f}_t(\zeta) = C(t+i\zeta)^{-3/2},$$

with the branch of the square root such that $t^{\frac{3}{2}}$ is positive for $t > 0$. Going back to h_t we have therefore proved that for $\xi \in \mathbb{R}$

$$C^{-1}\hat{h}_t(\xi) = (t+i\xi)^{-3/2} + (t-i\xi)^{-3/2}.$$

Fix $R \in \mathbb{R}$, $R > 0$. We define a bounded linear operator $U_R \colon \mathcal{L}^2(M) \to \mathcal{L}^2(\mathbb{R}_\geqslant \times M)$ as follows:

$$(U_R s)(t) = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{2\pi}} \displaystyle\int_{-R^4}^{R^4} \hat{h}_t(\xi)\, e^{i\xi D} dx & \text{if } t \leqslant R, \\ 0 & \text{if } t > R. \end{cases}$$

It is clear that U_R satisfies the propagation condition. We have

$$\|(U_R - V)(s)\|^2 \leqslant \sqrt{2}\|\sqrt{|D|}e^{-R|D|}\|^2\|s\|^2 + \sqrt{2}\int_0^R \left\|\int_{|\xi|>R^4} \hat{h}_t(\xi)\, e^{i\xi D}\, d\xi(s)\right\|^2 dt.$$

For the second summand on the right-hand side we can use the explicit description of $\hat{h}_t(\xi)$, to get the estimates

$$
\begin{aligned}
\int_0^R \left\| \int_{|\xi|>R^4} \hat{h}_t(\xi) e^{i\xi D}(s) \right\|^2 dt &\leqslant \|s\|^2 \int_0^R \left(\int_{|\xi|>R^4} |\hat{h}_t(\xi)|\, d\xi \right)^2 dt \\
&\leqslant \|s\|^2 \int_0^R \left(\int_{|\xi|>R^4} (C|t+i\xi|^{-3/2} + C|t-i\xi|^{-3/2})\, d\xi \right)^2 dt \\
&\leqslant 2C\|s\|^2 \int_0^R \left(\int_{|\xi|>R^4} |\xi|^{-\frac{3}{2}}\, d\xi \right)^2 dt \\
&\leqslant C' \frac{1}{R} \|s\|^2.
\end{aligned}
$$

Summarizing, we have shown that there exists a positive $C > 0$ such that

$$
\|(U_R - V)(s)\|^2 \leqslant C(\|\sqrt{|D|} e^{-R|D|}\|^2 + R^{-1}) \cdot \|s\|^2
$$

proving that $U_R \to V$ in operator norm as $R \to +\infty$.

Next we need to show that $\phi U_R - U_R(\phi \circ i)$ is compact for each $\phi \in C_0(\mathbb{R}_{\geqslant} \times M)$. For notational convenience we deal with V instead of U_R. One checks immediately that the arguments also work for U_R. By the Kasparov Lemma 3.2 it suffices to prove that $\phi_1 V \psi_2$ is compact whenever $\phi_1 \in C_0([0,\infty) \times M)$ $\psi_2 \in C_0(M)$ and the image of the support of ψ_2 through the inclusion map i is disjoint from the support of ϕ_1. Clearly, it suffices to prove that $\phi_1 V \psi_2$ is the norm limit of compact operators. We can obviously consider $\phi_1 = \alpha \otimes \psi_1$ with $\psi_1 \in C_0(M)$ and $\alpha \in C_0[0,\infty)$. There are then two cases:

(1) $\alpha(0) = 0$, and we may as well assume that α is supported away from $t = 0$;
(2) α is not supported away from $t = 0$ but $d(\operatorname{supp}\psi_1, \operatorname{supp}\psi_2) \geqslant \delta > 0$.

