# Dirac operators, heat kernels and microlocal analysis. Part I: Family index theory.

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Let $(M^n, g)$ be an n-dimensional riemannian manifold. The metric $g$ defines in a natural way a class of first order differential operators, the *generalized Dirac operators*. The main goal of this survey is to explain how the *analytic* properties of these operators can be used to investigate the *geometry* of the manifold $M$. As the title suggests, we shall concentrate on those analytic properties that can be derived from the study of the heat equation on the manifold $M$ and the use of techniques from microlocal analysis, more precisely the use of pseudodifferential operators. Many of the results presented below are nowadays considered as classical, others, expecially those dealing with manifolds with boundary, are very recent. In this first part$^{(*)}$ we shall present the heat-kernel approach to the Atiyah-Singer index theorem, Bismut's proof of the family index theorem, the Atiyah-Patodi-Singer index theorem on manifolds with boundary, the $b$-pseudodifferential calculus and the family index theorem on manifolds with boundary.

## 1. Generalized Dirac operators

We consider the Clifford bundle $\mathrm{Cl}(M)$ associated to the riemannian manifold $(M, g)$. This is the bundle with fibre at $x \in M$ equal to the complex Clifford algebra $\mathrm{Cl}(T^*_x M, g^*_x)$. $\mathrm{Cl}(M)$ is a smooth bundle of complex unital associative algebras. For the definition of Clifford algebra associated to a vector space $V$ endowed with a symmetric bilinear form $g$ we follow the convention that $\alpha \cdot \beta + \beta \cdot \alpha = 2g(\alpha, \beta)$ for each $\alpha, \beta \in V$.

Let $E$ be a smooth vector bundle on $M$. We assume that $E$ is a bundle of Clifford modules; thus for each $x \in M$ there exists a linear map $\mathrm{cl}_x : \mathrm{Cl}(T^*_x M) \to \mathrm{End}(E_x)$ satisfying $\mathrm{cl}_x(\alpha \cdot \beta) = \mathrm{cl}_x(\alpha) \circ \mathrm{cl}_x(\beta)$ and varying smoothly with $x$. We obtain a bundle map $\mathrm{cl} : \mathrm{Cl}(M) \to \mathrm{End}(E)$ which is a representation when restricted to each fibre. In particular we obtain a map $C^\infty(M, T^*M \otimes E) \to C^\infty(M, E)$, $\;\xi \otimes e \to \mathrm{cl}(\xi)e$. We denote this map by $c$.

Let $\nabla^E : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ be a connection on the bundle $E$ and let us consider the operator $\frac{1}{i}\nabla^E$. A generalized Dirac operator $D\!\!\!/$ on $M$ is obtained as the composition

$$C^\infty(M, E) \xrightarrow{\frac{1}{i}\nabla^E} C^\infty(M, T^*M \otimes E) \xrightarrow{c} C^\infty(M, E).$$

$^{(*)}$ Part II and Part III will deal respectively with the surgery problem for heat-kernel invariants and noncommutative geometry on Galois coverings.

Thus $\not{D} = c \circ \frac{1}{i}\nabla^E$ is a first order differential operator acting on the smooth sections of $E$. Here are some of the most basic properties of $\not{D}$. If $\{e_j\}$ is a local orthonormal frame of $TM$ and $\{e^j\}$ is the dual orthonormal frame defined by the metric then

$$\not{D} = \sum_j \mathrm{cl}(e^j)\nabla^E_{ie_j}.$$

The principal symbol of $\not{D}$ is given by Clifford multiplication: $\sigma_1(\not{D})(\xi) = \mathrm{cl}(\xi), \xi \in T^*M$. This implies that $\not{D}$ is an *elliptic* operator: $\sigma_1(\not{D})(\xi) \in \mathrm{Iso}(E_x, E_x)\, \forall \xi \neq 0$ in $T^*_x M$. If $E$ is endowed with a hermitian metric $h$ and the Clifford action is unitary then for the square of the Dirac operator we obtain $\sigma_2(\not{D} \circ \not{D}^*) = \sigma_2(\not{D}^2)(\xi) = ||\xi||^2 \mathrm{Id}_{E_x}$, $\xi \in T^*_x M$. Second order differential operators satisfying the second equation are called *generalized laplacians*. We shall also assume that $\nabla^E$ is unitary with respect to $h$ and that it is a *Clifford connection*, this means that it is compatible with respect to the Clifford action: $[\nabla^E_X, \mathrm{cl}(\alpha)](e) = \mathrm{cl}(\nabla^M_X \alpha)(e)\, \forall e \in C^\infty(M,E)$, $\forall X \in C^\infty(M,TM)$, $\forall \alpha \in C^\infty(M,T^*M)$ and with $\nabla^M$ denoting the Levi-Civita connection. Under the above assumptions the Dirac operator $\not{D}$ is formally self-adjoint with respect to the $L^2$-scalar product induced by $g$ and $h$ on $C^\infty(M,E)$: $(\not{D})^* = \not{D}$. Thus, thought as an unbounded operator on $L^2(M,E)$ with domain equal to $H^1(M,E)$, the Dirac operator is self-adjoint. From now on we shall always assume the Clifford action unitary and the connection $\nabla^E$ unitary and Clifford. If $M$ is even dimensional it is natural to consider Clifford modules which are graded. This means that $E = E^+ \oplus E^-$ and that $\mathrm{cl}_x(\xi) \in \mathrm{Hom}(E^\pm_x, E^\mp_x)$ for each $x \in M$ and for each $\xi \in T^*_x M$. A generalized Dirac operator as above will then be of the form

$$\not{D} = \begin{pmatrix} 0 & \not{D}^- \\ \not{D}^+ & 0 \end{pmatrix}, \quad (\not{D}^+)^* = \not{D}^- \tag{1.1}$$

