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Relative pairings and the Atiyah-Patodi-Singer index formula for the Godbillon-Vey cocycle

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Dedicated to Henri Moscovici on the occasion of his sixty-fifth anniversary

Abstract. We describe a Godbillon-Vey index formula for longitudinal Dirac operators on a foliated bundle $(X, \mathcal{F})$ with boundary; in particular, we define a *Godbillon-Vey eta invariant* on $(\partial X, \mathcal{F}_\partial)$, that is, a secondary invariant for longitudinal Dirac operators on type III foliations. Our theorem generalizes the classic Atiyah-Patodi-Singer index formula for $(X, \mathcal{F})$. Moreover, employing the Godbillon-Vey index as a pivotal example, we explain a new approach to higher index theory on geometric structures with boundary. This is heavily based on the interplay between the absolute and relative pairing of K-theory and cyclic cohomology for an exact sequence of Banach algebras, which in the present context takes the form $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{B} \to 0$ with $\mathfrak{J}$ dense and holomorphically closed in $C^*(X, \mathcal{F})$ and $\mathfrak{B}$ depending only on boundary data.

1. Introduction

Connes' index theorem for G-proper manifolds [**1**], with G an étale groupoid, unifies under a single statement most of the existing (longitudinal) index theorems. We shall focus on a particular case of such a theorem, that of foliated bundles. Thus, let N be a closed compact manifold. Let $\Gamma \to \tilde{N} \to N$ be a Galois Γ-cover. Let T be a smooth oriented compact manifold with an action of Γ which is assumed to be by diffeomorphisms, orientation preserving and locally faithful, as in [**14**]. Let $Y = \tilde{N} \times_\Gamma T$ and let $(Y, \mathcal{F})$ be the associated foliation. (This is an example of G-proper manifold with G equal to the groupoid $T \rtimes \Gamma$.) Let D be a Γ-equivariant family of Dirac operators on the fibration $\tilde{N} \times T \to T$; such a family induces a longitudinal Dirac operator on $(Y, \mathcal{F})$.

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If T = point and $\Gamma = \{1\}$ we have a compact manifold and Connes' index theorem reduces to the Atiyah-Singer index theorem. If $\Gamma = \{1\}$ we simply have a fibration and the theorem reduces to the Atiyah-Singer family index theorem. If T = point then we have a Galois covering and Connes' index theorem reduces to the Connes-Moscovici higher index theorem. If $\dim T > 0$ and $\Gamma \neq \{1\}$, then Connes' index theorem is a higher foliation index theorem on the foliated manifold $(Y, \mathcal{F})$.

One particularly interesting higher index is the so-called Godbillon-Vey index; an alternative treatment of Connes' index formula in this particular case was given by Moriyoshi-Natsume in [**14**]. Subsequently, Gorokhovsky and Lott [**4**] gave a superconnection proof of Connes' index theorem, including an explicit formula for the Godbillon-Vey higher index. Leichtnam and Piazza [**7**] extended Connes' index theorem to foliated bundles with boundary, using an extension of Melrose b-calculus and the Gorokhovsky-Lott superconnection approach. Unfortunately, a key assumption in [**7**] is that the group Γ be of polynomial growth. This excludes many interesting examples and higher indeces; in particular it excludes the possibility of proving a Atiyah-Patodi-Singer formula for the Godbillon-Vey higher index.

One primary objective of this article is to illustrate such a result. Complete proofs will appear in [**16**].

In tackling the problem we develop what we believe is a new approach to index theory on manifolds with boundary. This can be summarized as follows. We define a short exact sequence of Banach algebras

$$0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{B} \to 0$$

with $\mathfrak{J}$ dense and holomorphically closed in $C^*(X, \mathcal{F})$ and with $\mathfrak{B}$ depending only on boundary data. We prove that there are well defined Dirac index classes, denoted respectively $\mathrm{Ind}(D, D^\partial) \in K_*(\mathfrak{A}, \mathfrak{B})$ and $\mathrm{Ind}(D) \in K_*(\mathfrak{J})$, and that these index classes correspond under excision; the relative class $\mathrm{Ind}(D, D^\partial) \in K_*(\mathfrak{A}, \mathfrak{B})$ is obtained by using the graph projection of D and of D^{cyl} whereas the index class $\mathrm{Ind}(D) \in K_*(\mathfrak{J})$ is obtained through the parametrix of D and the associated remainders. Next, for (certain) cyclic k-cocycles defining a higher index in the closed case, let us name one of such cocycles τ_k, we define

- a cyclic k-cocycle on $\mathfrak{J}$, still denoted τ_k;
- a eta cyclic cocycle σ_{k+1} on $\mathfrak{B}$; σ_{k+1} (which thus depends solely on boundary data) is obtained by a sort of suspension procedure involving τ_k and a specific 1-cocycle σ_1 (Roe's 1-cocycle);
- a *relative* cyclic k-cocyle (τ_k^r, σ_{k+1}), with τ_k^r a cyclic cochain defined from τ_k through a regularization à la Melrose.

The index formula in this context is obtained by establishing the equality

$$\langle \mathrm{Ind}(D), [\tau_k] \rangle = \langle \mathrm{Ind}(D, D^\partial), [\tau_k^r, \sigma_{k+1}] \rangle .$$

On the left hand side we have the absolute pairing, which is by definition the higher index. On the right hand side we have the relative pairing; multiplying the operator by $s > 0$, using the definition of the relative pairing and taking the limit as $s \downarrow 0$ we obtain the right hand side of the Atiyah-Patodi-Singer index formula. The eta-correction term is obtained through the eta cocycle σ_{k+1}.

We end this brief introduction by pointing out that relative pairings in K-theory and cyclic cohomology have already been successfully employed in the study of geometric and topological invariants of elliptic operators. We particularly wish to mention here the paper by Lesch, Moscovici and Pflaum [**9**]; in this interesting article the absolute and relative pairings associated to a suitable short exact sequence of algebras (this is a short exact sequence of parameter dependent pseudodifferential operators) are used in order to define and study a generalization of the divisor flow of Melrose on a closed compact manifold, see [**12**] and also [**10**].

The results of this article first appeared in [**15**].

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2. Geometry of foliated bundles.

2.1. Manifolds with boundary. Let now (M, g) be a riemannian manifold with boundary; the metric is assumed to be of product type in a collar neighborhood $U \simeq [0,1] \times \partial M$ of the boundary. Let $\tilde{M}$ be a Galois Γ-cover of M; we let $\tilde{g}$ be the lifted metric. We also consider $\partial\tilde{M}$, the boundary of $\tilde{M}$. Let T be a smooth oriented compact manifold with an action of Γ by orientation-preserving diffeomorphisms. We assume that this action is locally faithful, as in [**14**], that is: if $\gamma \in \Gamma$ acts as the identity map on an open set in T, then γ is the identity element in Γ.

Let $X_0 = \tilde{M} \times_\Gamma T$; this is a manifold with boundary and the boundary ∂X_0 is equal to $\partial\tilde{M} \times_\Gamma T$. $(X_0, \mathcal{F}_0)$ denotes the associated foliated bundle. The leaves of $(X_0, \mathcal{F}_0)$ are manifolds with boundary endowed with a product-type metric. The boundary ∂X_0 inherits a foliation $\mathcal{F}_\partial$. The cylinder $\mathbb{R} \times \partial X_0$ also inherits a foliation $\mathcal{F}_{\mathrm{cyl}}$, obtained by crossing the leaves of $\mathcal{F}_\partial$ with $\mathbb{R}$. Similar considerations apply to the half cylinders $(-\infty, 0] \times \partial X_0$ and $\partial X_0 \times [0, +\infty)$.

2.2. Manifolds with cylindrical ends. **Notation.** We consider $\tilde{V} := \tilde{M} \cup_{\partial\tilde{M}} \left((-\infty, 0] \times \partial\tilde{M}\right)$, endowed with the extended metric and the obviously extended Γ action along the cylindrical end. We consider $X := \tilde{V} \times_\Gamma T$; this is a foliated bundle, with leaves manifolds with cylindrical ends. We denote by $(X, \mathcal{F})$ this foliation. Notice that $X = X_0 \cup_{\partial X_0} ((-\infty, 0] \times \partial X_0)$; moreover the foliation $\mathcal{F}$ is obtained by extending $\mathcal{F}_0$ on X_0 to X via the product cylindrical foliation $\mathcal{F}_{\mathrm{cyl}}$ on $(-\infty, 0] \times \partial X_0$. We can write more suggestively: $(X, \mathcal{F}) = (X_0, \mathcal{F}_0) \cup_{(\partial X_0, \mathcal{F}_\partial)} (((-\infty, 0] \times \partial X_0, \mathcal{F}_{\mathrm{cyl}}))$. For $\lambda > 0$ we shall also consider the finite cyclinder $\tilde{V}_\lambda = \tilde{M} \cup_{\partial\tilde{M}} \left([-\lambda, 0] \times \partial\tilde{M}\right)$ and the resulting foliated manifold $(X_\lambda, \mathcal{F}_\lambda)$. Finally, with a small abuse, we introduce the notation: $\mathrm{cyl}(\partial X) :=$

$\mathbb{R} \times \partial X_0$, $\mathrm{cyl}^-(\partial X) := (-\infty, 0] \times \partial X_0$ and $\mathrm{cyl}^+(\partial X) := \partial X_0 \times [0, +\infty)$ The foliations induced on $\mathrm{cyl}(\partial X)$, $\mathrm{cyl}^{\pm}(\partial X)$ by $\mathcal{F}_\partial$ will be denoted by $\mathcal{F}_{\mathrm{cyl}}$, $\mathcal{F}^{\pm}_{\mathrm{cyl}}$.

2.3. Holonomy groupoid. We consider the groupoid $G := (\tilde{V} \times \tilde{V} \times T)/\Gamma$ with Γ acting diagonally; the source map and the range map are defined by $s[y, y', \theta] = [y', \theta]$, $r[y, y', \theta] = [y, \theta]$. Since the action on T is assumed to be locally faithful, we know that (G, r, s) is isomorphic to the holonomy groupoid of the foliation $(X, \mathcal{F})$. In the sequel, we shall simply call (G, r, s) *the holonomy groupoid.* If $E \to X$ is a hermitian vector bundle on X, with product structure along the cylindrical end, then we can consider the bundle over G equal to $(s^*E)^* \otimes r^*E$.

3. Wiener-Hopf extensions

3.1. Foliation C^*-algebras. We consider $C_c(X, \mathcal{F}) := C_c(G)$. $C_c(X, \mathcal{F})$ can also be defined as the space of Γ-invariant continuous functions on $\tilde{V} \times \tilde{V} \times T$ with Γ-compact support. More generally we consider $C_c(X, \mathcal{F}; E) := C_c(G, (s^*E)^* \otimes r^*E)$ with its well known *-algebra structure given by convolution. We shall often omit the vector bundle E from the notation.

The foliation C^*-algebra $C^*(X, \mathcal{F}; E)$ is defined by completion of $C_c(X, \mathcal{F}; E)$. See for example [**14**] where it is also proved that $C^*(X, \mathcal{F}; E)$ is isomorphic to the C^*-algebra of compact operators of the Connes-Skandalis $C(T) \rtimes \Gamma$-Hilbert module $\mathcal{E}$ (this is also described in [**14**]). Summarizing: $C^*(X, \mathcal{F}; E) \cong \mathbb{K}(\mathcal{E}) \subset \mathcal{L}(\mathcal{E})$.