Let us treat (1) first. Take $\Lambda \gg 0$ and consider $\chi_{[-\Lambda,\Lambda]}$. Then we can consider $(\alpha \otimes \psi_1) V_\Lambda \psi_2$ with $(V_\Lambda s)(t) := \sqrt{2|D|} e^{-t|D|} \chi_{[-\Lambda,\Lambda]}(D)(s)$. The operator $(\alpha \otimes \psi_1) V_\Lambda \psi_2$ is compact; indeed $\chi_{[-\Lambda,\Lambda]}(D)$ is in $C^*(M)$, so that $\chi_{[-\Lambda,\Lambda]}(D)\psi$ is compact for each $\psi \in C_0(M)$. We are considering $(\alpha \otimes \psi_1) V \chi_{[-\Lambda,\Lambda]}(D)\psi_2$; since $\chi_{[-\Lambda,\Lambda]}(D)\psi_2$ is compact and since V is an isometry (and the composition of a compact operator with a bounded operator is again compact), we conclude that $(\alpha \otimes \psi_1) V_\Lambda \psi_2$ is compact. It remains to show that the operator norm of $(\alpha \otimes \psi_1) V \psi_2 - (\psi_1 \otimes \alpha) V_\Lambda \psi_2$ is small. We shall achieve this by proving that $\alpha V - \alpha V_\Lambda$ is small in norm. Consider the spectral transform (4.37); under this transformation, which is an isometry, the operators V and V_Λ diagonalize as the direct sum of bounded operators $V_j : L^2(\mathbb{R}_\lambda, d\mu_j) \to L^2([0,\infty), L^2(\mathbb{R}_\lambda, d\mu_j))$ and similarly for V_Λ. We have: $(V_j \sigma)(t, \lambda) = e^{-t|\lambda|} \sqrt{2|\lambda|} \sigma(\lambda)$ and similarly for $V_{\Lambda,j}$. Recall that we are under the assumption that the L^2-spectrum of D does not contain 0. Thus 0 is never in the support of any of the measures μ_j. We are also under the assumption that α is supported away from 0. Using this and some elementary computation one proves that $\|\alpha V_j \psi - \alpha V_{\Lambda,j}\|$ is exponentially decreasing in Λ. Thus $\alpha V_{\Lambda,j} \xrightarrow{\Lambda \to \infty} \alpha V_j$ in norm and therefore $(\alpha \otimes \psi_1) V_\Lambda \psi_2 \xrightarrow{\Lambda \to \infty} (\alpha \otimes \psi_1) V \psi_2$ in norm, which is what we wanted to show.

Next we tackle (2). It suffices to work under the assumption that $\alpha \equiv 1$, so that we are looking at $\psi_1 V \psi_2$ with $d(\operatorname{supp}\psi_1, \operatorname{supp}\psi_2) \geqslant \delta > 0$. Let $P : \mathcal{L}^2(\mathbb{R}_{\geqslant} \times M) \to \mathcal{L}^2(\mathbb{R}_{\geqslant} \times M)$ be the operator of multiplication by the characteristic function of the t-interval $[0, \epsilon\delta]$. Then $\psi_1 V \psi_2 = P \psi_1 V \psi_2 + (\operatorname{Id} - P)\psi_1 V \psi_2$. The second summand on the right-hand side is compact by the same argument we have employed for (1). Thus, it suffices to show that the norm of

$P\psi_1 V\psi_2$, as an operator from $\mathcal{L}^2(M) \to \mathcal{L}^2(\mathbb{R}_\geqslant \times M)$, is less than ϵ. Write $\psi_1 V\psi_2$ as

$$\psi_1 \left(\frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \hat{h}_t(\omega)\, e^{i\omega D}\, d\omega \right) \psi_2$$

with $h_t(\lambda) := \sqrt{2|\lambda|}e^{-t|\lambda|}$. From the assumption $d(\mathrm{supp}\psi_1, \mathrm{supp}\psi_2) \geqslant \delta > 0$ and the propagation ω of $e^{i\omega D}$, this is equal to

$$\psi_1 \left(\frac{1}{\sqrt{2\pi}} \int_{|\omega|\geqslant\delta} \hat{h}_t(\omega)e^{i\omega D}\, d\omega \right) \psi_2.$$

Fix $s \in \mathcal{L}^2(M)$. Then

$$\|P\psi_1 V\psi_2(s)\|^2_{\mathcal{L}^2(\mathbb{R}_\geqslant \times M)} \leqslant \int_0^{\epsilon\delta} \left(\frac{1}{\sqrt{2\pi}} \int_{|\omega|\geqslant\delta} |\hat{h}_t(\omega)|d\omega \right)^2 \|\psi_1\|^2_\infty \|\psi_2\|^2_\infty \|s\|^2_{\mathcal{L}^2}\, dt.$$

Ir remains to show that $\int_0^{\epsilon\delta} ((1/\sqrt{2\pi}) \int_{|\omega|\geqslant\delta} |\hat{h}_t(\omega)|d\omega)^2 dt$ is small. However, from the explicit computation of $\hat{h}_t$ we see that this is less than

$$C \int_0^{\epsilon\delta} \left(\int_{|\omega|\geqslant\delta} |\omega|^{-3/2}\, d\omega \right)^2 dt \quad \text{with } C > 0.$$

Because this latter term is equal to $C\epsilon\delta(16/\delta) = 16C\epsilon$, the proof of Proposition 4.18 is complete.