**Examples.** Among the standard examples of generalized Dirac operators of the above type are: the de Rham operator $d + d^*$, the Dolbeault operator $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ on a Kähler manifold, the Dolbeault operator on a Kähler manifold with values in a holomorphic hermitian vector bundle $W$, $\sqrt{2}(\overline{\partial}_W + \overline{\partial}^*_W)$, the signature operator and *the* Dirac operator $\not{D}_{\mathrm{spin}}$ on a spin manifold. In the first 3 examples we take as $E$ the obvious exterior powers of the cotangent bundle and $\nabla^E$ the induced Levi-Civita connection. The Clifford action is given in terms of exterior and interior multiplication. The signature operator $\not{D}_{\mathrm{sign}}$ on a $4k$-dimensional manifold $M$ is defined as the de Rham operator $d + d^*$ acting on the vector bundles $E^\pm$ equal to $\pm 1$-eigenspaces of the involution $\tau : \Lambda(T^*M) \to \Lambda(T^*M)$ defined as $i^{p(p-1)+k}\star$ on $p$-forms. The signature operator is graded in the sense of (1.1).

## 2. The Atiyah-Singer index theorem: the heat-kernel proof.

Let $M$ be a *closed compact oriented* manifold and let $\not{D}$ be a generalized Dirac operator. As already remarked the operator $\not{D}$ is elliptic. Elliptic differential operator of order $k$ on closed compact manifolds are *Fredholm*$^{(*)}$ when considered as bounded operator between Sobolev spaces: $H^m(M,E) \to H^{(m-k)}(M,E)$. The index is independent of $m$ because of *elliptic regularity.* Of course since $\not{D}$ is formally self-adjoint we find easily that $\mathrm{ind}(\not{D}) = 0$. In order to have a non-zero index we need to consider *even* dimensional manifolds and $\mathbb{Z}_2$-graded Clifford modules $E = E^+ \oplus E^-$. In this case we can consider

$$\mathrm{ind}(\not{D}^+) = \dim(\ker(\not{D}^+)) - \dim(\mathrm{coker}(\not{D}^+)) = \dim(\ker(\not{D}^+)) - \dim(\ker(\not{D}^-)).$$

Elliptic regularity implies that $\mathrm{Ker}(\not{D}^\pm) \subset C^\infty(M, E^\pm)$. Since the index does not change under a homotopy through Fredholm operators it is conceivable that this number will only depend on the *topological properties* of the manifold $M$ and of the vector bundle $E$. The celebrated Atiyah-Singer index theorem gives a precise formula for $\mathrm{ind}(\not{D}^+)$ in terms of characteristic cohomology classes associated to the manifold $M$ and the bundle $E$. We shall now describe the content of this beautiful result and its proof using heat-kernel techniques.

Using the functional calculus associated to the essentially self-adjoint operator $\not{D}^2$ one can define the heat operator $\exp(-t\not{D}^2)$. One of the fundamental properties of $\exp(-t\not{D}^2)$ is that for each $t > 0$ it is an integral operator with smooth Schwartz kernel (it is a *smoothing operator*). This means that

$$(\exp(-t\not{D}^2)f)(x) = \int_M K_t(x,y)f(y) \quad \forall f \in L^2(M,E)$$

with the Schwartz kernel $K_t$, the *heat kernel*, a smooth section of the bundle $\mathrm{HOM}(E,E) \to M \times M$ with fiber at $(x,y)$ equal to $\mathrm{Hom}(E_x, E_y)$. Notice that if $\Delta \leftrightarrow M$ is the diagonal in $M \times M$, then $K_t|_\Delta \in C^\infty(M, \mathrm{End}(E))$. Another fundamental property of the heat operator, that justifies its name, is that $F(t,\cdot) \equiv \exp(-t\not{D}^2)f$ provides a solution of the heat equation with initial condition at $t = 0$ equal to $f \in C^\infty(M,E)$. We shall often use the notation $\exp(-t\not{D}^2)$ for both the heat operator and the heat kernel. We use the volume form associated to $g$ to identify $C^\infty$-functions and smooth densities on $M$. In general a smoothing operator $A$ acting on the Hilbert space $L^2(M,E)$ is *trace class*, with trace equal to the integal of the kernel $K(A)$ over the diagonal in $M \times M$ (Lidski's theorem). This important remark applies to both $\exp(-t\not{D}^-\not{D}^+)$ and $\exp(-t\not{D}^+\not{D}^-)$, thus $\mathrm{Tr}(\exp(-t\not{D}^-\not{D}^+)) = \sum_{\lambda \in \mathrm{spec}(\not{D}^-\not{D}^+)} e^{-t\lambda}$ is finite and similarly for $\mathrm{Tr}(\exp(-t\not{D}^+\not{D}^-))$; moreover, by Lidski's theorem, they are equal to the integral over

$^{(*)}$ Recall that a bounded operator between Banach spaces is Fredholm if the kernel and cokernel are finite dimensional. The index of a Fredholm operator $A$ is by definition $\mathrm{ind}(A) = \dim(\ker A) - \dim(\mathrm{coker}\, A)$. The index is a *continuous* integer valued function on the space of Fredholm operators.

$\Delta \leftrightarrow M$ of $\mathrm{tr}_{E^+}\exp(-t\,\not{D}^-\not{D}^+)|_\Delta$ and $\mathrm{tr}_{E^-}\exp(-t\not{D}^+\not{D}^-)|_\Delta$ respectively. Here $\mathrm{tr}_{E^\pm}$ denotes the vector bundle trace. Consider the *supertrace* of $\exp(-tD^2)$, $\mathrm{STr}\,(\exp(-t\not{D}^2)) \equiv \mathrm{Tr}(\exp(-t\,\not{D}^-\not{D}^+)) - \mathrm{Tr}(\exp(-t\,\not{D}^+\not{D}^-))$. The heat equation, together with the vanishing of the trace on commutators, implies that this difference does not depend on $t$; more precisely

$$\frac{d}{dt}(\mathrm{STr}\,(\exp(-tD^2))) = -\,\mathrm{Tr}[\not{D}^-, \not{D}^+\exp(-\tau\not{D}^-\not{D}^+)] = 0 \qquad (2.1).$$