3.2. Foliation von Neumann algebras. Consider the family of Hilbert spaces $\mathcal{H} := (\mathcal{H}_\theta)_{\theta \in T}$, with $\mathcal{H}_\theta := L^2(\tilde{V} \times \{\theta\}, E_\theta)$. Then $C_c(\tilde{V} \times T)$ is a continuous field inside $\mathcal{H}$, that is, a linear subspace in the space of measurable sections of $\mathcal{H}$. Let $\mathrm{End}(\mathcal{H})$ the space of measurable families of bounded operators $T = (T_\theta)_{\theta \in T}$, where bounded means that each T_θ is bounded on $\mathcal{H}_\theta$. Then $\mathrm{End}(\mathcal{H})$ is a C^*-algebra, in fact a von Neumann algebra, equipped with the norm

$$\|T\| := \operatorname{ess.sup}\{\|T_\theta\|\,; \theta \in T\}$$

with $\|T_\theta\|$ the operator norm. We also denote by $\mathrm{End}_\Gamma(\mathcal{H})$ the C^*-subalgebra of $\mathrm{End}(\mathcal{H})$ consisting of Γ-equivariant measurable families of operators. This is often denoted $W^*(X, \mathcal{F})$ and named the *foliation von Neumann algebra* associated to $(X, \mathcal{F})$. We set $C^*_\Gamma(\mathcal{H})$ the closure of Γ-equivariant families $T = (T_\theta)_{\theta \in T} \in \mathrm{End}_\Gamma(\mathcal{H})$ preserving the continuous field $C_c(\tilde{V} \times T)$. In [**14**], Section 2 it is proved that the foliation C^*-algebra $C^*(X, \mathcal{F})$ is isomorphic to a C^*-subalgebra of $C^*_\Gamma(\mathcal{H}) \subset \mathrm{End}_\Gamma(\mathcal{H})$ [1]. Notice, in particular, that an element in $C^*(X, \mathcal{F})$ can be considered as a Γ-equivariant family of operators.

3.3. Translation invariant operators. Recall $\mathrm{cyl}(\partial X) := \mathbb{R} \times \partial X_0 \equiv (\mathbb{R} \times \partial\tilde{M}) \times_\Gamma T$ with Γ acting trivially in the $\mathbb{R}$-direction of $(\mathbb{R} \times \partial\tilde{M})$. We consider the foliated cylinder $(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}})$ and its holonomy groupoid $G_{\mathrm{cyl}} := ((\mathbb{R} \times \partial\tilde{M}) \times (\mathbb{R} \times \partial\tilde{M}) \times T)/\Gamma$ (source and range maps are clear). Let $\mathbb{R}$ act trivially on T; then $(\mathbb{R} \times \partial\tilde{M}) \times (\mathbb{R} \times \partial\tilde{M}) \times T$ has a $\mathbb{R} \times \Gamma$-action, with $\mathbb{R}$ acting by translation on itself. We consider the *-algebra $B_c(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}) \equiv B_c$ defined as

$$\{k \in C((\mathbb{R}\times\partial\tilde{M})\times(\mathbb{R}\times\partial\tilde{M})\times T); k \text{ is } \mathbb{R}\times\Gamma\text{-invariant}, k \text{ has } \mathbb{R}\times\Gamma\text{-compact support}\} \tag{3.1}$$

[1]The C^*-algebra $C^*_\Gamma(\mathcal{H})$ was denoted $\mathfrak{B}$ in [**14**]

The product is by convolution. An element ℓ in B_c defines a Γ-equivariant family $(\ell(\theta))_{\theta\in T}$ of translation-invariant operators. The completion of B_c with respect to the obvious C^*-norm (the sup over θ of the operator-L^2-norm of $\ell(\theta)$) gives us a C^*-algebra that will be denoted $B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}})$ or more briefly B^*.

3.4. Wiener-Hopf extensions. Recall the Hilbert $C(T) \rtimes \Gamma$-module $\mathcal{E}$ and the C^*-algebras $\mathbb{K}(\mathcal{E})$ and $\mathcal{L}(\mathcal{E})$. Since the $C(T) \rtimes \Gamma$-compact operators $\mathbb{K}(\mathcal{E})$ are an ideal in $\mathcal{L}(\mathcal{E})$ we have the classical short exact sequence of C^*-algebras

$$0 \to \mathbb{K}(\mathcal{E}) \hookrightarrow \mathcal{L}(\mathcal{E}) \xrightarrow{\pi} \mathcal{Q}(\mathcal{E}) \to 0$$

with $\mathcal{Q}(\mathcal{E}) = \mathcal{L}(\mathcal{E})/\mathbb{K}(\mathcal{E})$ the Calkin algebra. Let $\chi^0_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ be the characteristic function of $(-\infty, 0]$; let $\chi_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ be a smooth function with values in $[0, 1]$ such that $\chi(t) = 1$ for $t \le -\epsilon$, $\chi(t) = 0$ for $t \ge 0$. Let χ^0 and χ be the functions induced by $\chi^0_{\mathbb{R}}$ and $\chi_{\mathbb{R}}$ on X. Similarly, introduce χ^0_{cyl} and χ_{cyl}.

Lemma 3.2. *There exists a bounded linear map*

$$s : B^* \to \mathcal{L}(\mathcal{E}) \tag{3.3}$$

extending $s_c : B_c \to \mathcal{L}(\mathcal{E})$, $s_c(\ell) := \chi^0 \ell \chi^0$. Moreover, the composition $\rho = \pi s$ induces an **injective** *C^*-homomorphism*

$$\rho : B^* \to \mathcal{Q}(\mathcal{E}). \tag{3.4}$$

We consider $\mathrm{Im}\,\rho$ as a C^*-subalgebra in $\mathcal{Q}(\mathcal{E})$ and identify it with $B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}})$ via ρ. Set

$$A^*(X; \mathcal{F}) := \pi^{-1}(\mathrm{Im}\,\rho) \quad \text{with } \pi \text{ the projection} \quad \mathcal{L}(\mathcal{E}) \to \mathcal{Q}(\mathcal{E}).$$

Recalling the identification $C^*(X, \mathcal{F}) = \mathbb{K}(\mathcal{E})$, we thus obtain a short exact sequence of C^*-algebras:

$$0 \to C^*(X, \mathcal{F}) \to A^*(X; \mathcal{F}) \xrightarrow{\pi} B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}) \to 0 \tag{3.5}$$

where the quotient map is still denoted by π. Notice that (3.5) splits as a short exact sequence of *Banach spaces*, since we can choose $s : B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}) \to A^*(X; \mathcal{F})$ the map in (3.3) as a section. So

$$A^*(X; \mathcal{F}) \cong C^*(X, \mathcal{F}) \oplus s(B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}))$$

as Banach spaces.

There is also a linear map $t : A^*(X, \mathcal{F}) \to C^*(X, \mathcal{F})$ which is obtained as follows: if $k \in A^*(X; \mathcal{F})$, then k is uniquely expressed as $k = a + s(\ell)$ with $a \in C^*(X, \mathcal{F})$ and $\pi(k) = \ell \in B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}})$. Thus, $\pi(k) = [\chi^0 \ell \chi^0] \in \mathcal{Q}(\mathcal{E})$ for one (and only one) $\ell \in B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}})$ since ρ is injective. We set

$$t(k) := k - s\pi(k) = k - \chi^0 \ell \chi^0 \tag{3.6}$$

Then $t(k) \in C^*(X, \mathcal{F})$.

4. Relative pairings and the eta cocycle: the algebraic theory

4.1. Introductory remarks. On a *closed* foliated bundle $(Y, \mathcal{F})$, the Godbillon-Vey cyclic cocycle is initially defined on the "small" algebra $\mathcal{A}_c \subset C^*(Y, \mathcal{F})$ of Γ-equivariant smoothing operators of Γ-compact support (viz. $\mathcal{A}_c := C^\infty_c(G, (s^*E)^* \otimes r^*E)$). Since the index class defined using a pseudodifferential parametrix is already well defined in $K_*(\mathcal{A}_c)$, the pairing between the the Godbillon-Vey cyclic cocycle and the index class is well-defined.

In a second stage, the cocycle is continuously extended to a dense holomorphically closed subalgebra $\mathfrak{A} \subset C^*(Y, \mathcal{F})$; there are at least two reasons for doing this. First, it is only by going to the C^*-algebraic index that the well known properties for the signature and the spin Dirac operator of a metric of positive scalar curvature hold. The second reason for this extension rests on the structure of the index class *which is employed in the proof of the higher index formula*, i.e. either the graph projection or the Wassermann projection; in both cases $\mathcal{U}_c$ is too small to contain the index class and one is therefore forced to find an intermediate subalgebra $\mathfrak{A}$, $\mathcal{A}_c \subset \mathfrak{A} \subset C^*(Y, \mathcal{F})$; $\mathfrak{A}$ *is big enough for the two particular index classes to belong to it but small enough for the Godbillon-Vey cyclic cocycle to extend; moreover, being dense and holomorphically closed it has the same* K*-theory as* $C^*(Y, \mathcal{F})$.

Let now $(X, \mathcal{F})$ be a foliated bundle with cylindrical ends; in this section we shall select "small" subalgebras

$$J_c \subset C^*(X, \mathcal{F})\,, \quad A_c \subset A^*(X, \mathcal{F})\,, \quad B_c \subset B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}})\,,$$

with J_c an ideal in A_c, so that there is a short exact sequence $0 \to J_c \hookrightarrow A_c \xrightarrow{\pi_c} B_c \to 0$ which is a subsequence of $0 \to C^*(X, \mathcal{F}) \hookrightarrow A^*(X; \mathcal{F}) \xrightarrow{\pi} B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}) \to 0$. We shall then proceed to define the relevant cyclic cocycles, relative and absolute, and study, algebraically, their main properties. As in the closed case, we shall eventually need to find an intermediate short exact sequence, sitting between the two, $0 \to \mathfrak{J} \hookrightarrow \mathfrak{A} \to \mathfrak{B} \to 0$, with constituents big enough for the the two index classes we shall define to belong to them but small enough for the cyclic cocycles (relative and absolute) to extend; this is quite a delicate point and it will be explained in Section 5. We anticipate that in contrast with the closed case the ideal J_c in the small subsequence will be too small even for the index class defined by a pseudodifferential parametrix. This has to do with the non-locality of the parametrix on manifolds with boundary; it is a phenomenon that was explained in detail in [**7**].

4.2. Small dense subalgebras. Define $J_c := C_c^\infty(X, \mathcal{F})$; see subsection 3.1. Redefine B_c as

$$\{k \in C^\infty((\mathbb{R}\times\partial\tilde{M})\times(\mathbb{R}\times\partial\tilde{M})\times T); k \text{ is } \mathbb{R}\times\Gamma\text{-invariant}, k \text{ has } \mathbb{R}\times\Gamma\text{-compact support}\}$$

see subsection 3.3. We now define A_c; consider the functions χ^λ, $\chi^\lambda_{\mathrm{cyl}}$ induced on X and $\mathrm{cyl}(\partial X)$ by the real function $\chi^{\mathbb{R}}_{(-\infty,-\lambda]}$. We shall say that k is in A_c if it is a smooth function on $\tilde{V} \times \tilde{V} \times T$ which is Γ-invariant and for which there exists $\lambda \equiv \lambda(k) > 0$, such that

- $k - \chi^\lambda k \chi^\lambda$ is of Γ-compact support
- there exists $\ell \in B_c$ such that $\chi^\lambda k \chi^\lambda = \chi^\lambda_{\mathrm{cyl}} \ell \chi^\lambda_{\mathrm{cyl}}$ on $((-\infty, -\lambda] \times \partial\tilde{M}) \times ((-\infty, -\lambda] \times \partial\tilde{M}) \times T$

LEMMA 4.1. *A_c is a *-subalgebra of $A^*(X, \mathcal{F})$. Let $\pi_c := \pi|_{A_c}$; there is a short exact sequence of *-algebras*

$$0 \to J_c \hookrightarrow A_c \xrightarrow{\pi_c} B_c \to 0\,. \tag{4.2}$$

REMARK 4.3. *Notice that the image of A_c through $t|_{A_c}$ is not contained in J_c since χ^0 is not even continuous. Similarly, the image of B_c through $s|_{B_c}$ is not contained in A_c.*

4.3. Relative cyclic cocycles. Let A be a $k-$algebra over $k = \mathbb{C}$. Recall the cyclic cohomology groups $HC^*(A)$ [**1**]. Given a second algebra B together with a surjective homomorphism $\pi : A \to B$, one can define the relative cyclic complex $C^n_\lambda(A, B) := \{(\tau, \sigma) : \tau \in C^n_\lambda(A), \sigma \in C^{n+1}_\lambda(B)\}$ with coboundary map given by $(\tau, \sigma) \longrightarrow (\pi^*\sigma - b\tau, b\sigma)$. A relative cochain (τ, σ) is thus a cocycle if $b\tau = \pi^*\sigma$ and $b\sigma = 0$. One obtains in this way the relative cyclic cohomology groups $HC^*(A, B)$. If A and B are Fréchet algebra, then we can also define the topological (relative) cyclic cohomology groups. More detailed information are given, for example, in [**9**].