We now prove Proposition 4.19. We want to show that $(|D| + \partial_t)/(D - \partial_t)$ belongs to $D^*(\mathbb{R} \times M)^\Gamma$. We must prove that this operator is a norm limit of operators that are pseudo-local and of finite propagation.

We write $(|D| + \partial_t)/(D - \partial_t)$ as $|D|/(D - \partial_t) + \partial_t/(D - \partial_t)$ and deal with the two summands separately.

We think of $(D - \partial_t)^{-1} \colon L^2 \to H^1$ as bounded operator from L^2 to the Sobolev space H^1, and we will compose it with $\partial_t \colon H^1 \to L^2$ or $|D| \colon H^1 \to L^2$ as bounded operator from H^1 to L^2.

We will show that for $\mathcal{D} := \left(\begin{smallmatrix} 0 & D+\partial_t \\ D-\partial_t & 0 \end{smallmatrix} \right)$ the operator $\mathcal{D}^{-1} \colon L^2 \to H^1$ can be approximated by operators F_ϵ of finite propagation in the operator norm from L^2 to the Sobolov space H^1, such that the commutator $[F_\epsilon, \phi] \colon L^2 \to H^1$ is compact as an operator from L^2 to H^1 whenever ϕ is (multiplication by) a compactly supported continuous function. The same is then true for its corner $(D - \partial_t)^{-1}$.

Secondly, we will show that also $\partial_t \colon H^1 \to L^2$ and $|D| \colon H^1 \to L^2$ can be approximated as operators from H^1 to L^2 by finite propagation operators such that the commutator of the approximating operators with compactly supported functions is compact as operator from H^1 to L^2.

Having achieved this, the compositions will have the same required property.

Of the three operators to study, ∂_t itself has propagation zero, and the commutator $[\partial_t, \phi]$ is multiplication with the compactly supported function $\partial_t\phi$, which as operator from H^1 to L^2 is compact by the Rellich lemma.

Next we study $\mathcal{D}^{-1}$. As $\mathcal{D}$ is an invertible elliptic operator of first order, we can and will choose on H^1 the norm such that $\mathcal{D} \colon H^1 \to L^2$ is an isometry. Let f be an odd smooth bounded function equal to $1/x$ on the spectrum of the invertible self-adjoint operator $\mathcal{D}$. Note that f, and thus its Fourier transform $\hat{f}$, lies in $L^2(\mathbb{R})$. We consider the function $x \mapsto xf(x)$ and we arrange that $g(x) := xf(x) - 1$ is compactly supported. Thus its Fourier transform $\hat{g}(\xi)$ will be in the Schwartz space $\mathcal{S}$. This means that $\partial_\xi \hat{f} - \frac{1}{\sqrt{2\pi}}\delta_0$ is an element in $\mathcal{S}$; we deduce from this that $\hat{f}$ is bounded, smooth outside 0, odd and of Schwartz class as $|\xi| \to +\infty$. Write $\hat{f} = g_\epsilon + v_\epsilon$ with g_ϵ odd and compactly supported and $v_\epsilon \in \mathcal{S}$ with the property that $|\partial_x v_\epsilon|_{L^1(\mathbb{R})} < \epsilon$.

We achieve this by setting $v_\epsilon := \phi_\epsilon \hat{f}$ with an even smooth cutoff function ϕ which vanishes in a sufficiently large neighborhood of 0 and which has uniformly small derivative (using that $\hat{f}$ is of Schwartz class at $\pm\infty$). Then $f = h_\epsilon + w_\epsilon$ with h_ϵ having compactly supported Fourier transform g_ϵ and $w_\epsilon \in \mathcal{S}$ with the property that $|xw_\epsilon(x)|_\infty < 2\pi\epsilon$. We deduce that $f(\mathcal{D}) = \mathcal{D}^{-1}$ and

$$\|f(\mathcal{D}) - h_\epsilon(\mathcal{D})\|_{L^2 \to H^1} = \|w_\epsilon(\mathcal{D})\|_{L^2 \to H^1} = \|\mathcal{D}w_\epsilon(\mathcal{D})\|_{L^2 \to L^2} \leqslant |xw_\epsilon(x)|_\infty < 2\pi\epsilon.$$

Finally, $h_\epsilon(\mathcal{D})$ is of finite propagation by unit propagation speed for the Dirac type operator $\mathcal{D}$.