As $t \to +\infty$ the heat operator converges exponentially, within smoothing operators, to the orthogonal projection onto the null space of $\not{D}^2$. This implies that $\mathrm{STr}\,(\exp(-t\not{D}^2))$ converges exponentially to the supertrace of the projection onto the null space of $\not{D}^2$ which is easily seen to be the index of $\not{D}^+$. Integrating (2.1) from $t$ to $+\infty$ we thus obtain the *McKean-Singer formula*:

$$\mathrm{ind}(\not{D}^+) = \mathrm{STr}\,(\exp(-t\not{D}^2)) \equiv \int_M \mathrm{str}_E(K_t|_\Delta) \quad \forall t > 0. \qquad (2.2)$$

The crucial step in the proof of the index theorem is in showing that as $t \to 0^+$ the vector bundle supertrace of the heat kernel restricted to the diagonal, $\mathrm{str}_E(K_t|_\Delta)$, converges itself to a density on $\Delta \equiv M$ which is *explicitly computable.* We denote this density by $\mathrm{AS}(M, \nabla^M)_{[n]}$, $n$ being the dimension of the manifold $M$ and $\mathrm{AS}(M,\nabla^M) \equiv \mathrm{AS}$ being an explicit *closed* differential form constructed out of the curvature tensors of $M$ and $E$ (*). We denote by [AS] the cohomology class associated to $\mathrm{AS} \in C^\infty(M, \Lambda^* M)$. The Atiyah-Singer index theorem for $\not{D}^+$ then follows by letting $t \to 0^+$ on the right hand side of (2.2), thus equating the integral over $\Delta \equiv M$ of this explicit geometric expression with the index of $\not{D}^+$:

$$\mathrm{ind}(\not{D}^+) \;=\; \int_M \mathrm{AS}_{[n]} \;=\; \int_M \mathrm{AS} \;= <[\mathrm{AS}],[M]>. \qquad (2.3)$$

The last equality in (2.3) makes use of the pairing between de Rham cohomology and singular homology. The equality $\mathrm{ind}(D^+) =< [\mathrm{AS}],[M] >$ is the original form of the Atiyah-Singer index theorem (see [AS 1]); the fact that the right hand-side is obtained by precisely integrating a local form coming from the short-time asymptotic of the heat-kernel is additional information that is crucial in many developments of the theory. One speaks in these cases of *local* Atiyah-Singer index theory (see [ABP][G][BGV]). The index formula can in fact be proved for any elliptic pseudodifferential operator using $K$-theory [AS 1]. However at this level of generality the theorem is not local. Locality is a special feature of generalized Dirac operators associated to unitary Clifford modules endowed with unitary Clifford connections.

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(*) If $\alpha \in C^\infty(M, \Lambda^* M)$ is a differential form on $M$ we denote by $\alpha_{[k]}$ the component of $\alpha$ in $C^\infty(M, \Lambda^k M)$

**3. Geometric applications.**

One can apply the Atiyah-Singer index formula to the examples presented in the first section. Defining $(d+d^*)^+$ and $(\overline{\partial}_W+\overline{\partial}^*_W)^+$ as the restriction of the de Rham and the Dolbeault operator to even differential forms and computing AS in each case gives

$$\mathrm{ind}(d+d^*)^+ =< e(M),[M]> \quad \mathrm{ind}(\not{D}^+_{\mathrm{sign}}) =< L(M),[M]>$$
$$\mathrm{ind}(\overline{\partial}_W+\overline{\partial}^*_W)^+ =< \mathrm{Td}(M)\wedge \mathrm{Ch}(W),[M]> \quad \mathrm{ind}(\not{D}^+_{\mathrm{spin}}) =< \widehat{A}(M),[M]>$$

In these formulas the Euler class, the Hirzebruch L-class, the Todd class, the Chern character of a vector bundle and the A-roof class appear ([Hi]). The last equality explains the *integrality* of the $\widehat{A}$-genus on a spin manifold (this being in fact one of the original motivations of Atiyah and Singer). For the other examples de-Rham-Dolbeault-Hodge theory can be used giving $\mathrm{ind}(d+d^*)^+=\chi(M)$, the Euler characteristic of the manifold $M$, $\mathrm{ind}(\not{D}^+_{\mathrm{sign}})=\mathrm{sign}(M)$, the signature (*) of the manifold $M$, $\mathrm{ind}(\overline{\partial}_W+\overline{\partial}^*_W)^+=\chi(M,W)$, the Euler characteristic of the sheaf of germs of holomorphic sections of $W$. Thus, finally, $\chi(M) =< e(M),[M]>$ $\mathrm{sign}(M) =< L(M),[M]>$, $\chi(M,W) < \mathrm{Td}(M)\wedge\mathrm{Ch}(W),[M]>$. In other words the Atiyah-Singer index theorem implies at once the Chern-Gauss-Bonnet theorem, the Hirzebruch signature theorem and the Riemann-Roch theorem. Other classical applications of the Atiyah-Singer index theorem are to the existence of metrics of positive scalar curvature [GL] and to the computation of the dimension of moduli spaces (most notably the moduli space of anti-self-dual connections on a 4-manifold [DK], of pseudoholomorphic curves on a symplectic manifold [Gr] and of solutions of the Seiberg-Witten equations [KM]).

**4. Bismut's heat-kernel proof of the family index theorem.**

We come now to Bismut's heat-kernel proof of the Atiyah-Singer family index theorem for Dirac operators. Let $\not{D}=(\not{D}_z)_{z\in B}$ be a family of Dirac operators acting on $C^\infty(M_z,E_z)$ and parametrized by a compact manifold $B$. We assume that $(\not{D}_z)_{z\in B}$ depends smoothly on $z$ and that each manifold $M_z$ is closed and compact. Families of this kind arise from smooth fibrations $M\to B$ with fibre $M_z$ over $z\in B$ of the above type and endowed with a vertical riemannian metric $g_{M/B}$. Of course we also need a vector bundle $E$ over $M$ with a vertical Clifford module structure.