4.4. Roe's 1-cocycle. In this subsection, and in the next two, we study a particular but important example. We assume that T is a point and that $\Gamma = \{1\}$, so that we are really considering a compact manifold X_0 with boundary ∂X_0 and associated manifold with cylindrical ends X; we keep denoting the cylinder $\mathbb{R}\times\partial X_0$ by $\mathrm{cyl}(\partial X)$ (thus, as before, we don't write the subscript 0). The algebras appearing in the short exact sequence (4.2) are now given by $J_c = C^\infty_c(X \times X)$,

$$B_c = \{k \in C^\infty((\mathbb{R}\times\partial X_0)\times(\mathbb{R}\times\partial X_0)); k \text{ is } \mathbb{R}\text{-invariant}, k \text{ has compact } \mathbb{R}\text{-support}\}\,.$$

Finally, a smooth function k on $X \times X$ is in A_c if there exists a $\lambda \equiv \lambda(k) > 0$ such that

(i) $k - \chi^\lambda k \chi^\lambda$ is of compact support on $X \times X$;

(ii) $\exists\, \ell \in B_c$ such that $\chi^\lambda k \chi^\lambda = \chi^\lambda_{\mathrm{cyl}} \ell \chi^\lambda_{\mathrm{cyl}}$ on $((-\infty, -\lambda]\times\partial X_0)\times(-\infty, -\lambda]\times\partial X_0)$.

For such a $k \in A_c$ we define $\pi_c(k) = \ell$ and we have the short exact sequence of $*$-algebras $0 \to J_c \hookrightarrow A_c \xrightarrow{\pi_c} B_c \to 0$. Incidentally, in the Wiener-Hopf short exact sequence (3.5), which now reads as $0 \to C^*(X) \to A^*(X) \xrightarrow{\pi} B^*(\mathrm{cyl}(\partial X)) \to 0$, the left term $C^*(X)$ is clearly given by the compact operators on $L^2(X)$.

We shall define below a 0-*relative* cyclic cocycle associated to the homomorphism $\pi_c : A_c \to B_c$. To this end we start by defining a cyclic 1-cocycle σ_1 for the algebra B_c; this is directly inspired from work of John Roe (indeed, a similarly defined 1-cocycle plays a fundamental role in his index theorem on partioned manifolds [**17**]).

Consider the characteristic function $\chi^\lambda_{\mathrm{cyl}}$, $\lambda > 0$, induced on the cylinder by the real function $\chi^{\mathbb{R}}_{(-\infty,-\lambda]}$. For notational convenience, unless absolutely necessary, we shall not distinguish between $\chi^\lambda_{\mathrm{cyl}}$ on the cylinder $\mathrm{cyl}(\partial X)$ and χ^λ on the manifold with cylindrical ends X.

We define $\sigma^\lambda_1 : B^{\mathbb{R}}_c \times B_c \to \mathbb{C}$ as

$$\sigma^\lambda_1(\ell_0, \ell_1) := \mathrm{Tr}(\ell_0[\chi^\lambda\,,\, \ell_1])\,. \tag{4.4}$$

One can check that the operators $[\chi^\lambda\,,\, \ell_0]$ and $\ell_0[\chi^\lambda\,,\, \ell_1]$ are trace class $\forall \ell_0, \ell_1 \in B_c$ (and $\mathrm{Tr}[\chi^\lambda\,,\, \ell_0] = 0$). In particular $\sigma^\lambda_1(\ell_0, \ell_1)$ is well defined.

Proposition 4.5. *The value* $\mathrm{Tr}(\ell_0[\chi^\lambda\,,\, \ell_1])$ *is independent of* λ *and will simply be denoted by* $\sigma_1(\ell_0, \ell_1)$. *The functional* $\sigma_1 : B_c \times B_c \to \mathbb{C}$ *is a cyclic 1-cocycle.*

4.5. Melrose' regularized integral. Recall that our immediate goal is to define a *relative* cyclic 0-cocycle for the homomorphism $\pi_c : A_c \to B_c$ appearing in the short exact sequence of the previous section. Having defined a 1-cocycle σ_1 on B_c we now need to define a 0-cochain on A_c. Our definition will be a simple adaptation of the definition of the b-trace in Melrose' b-calculus [**11**] (but since our algebra A_c is very small, we can give a somewhat simplified treatment). Recall that for $\lambda > 0$ we are denoting by X_λ the compact manifold obtained attaching $[-\lambda, 0] \times \partial X_0$ to our manifold with boundary X_0.

So, let $k \in A_c$ with $\pi_c(k) = \ell \in B_c$. Since ℓ is $\mathbb{R}$-invariant on the cylinder $\mathrm{cyl}(\partial X) = \mathbb{R} \times \partial X_0$ we can write $\ell(y, y', s)$ with $y, y' \in \partial X_0$, $s \in \mathbb{R}$. Set

$$\tau_0^r(k) := \lim_{\lambda \to +\infty} \left(\int_{X_\lambda} k(x,x) \mathrm{dvol}_g - \lambda \int_{\partial X_0} \ell(y,y,0) \mathrm{dvol}_{g_\partial} \right) \tag{4.6}$$

where the superscript r stands for *regularized*. (The b-superscript would be of course more appropiate; unfortunately it gets confused with the b operator in cyclic cohomology.) It is elementary to see that the limit exists; in fact, because of the very particular definition of A_c the function $\varphi(\lambda) := \int_{X_\lambda} k(x,x) \mathrm{dvol}_g - \lambda \int_{\partial X_0} \ell(y,y,0) \mathrm{dvol}_{g_\partial}$ becomes constant for large values of λ. The proof is elementary. τ_0^r defines a 0-cochain on A_c.

Remark 4.7. *Notice that* (4.6) *is nothing but Melrose' regularized integral* [**11**], *in the cylindrical language, for the restriction of k to the diagonal of $X \times X$.*

We shall also need the following

Lemma 4.8. *If $k \in A_c$ then $t(k)$, which is a priori a compact operator, is in fact trace class and $\tau_0^r(k) = \mathrm{Tr}(t(k))$.*

We remark once again that $t(k)$ is not an element in J_c.

4.6. The regularized integral and Roe's 1-cocycle define a relative 0-cocycle. We finally consider the relative 0-cochain (τ_0^r, σ_1) for the pair $A_c \xrightarrow{\pi_c} B_c$.

Proposition 4.9. *The relative 0-cochain (τ_0^r, σ_1) is a relative 0-cocycle. It thus defines an element $[(\tau_0^r, \sigma_1)]$ in the relative group $HC^0(A_c, B_c)$.*

There are several proofs of this Proposition; we have stated that σ_1 is a cocycle and what needs to be proved now is that $b\tau_0^r = (\pi_c)^* \sigma_1$. One proof of this equality employs Lemma 4.8; another one use the Hilbert transform and Melrose' formula for the b-trace of a commutator [**11**], see the next Subsection.

4.7. Melrose' 1-cocycle and the relative cocycle condition via the b-trace formula. As we have anticipated in the previous subsection, the equation $b\tau_0^r = \pi_c^* \sigma_1$ is nothing but a compact way of rewriting Melrose' formula for the b-trace of a commutator. We wish to explain this point here. Following now the notations of the b-calculus, we consider the sligthy larger algebras

$$A_c^b := \Psi_{b,c}^{-\infty}(X, E)\,, \quad B_c^b := \Psi_{b,I,c}^{-\infty}(\overline{N_+ \partial X}, E|_\partial)\,, \quad J_c^b := \rho_{\mathrm{ff}} \Psi_{b,c}^{-\infty}(X, E)$$

and $0 \longrightarrow J_c^b \longrightarrow A_c^b \xrightarrow{\pi_c^b} B_c^b \longrightarrow 0$, with π_c^b equal to Melrose' indicial operator $I(\cdot)$. Let τ_0^r be equal to the b-Trace: $\tau_0^r := {}^b\,\mathrm{Tr}$. Observe that σ_1 also defines a 1-cocyle on B_c^b. We can thus consider the relative 0-cochain (τ_0^r, σ_1) for the homomorphism $A_c^b \xrightarrow{I(\cdot)} B_c^b$; in order to prove that this is a relative 0-cocycle it remains to to show that $b\tau_0^r(k, k') = \sigma_1(I(k), I(k'))$, i.e.

$${}^b\,\mathrm{Tr}[k, k'] = \mathrm{Tr}(I(k)[\chi^0, I(k')]) \tag{4.10}$$

Recall here that Melrose' formula for the b-trace of a commutator is

$${}^b\,\mathrm{Tr}[k, k'] = \frac{i}{2\pi} \int_{\mathbb{R}} \mathrm{Tr}_{\partial X} \left(\partial_\mu I(k, \mu) \circ I(k', \mu) \right) d\mu \tag{4.11}$$

with $\mathbb{C} \ni z \to I(k,z)$ denoting the indicial family of the operator k, i.e. the Fourier transform of the indicial operator $I(k)$.

Inspired by the right hand side of (4.11) we consider an arbitrary compact manifold Y, the algebra $B^b_c(\mathrm{cyl}(Y))$ and the 1-cocycle

$$\mathfrak{s}_1(\ell,\ell') := \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}_Y\left(\partial_\mu \hat{\ell}(\mu)\circ \hat{\ell}'(\mu)\right) d\mu \tag{4.12}$$

That this is a cyclic 1-cocycle follows by elementary arguments. Formula (4.12) defines what should be called Melrose' 1-cocycle

PROPOSITION 4.13. *Roe's 1-cocycle σ_1 and Melrose 1-cocycle $\mathfrak{s}_1$ coincide:*

$$\sigma_1(\ell,\ell') := \mathrm{Tr}(\ell[\chi^0,\ell']) = \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}_Y\left(\partial_\mu \hat{\ell}(\mu)\circ \hat{\ell}'(\mu)\right) d\mu =: \mathfrak{s}_1(\ell,\ell') \tag{4.14}$$

Proposition 4.13 and Melrose' formula imply at once the relative 0-cocyle condition for (τ_0^r,σ_1): indeed using first Proposition 4.13 and then Melrose' formula we get:

$$\begin{aligned}\sigma_1(I(k),I(k')) := \mathrm{Tr}(I(k)[\chi^0,I(k')]) &= \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}_{\partial X}\left(\partial_\mu I(k,\mu)\circ I(k',\mu)\right) d\mu \\ &= {}^b\,\mathrm{Tr}[k,k'] = b\tau_0^r(k,k')\,.\end{aligned}$$

Thus $I^*(\sigma_1) = b\tau_0^r$ as required.

Conclusions. We have seen the following:

- the right hand side of Melrose' formula defines a 1-cocycle $\mathfrak{s}_1$ on $B_c(\mathrm{cyl}(Y))$, Y any closed compact manifold;
- Melrose 1-cocycle $\mathfrak{s}_1$ equals Roe's 1-cocycle σ_1
- Melrose' formula itself can be interpreted as a *relative* 0-cocyle condition for the 0-cochain $(\tau_0^r,\mathfrak{s}_1) \equiv (\tau_0^r,\sigma_1)$.

4.8. Philosophical remarks. We wish to recollect the information obtained in the last three subsections and start to explain our approach to Atiyah-Patodi-Singer higher index theory.

On a closed compact orientable riemannian smooth manifold Y let us consider the algebra of smoothing operators $J_c(Y) := C^\infty(Y\times Y)$. Then the functional $J_c(Y) \ni k \to \int_Y k|_\Delta \mathrm{dvol}$ defines a 0-cocycle τ_0 on $J_c(Y)$; indeed by Lidski's theorem the functional is nothing but the functional analytic trace of the integral operator corresponding to k and we know that the trace vanishes on commutators; in formulae, $b\tau_0 = 0$. The 0-cocycle τ_0 plays a fundamental role in the proof of the Atiyah-Singer index theorem, but we leave this aside for the time being.

Let now X be a smooth orientable manifold with cyclindrical ends, obtained from a manifold with boundary X_0; let $\mathrm{cyl}(\partial X) = \mathbb{R}\times\partial X_0$. We have then defined algebras $A_c(X)$, $B_c(\mathrm{cyl}(\partial X))$ and $J_c(X)$ fitting into a short exact sequence $0 \to J_c(X) \to A_c(X) \xrightarrow{\pi_c} B_c(\mathrm{cyl}(\partial X)) \to 0$.