Let now ϕ be a compactly supported smooth function on $\mathbb{R} \times M$. Bearing in mind the finite propagation (say R_ϵ) of $h_\epsilon(\mathcal{D})$, we choose a compactly supported function ψ which takes the value 1 on the R_ϵ-neighborhood of the support of ϕ. Then $h_\epsilon(\mathcal{D})\phi = \psi h_\epsilon(\mathcal{D})\phi$ and $\phi h_\epsilon(\mathcal{D}) = \phi h_\epsilon(\mathcal{D})\psi$. Choose a compact spin manifold X with an open subset U which is isometric (preserving the spin structure) to an open neighborhood V of the support of ψ. To construct X, take, for example, the double of a compact 0-codimensional submanifold with boundary of $\mathbb{R} \times M$ containing the support of ψ. Then (again by unit propagation speed for Dirac operators) the operators $\phi h_\epsilon(\mathcal{D})\psi$ and $\psi h_\epsilon(\mathcal{D})\phi$ are unitarily equivalent to the corresponding operators on the compact manifold X; we see in this way that the commutator $[\phi, h_\epsilon(\mathcal{D})]$ is unitarily equivalent to $[\phi_X, h_\epsilon(D_X)]\colon H^1(X) \to L^2(X)$. Here ϕ_X is the function ϕ transported to X via the isometry, and D_X is the Dirac operator on X. Now it is a classical fact that $h_\epsilon(D_X)$ is a pseudodifferential operator of order 1. This follows, for example, from [**41**, Theorem XII.1.3]. Strictly speaking, we write $h_\epsilon(D_X) = D_X w_\epsilon(\sqrt{D_X^2})$ which is possible because we made sure that $h_\epsilon(x)$ is an odd function. Our original $f(x)$ is smooth and equal to $1/x$ for x large, hence is a symbol of order -1. More precisely, it belongs to $S_{1,0}^{-1}(\mathbb{R})$ in the sense of [**41**, Lemma XII.1.2]. Now $h_\epsilon(x)$ differs from $f(x)$ by the Fourier transform of a Schwartz function, that is, by a Schwartz function, that is, also belongs to $S_{1,0}^{-1}$. As h_ϵ is odd, w_ϵ (satisfying $xw_\epsilon(|x|) = h_\epsilon(x)$) is smooth and belongs to $S_{1,0}^{-2}$. By Seeley's theorem on complex powers of elliptic operators (or the special proof given in [**41**, Section XII.1]), $\sqrt{D_X^2}$ is a positive pseudodifferential operator of order 1 with scalar valued principal symbol on the compact manifold X. Thus, all the hypotheses of [**41**, Theorem XII.1.3] are fulfilled and we conclude that $w_\epsilon(\sqrt{D_X^2})$ is a pseudodifferential operator of order -2 and $h_\epsilon(D_X) = D_X w_\epsilon(\sqrt{D_X^2})$ is a pseudodifferential operator of order -1. By standard results of the pseudodifferential calculus this implies that its commutator with the smooth function ϕ_X is a pseudodifferential operator of order -2 (this is a direct consequence of the short exact sequence defined by the principal symbol and the formula for the principal symbol of a composition). So, up to unitary equivalence, $[h_\epsilon(\mathcal{D}), \phi]$ can be written as composition of the bounded operators $[h_\epsilon(D_X), \phi_X]\colon L^2 \to H^2$ and $i\colon H^2 \to H^1$ where the latter operator is compact by the Rellich lemma on the compact manifold X. Therefore $[h_\epsilon(\mathcal{D}), \phi]$ indeed is compact, as we had to show. Then also the commutators with arbitrary continuous compactly supported functions are compact because the smooth functions are dense in sup-norm in C_0.

Finally, we treat $|D|\colon H^1 \to L^2$. Note that this should really be written as $\mathrm{id}_{L^2(\mathbb{R})} \otimes |D|$, which is not a function of the Dirac operator on $\mathbb{R} \times M$; this makes the analysis slightly more complicated.