**Example.** As a simple example we can consider the family

$$\not{D}_z=\frac{1}{i}\frac{\partial}{\partial\theta}+\frac{z-\frac{1}{2}}{\pi}, \quad z\in S^1=\mathbb{R}/\pi\mathbb{Z}. \tag{4.1}$$

Each operator $\not{D}_z$ acts on $S^1$; the fibration here is $S^1\times S^1\to S^1$ and the bundle $E$ is the line bundle over the torus $S^1\times S^1$ obtained from the trivial line bundle by identifying the

(*) The signature of an orientable $4k$-dimensional manifold is by definition the signature of the symmetric non-degenerate bilinear form $H^{2k}(M,\mathbb{R})\times H^{2k}(M,\mathbb{R})\longrightarrow\mathbb{R}$ sending $(\alpha,\beta)$ into $<\alpha\wedge\beta,[M]>$.

points $(\theta, 0, v)$ and $(\theta, \pi, \exp(-i\theta)v)$. The twisting is necessary in order to obtain a family which is globally defined (i.e. periodic) on $S^1 = \mathbb{R}/\pi\mathbb{Z}$. Later we shall also consider the family $\widetilde{D\!\!\!/}_z$ obtained by changing the sign of the second summand on the right-hand-side of (4.1) and of the twisting in the definition of the bundle $E$.

Suppose now that the fibres of $M \to B$ are of even dimension $n = 2k$ and assume for the time being that the dimensions of $\mathrm{Ker}(D\!\!\!/_z^+)$ and $\mathrm{Ker}(D\!\!\!/_z^-)$ are constant in $z \in B$. We obtain two vector bundles: $\mathrm{Ker}(D\!\!\!/^\pm) = \cup_{z\in B} \mathrm{Ker}(D\!\!\!/_z^\pm)$. The index bundle associated to the family $D\!\!\!/ = (D\!\!\!/_z)_{z\in B}$ is by definition the formal difference $\mathrm{Ind}(D\!\!\!/) = [\mathrm{null}(D\!\!\!/^+)] - [\mathrm{null}(D\!\!\!/^-)]$, an element in the $K$-theory $K^0(B)^{(*)}$ of the base $B$. The cohomology class $\mathrm{Ch}(\mathrm{Ind}(D\!\!\!/))$ is thus well defined at least if the kernel of $D\!\!\!/_z^\pm$ are of constant dimension. On the other hand it can be proved that this latter property can always be ensured up to a smoothing perturbation which does not affect the final answer. Summarizing, given a smooth family $D\!\!\!/ = (D\!\!\!/_z)_{z\in B}$ of Dirac operators on the even dimensional manifolds which are the fibres of a fibration, the cohomology class $\mathrm{Ch}(\mathrm{Ind}(D\!\!\!/))$ is well defined. The problem is once again to give a geometric formula for $\mathrm{Ch}(\mathrm{Ind}(D\!\!\!/))$, an *a priori* analytic object. Notice however that, as before, the class $\mathrm{Ind}(D\!\!\!/) \in K^0(B)$ depends only on the homotopy class of the given family of Fredholm operators.$^{(\dagger)}$

The cornerstone of Bismut's [B] treatment of the family index theorem is the use of an infinite-dimensional version of the superconnection formalism of Quillen [Q]. This formalism suggests the following: instead of considering the family of generalized laplacians $(D\!\!\!/_z^2)$ one considers a family of generalized laplacians *with $B$-differential form coefficients.* We denote by $K_t$ the family of heat-kernels associated to this new family of generalized laplacians: hence for each $z \in B$ and each $t > 0$, $K_t(z) \in C^\infty(M_z \times M_z, \mathrm{HOM}(E_z, E_z) \otimes \Lambda(T_z^*B))$. More globally $K_t \in C^\infty(M \times_\pi M, \pi^*\Lambda^*B \otimes \mathrm{HOM}(E, E))$. The fiber supertraces of these heat-kernels combine to give a *smooth* differential form on $B$. We denote by $\mathrm{STr}(K_t)$ this smooth differential form; thus $(\mathrm{STr}(K_t))(z) \in \Lambda(T_z^*B)$ is obtained by taking the integral over $\Delta_z \leftrightarrow M_z$ of $\overline{\kappa}_t(z)$, with $\overline{\kappa}_t(z)$ defined as the vector-bundle-supertrace of the restriction of the heat-kernel to the diagonal of $M_z \times M_z$. Bismut then proves that $\mathrm{STr}(K_t)$ satisfies the following properties

(i) It is a closed differential form $\forall\, t > 0$.

(ii) It does not depend on $t$ modulo exact forms: $d/dt(\mathrm{STr}(K_t)) = d_B\, \alpha_t$.

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$^{(*)}$ By definition $K^0(B)$ is the Grothendieck group associated to the semigroup of isomorphism classes of vector bundles on $B$. An element $\alpha \in K^0(B)$ should be thought as a formal difference of isomorphism classes of vector bundles: $\alpha = [E] - [F]$. There is a map $\mathrm{Ch} : K^0(B) \otimes \mathbb{R} \to H^{\mathrm{even}}(B, \mathbb{R})$ obtained by associating to $\alpha = [E] - [F]$ the cohomology class $\mathrm{Ch}(E) - \mathrm{Ch}(F)$. This map is an isomorphism.

$^{(\dagger)}$ In fact it can be proved (see [A]) that the index bundle construction explained here provides an isomorphism $[B, \mathcal{F}(H)] \longrightarrow K^0(B)$, with $[B, \mathcal{F}(H)]$ denoting the *homotopy classes* of continuous maps from the parameter space $B$ to the space of Fredholm operators on a Hilbert space $H$.