Corresponding to the 0-cocycle τ_0 in the closed case we can define two important 0-cocycles on a manifold with cyclindrical ends X:

- We can consider τ_0 on $J_c(X) = C^\infty_c(X\times X)$; this is well defined and does define a 0-cocycle . We shall refer to τ_0 on $J_c(X)$ as an *absolute* 0-cocycle.
- Starting with the absolute 0-cocycle τ_0 on $J_c(X)$ we define a *relative* 0-cocycle (τ_0^r,σ_1) for $A_c(X) \xrightarrow{\pi_c} B_c(\mathrm{cyl}(\partial X))$. The relative 0-cocycle (τ_0^r,σ_1) is obtained through the following two fundamental steps.

(1) We define a 0-*cochain* τ_0^r on $A_c(X)$ by replacing the integral with Melrose' regularized integral.
(2) We define a 1-cocycle σ_1 on $B_c(\mathrm{cyl}(\partial X))$ by taking a *suspension* of τ_0 through the linear map $\delta(\ell) := [\chi^0, \ell]$. In other words, $\sigma_1(\ell_0, \ell_1)$ is obtained from $\tau_0 \equiv \mathrm{Tr}$ by considering $(\ell_0, \ell_1) \to \tau_0(\ell_0[\chi^0, \ell_1]) \equiv \tau_0(\ell_0 \delta(\ell_1))$.

DEFINITION 4.1. *We shall also call Roe's 1-cocycle σ_1 the eta 1-cocycle corresponding to the absolute 0-cocycle τ_0.*

In order to justify the wording of this definition we need to show that all this has something to do with the eta invariant and its role in the Atiyah-Patodi-Singer index formula. This will be explained in Section 6 and Section 7.

4.9. The absolute Godbillon-Vey 2-cyclic cocycle τ_{GV}. Let $(Y, \mathcal{F})$, $Y = \tilde{N} \times_\Gamma T$, be a foliated bundle *without* boundary. We assume that $T = S^1$. Let $E \to Y$ an hermitian complex vector bundle on Y. Let $\widehat{E}$ be the Γ-equivariant lift of E to $\tilde{N} \times T$. Let $(G, s : G \to Y, r : G \to Y)$ be the holonomy groupoid associated to Y, $G = (\tilde{N} \times \tilde{N} \times T)/\Gamma$. Consider again the convolution algebra $C_c^\infty(G, (s^*E)^* \otimes r^*E)$, of equivariant smoothing families with Γ-compact support. The notation $\Psi_c^{-\infty}(G, E)$ is also employed. On $\Psi_c^{-\infty}(G, E) \equiv C_c^\infty(G, (s^*E)^* \otimes r^*E)$ we can define a remarkable 2-cocycle, denoted τ_{GV} and known as the Godbillon-Vey cyclic cocycle. First of all, recall that there is a weight ω_Γ defined on the algebra $\Psi_c^{-\infty}(G; E)$,

$$\omega_\Gamma(k) = \int_{Y(\Gamma)} \mathrm{Tr}_{(\tilde{n},\theta)} k(\tilde{n}, \tilde{n}, \theta) d\tilde{n}\, d\theta\,. \tag{4.15}$$

In this formula $Y(\Gamma)$ is the fundamental domain in $\tilde{N} \times T$ for the free diagonal action of Γ on $\tilde{N} \times T$ and we have restricted the kernel k to $\Delta_{\tilde{N}} \times T \subset \tilde{N} \times \tilde{N} \times T$, $\Delta_{\tilde{N}}$ denoting the diagonal set in $\tilde{N} \times \tilde{N}$, $\Delta_{\tilde{N}} \times T \equiv \tilde{N} \times T$; $\mathrm{Tr}_{(\tilde{n},\theta)}$ denotes the trace on $\mathrm{End}(\widehat{E}_{(\tilde{n},\theta)})$. (If the measure on T is Γ-invariant, then this weight is a trace; however, we don't want to make this assumption here.) Recall then the bundle $\widehat{E}'$ on $Y \times T$: this is the same vector bundle as $\widehat{E}$ but with a different Γ-action. See [**14**] for details. There is a natural identification $\Psi_c^{-\infty}(G; E) \equiv \Psi_c^{-\infty}(G; E')$. We shall consider the linear space $\Psi_c^{-\infty}(G; E, E')$; using the above identification we can give $\Psi_c^{-\infty}(G; E, E')$ a natural bimodule structure over $\Psi_c^{-\infty}(G; E)$. We shall be interested in the linear functional [2] defined on the bimodule $\Psi_c^{-\infty}(G; E, E')$ by the analogue of (4.15). To be quite explicit

$$\omega_\Gamma(k) = \int_{Y(\Gamma)} \mathrm{Tr}_{(\tilde{n},\theta)} k(\tilde{n}, \tilde{n}, \theta) d\tilde{n}\, d\theta\,, \quad k \in \Psi_c^{-\infty}(G; E, E') \tag{4.16}$$

where we now identify $\widehat{E}_{(\tilde{n},\theta)}$ and $\widehat{E}'_{(\tilde{n},\theta)}$ given that thet are identical vector spaces (it is only the Γ-actions that are different). We call (4.16) the *bimodule trace.* (This name come from the following fundamental property: if $k \in \Psi_c^{-\infty}(G; E, E')$, $k' \in \Psi_c^{-\infty}(G; E) \equiv \Psi_c^{-\infty}(G; E')$ then $\omega_\Gamma(kk') = \omega_\Gamma(k'k)$.)

Recall now the two derivations $\delta_2 := [\phi, \quad]$ and $\delta_1 := [\dot{\phi}, \quad]$ coming from the modular automorphism group described in [**14**]. More precisely, we have a

[2] This will not be a weight, given that on a bimodule there is no notion of positive element

derivation δ_2 and a bimodule derivation δ_1,

$$\delta_2 : \Psi_c^{-\infty}(G,E) \to \Psi_c^{-\infty}(G;E)\,, \quad \delta_1 : \Psi_c^{-\infty}(G,E) \to \Psi_c^{-\infty}(G;E,E')\,, \tag{4.17}$$

DEFINITION 4.2. *With $1 = \dim T$, the Godbillon-Vey cyclic 2-cocycle on $C_c^\infty(G,(s^*E)^*\otimes r^*E)$ is defined to be*

$$\tau_{GV}(a_0,a_1,a_2) = \frac{1}{2!}\sum_{\alpha\in\mathfrak{S}_2} \operatorname{sign}(\alpha)\,\omega_\Gamma(a_0\;\delta_{\alpha(1)}a_1\;\delta_{\alpha(2)}a_2) \tag{4.18}$$

with ω_Γ the bimodule trace in (4.16).

The fact that this 3-linear functional is indeed a cyclic 2-cocycle is proved in [**14**]. We now go back to a foliated bundle $(X,\mathcal{F})$ with cylindrical ends, with $X := \tilde{M}\times_\Gamma T$, as in Section 2. We consider the small subalgebras introduced in Subsection 4.2. The weight ω_Γ is still well defined on $J_c(X,\mathcal{F})$; the 2-cocycle τ_{GV} can thus be defined on $J_c(X,\mathcal{F})$, giving us the *absolute* Godbillon-Vey cyclic cocycle.

4.10. The eta 3-coycle σ_{GV} corresponding to τ_{GV}. Now we apply the general philosophy explained at the end of the previous Section. Let χ^0 be the usual characteristic function of $(-\infty,0]\times\partial X_0$ in $\mathrm{cyl}(\partial X) = \mathbb{R}\times\partial X_0$. Write $\mathrm{cyl}(\partial X) = (\mathbb{R}\times\partial\tilde{M})\times_\Gamma T$ with Γ acting trivially on the $\mathbb{R}$ factor. Let $\mathrm{cyl}(\Gamma)$ be a fundamental domain for the action of Γ on $(\mathbb{R}\times\partial\tilde{M})\times T$; finally, let $\omega_\Gamma^{\mathrm{cyl}}$ be the corresponding weight. We keep denoting this weight by ω_Γ. Recall the derivation $\delta(\ell) := [\chi^0,\ell]$; recall that we passed from the absolute 0-cocycle $\tau_0\equiv\mathrm{Tr}$ to the 1-eta cocycle on the cylindrical algebra B_c by considering $(\ell_0,\ell_1)\to\tau_0(\ell_0\delta(\ell_1))$. We referred to this operation as a *suspension*.

We are thus led to *suspend* definition 4.2, thus defining the following 4-linear functional on the algebra B_c.

DEFINITION 4.3. *The eta cochain σ_{GV} associated to the absolute Godbillon-Vey 2-cocycle $\tau_{GV}(a_0,a_1,a_2)$ is by definition*

$$\sigma_{GV}(\ell_0,\ell_1,\ell_2,\ell_3) = \frac{1}{3!}\sum_{\alpha\in\mathfrak{S}_3} \operatorname{sign}(\alpha)\,\omega_\Gamma(\ell_0\;\delta_{\alpha(1)}\ell_1\;\delta_{\alpha(2)}\ell_2\;\delta_{\alpha(3)}\ell_3) \tag{4.19}$$

with $\delta_3(\ell) := [\chi^0,\ell]$. The eta cochain is a 4-linear functional on $B_c(\mathrm{cyl}(\partial X),\mathcal{F}_{\mathrm{cyl}})$)

In fact, we can define, as we did for σ_1, the 3-cochain σ_{GV}^λ by employing the characteristic function χ^λ. However, one checks easily that the value of σ_{GV}^λ does not depend on λ. One can prove that this definition is well posed, namely that each term $(\ell_0\;\delta_{\alpha(1)}(\ell_1)\;\delta_{\alpha(2)}(\ell_2)\;\delta_{\alpha(3)}(\ell_3))$ is of finite weight. We then have the important

PROPOSITION 4.20. *The eta functional σ_{GV} is cyclic and it is a 3-cocycle: $b\,\sigma_{GV} = 0$.*

4.11. The relative Godbillon-Vey cyclic cocycle $(\tau_{GV}^r,\sigma_{GV})$. We now apply our strategy as in Subsection 4.8. Thus starting with the absolute cyclic cocycle τ_{GV} on $J_c(X,\mathcal{F})$ we first consider the 3-linear functional on $A_c(X,\mathcal{F})$ given by $\psi_{GV}^r(k_0,k_1,k_2) := \frac{1}{2!}\sum_{\alpha\in\mathfrak{S}_2}\operatorname{sign}(\alpha)\,\omega_\Gamma^r(k_0\;\delta_{\alpha(1)}k_1\;\delta_{\alpha(2)}k_2)$ with ω_Γ^r the regularized weight corresponding to ω_Γ

Next we consider the *cyclic* cochain associated to ψ^r_{GV}:

$$\tau^r_{GV}(k_0,k_1,k_2) := \frac{1}{3}\left(\psi^r_{GV}(k_0,k_1,k_2) + \psi^r_{GV}(k_1,k_2,k_0) + \psi^r_{GV}(k_2,k_0,k_1)\right) . \tag{4.21}$$

The next Proposition is crucial:

PROPOSITION 4.22. *The relative cyclic cochain $(\tau^r_{GV}, \sigma_{GV}) \in C^2_\lambda(A_c, B_c)$ is a relative 2-cocycle: thus $b\sigma_{GV} = 0$ (which we already know) and $b\tau^r_{GV} = (\pi_c)^*\sigma_{GV}$.*

For later use we also state the analogue of Lemma 4.8:

PROPOSITION 4.23. *Let $t : A^*(X,\mathcal{F}) \to C^*(X,\mathcal{F})$ be the section introduced in Subsection 3.4. If $k \in A_c \subset A^*(X,\mathcal{F})$ then $t(k)$ has finite weight. Moreover, for the regularized weight $\omega^r_\Gamma : A_c \to \mathbb{C}$ we have*

$$\omega^r_\Gamma = \omega_\Gamma \circ t \tag{4.24}$$

5. Smooth subalgebras

In this section we select important subsequences of $0 \to C^*(X,\mathcal{F}) \to A^*(X;\mathcal{F}) \to B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}) \to 0$.