We begin by analyzing the operator $|D|$ acting on M. We keep considering $|D|$ as a bounded operator from H^1 to L^2. As above, we can write $|D| = k_\epsilon(D) + u_\epsilon(D)$ where u_ϵ now is an even Schwartz function such that $\|u_\epsilon(D)\| < \epsilon$ (even when considered as operator $L^2 \to L^2$) and such that $k_\epsilon(D)$ has finite propagation, say R_ϵ. Then, as above, for a smooth compactly supported function ϕ_1 on M, the commutator $[k_\epsilon(D), \phi_1]$ is unitarily equivalent to $[k_\epsilon(D_X), \phi_{1,X}]$ for a compact manifolfd X. And, exactly with the same reasoning as above, $k_\epsilon(x)$ is an even symbol of order 1, so that $k_\epsilon(D_X)$ is a pseudodifferential operator of order 1. Therefore $[k_\epsilon(D_X), \phi_{1,X}]$

is a pseudodifferential operator of order 0, defining a bounded operator $L^2 \to L^2$. So the same is true for $[k_\epsilon(D), \phi_1]$ (which is additionally supported on a compact subset of M).

Now we return to $\mathbb{R} \times M$. Note that the precise meaning of $\mathrm{id}_{L^2(\mathbb{R})} \otimes D \colon H^1(\mathbb{R} \times M) \to L^2(\mathbb{R} \times M)$ is the composition of the (bounded) embedding $H^1(\mathbb{R} \times M) \hookrightarrow L^2(\mathbb{R}) \otimes H^1(M)$ with the bounded operator $\mathrm{id}_{L^2(\mathbb{R})} \otimes D \colon L^2(\mathbb{R}) \otimes H^1(M) \to L^2(\mathbb{R} \times M)$. We will use this notation throughout. We write

$$\mathrm{id}_{L^2(\mathbb{R})} \otimes |D| = \mathrm{id}_{L^2(\mathbb{R})} \otimes k_\epsilon(D) + \mathrm{id}_{L^2(\mathbb{R})} \otimes u_\epsilon(D) \colon H^1(\mathbb{R} \times M) \longrightarrow L^2(\mathbb{R} \times M).$$

The first summand on the right-hand side, a bounded operator $H^1 \to L^2$, has finite propagation whereas the second has small norm as on operator from $H^1 \to L^2$. Thus, we are left with the task of proving that $\mathrm{id}_{L^2(\mathbb{R})} \otimes k_\epsilon(D)$ is pseudolocal as an operator from H^1 to L^2. Given compactly supported smooth functions ϕ_2 on $\mathbb{R}$ and ϕ_1 on M, the commutator $[\mathrm{id}_{L^2(\mathbb{R})} \otimes k_\epsilon(D), \phi_2\phi_1]$ equals $\phi_2 \otimes [k_\epsilon(D), \phi_1]$ which factors as the inclusion $H^1 \to L^2$ composed with the bounded operator $\phi_2 \otimes [k_\epsilon(D), \phi_1] \colon L^2 \to L^2$. As, in addition, this commutator is compactly supported, the Rellich lemma implies that this composition is compact as an operator from H^1 to L^2. As smooth compactly supported functions of the form $\phi_1 \otimes \phi_2$ are dense in all continuous functions of compact support, this finishes the proof of Proposition 4.19.

REMARK 4.21. We use the calculus of pseudodifferential operators here just for convenience. In [**29**], we generalize the assertions to perturbations of Dirac type operators which are not necessarily pseudodifferential, replacing the pseudodifferential arguments by purely functional analytic ones.

5. *Mapping the positive scalar curvature sequence to analysis*

In this section, we finally prove Theorem 1.28. One part of this theorem is the construction and commutativity of the following diagram (1.15).

$$
\begin{array}{ccccccccc}
\longrightarrow & \Omega^{\mathrm{spin}}_{n+1}(B\Gamma) & \longrightarrow & R^{\mathrm{spin}}_{n+1}(B\Gamma) & \longrightarrow & \mathrm{Pos}^{\mathrm{spin}}_{n}(B\Gamma) & \longrightarrow & \Omega^{\mathrm{spin}}_{n}(B\Gamma) & \longrightarrow \\
& \downarrow \beta & & \downarrow \mathrm{Ind}_\Gamma & & \downarrow \rho_\Gamma & & \downarrow \beta & \\
\longrightarrow & K_{n+1}(B\Gamma) & \longrightarrow & K_{n+1}(C^*_r\Gamma) & \longrightarrow & K_{n+1}(D^*_\Gamma) & \longrightarrow & K_n(B\Gamma) & \longrightarrow
\end{array}
$$