(iii) It represents $\mathrm{Ch}(\mathrm{Ind}(D))$ in $H^{\mathrm{even}}(B,\mathbb{R})$.
This last property can be proved directly as in Bismut's original argument or by showing, as in [BGV], that the limit as $t \to +\infty$ of $\mathrm{STr}(K_t)$ converges as a differential form on $B$ to the Chern character of the index bundle (a result which is due to Berline and Vergne). Finally, as in the previous section, the limit as $t \to 0^+$ of $\overline{\kappa}_t$ (the vector-bundle-supertrace of the heat kernel $K_t$ restricted to the fiber-diagonal) can be computed , giving an explicit differential form $\omega = \mathrm{AS}_{[n,*]}$ on the total space of the fibration $M \to B$ which is of degree $n$ in the fibre direction. As in the closed case the form $\omega$ comes from a *closed* differential form $\mathrm{AS} \in C^\infty(M, \Lambda^* M)$. Putting everything together we obtain Bismut's proof of the *local* family index formula for Dirac operators (see ref. [AS 2]):

$$\mathrm{Ch}(\mathrm{Ind}(\not{D})) = \left[\int_{\mathrm{fibre}} \omega\right] = \int_{\mathrm{fibre}} [\mathrm{AS}] \quad \text{in } H^{\mathrm{even}}(B,\mathbb{R}).$$

Bismut's treatment of the family index theorem has opened the way to a variety of new results. Besides the case of manifolds with boundary, treated below, we would like to mention the theory of determinant bundles, a topic with fascinating applications ranging from gauge theory to arithmetic algebraic geometry (see [B2] for a very complete survey). Determinant bundles will also be treated in Part II of this survey, in connection with the surgery problem.

**5. Manifolds with boundary and Atiyah-Patodi-Singer index theory.**

Let $M$ be a compact riemannian manifold *with* boundary. Simple examples (such as the $\overline{\partial}$-operator on the disc) show that, in general, elliptic operators on $M$ are *not* Fredholm on $L^2$. In order to obtain a finite dimensional kernel and cokernel it is necessary to impose *boundary conditions*. Among the simplest boundary conditions are those of *local type*, Dirichlet, Neumann or more generally Lopatinski boundary conditions (see [Hö]). Notice that as soon as the boundary $\partial M$ has some non-trivial topology it is not at all clear that these classical local boundary conditions should exist. And in fact Atiyah and Bott [AB] showed that there exist topological obstructions to the existence of classical local boundary conditions for an elliptic operator on a manifold with boundary. Now, precisely because of their geometric nature, Dirac operators are among those operators for which these obstructions are almost always *non-zero*. All these statements can be made very precise and rigorous by employing K-theory but we shall not enter into this; suffice it to observe that for interesting operators, such as the signature operator, it is not possible to impose classical boundary conditions (and thus to obtain Fredholm operators).

In trying to prove a signature theorem on manifolds with boundary Atiyah, Patodi and Singer found *non-local* boundary conditions for which generalized Dirac operators become Fredholm. In a fundamental series of papers [APS 1,2,3] they investigated the index theory of such boundary value problems and its applications to geometric and topological questions.

Thus suppose that $M$ is orientable even dimensional and that the riemannian metric $g$ is of product-type near the boundary. The Dirac operators $\not{D}^{\pm}$ can be written, near the boundary, as $\pm\,\partial/\partial u + \not{D}_0$ with $u$ equal to the inward normal variable to the boundary and $\not{D}_0$ the generalized Dirac operator induced on $\partial M$. The operator $\not{D}_0$ is an elliptic and essentially self-adjoint operator on the closed compact manifold $\partial M$. The $L^2$-spectrum is therefore discrete and real. Let $\{e_\lambda\}$ be an $L^2$-orthonormal basis of eigenfunctions of $\not{D}_0$. Let $\Pi_\geq$ be the spectral projection corresponding to the non-negative eigenvalues of $\not{D}_0$ and let

$$C^\infty(M, E^+, \Pi_\geq) = \{\, s \in C^\infty(M, E^+) \mid \Pi_\geq(s|_{\partial M}) = 0 \,\}.$$

Thus a section $s$ belongs to $C^\infty(M, E^+, \Pi_\geq)$ if $s|_{\partial M} = \sum_{\lambda<0} s^\lambda_{\partial M} e_\lambda$. The Atiyah-Patodi-Singer theorem [APS 1] states that the operator $\not{D}^+$ acting on the Sobolev completion of $C^\infty(M, E^+, \Pi_\geq)$ (we denote this operator by $\not{D}^+_{\Pi_\geq}$) is a Fredholm operator with index

$$\operatorname{ind}(\not{D}^+_{\Pi_\geq}) = \int_M \mathrm{AS} - \frac{1}{2}(\eta(\not{D}_0) + \dim \operatorname{Ker} \not{D}_0). \tag{5.1}$$

Here $\eta(\not{D}_0)$ is the *eta invariant* of the self-adjoint operator $\not{D}_0$. It is a *spectral invariant* of $\not{D}_0$ that measures the *asymmetry* of $\operatorname{spec}(\not{D}_0)$ with respect to $0 \in \mathbb{R}$. It is defined as the value at $s = 0$ of the meromorphic continuation of the complex function $\sum_{\lambda\neq 0} \operatorname{sign}\lambda\, |\lambda|^{-s} \quad \Re s >> 0$ with $\lambda$ running over the eigenvalues of $\not{D}_0$. Equivalently, using the Mellin transform,

$$\eta(\not{D}_0) = \int_0^\infty \frac{2}{\sqrt{\pi}} \operatorname{Tr}(\not{D}_0\, e^{-(t\,\not{D}_0)^2}) dt \equiv \int_0^\infty \eta(\not{D}_0)(t) dt. \tag{5.2}$$

The proof of the Atiyah-Patodi-Singer ($\equiv$ APS) index theorem relies heavily on the heat-kernel method. The second summand on the right hand side of (5.1) should be thought as the *boundary correction term* in the index formula. Applying the APS index formula to the signature operator $\not{D}^+_{\mathrm{sign}}$ and applying de Rham-Hodge-Kodaira theory one finally obtaines the *signature theorem on manifolds with boundary* [APS 1]:

$$\operatorname{sign}(M, \partial M) = \int_M L(M, \nabla^M) - \frac{1}{2}\eta(\not{D}_{\mathrm{sign},0}).$$

Later this formula was used to establish in the positive sense the Hirzebruch conjecture, stating the equality of the signature defect of a cusp singularity of a Hilbert modular variety with the value at zero of a certain $L$-function attached to the cusp. This deep theorem, a result that puts together almost all branches of mathematics, is due to Atiyah-Donnelly-Singer and Müller. See [Mü] for a nice survey on both the original results and what came afterward.