5.1. Shatten ideals. Let χ_Γ be a characteristic function for a fundamental domain of $\Gamma \to \tilde{M} \to M$. Consider $\Psi^{-\infty}_c(G;E) =: J_c(X\,\mathcal{F}) \equiv J_c$. We shall often omit the bundle E from the notation.

DEFINITION 5.1. *Let $k \in J_c$ be positive and self-adjoint. The Schatten norm $||k||_m$ of k is defined as*

$$(||k||_m)^m := \sup_{\theta \in T} ||\chi_\Gamma\,(k(\theta))^m \chi_\Gamma||_1 \tag{5.1}$$

with the $||\ ||_1$ denoting the usual trace-norm on the Hilbert space $\mathcal{H}_\theta = L^2(\tilde{V} \times \{\theta\}$. Equivalently

$$(||k||_m)^m = \sup_{\theta \in T} ||\chi_\Gamma\,(k(\theta))^{m/2}||^2_{HS} . \tag{5.2}$$

with $||\ ||_{HS}$ denoting the usual Hilbert-Schmidt norm. In general, we set $||k||_m := ||\ (kk^*)^{1/2}\ ||_m$. The Schatten norm of $k \in J_c$ is easily seen to be finite for any $m \geq 1$; we define $\mathcal{I}_m(X,\mathcal{F}) \equiv \mathcal{I}_m$ as the completion of J_c with respect to $||\ \ ||_m$

One can prove that $\mathcal{I}_m$ is a Banach algebra and an ideal inside $C^*(X,\mathcal{F})$. Moreover:

PROPOSITION 5.3. *The weight ω_Γ extends continuously from $J_c \equiv C^\infty_c(G)$ to $\mathcal{I}_1$.*

We shall now introduce the subalgebra of $C^*(X,\mathcal{F})$ that will be used in the proof of our index theorem. Consider on the cylinder $\mathbb{R} \times Y$ (with cylindrical variable s) the functions

$$f_{\mathrm{cyl}}(s,y) := \sqrt{1+s^2} \qquad g_{\mathrm{cyl}}(s,y) = 1 + s^2 . \tag{5.4}$$

We denote by f and g smooth functions on X equal to f_{cyl} and g_{cyl} on the open subset $(-\infty,0)\times Y$; f and g are well defined up to a compactly supported function. We set

$$\mathcal{J}_m(X,\mathcal{F}) := \{k \in \mathcal{I}_m \mid gk \text{ and } kg \text{ are bounded}\} \tag{5.5}$$

We shall often simply write $\mathcal{J}_m$.

PROPOSITION 5.6. *$\mathcal{J}_m$ is a subalgebra of $\mathcal{I}_m$ and a Banach algebra with the norm*

$$\|k\|_{\mathcal{J}_m} := \|k\|_m + \|gk\|_{C^*} + \|kg\|_{C^*} \,. \tag{5.7}$$

Moreover $\mathcal{J}_m$ is holomorphically closed in $\mathcal{I}_m$ (and, therefore, in $C^*(X,\mathcal{F})$).

5.2. Schatten extensions. Let $(Y,\mathcal{F})$, $Y := \tilde{N}\times_\Gamma T$, be a foliated T-bundle *without* boundary; for example $Y = \partial X \equiv \partial X_0$. Consider $(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$ the associated foliated cylinder. Recall the function χ^0_{cyl} (often just χ^0), the function on the cylinder induced by the characteristic function of $(-\infty, 0]$ in $\mathbb{R}$. Notice that the definition of Schatten norm also apply to $(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$, viewed as a foliated T-bundle with cylindrical ends. Let $\Psi^{-p}_{\mathbb{R},c}(G_{\mathrm{cyl}}) \equiv \Psi^{-p}_c(G_{\mathrm{cyl}}/\mathbb{R}_\Delta)$ be the space of $\mathbb{R}\times\Gamma$-equivariant families of pseudodifferential operators of order $-p$ on the fibration $(\mathbb{R}\times\tilde{N})\times T \to T$ with $\mathbb{R}\times\Gamma$-compact support. Consider an element $\ell \in \Psi^{-p}_c(G_{\mathrm{cyl}}/\mathbb{R}_\Delta)$; then we know that ℓ defines a bounded operator from the Sobolev field $\mathcal{E}^k$ to the Sobolev field $\mathcal{E}^{k+p}$. See [**14**], Section 3. Let us denote, as in [**14**], the operator norm of a bounded operator L from $\mathcal{E}^k$ to $\mathcal{E}^j$ as $\|L\|_{j,k}$; notice the reverse order. For a $\mathbb{R}\times\Gamma$-invariant, $\mathbb{R}\times\Gamma$-compactly supported pseudodifferential operator of order $(-p)$, P, we consider the norm

$$|||P|||_p := \max(\|P\|_{-n,-n-p}\,,\, \|P\|_{n+p,n}) \tag{5.8}$$

with n a fixed integer strictly greater than $\dim N$. We denote the closure of $\Psi^{-p}_c(G_{\mathrm{cyl}}/\mathbb{R}_\Delta)$ with respect to the norm $|||\cdot|||_p$ by $\mathrm{OP}^{-p}(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$. We shall often write OP^{-p}.

PROPOSITION 5.9. $\mathrm{OP}^{-p}(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$ *is a Banach algebra and a subalgebra of* $B^*(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$

Consider now the bounded linear map $\partial_3^{\max} : B^* \to \mathrm{End}_\Gamma \mathcal{H}$ given by $\partial_3^{\max}\ell := [\chi^0, \ell]$. Consider in B^* the Banach subalgebra $\mathrm{OP}^{-1}(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$ and consider in $\mathrm{End}_\Gamma \mathcal{H}$ the subalgebra $\mathcal{J}_m(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$. Let ∂_3 be the restriction of $\partial_3^{\max}$ to $\mathrm{OP}^{-1}(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$. Since $\|\cdot\| \le |||\cdot|||$ we see that ∂_3 is also bounded. Let $\mathcal{D} := \{\ell \in \mathrm{OP}^{-1}(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}) \mid \partial_3(\ell) \in \mathcal{J}_m(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})\}$. One can prove that $\partial_3|_{\mathcal{D}}$ induces a closed derivation $\overline{\delta}_3$ with domain $\mathcal{D}$. This is clearly a closed extension of the derivation δ_3, $\delta_3(\ell) = [\chi^0, \ell]$, considered in Subsection 4.10.

DEFINITION 5.2. *If $m \ge 1$ we define $\mathcal{D}_m(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$ as $\mathrm{Dom}\,\overline{\delta}_3$ endowed with norm*

$$\|\ell\|_{\mathcal{D}_m} := |||\ell||| + \|[\chi^0_{\mathrm{cyl}}, \ell]\|_{\mathcal{J}_m} \,. \tag{5.10}$$

We shall often simply write $\mathcal{D}_m$ instead of $\mathcal{D}_m(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$.

PROPOSITION 5.11. *Let $m \ge 1$, then $\mathcal{D}_m$ is a Banach algebra with respect to* (5.10) *and a subalgebra of $B^* \equiv B^*(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}})$. Moreover, $\mathcal{D}_m$ is holomorphically closed in B^*.*

The Banach algebra we have defined is still too large for the purpose of extending the eta cocycle. We shall first intersect it with another holomophically closed Banach subalgebra of B^*.

Observe that there exists an action of $\mathbb{R}$ on $\Psi^{-1}_c(G_{\mathrm{cyl}}/\mathbb{R}_\Delta) \subset \mathrm{OP}^{-1}(\mathrm{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}) \subset B^*$ defined by

$$\alpha_t(\ell) := e^{its}\ell e^{-its}\,, \tag{5.12}$$

with $t \in \mathbb{R}$, s the variable along the cylinder and $\ell \in \Psi_c^{-1}(G_{\rm cyl}/\mathbb{R}_\Delta)$. Note that $\alpha_t(\ell)$ is again $(\mathbb{R} \times \Gamma)$-equivariant. It is clear that $|||\alpha_t(\ell)||| = |||\ell|||$; thus, by continuity, $\{\alpha_t\}_{t\in\mathbb{R}}$ yields a well-defined action, still denoted $\{\alpha_t\}_{t\in\mathbb{R}}$, of $\mathbb{R}$ on the Banach algebra $\mathrm{OP}^{-1}(\mathrm{cyl}(Y), \mathcal{F}_{\rm cyl})$. Note that this action is only strongly continuous. Let $\partial_\alpha : \mathrm{OP}^{-1} \to \mathrm{OP}^{-1}$ be the unbounded derivation associated to $\{\alpha_t\}_{t\in\mathbb{R}}$

$$\partial_\alpha(\ell) := \lim_{t\to 0} \frac{(\alpha_t(\ell) - \ell)}{t} \tag{5.13}$$

By definition

$$\mathrm{Dom}(\partial_\alpha) = \{\ell \in \mathrm{OP}^{-1} \mid \partial_\alpha(\ell) \text{ exists in } \mathrm{OP}^{-1}\}.$$

One can prove that the derivation ∂_α is in fact a *closed* derivation.

We endow $\mathrm{Dom}(\partial_\alpha)$ with the graph norm

$$|||\ell||| + |||\partial_\alpha(\ell)||| \,. \tag{5.14}$$

It is not difficult to see that $\mathrm{Dom}(\partial_\alpha)$ is a Banach algebra with respect to (5.14) and, obviously, a subalgebra of $B^* \equiv B^*(\mathrm{cyl}(Y), \mathcal{F}_{\rm cyl})$; moreover it is holomorphically closed in B^*.

We can now take the intersection of the Banach subalgebras $\mathcal{D}_m(\mathrm{cyl}(Y), \mathcal{F}_{\rm cyl})$ and $\mathrm{Dom}(\partial_\alpha)$:

$$\mathcal{D}_{m,\alpha}(\mathrm{cyl}(Y), \mathcal{F}_{\rm cyl}) := \mathcal{D}_m(\mathrm{cyl}(Y), \mathcal{F}_{\rm cyl}) \cap \mathrm{Dom}(\partial_\alpha)$$

and we endow it with the norm

$$\|\ell\|_{m,\alpha} := |||\ell||| + \|[\chi^0_{\rm cyl}, \ell]\|_{\mathcal{J}_m} + |||\partial_\alpha \ell||| \,. \tag{5.15}$$

Being the intersection of two holomorphically closed dense subalgebras, also the Banach algebra $\mathcal{D}_{m,\alpha}(\mathrm{cyl}(Y), \mathcal{F}_{\rm cyl})$ enjoys this property.
We are finally ready to define the subalgebra we are interested in. Recall the function $f_{\rm cyl}(s, y) = \sqrt{1 + s^2}$.

Definition 5.3. *If $m \geq 1$ we define*

$$\mathcal{B}_m(\mathrm{cyl}(Y), \mathcal{F}_{\rm cyl}) := \{\ell \in \mathcal{D}_{m,\alpha}(\mathrm{cyl}(Y), \mathcal{F}_{\rm cyl}) \mid [f, \ell] \;\; \textit{and} \;\; [f, [f, \ell]] \;\; \textit{are bounded}\} \,. \tag{5.16}$$

This will be endowed with norm

$$\begin{aligned} \|\ell\|_{\mathcal{B}_m} :=& \|\ell\|_{m,\alpha} + 2\|[f, \ell]\|_{B^*} + \|[f, [f, \ell]]\|_{B^*} \\ =& |||\ell||| + \|[\chi^0_{\rm cyl}, \ell]\|_{\mathcal{J}_m} + |||\partial_\alpha \ell||| + 2\|[f, \ell]\|_{B^*} + \|[f, [f, \ell]]\|_{B^*} \,. \end{aligned}$$

One can prove that $\mathcal{B}_m(\mathrm{cyl}(Y), \mathcal{F}_{\rm cyl})$ is a holomorphically closed dense subalgebra of B^*. We shall often simply write $\mathcal{B}_m$ instead of $\mathcal{B}_m(\mathrm{cyl}(Y), \mathcal{F}_{\rm cyl})$.