First of all, we need to give a precise definition for the vertical homomorphisms. Consider an element $[Y, f \colon Y \to B\Gamma, g_\partial] \in R^{\mathrm{spin}}_{n+1}(B\Gamma)$. Let g_Y be a Riemannian metric on Y extending g_∂. We consider the Galois Γ-covering $W := f^* E\Gamma$, endowed with the lifted metric g_W. We consider (W_∞, g), the complete Riemannian manifold with cylindrical ends associated to W. We wish to define $\mathrm{Ind}_\Gamma([Y, f \colon Y \to B\Gamma, g_\partial]) \in K_{n+1}(C^*_\Gamma)$; to this end consider the relative coarse index class $\mathrm{Ind}^{\mathrm{rel}}(D_{W_\infty}) \in K_{n+1}(C^*(W \subset W_\infty)^\Gamma)$ and its image $\mathrm{Ind}(D_W) \in K_{n+1}(C^*(W)^\Gamma)$ through the canonical isomorphism $K_{n+1}(C^*(W \subset W_\infty)^\Gamma) \simeq K_{n+1}(C^*(W)^\Gamma)$. We then consider the image of this class through the canonical isomorphism $u_* \colon K_{n+1}(C^*(W \subset W_\infty)^\Gamma) \simeq K_{n+1}(C^*_\Gamma)$ induced by the classifying map $u \colon W \to E\Gamma$. We have denoted this image by $\mathrm{Ind}_\Gamma(D_W)$, see 1.15. We set

$$\mathrm{Ind}_\Gamma([Y, f \colon Y \longrightarrow B\Gamma, g_\partial]) := \mathrm{Ind}_\Gamma(D_W) \in K_{n+1}(C^*_\Gamma).$$

That this index map $R^{\mathrm{spin}}_{n+1}(X) \xrightarrow{\mathrm{Ind}_\Gamma} K_{n+1}(C^*_r\Gamma)$ is well defined, that is, that $\mathrm{Ind}_\Gamma(D_W)$ is bordism invariant, can be proved in many different ways. In future work, we plan to give a treatment of bordism invariance in the spirit of coarse index theory. Alternatively, relying on published previous work, it follows from the compatibility between the coarse index class $\mathrm{Ind}^{\mathrm{rel}}(D_{W_\infty})$ and the Mishchenko–Fomenko index class, either obtained on the

associated manifold with cylindrical ends or à la APS, see Proposition 2.4, then applying to the Mishchenko–Fomenko index class [**3**] or [**21**], where [**3**] employs a relative index theorem and [**21**] is based on a gluing formula for index classes. We remark that it would also be possible to state and prove a relative index theorem similar to [**3**] but in coarse geometry and then apply Bunke's argument directly to the coarse index class $\mathrm{Ind}_\Gamma(D_W)$.

Consider now an element $[Z, f\colon Z \to B\Gamma, g_Z] \in \mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma)$; we consider the Γ-covering $M := f^*B\Gamma$ and we endow it with the lifted metric g. Then, by definition,

$$\rho_\Gamma[Z, f\colon Z \longrightarrow B\Gamma, g_Z] = \rho_\Gamma(g) \in K_{n+1}(D_\Gamma^*).$$

The fact that ρ_Γ is well defined follows from Corollary 1.16. Finally, let us recall the definition of the map $\beta\colon \Omega_{n+1}^{\mathrm{spin}}(B\Gamma) \to K_{n+1}(B\Gamma)$, as given by Higson and Roe [**11**–**13**]. Consider an element $[Z, f\colon Z \to B\Gamma] \in \Omega_{n+1}^{\mathrm{spin}}(B\Gamma)$ and let $M := f^*E\Gamma$, endowed with any Γ-invariant Riemannian metric g. We consider the class $[D_M] \in K_n(D^*(M)^\Gamma/C^*(M)^\Gamma) \simeq K_{n+1}(M/\Gamma) \equiv K_{n+1}(Z)$ and we push it forward through f_* to $K_{n+1}(B\Gamma)$:

$$\beta[Z, f\colon Z \longrightarrow B\Gamma] := f_*[D_M] \in K_{n+1}(B\Gamma).$$

We must now tackle the commutativity of the diagram. We consider the three distinct squares of the diagram from left to right. The commutativity of the first square, which is implicitly discussed in the work of Higson–Roe, follows from the definition of the $C_r^*\Gamma$-index class, as given in Subsections 1.3 and 1.4. The commutativity of the second square is a direct consequence of our APS index theorem, see Corollary 1.15 and more precisely formula (1.10). The commutativity of the third square is again a direct consequence of the definitions.