**6. Manifolds with boundary and the $b$-pseudodifferential calculus.**

The Atiyah-Patodi-Singer index theorem has seen a number of reformulations and alternative proofs. Cheeger was the first to find a different route to the formula, using analysis on spaces with cone-like singularities [C]. Among the latest contributions to the subject is the one that makes use of the $b$-pseudodifferential calculus. In this new approach (see [Me]) microlocal techniques are used in order to give an elegant and conceptually simple proof of the original result of Atiyah-Patodi-Singer. The main point is, briefly, the following. We assume, for simplicity, that the boundary operator is *invertible*. Consider the manifold with cylindrical ends, $\widehat{M}$, obtained by attaching the infinite cylinder $(-\infty, 0] \times \partial M$ to $M$. The metric $g$, being of product-type near the boundary, extends in a natural way to a metric $\widehat{g}$ on $\widehat{M}$. The Dirac operator $D\!\!\!\!/^{\pm}$ also extends, giving $\widehat{D\!\!\!\!/}^{\pm}$. It was already observed by Atiyah-Patodi-Singer that there exists a natural identification $\mathrm{Ker}(D\!\!\!\!/^{+}_{\Pi_{\geq}}) \longleftrightarrow \mathrm{Ker}_{L^2}(\widehat{D\!\!\!\!/}^{+})$ and similarly for the adjoint. In other words the APS-index on $M$ coincides with the $L^2$-index on the non-compact manifold $\widehat{M}$. The change of variable $x = \exp(u)$ compactifies $\widehat{M}$ but transforms $\widehat{g}$ into a metric which is singular near $\partial M$: $du^2 + g_{\partial M} \longrightarrow dx^2/x^2 + g_{\partial M}$. Metrics of this kind are called $b$-metrics. The Dirac operator $\widehat{D\!\!\!\!/}^{\pm}$ transforms accordingly, becoming $\pm x(\frac{\partial}{\partial x}) + D\!\!\!\!/_0$ near the boundary $\partial M = \{x = 0\}$. We shall use again the notation $M, g, D\!\!\!\!/$ for the compactified manifold, the $b$-metric and the associated Dirac operator. Define the $b$-differential operators on $M$, $\mathrm{Diff}^*_b(M)$, as the differential operators generated by the vector fields $\mathcal{V}_b(M) = \{V \in C^\infty(M, TM)$ such that $V$ is tangent to $\partial M\}$. In a local chart near the boundary $\partial M = \{x = 0\}$ the space $\mathcal{V}_b(M)$ is generated over $C^\infty(M)$ by the local vector fields of $\partial M$ and the vector field $x(\partial/\partial x)$. Clearly then the Dirac operator associated to a $b$-metric belongs to $\mathrm{Diff}^1_b(M, E)$. It is a $b$-elliptic operator.

On a *closed* manifold $N$ the algebra $\Psi^*(N) = \cup_{m \in \mathbb{R}} \Psi^m(N)$ of pseudodifferential operators [Hö 2] arises naturally when one considers the *Green operator* $G$ of an elliptic element $D \in \mathrm{Diff}^k(N)$ or the powers $A^s, s \in \mathbb{R}$, of a formally self-adjoint elliptic element in $\mathrm{Diff}^k(N)$. In these two cases it turns out that $G \in \Psi^{-k}(N)$ and that $A^s \in \Psi^{sk}(N)$. The property of being an *algebra* together with very natural *mapping properties* with respect to Sobolev spaces makes pseudodifferential operators a very effective tool in the study of the spectral theory of elliptic differential operators. An element $P \in \Psi^k(N)$ is an integral operator with a Schwartz kernel having distributional singularities along the diagonal $\Delta \subset N \times N$ (specifying which singularities are allowed amounts to a precise definition of $\Psi^*(N)$). The more *negative* is $k$ the more *regular* is the Schwartz kernel of the operator. In particular pseudodifferential operators of order $(-\infty)$ are smoothing operators. For example $\exp(-tD\!\!\!\!/^2) \in \Psi^{-\infty}(N, E)$.

On a manifold with boundary $M$ $b$-pseudodifferential operators $\Psi^*_b(M)$ have been introduced by Melrose with the same motivations as in the closed case (i.e. to study $b$-elliptic differential operators and their spectral theory). Simple examples, such as the Green operator associated to the $b$-differential operator $P = x(1-x)\partial/\partial x + c$, $c \in \mathbb{R}$, on the manifold with boundary $M = [0, 1]$, show that the Schwartz kernels of the putative

$b$-pseudodifferential operators must have more complicated singularities then in the closed case. This makes the theory richer but more difficult (see [Me] for a thorough account). In any case it is still true that $\exp(-\not{D}^2) \in \Psi_b^{-\infty}(M,E)$ (although these are not really smoothing operators) and that $\mathrm{str}_E(\exp(-t\not{D}^2)|_\Delta)$ can be studied, as $t \downarrow 0^+$, in terms of local geometric data. Moreover, as $t \to \infty$ the heat-kernel converges in a suitable sense to the orthogonal projection onto $\mathrm{Ker}(\not{D}^+) \oplus \mathrm{Ker}(\not{D}^-)$ which is, in turn, finite dimensional.