Let us go back to the foliated bundle with cylindrical end $(X, \mathcal{F})$. We now define

$$\mathcal{A}_m(X, \mathcal{F}) := \{k \in A^*(X, \mathcal{F}); \pi(k) \in \mathcal{B}_m(\mathrm{cyl}(\partial X), \mathcal{F}_{\rm cyl}), t(k) \in \mathcal{J}_m(X, \mathcal{F})\} \tag{5.17}$$

Lemma 5.18. *$\mathcal{A}_m(X, \mathcal{F})$ is a subalgebra of $A^*(X, \mathcal{F})$.*

Now we observe that, as vector spaces,

$$\mathcal{A}_m \cong \mathcal{J}_m \oplus s(\mathcal{B}_m) \,. \tag{5.19}$$

Granted this result, we endow $\mathcal{A}_m$ with the direct-sum norm:

$$||k||_{\mathcal{A}_m} := ||t(k)||_m + ||\pi(k)||_{\mathcal{B}_m} \tag{5.20}$$

Obviously s induces a bounded linear map $\mathcal{B}_m \to \mathcal{A}_m$ of Banach spaces.

PROPOSITION 5.21. *$(\mathcal{A}_m, ||\ \ ||_{\mathcal{A}_m})$ is a Banach algebra. Moreover, $\mathcal{J}_m$ is an ideal in $\mathcal{A}_m$ and there is a short exact sequence of Banach algebras:*

$$0 \to \mathcal{J}_m(X,\mathcal{F}) \to \mathcal{A}_m(X;\mathcal{F}) \xrightarrow{\pi} \mathcal{B}_m(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}) \to 0\,. \tag{5.22}$$

Finally, $t : A^*(X,\mathcal{F}) \to C^*(X,\mathcal{F})$ restricts to a bounded section $t : \mathcal{A}_m(X,\mathcal{F}) \to \mathcal{J}_m(X,\mathcal{F})$

5.3. Derivations. In order to extend continuously the cyclic cocycles τ_{GV} and $(\tau^r_{GV}, \sigma_{GV})$ we need to take into account the modular automorphism group, thus decreasing further the size of the short exact sequence $0 \to \mathcal{J}_m \to \mathcal{A}_m \xrightarrow{\pi} \mathcal{B}_m \to 0$. Consider the two derivations δ_1 and δ_2 introduced in Subsection 4.9. Let us consider first δ_2. Recall the C^*-algebra $C^*_\Gamma(\mathcal{H}) \supset C^*(X,\mathcal{F})$; it is obtained, by definition, by closing up the subalgebra $C_{\Gamma,c}(\mathcal{H}) \subset \mathrm{End}_\Gamma(\mathcal{H})$ consisting of those elements that preserve the continuous field $C^\infty_c(\tilde{V} \times T, E)$. We set $\mathrm{Dom}\,(\delta_2^{\max}) = \{k \in C_{\Gamma,c}(\mathcal{H}) \,|\, [\phi, k] \in C^*_\Gamma(\mathcal{H})\}$ and

$$\delta_2^{\max} : \mathrm{Dom}\,(\delta_2^{\max}) \to C^*_\Gamma(\mathcal{H}), \quad \delta_2^{\max}(k) := [\phi, k]\,.$$

One can prove that $\delta_2^{\max}$ is closable. Similarly, with self-explanatory notation, the bimodule derivation

$$\delta_1^{\max} : \mathrm{Dom}\,(\delta_1^{\max}) \to C^*_\Gamma(\mathcal{H},\mathcal{H}'), \quad \delta_1^{\max}(k) := [\dot{\phi}, k]\,,$$

with $\mathrm{Dom}\,(\delta_1^{\max}) := \{k \in C_{\Gamma,c}(\mathcal{H}) \,|\, [\dot{\phi}, k] \in C^*_\Gamma(\mathcal{H},\mathcal{H}')\}$ is closable. Let $\overline{\delta}_j^{\max}$ be their respective closures; thus, for example,

$$\overline{\delta}_2^{\max} : \mathrm{Dom}\,\overline{\delta}_2^{\max} \subset C^*_\Gamma(\mathcal{H}) \longrightarrow C^*_\Gamma(\mathcal{H})$$

and similarly for $\delta_1^{\max}$. Define now

$$\mathcal{D}_2 := \{a \in \mathrm{Dom}\,\overline{\delta}_2^{\max} \cap \mathcal{J}_m(X,\mathcal{F}) \mid \overline{\delta}_2^{\max} a \in \mathcal{J}_m(X,\mathcal{F})\}$$

and $\overline{\delta}_2 : \mathcal{D}_2 \to \mathcal{J}_m(X,\mathcal{F})$ as the restriction of $\overline{\delta}_2^{\max}$ to $\mathcal{D}_2$ with values in $\mathcal{J}_m(X,\mathcal{F})$. One can show that $\overline{\delta}_2$ is a closed derivation. Define similarly $\mathcal{D}_1$ and the closed derivation $\overline{\delta}_1$. We set

$$\mathfrak{J}_{\mathbf{m}} := \mathcal{J}_m \cap \mathrm{Dom}(\overline{\delta}_1) \cap \mathrm{Dom}(\overline{\delta}_2)\,. \tag{5.23}$$

with $\mathrm{Dom}(\overline{\delta}_1) = \mathcal{D}_1$ and $\mathrm{Dom}(\overline{\delta}_2) = \mathcal{D}_2$.

Consider next $\mathcal{B}_m$; we consider the derivations $\delta_1 := [\dot{\phi}_\partial, \ \ ]$, $\delta_2 := [\phi_\partial, \ \ ]$ on the cylinder $\mathbb{R} \times \partial X_0$; we have already encountered these derivations in the definition of the eta cocycle σ_{GV}; see more precisely Definition 4.3. Consider first δ_2. Define a closed derivation $\overline{\partial}_2$ by taking the closure of the closable derivation $\Psi_c^{-1}(G_{\mathrm{cyl}}/\mathbb{R}_\Delta) \xrightarrow{\partial_2} B^*$, with $\partial_2(\ell) := [\phi_\partial, \ell]$ and with $\Psi_c^{-1}(G_{\mathrm{cyl}}/\mathbb{R}_\Delta)$ endowed with the norm $|||\cdot|||$. One can prove that $\overline{\partial}_2|_{\mathfrak{D}_2}$, with

$$\mathfrak{D}_2 = \{b \in \mathrm{Dom}(\overline{\partial}_2) \;\mid\; \overline{\partial}_2(b) \in \mathcal{B}_m\}$$

is a closed derivation with values in $\mathcal{B}_m$. We set $\overline{\delta}_2 := \overline{\partial}_2|_{\mathfrak{D}_2}$; thus $\mathrm{Dom}(\overline{\delta}_2) = \mathfrak{D}_2$ and $\overline{\delta}_2 := \overline{\partial}_2|_{\mathfrak{D}_2}$. A similarly definition of $\overline{\delta}_1$ and $\mathrm{Dom}(\overline{\delta}_1)$ can be given. We set

$$\mathfrak{B}_{\mathbf{m}} := \mathcal{B}_m \cap \mathrm{Dom}(\overline{\delta}_1) \cap \mathrm{Dom}(\overline{\delta}_2) \equiv \mathcal{B}_m \cap \mathfrak{D}_1 \cap \mathfrak{D}_2\,. \tag{5.24}$$

We endow $\mathfrak{B}_{\mathbf{m}}$ with the norm

$$\|\ell\|_{\mathfrak{B}_{\mathbf{m}}} := \|\ell\|_{\mathcal{B}_m} + \|\overline{\delta}_1 \ell\|_{\mathcal{B}_m} + \|\overline{\delta}_2 \ell\|_{\mathcal{B}_m} \tag{5.25}$$

PROPOSITION 5.26. *$\mathfrak{B}_{\mathbf{m}}$ is holomorphically closed in $B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}})$.*

Finally, we consider the Banach algebra $\mathcal{A}_m(X, \mathcal{F})$ which is certainly contained in $C^*_\Gamma(\mathcal{H})$, given that $A_c(X, \mathcal{F})$ is contained in $C_{\Gamma,c}(\mathcal{H})$. Consider again $\overline{\delta}_j^{\max}$ and restrict it to a derivation with values in $\mathcal{A}_m(X, F)$:

$$\overline{\delta}_2 : \mathcal{D}_2 \to \mathcal{A}_m(X, F)$$

with $\mathcal{D}_2 = \{a \in \mathrm{Dom}\, \overline{\delta}_2^{\max} \mid \overline{\delta}_2^{\max} a \in \mathcal{A}_m(X, F)\}$ and similarly for $\overline{\delta}_1$. We obtain in this way closed derivations $\overline{\delta}_1$ and $\overline{\delta}_2$ with domains $\mathrm{Dom}\overline{\delta}_1 = \mathcal{D}_1$ and $\mathrm{Dom}\overline{\delta}_2 = \mathcal{D}_2$. We set

$$\mathfrak{A}_{\mathbf{m}} := \mathcal{A}_m \cap \mathrm{Dom}(\overline{\delta}_1) \cap \mathrm{Dom}(\overline{\delta}_2) \cap \pi^{-1}(\mathfrak{B}_{\mathbf{m}})\,. \tag{5.27}$$

LEMMA 5.28. *The map π sends $\mathfrak{A}_{\mathbf{m}}$ into $\mathfrak{B}_{\mathbf{m}}$; $\mathfrak{J}_{\mathbf{m}}$ is an ideal in $\mathfrak{A}_{\mathbf{m}}$ and we obtain a short exact sequence of Banach algebras*

$$0 \to \mathfrak{J}_{\mathbf{m}} \to \mathfrak{A}_{\mathbf{m}} \xrightarrow{\pi} \mathfrak{B}_{\mathbf{m}} \to 0 \tag{5.29}$$

The section s and t restricts to bounded sections $s : \mathfrak{B}_{\mathbf{m}} \to \mathfrak{A}_{\mathbf{m}}$ and $t : \mathfrak{A}_{\mathbf{m}} \to \mathfrak{J}_{\mathbf{m}}$. Finally, $\mathfrak{J}_{\mathbf{m}}$ is holomorphically closed in $C^*(X, \mathcal{F})$.

5.4. Isomorphism of K-groups. Let $0 \to J \to A \xrightarrow{\pi} B \to 0$ a short exact sequence of Banach algebras. Recall that $K_0(J) := K_0(J^+, J) \simeq \mathrm{Ker}(K_0(J^+) \to \mathbb{Z})$ and that $K(A^+, B^+) = K(A, B)$. For the definition of relative K-groups we refer, for example, to [**5**], [**9**]. Recall that a relative K_0-element for $A \xrightarrow{\pi} B$ is represented by a triple (P, Q, p_t) with P and Q idempotents in $M_{k\times k}(A)$ and $p_t \in M_{k\times k}(B)$ a path of idempotents connecting $\pi(P)$ to $\pi(Q)$. The excision isomorphism

$$\alpha_{\mathrm{ex}} : K_0(J) \longrightarrow K_0(A, B) \tag{5.30}$$

is given by

$$\alpha_{\mathrm{ex}}([(P, Q)]) = [(P, Q, \mathbf{c})]$$

with $\mathbf{c}$ denoting the constant path. Consider also $\mathfrak{J}_{\mathbf{m}} := \mathcal{J}_m \cap \mathrm{Dom}(\overline{\delta}_1) \cap \mathrm{Dom}(\overline{\delta}_2)$ and recall that this is a smooth subalgebra of $C^*(X, \mathcal{F})$: using also the excision isomorphism, we obtain

$$K_0(A^*, B^*) \simeq K_0(C^*(X, \mathcal{F})) \simeq K_0(\mathfrak{J}_{\mathbf{m}}) \simeq K_0(\mathfrak{A}_{\mathbf{m}}, \mathfrak{B}_{\mathbf{m}})\,. \tag{5.31}$$

5.5. Extended cocycles. Recall, from general theory, that $[\tau_{GV}] \in HC^2(J_c)$ and $[(\tau^r_{GV}, \sigma_{GV})] \in HC^2(A_c, B_c)$ can be paired with elements in $K_0(J_c)$ and $K_0(A_c, B_c)$ respectively. See the proof of our index formula below for the definition of the relative pairing. Introduce now the S^{p-1} operation and

$$S^{p-1}\tau_{GV} =: \tau_{2n} \quad \text{and} \quad (S^{p-1}\tau^r_{GV}, \frac{3}{2p+1} S^{p-1}\sigma_{GV}) =: (\tau^r_{2p}, \sigma_{(2p+1)}).$$

We obtain in this way cyclic cocycles and thus classes $[\tau_{2p}] \in HC^{2p}(J_c)$ and $[(\tau^r_{2p}, \sigma_{(2p+1)})] \in HC^{2p}(A_c, B_c)$.