The remaining part of Theorem 1.28 deals with a compact space X with fundamental group Γ and universal covering $\tilde{X}$. Here one uses the canonical isomorphisms $R_*^{\mathrm{spin}}(X) = R_*^{\mathrm{spin}}(\Gamma)$ (the structure groups depend only on the fundamental group), $K_*(X) = K_{*+1}(D^*(\tilde{X})^\Gamma/C^*(\tilde{X})^\Gamma)$ and $K_*(C^*(\tilde{X})) = K_*(C_r^*\Gamma)$. Then the proof of (1.14) is exactly parallel to the proof of (1.15) once we use the results explained in Remark 1.17. This finishes the proof of Theorem 1.28.

References

1. P. ANTONINI, 'The Calderon projection over c^*-algebras', Preprint, 2013, arXiv:1307.1968.
2. B. BLACKADAR, *K-theory for operator algebras*, 2nd edn, Mathematical Sciences Research Institute Publications 5 (Cambridge University Press, Cambridge, 1998).
3. U. BUNKE, 'A K-theoretic relative index theorem and Callias-type Dirac operators', *Math. Ann.* 303 (1995) 241–279.
4. A. CONNES and H. MOSCOVICI, 'Cyclic cohomology, the Novikov conjecture and hyperbolic groups', *Topology* 29 (1990) 345–388.
5. B. HANKE, D. PAPE and T. SCHICK, 'Codimension two index obstructions to positive scalar curvature', Preprint, 2014, arXiv:1402.4094.
6. N. HIGSON, 'A note on the cobordism invariance of the index', *Topology* 30 (1991) 439–443.
7. N. HIGSON, E. K. PEDERSEN and J. ROE, 'C^*-algebras and controlled topology', *K-Theory* 11 (1997) 209–239.
8. N. HIGSON and J. ROE, 'A homotopy invariance theorem in coarse cohomology and K-theory', *Trans. Amer. Math. Soc.* 345 (1994) 347–365.
9. N. HIGSON and J. ROE, 'On the coarse Baum–Connes conjecture', *Novikov conjectures, index theorems and rigidity, vol. 2 (Oberwolfach, 1993)*, London Mathematical Society Lecture Note Series 227 (Cambridge University Press, Cambridge, 1995) 227–254.

10. N. HIGSON and J. ROE, *Analytic K-homology*, Oxford Mathematical Monographs (Oxford University Press, Oxford, 2000). Oxford Science Publications.

11. N. HIGSON and J. ROE, 'Mapping surgery to analysis. I. Analytic signatures', *K-Theory* 33 (2005) 277–299.

12. N. HIGSON and J. ROE, 'Mapping surgery to analysis. II. Geometric signatures', *K-Theory* 33 (2005) 301–324.

13. N. HIGSON and J. ROE, 'Mapping surgery to analysis. III. Exact sequences', *K-Theory* 33 (2005) 325–346.

14. N. HIGSON and J. ROE, 'K-homology, assembly and rigidity theorems for relative eta invariants', *Pure Appl. Math. Q.* 6 (2, Special Issue: In honor of Michael Atiyah and Isadore Singer) (2010) 555–601.

15. N. HIGSON, J. ROE and G. YU, 'A coarse Mayer–Vietoris principle', *Math. Proc. Cambridge Philos. Soc.* 114 (1993) 85–97.

16. D. W. KAMMLER, *A first course in Fourier analysis*, 2nd edn (Cambridge University Press, Cambridge, 2007).

17. G. G. KASPAROV, 'Topological invariants of elliptic operators. I. K-homology', *Izv. Akad. Nauk SSSR Ser. Mat.* 39 (1975) 796–838.

18. E. LEICHTNAM, J. LOTT and P. PIAZZA, 'On the homotopy invariance of higher signatures for manifolds with boundary', *J. Differential Geom.* 54 (2000) 561–633.

19. E. LEICHTNAM and P. PIAZZA, 'The b-pseudodifferential calculus on Galois coverings and a higher Atiyah–Patodi–Singer index theorem', *Mém. Soc. Math. Fr. (N.S.)* 68 (1997) iv+121.