What is missing is the trace class property for elements in $\Psi_b^{-\infty}$. However it is still possible to define a regularized trace [Me], the $b$-trace $b-\mathrm{Tr}(\cdot)$ . Of course if $A \in \Psi_b^*$ happens to be trace class then $b-\mathrm{Tr}(A) = \mathrm{Tr}(A)$; this is the case of the orthogonal projection onto $\mathrm{Ker}(\not{D}) = \mathrm{Ker}(\not{D}^+) \oplus \mathrm{Ker}(\not{D}_-)$. The main feature of the $b$-trace is that it is not zero on commutators. This "defect" can be computed in a precise way; *it only depends on the boundary data.* Following now the heat-kernel proof of the Atiyah-Singer index theorem (see Sect. 2) we see that the analogue of (2.1) for the $b$-supertrace *cannot* hold. Using the formula for the $b$-trace of a commutator it is possible to show that (2.1) is now replaced by

$$\frac{d}{dt}(\mathrm{b}-\mathrm{STr}\,(\exp(-t\not{D}^2))) = \eta(\not{D}_0)(t) \qquad (6.1).$$

Integrating (6.1) from 0 to $+\infty$ and using the above remarks on the short and large-time behaviour of the heat-kernel (together with the equality $\mathrm{ind}_{\mathrm{APS}}(\not{D}^+) = \mathrm{ind}_{L^2}(\not{D}^+)$) it is now easy to recover the Atiyah-Patodi-Singer index theorem under the assumption $\mathrm{Ker}(\not{D}_0) = \{0\}$. A slightly more delicate analysis is needed in the non-invertible case. For more on the $b$-calculus proof of the APS index theorem we refer the reader to [Me]. See also [P] for a different proof which also covers the case of *b-pseudodifferential* operators. The best results in the pseudodifferential context are given in [MeNi].

## 7. Spectral sections and Atiyah-Patodi-Singer family index theory.

Among the many implications of Bismut's heat-kernel treatment of the family index theorem we concentrate now on the family version of the Atiyah-Patodi-Singer index theorem. The first result in this direction is due to Bismut and Cheeger [BC 1,2,3]; $\not{D} = (\not{D}_z)_{z\in B}$ is now a family of Dirac operators on *even dimensional* manifolds with boundary, parametrized by a compact smooth manifold $B$. Certainly for each *fixed* $z \in B$ the operator $\not{D}^+_{z,\Pi_\geq}$ is Fredholm. However in order for the family of APS boundary value problems $\not{D}^+_{z,\Pi_\geq}$ to define a *smooth* (or even *continuous*) family of Fredholm operators it is necessary to assume that the boundary operators $\not{D}_{0,z}$ are all *invertible*, or, more generally, that the kernels $\mathrm{Ker}(\not{D}_{0,z})$ are of *constant dimension* in $z$. We shall refer to these as the invertible and constant rank case respectively. Intuitively we are ruling out the possibility that the eigenvalues of the boundary operators cross 0 when thought as functions of $z \in B$. In fact it is not difficult to see that $\mathrm{ind}(\not{D}^+_{z,\Pi_\geq})$ will jump each time a non-zero eigenvalue of $\not{D}_{0,z}$ reaches 0 (whereas we know that a continuous family of Fredholm operators must have *constant* index). Under the above assumption on the boundary family Bismut and Cheeger prove that the index bundle $\mathrm{Ind}(\not{D}_{\Pi_\geq})$ is well defined and that the following formula holds

$$\mathrm{Ch}(\mathrm{Ind}(\not{D}_{\Pi_\geq})) = \int_{\text{fibre}} \mathrm{AS} \;-\; \frac{1}{2}(\widehat{\eta} + \mathrm{Ch}(\mathrm{Ker}(\not{D}_0))) \quad \text{in } H^*(B,\mathbb{R}) \tag{7.1}$$

(the formula is fully proved in the invertible case in [BC 1,2] and stated in the constant rank case in [BC 3]).

In this formula $\widehat{\eta} \in C^\infty(B, \Lambda^* B)$ is the eta form of Bismut-Cheeger; it is a higher version of the eta invariant, in the sense that the 0-degree component of $\widehat{\eta}$ computed at $z \in B$ is equal to $\eta(\not{D}_{0,z})$. The Bismut-Cheeger eta form is defined in terms of the superconnection formalism by a formula similar to (5.2). The assumption that the operators of the family $\not{D}_0$ have kernels of constant dimension plays a crucial role in the proof of the convergence of the integral.

The results of Bismut-Cheeger were significantly improved in [MP 1,2]. One of the main themes of [MP 1,2] is that in order to develop a *general APS* family index theory, i.e. with *no assumptions* on the boundary family, it is necessary to consider *generalized* APS boundary conditions. The key concept in this direction, that of *spectral section*, was introduced in [MP 1,2]. We now recall the definition. Let $D = (D_z)_{z\in B}$ be a smooth family of formally self-adjoint elliptic differential operators. Each operator $D_z$ acts on the sections of a hermitian vector bundle $F_z$ over a closed riemannian manifold $N_z$. The boundary family $\not{D}_0$ associated to a family $\not{D}$ of Dirac operators on even dimensional manifolds with boundary is a fundamental and motivating example. A spectral section $P$ for $D$ is a *smooth* family $P = (P_z)$ of self-adjoint projections with $P_z \in \Psi^0(N_z; F_z)$ and satisfying the following property:

$$\exists\, R \in \mathbb{R}\,\text{such that} : D_z u = \lambda u \Rightarrow \begin{cases} Pu = u & \text{if } \lambda > R \\ Pu = 0 & \text{if } \lambda < -R. \end{cases} \tag{7.2}$$

This means that each operator $P_z$ is equal to the identity on the eigenfunctions of $D_z$ corresponding to large positive eigenvalues and equal to 0 on those corresponding to large negative eigenvalues. Each $P_z$ is a finite rank perturbation of the spectral projection $\Pi_{\geq,z}$ corresponding to the non-negative eigenvalues of $D_z$ as $z$ varies in $B$. Of course, in general, the family of self-adjoint projections $\Pi_\geq = (\Pi_{\geq,z})_{z\in B}$ does *not* vary continuously with the parameter $z \in B$ and this is the reason why the more general notion of spectral section is introduced.