PROPOSITION 5.32. *Let $2n$ equal to the dimension of the leaves in $X = \tilde{M} \times_\Gamma S^1$. Then the absolute cocycle τ_{2n} extends to a bounded cyclic cocycle on $\mathfrak{J}_{\mathbf{2n+1}}$ and the eta cocycle $\sigma_{(2n+1)}$ extends to a bounded cyclic cocycle on $\mathfrak{B}_{2n+1}$.*

PROPOSITION 5.33. *Let $\deg S^{p-1}\tau^r_{GV} = 2p > m(m-1)^2 - 2 = m^3 - 2m^2 + m - 2$, with $m = 2n+1$ and $2n$ equal to the dimension of the leaves in $(X, \mathcal{F})$. Then the regularized Godbillon-Vey cochain $S^{p-1}\tau^r_{GV}$, which is by definition τ^r_{2p}, extends to a bounded cyclic cochain on $\mathfrak{A}_m$.*

Summarizing: fix $m = 2n+1$, with $2n$ equal to dimension of the leaves and set

$$\mathfrak{J} := \mathfrak{J}_{\mathbf{m}}\,, \quad \mathfrak{A} := \mathfrak{A}_{\mathbf{m}}\,, \quad \mathfrak{B} := \mathfrak{B}_{\mathbf{m}}$$

Using the above two Propositions we see that there are well defined classes

$$[\tau_{2p}] \in HC^{2p}(\mathfrak{J}) \quad \text{for} \quad 2p \geq 2n \tag{5.34}$$

$$[(\tau^r_{2p}, \sigma_{(2p+1)})] \in HC^{2p}(\mathfrak{A}, \mathfrak{B}) \quad \text{for} \quad 2p > m(m-1)^2 - 2\,. \tag{5.35}$$

6. C^*-index classes. Excision

6.1. Dirac operators. We begin with a closed foliated bundle $(Y, \mathcal{F})$, with $Y = \tilde{N} \times_\Gamma T$. We are also given a Γ-equivariant complex vector bundle $\widehat{E}$ on $\tilde{N} \times T$, or, equivalently, a complex vector bundle on Y. We assume that $\widehat{E}$ has a Γ-equivariant vertical Clifford structure. We obtain in this way a Γ-equivariant family of Dirac operators $(D_\theta)_{\theta \in T}$ that will be simply denoted by D. If $(X_0, \mathcal{F}_0)$, $X_0 = \tilde{M} \times_\Gamma T$, is a foliated bundle with boundary, as in the previous sections, then we shall assume the relevant geometric structures to be of product-type near the boundary. If $(X, \mathcal{F})$ is the associated foliated bundle with cylindrical ends, then we shall extend all the structure in a constant way along the cylindrical ends. We shall eventually assume $\tilde{M}$ to be of even dimension, the bundle $\widehat{E}$ to be $\mathbb{Z}_2$-graded and the Dirac operator to be odd and formally self-adjoint. We denote by $D^\partial \equiv (D^\partial_\theta)_{\theta \in T}$ the boundary family defined by D^+. This is a Γ-equivariant family of formally self-adjoint first order elliptic differential operators on a complete manifold. We denote by $D^{\rm cyl}$ the operator induced by $D^\partial \equiv (D^\partial_\theta)_{\theta \in T}$ on the cylindrical foliated manifold $(\mathrm{cyl}(\partial X), \mathcal{F}_{\rm cyl})$; $D^{\rm cyl}$ is $\mathbb{R} \times \Gamma$-equivariant. We refer to [**14**] [**7**] for precise definitions. In all of this section we shall make the following fundamental

Assumption. There exists $\epsilon > 0$ such that $\forall \theta \in T$

$$L^2 - \mathrm{spec}(D^\partial_\theta) \cap (-\epsilon, \epsilon) = \emptyset \tag{6.1}$$

For specific examples where this assumption is satisfied, see [**7**]. We shall concentrate on the spin-Dirac case, but it will be clear how to extend the results to general Dirac-type operators.

6.2. Index class in the closed case. Let $(Y, \mathcal{F})$ be a closed foliated bundle. First, we need to recall how in the closed case we can define an index class $\mathrm{Ind}(D) \in K_*(C^*(Y, \mathcal{F}))$. There are in fact three equivalent description of $\mathrm{Ind}(D)$, each one with its own interesting features:

- the Connes-Skandalis index class, defined by the Connes-Skandalis projector P_Q associated to a pseudodifferential parametrix Q for D; Q can be chosen of Γ-compact support;
- the Wassermann index class, defined by the Wassermann projector W_D;
- the index class of the graph projection, defined by the graph projection e_D.

It is well known that the three classes introduced above are equal in $K_0(C^*(Y, \mathcal{F}))$.

6.3. The relative index class $\mathrm{Ind}(D, D^\partial)$**.** Let now $(X, \mathcal{F})$ be a foliated bundle with cylindrical ends. Let $(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}})$ the associated foliated cylinder. Recall $0 \to C^*(X, \mathcal{F}) \to A^*(X; \mathcal{F}) \xrightarrow{\pi} B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}) \to 0$, the Wiener-Hopf extension of the C^*-algebra of translation invariant operators $B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}})$; see Subsection 3.4. We shall be concerned with the K-theory group $K_*(C^*(X, \mathcal{F}))$ and with the relative group $K_*(A^*(X; \mathcal{F}), B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}))$. We shall write more briefly $0 \to C^* \to A^* \xrightarrow{\pi} B^* \to 0$, and $K_*(A^*, B^*)$. Recall that a relative K_0-cycle for $A^* \xrightarrow{\pi} B^*$ is a triple (P, Q, p_t) with P and Q idempotents in $M_{k\times k}(A^*)$ and $p_t \in M_{k\times k}(B^*)$ a path of idempotents connecting $\pi(P)$ to $\pi(Q)$.

Proposition 6.2. *Let $(X, \mathcal{F})$ be a foliated bundle with cyclindrical ends, as above. Consider the Dirac operator on X, $D = (D_\theta)_{\theta \in T}$. Assume* (6.1)*. Then the graph projection e_D and the Wassermann projection W_D define two relative classes in $K_0(A^*, B^*)$. These two classes are equal and fix the relative index class* $\mathrm{Ind}(D, D^\partial)$.

The relative classs of Proposition 6.2 are more precisely given by the triples
(6.3)
$$(e_D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, p_t) \text{ with } p_t := e_{tD^{\mathrm{cyl}}} \text{ and } (W_D, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, q_t) \text{ with } q_t := W_{tD^{\mathrm{cyl}}}\,,$$

with $t \in [1, +\infty]$. The content of the Proposition is that these two triples do define elements in $K_0(A^*, B^*)$ and that these two elements are equal.

6.4. The index class $\mathrm{Ind}(D)$**.** Recall the results in [**7**] where it is proved that there is a well defined parametrix Q for D^+, $QD^+ = \mathrm{Id} - S_+$, $D^+Q = \mathrm{Id} - S_-$, with remainders $S_\pm$ in $\mathbb{K}(\mathcal{E}) \equiv C^*(X, \mathcal{F})$. Consequently, there is a well defined Connes-Skandalis projector P_Q. The construction explained in [**7**] is an extension to the foliated case of the parametrix construction of Melrose, with particular care devoted to the non-compactness of the leaves.

Definition 6.1. *The index class associated to a Dirac operator on $(X, \mathcal{F})$ satisfying assumption* (6.1) *is the Connes-Skandalis index class associated to the Connes-Skandalis projector P_Q. It is denoted by* $\mathrm{Ind}(D) \in K_0(C^*(X, \mathcal{F}))$.

6.5. Excision for index classes. The following Proposition plays a fundamental role in our approach to higher APS index theory:

Proposition 6.4. *Let $D = (D_\theta)_{\theta \in T}$ be a Γ-equivariant family of Dirac operators on a foliated manifold with cylindrical ends $X = \tilde{M} \times_\Gamma T$. Assume that $\tilde{M}$ is even dimensional. Assume* (6.1)*. Then*

$$\alpha_{\mathrm{ex}}\,(\,\mathrm{Ind}(D)\,) = \mathrm{Ind}(D, D^\partial) \tag{6.5}$$

7. Index theorems

7.1. Notation. From now on we shall fix the dimension of the leaves of $(X, \mathcal{F})$, equal to $2n$, and set

$$\mathfrak{J} := \mathfrak{J}_{\mathbf{2n+1}}\,, \quad \mathfrak{A} := \mathfrak{A}_{\mathbf{2n+1}} \quad \text{and} \quad \mathfrak{B} := \mathfrak{B}_{\mathbf{2n+1}} \tag{7.1}$$

so that the short exact sequence in (5.29), for $m = 2n + 1$, is denoted simply as

$$0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{B} \to 0 \tag{7.2}$$

This is the intermediate subsequence, between $0 \to J_c \to A_c \to B_c \to 0$ *and* $0 \to C^*(X,\mathcal{F}) \to A^*(X,\mathcal{F}) \to B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}) \to 0$*, that we have mentioned in the introductory remarks in Subsection 4.1.*

7.2. Smooth index classes. In Sections 6.3 and 6.4 we stated the existence of two C^*-algebraic index classes: the index class and the *relative* index class. We have also seen in Subsection 5.5 that the absolute and relative cyclic cyclic cocycles τ_{GV} and $(\tau^r_{GV}, \sigma_{GV})$ extend from J_c and $A_c \xrightarrow{\pi_c} B_c$ to the smooth subalgebras $\mathfrak{J}$ and $\mathfrak{A} \xrightarrow{\pi} \mathfrak{B}$. In order to make use of the latter information, we need to *smooth-out* our index classes. This is the content of the following

THEOREM 7.3.
1)The Connes-Skandalis projector defines a smooth index class $\mathrm{Ind}^s(D) \in K_0(\mathfrak{J})$*; moreover, if* $\iota_* : K_0(\mathfrak{J}) \to K_0(C^*(X,\mathcal{F}))$ *is the isomorphism induced by the inclusion* ι*, then* $\iota_*(\mathrm{Ind}^s(D)) = \mathrm{Ind}(D)$*.*
2)The graph projections on $(X,\mathcal{F})$ *and* $(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}})$ *define a smooth relative index class* $\mathrm{Ind}^s(D, D^\partial) \in K_0(\mathfrak{A},\mathfrak{B})$*; moreover, if* $\iota_* : K_0(\mathfrak{A},\mathfrak{B}) \to K_0(A^*, B^*)$ *is the isomorphism induced by the inclusion* ι*, then* $\iota_*(\mathrm{Ind}^s(D, D^\partial)) = \mathrm{Ind}(D, D^\partial)$*.*
3)Finally, if $\alpha^s_{\mathrm{ex}} : K_0(\mathfrak{J}) \to K_0(\mathfrak{A},\mathfrak{B})$ *is the smooth excision isomorphism, then*

$$\alpha^s_{\mathrm{ex}}(\mathrm{Ind}^s(D)) = \mathrm{Ind}^s(D, D^\partial) \quad \text{in} \quad K_0(\mathfrak{A},\mathfrak{B})\,. \tag{7.4}$$

7.3. The higher APS index formula for the Godbillon-Vey cocycle. We can now state a APS formula for the Godbillon-Vey cocycle. Let us summarize our geometric data. We have a foliated bundle with boundary $(X_0, \mathcal{F}_0)$, $X_0 = \tilde{M} \times_\Gamma T$ with $T = S^1$ We assume that the dimension of $\tilde{M}$ is even and that all our geometric structures (metrics, connections, etc) are of product type near the boundary. We also consider $(X,\mathcal{F})$, the associated foliation with cylindrical ends. We are given a Γ-invariant $\mathbb{Z}_2$-graded hermitian bundle $\widehat{E}$ on the trivial fibration $\tilde{M} \times T$, endowed with a Γ-equivariant vertical Clifford structure. We have a resulting Γ-equivariant family of Dirac operators $D = (D_\theta)$.