20. E. LEICHTNAM and P. PIAZZA, 'Homotopy invariance of twisted higher signatures on manifolds with boundary', *Bull. Soc. Math. France* 127 (1999) 307–331.

21. E. LEICHTNAM and P. PIAZZA, 'On higher eta-invariants and metrics of positive scalar curvature', *K-Theory* 24 (2001) 341–359.

22. J. LOTT, 'Higher eta-invariants', *K-Theory* 6 (1992) 191–233.

23. J. LOTT, 'Superconnections and higher index theory', *Geom. Funct. Anal.* 2 (1992) 421–454.

24. R. B. MELROSE, *The Atiyah–Patodi–Singer index theorem*, Research Notes in Mathematics 4 (A K Peters Ltd, Wellesley, MA, 1993).

25. H. MORIYOSHI and T. NATSUME, 'The Godbillon–Vey cyclic cocycle and longitudinal Dirac operators', *Pacific J. Math.* 172 (1996) 483–539.

26. H. MORIYOSHI and P. PIAZZA, 'Eta cocycles, relative pairings and the Godbillon–Vey index theorem', *Geom. Funct. Anal.* 22 (2012) 1708–1813.

27. D. PAPE, 'Index theory and positive scalar curvature', PhD Thesis, Mathematisches Institut, Georg-August-Universität Göttingen, 2011.

28. P. PIAZZA and T. SCHICK, 'Bordism, rho-invariants and the Baum–Connes conjecture', *J. Noncommut. Geom.* 1 (2007) 27–111.

29. P. PIAZZA and T. SCHICK, 'The surgery sequence, k-theory and the signature operator', Preprint, 2013, arXiv 1309.4370.

30. M. RAMACHANDRAN, 'von Neumann index theorems for manifolds with boundary', *J. Differential Geom.* 38 (1993) 315–349.

31. J. ROE, 'Partitioning noncompact manifolds and the dual Toeplitz problem', *Operator algebras and applications*, vol. 1, London Mathematical Society Lecture Note Series 135 (Cambridge University Press, Cambridge, 1988) 187–228.

32. J. ROE, 'A note on the relative index theorem', *Quart. J. Math. Oxford Ser.* (2) 42 (1991) 365–373.

33. J. ROE, *Index theory, coarse geometry, and topology of manifolds*, CBMS Regional Conference Series in Mathematics 90 (Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996).

34. J. ROE, "Comparing analytic assembly maps', *Q. J. Math.* 53 (2002) 241–248.

35. J. ROE, 'Positive curvature, partial vanishing theorems and coarse indices', Preprint, 2012, arXiv:1210.6100v1.

36. J. ROSENBERG and S. STOLZ, 'Metrics of positive scalar curvature and connections with surgery', *Surveys on surgery theory*, vol. 2, Annals of Mathematics Studies 149 (Princeton University Press, Princeton, NJ, 2001) 353–386.

37. T. SCHICK, 'L^2-index theorems, KK-theory, and connections', *New York J. Math.* 11 (2005) 387–443 (electronic).

38. T. SCHICK and M. E. ZADEH, 'Large scale index of multi-partitioned manifolds', Preprint, 2013, arXiv:1308.0742.

39. P. SIEGEL, 'The Mayer–Vietoris sequence for the analytic structure group', Preprint, 2012, arXiv:1212.0241.

40. E. M. STEIN and R. SHAKARCHI, 'An introduction', *Fourier analysis*, Princeton Lectures in Analysis 1 (Princeton University Press, Princeton, NJ, 2003).

41. M. E. TAYLOR, *Pseudodifferential operators*, Princeton Mathematical Series 34 (Princeton University Press, Princeton, NJ, 1981).

42. C. WAHL, 'Higher ρ-invariants and the surgery structure set', *J. Topol.* 6 (2013) 154–192.

43. N. E. WEGGE-OLSEN, 'A friendly approach', *K-theory and C*-algebras*, Oxford Science Publications (The Clarendon Press Oxford University Press, New York, 1993).

44. F. WU, 'The higher Γ-index for coverings of manifolds with boundaries', *Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995)*, Fields Institute Communications 17 (American Mathematical Society, Providence, RI, 1997) 169–183.

45. Z. Xie and G. Yu, 'Positive scalar curvature, higher rho invariants and localization algebras', Preprint, 2013, arXiv:1302.4418.
46. M. E. Zadeh, 'Index theory and partitioning by enlargeable hypersurfaces', *J. Noncommut. Geom.* 4 (2010) 459–473.

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