When does a family of self-adjoint elliptic operators admit a spectral section? In order to answer this basic question recall [AS 3] that a family of *self-adjoint* Fredholm operators parametrized by $B$ defines an index class $\mathrm{Ind}(D) \in K^1(B)^{(*)}$. As a simple example we can consider the case $B = S^1 = \mathbb{R}/\mathbb{Z}$. Then $K^1(B) = \mathbb{Z}$ and the index class associated to a self-adjoint family $D = (D_z)_{z\in S^1}$ is nothing but its *spectral flow*, i.e. the *net number* of eigenvalues that change sign when $z$ varies over $S^1$. For example the family

$^{(*)}$ This is analogue to the construction described in Sect. 4 for Fredholm operators that are not self-adjoint.

$D \equiv \not{D} = (\not{D}_z)_{z\in S^1}$ introduced at the beginning of Sect. 4 (see (4.1)) has spectral flow equal to one since the eigenvalues of $\not{D}_z$ are $\{\lambda_k(z) = k + \frac{z-\frac{1}{2}}{\pi}\}$. On the other hand the family $P_z = \not{D}_z \oplus \widetilde{\not{D}}_z$ (see Sect. 5 for the definition) has spectral flow equal to 0.

One of the main results in [MP 1] asserts that a spectral section for $D$ exists if and only if $\mathrm{Ind}(D) = 0$ in $K^1(B)$ i.e. if and only if $D$ is homotopic through self-adjoint Fredholm families to an invertible family. If $\mathrm{Ind}(D) = 0$ then the set of spectral sections $\mathrm{SP}(D)$ associated to $D$ is infinite. $\mathrm{Ind}(D) = 0$ is a *topological* condition on the family $D$. For example a spectral section for an $S^1$-family exists if and only if the spectral flow is equal to zero. Of course this is a much weaker condition than requiring all the operators of the family to be invertible. The following picture illustrates the behaviour of the eigenvalues of an invertible $S^1$-family and of the two $S^1$-families considered above (thus with spectral flow equal to 0 and 1 respectively).

Now, by the cobordism invariance of the index we know that for a *boundary* Dirac family such as $\not{D}_0$ it is always the case that $\mathrm{Ind}(\not{D}_0) = 0$. Thus according to the above two results a boundary family always admits an infinite number of spectral sections. Once a spectral section $P$ for $\not{D}_0$ has been fixed, it is possible to define a *smooth* family of generalized APS boundary value problem $(\not{D}^+, P)$ and thus an index class $\mathrm{Ind}(\not{D}^+, P) \in K^0(B)$. The definition of the operator $(\not{D}^+, P)_z$, $z \in B$ is similar to that of the APS boundary value problem given in Sect. 5, namely $(\not{D}^+, P)_z$ is by definition $\not{D}^+_z$ with domain $C^\infty(M_z, E^+_z, P_z) = \{s \in C^\infty(M_z, E^+_z) \mid P_z(s|_{\partial M}) = 0\}$. The $b$-pseudodifferential calculus is heavily used in [MP 1] to give a formula for the Chern character of this index class:

$$\mathrm{Ch}(\mathrm{Ind}(\not{D}^+, P)) = \int_{\mathrm{fibre}} \mathrm{AS} - \frac{1}{2}\widehat{\eta}_P \tag{7.3}$$

with $\widehat{\eta}_P \in C^\infty(B, \Lambda^* B)$ a $P$-eta form generalizing the one of Bismut-Cheeger. If $\mathrm{null}(\not{D}_{0,z}) = \{0\}$ $\forall z \in B$ then we can choose $P = \Pi_{\geq}$; since it is possible to show that $\widehat{\eta}_{\Pi_{\geq}} = \widehat{\eta} + \mathrm{Ch}(\mathrm{Ker}(\not{D}_0))$ we see that in this case the index theorem in [MP 1] reduces to the Bismut-Cheeger family index theorem.

One can also compare index bundles corresponding to different spectral sections: if $P$ and $Q$ are two spectral sections for the family $D = \not{D}_0$ then there is a well defined *difference class* $[P - Q] \in K^0(B)$ and the following *relative index formula* holds: $\mathrm{Ind}(\not{D}, Q) - \mathrm{Ind}(\not{D}, P) = [P - Q]$ in $K^0(B)$. Applying the Chern character homomorphism $\mathrm{Ch} : K^0(B) \otimes$

$\mathbb{R} \to H^{\text{even}}(B,\mathbb{R})$ we obtain $\mathrm{Ch}(\mathrm{Ind}(\not{D},Q)) - \mathrm{Ch}(\mathrm{Ind}(\not{D},P)) = \mathrm{Ch}([P-Q])$ so that, using (7.3), we also get

$$\widehat{\eta}_P - \widehat{\eta}_Q = \mathrm{Ch}([P-Q]) \text{ in } H^{\text{even}}(M,\mathbb{R}).$$

This *jump* formula for the eta form can in fact be proved directly and without assuming that the family $D$ arises as a boundary family (of course in order to ensure the existence of the spectral sections we do have to assume that $\mathrm{Ind}(D)=0$ in $K^1(B)$).

The results of [MP 1], which are only valid for *even* dimensional manifolds with boundary, can be extended to *odd* dimensional manifolds with boundary using the notion of Cl(1)-spectral section, a *suspension* technique inspired by Bismut and Freed and the properties of the $b$-pseudodifferential calculus. For more on the odd case, which was left open by Bismut and Cheeger even in the invertible case, we refer the reader to [MP 2].

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