Fix $m = 2n+1$, with $2n$ equal to dimension of the leaves and set as before

$$\mathfrak{J} := \mathfrak{J}_{\mathbf{m}}\,, \quad \mathfrak{A} := \mathfrak{A}_{\mathbf{m}}\,, \quad \mathfrak{B} := \mathfrak{B}_{\mathbf{m}}$$

We know that there are well defined index classes

$$\mathrm{Ind}^s(D) \in K_0(\mathfrak{J})\,, \quad \mathrm{Ind}^s(D, D^\partial) \in K_0(\mathfrak{A},\mathfrak{B})\,,$$

the first given in terms of a parametrix Q and the second given in term of the graph projection e_D. Proposition 5.32 and Proposition 5.33 imply the existence of the following additive maps:

$$\langle\,\cdot\,, [\tau_{2p}]\rangle : K_0(\mathfrak{J}) \to \mathbb{C}\,, \quad 2p \geq 2n \tag{7.5}$$

$$\langle\,\cdot\,, [(\tau^r_{2p}, \sigma_{(2p+1)})]\rangle : K_0(\mathfrak{A},\mathfrak{B}) \to \mathbb{C}\,, \quad 2p > m(m-1)^2 - 2\,. \tag{7.6}$$

DEFINITION 7.1. *Let* $(X_0, \mathcal{F}_0)$*,* $X_0 = \tilde{M} \times_\Gamma S^1$*, as above and assume* (6.1)*. The Godbillon-Vey higher index is the number*

$$\mathrm{Ind}_{GV}(D) := \langle \mathrm{Ind}(D), [\tau_{2n}]\rangle. \tag{7.7}$$

Notice that, in fact, $\mathrm{Ind}_{GV}(D) := \langle \mathrm{Ind}^s(D), [\tau_{2p}]\rangle$ for each $p \geq n$.

The following theorem is the main results of this paper:

THEOREM 7.8. *Let $X_0 = \tilde{M} \times_\Gamma S^1$ be a foliated bundle with boundary and let $D := (D_\theta)_{\theta \in S^1}$ be a Γ-equivariant family of Dirac operators as above. Assume* (6.1) *on the boundary family. Fix $2p > m(m-1)^2 - 2$ with $m = 2n+1$ and $2n$ equal to the dimension of the leaves. Then the following two equalities hold*

$$\mathrm{Ind}_{GV}(D) = \langle \mathrm{Ind}^s(D, D^\partial), [(\tau^r_{2p}, \sigma_{2p+1}]\rangle = \int_{X_0} \mathrm{AS} \wedge \omega_{GV} - \eta_{GV} \tag{7.9}$$

with

$$\eta_{GV} := \frac{(2p+1)}{p!} \int_0^\infty \sigma_{(2p+1)}([\dot{p}_t, p_t], p_t, \ldots, p_t) dt\,, \quad p_t := e_{tD^{\mathrm{cyl}}}\,, \tag{7.10}$$

defining the Godbillon-Vey eta invariant *of the boundary family and* AS *denoting the form induced on X_0 by the (Γ-invariant) Atiyah-Singer form for the fibration $\tilde{M} \times S^1 \to S^1$ and the hermitian bundle $\widehat{E}$.*

Notice that using the Fourier transformation the Godbillon-Vey eta invariant η_{GV} does depend only on the boundary family $D^\partial \equiv (D^\partial_\theta)_{\theta \in S^1}$.

PROOF. For notational convenience we set $\tau_{2p} \equiv \tau_{GV}$, $\tau^r_{2p} \equiv \tau^r_{GV}$ and $\sigma_{(2p+1)} \equiv \sigma_{GV}$. We also write α_{ex} instead of α^s_{ex}. The left hand side of formula (7.9) is, by definition, the pairing $\langle [P_Q, e_1], \tau_{GV}\rangle$ with P_Q the Connes-Skandalis projection and $e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Here we have also used the remark that $\mathrm{Ind}_{GV}(D) := \langle \mathrm{Ind}^s(D), [\tau_{2p}]\rangle$ for each $p \geq n$. Recall that $\alpha_{\mathrm{ex}}([P_Q, e_1])$ is by definition $[P_Q, e_1, \mathbf{c}]$, with $\mathbf{c}$ the constant path with value e_1. Since the derivative of the constant path is equal to zero and since $\tau^r_{GV}|_{\mathfrak{J}} = \tau_{GV}$, using the obvious extension of (4.24), we obtain at once the crucial relation

$$\langle \alpha_{\mathrm{ex}}([P_Q, e_1]), [(\tau^r_{GV}, \sigma_{GV})]\rangle = \langle [P_Q, e_1], [\tau_{GV}]\rangle\,. \tag{7.11}$$

Now we use the excision formula, asserting that $\alpha_{\mathrm{ex}}([P_Q, e_1])$ is equal, *as a relative class*, to $[e_D, e_1, p_t]$ with $p_t := e_{tD^{\mathrm{cyl}}}$. Thus

$$\langle [e_D, e_1, p_t], [(\tau^r_{GV}, \sigma_{GV})]\rangle = \langle [P_Q, e_1], [\tau_{GV}]\rangle$$

which is the first equality in (7.9) (in reverse order). Using also the definition of the relative pairing we can summarize our results so far as follows:

$$\begin{aligned}
\mathrm{Ind}_{GV}(D) &:= \langle \mathrm{Ind}^s(D), [\tau_{GV}]\rangle \\
&\equiv \langle [P_Q, e_1], [\tau_{GV}]\rangle \\
&= \langle \alpha_{\mathrm{ex}}([P_Q, e_1]), [(\tau^r_{GV}, \sigma_{GV})]\rangle \\
&= \langle [e_D, e_1, p_t], [(\tau^r_{GV}, \sigma_{GV})]\rangle \\
&:= \frac{1}{p!}\tau^r_{GV}(e_D - e_1) + \frac{(2p+1)}{p!} \int_1^{+\infty} \sigma_{GV}([\dot{p}_t, p_t], p_t, \ldots, p_t) dt \\
&\equiv \frac{1}{p!}\tau^r_{GV}(\widehat{e}_D) + \frac{(2p+1)}{p!} \int_1^{+\infty} \sigma_{GV}([\dot{p}_t, p_t], p_t, \ldots, p_t) dt
\end{aligned}$$

with $\widehat{e}_D = (D + \mathfrak{s})^{-1}$. Notice that the convergence at infinity of the integral $\int_1^{+\infty} \sigma_{GV}([\dot{p}_t, p_t], p_t, \ldots, p_t) dt$ follows from the fact that the pairing is well defined.

Replace D by uD, $u > 0$. We obtain, after a simple change of variable in the integral,

$$\frac{(2p+1)}{p!}\int_u^{+\infty}\sigma_{GV}([\dot{p_t}, p_t], p_t, \ldots, p_t, p_t)dt = -\langle \mathrm{Ind}^s(uD), [\tau_{GV}]\rangle + \frac{1}{p!}\tau^r_{GV}(\widehat{e}_{uD})$$

But the absolute pairing $\langle \mathrm{Ind}^s(uD), [\tau_{GV}]\rangle$ in independent of u and of course equal to $\mathrm{Ind}_{GV}(D)$; thus

$$\frac{(2p+1)}{p!}\int_u^{+\infty}\sigma_{GV}([\dot{p_t}, p_t], p_t, \ldots, p_t, p_t)dt = -\mathrm{Ind}_{GV}(D) + \frac{1}{p!}\tau^r_{GV}(\widehat{e}_{uD})$$

The second summand of the right hand side can be proved to converge as $u \downarrow 0$ to $\int_{X_0} \mathrm{AS} \wedge \omega_{GV}$ (this employs Getzler rescaling exactly as in [**14**]). Thus the limit

$$\frac{(2p+1)}{p!}\lim_{u\downarrow 0}\int_s^{+\infty}\sigma_{GV}([\dot{p_t}, p_t], p_t, \ldots, p_t, p_t)dt$$

exists [3] and is equal to $\int_{X_0} \mathrm{AS} \wedge \omega_{GV} - \mathrm{Ind}_{GV}(D)$. The theorem is proved □

Remark. The classic Atiyah-Patodi-Singer index theorem in obtained proceeding as above, but pairing the index class with the absolute 0-cocycle τ_0 and the relative index class with the relative 0-cocycle (τ_0^r, σ_1). Equating the absolute and the relative pairing, as above, we obtain an index theorem. It can be proved that this is precisely the APS index theorem on manifolds with cylindrical ends; in other words, the eta-term we obtain is precisely the APS eta invariant for the boundary operator. The classic APS index theorem from the point of view of relating pairing was announced by the first author in [**13**]. This approach is also a Corollary of the main result of the recent preprint of Lesch, Moscovici and Pflaum [**8**], that is, the computation of the Connes-Chern character of the relative homology cycle associated to a Dirac operator on a manifold with boundary in terms of local data and higher eta cochain for the *commutative* algebra of smooth functions on the boundary (see also [**3**] and [**18**]). Needless to say, the results in [**8**] go well beyond the computation of the index; however, they don't have much in common with the non-commutative results presented in this paper.

7.4. Eta cocycles. The ideas explained in the previous sections can be extended to general cocycles $\tau_k \in HC^k(C^\infty_c(G, (s^*E)^* \otimes r^*E))$; we simply need to require that these cocycles are in the image of a suitable Alexander-Spanier homomorphism since we can then replace integrals with regularized integrals in the passage from absolute to relative cocycles. This general theory will be treated elsewhere. Here we only want to comment on the particular case of Galois coverings, since this case illustrates very well the general framework. In this important example the techniques of this paper can be used in order to give an alternative approach to the higher index theory developed in [**6**], much more in line with the original treatment given by Connes and Moscovici in their fundamental paper [**2**].

We now give a very short treatment of this important example, assuming a certain familiarity with the seminal work of Connes and Moscovici. Let $\Gamma \to \tilde{M} \to M$ be a Galois covering with boundary and let $\Gamma \to \tilde{V} \to V$ be the associated covering with cylindrical ends. In the closed case higher indeces for a Γ-equivariant Dirac

[3]the situation here is similar to the one for the eta invariant in the seminal paper of Atiyah-Patodi-Singer; the regularity there is a consequence of their index theorem

operator on $\tilde{M}$ are obtained through Alexander-Spanier cocycles, so we concentrate directly on these. Let ϕ be an Alexander-Spanier p-cocycle; for simplicity we assume that ϕ is the sum of decomposable elements given by the cup product of Alexander-Spanier 1-cochains: $\phi = \sum_i \delta f_1^{(i)} \cup \delta f_2^{(i)} \cup \cdots \cup \delta f_p^{(i)}$ where $f_j^{(i)} : \tilde{M} \to \mathbb{C}$ is continuous. Here we assume that $\delta f_j^{(i)}$, $\delta f_j^{(i)}(\tilde{m}, \tilde{m}') := (f_j^{(i)}(\tilde{m}') - f_j^{(i)}(\tilde{m}))$ is Γ-invariant with respect to the diagonal action of Γ on $\tilde{M} \times \tilde{M}$. This is a non-trivial assumption. We shall omit $\cup$ from the notation. The cochain ϕ is a cocycle (where we recall that for an Alexander-Spanier p-cochain given by a continuos function $\phi : \tilde{M}^{p+1} \to \mathbb{C}$ invariant under the diagonal Γ-action, one sets $\delta\phi(x_0, x_1, \dots, x_{p+1}) := \sum_0^{p+1} (-1)^i \phi(x_0, \dots, \hat{x}_i, \dots, x_{p+1})$). Always in the closed case we obtain a cyclic p-cocycle for the convolution algebra $C_c^\infty(\tilde{M} \times_\Gamma \tilde{M})$ by setting

$$\tau_\phi(k_0, \dots, k_p) = \frac{1}{p!} \sum_{\alpha \in \mathfrak{S}_p} \sum_i \operatorname{sign}(\alpha) \omega_\Gamma(k_0\, \delta^{(i)}_{\alpha(1)} k_1 \cdots \delta^{(i)}_{\alpha(p)} k_p)\,, \tag{7.12}$$

with $\delta_j^{(i)} k := [k, f_j^{(i)}]$. Notice that $[k, f_j^{(i)}]$ is the Γ-invariant kernel whose value at $(\tilde{m}, \tilde{m}')$ is given by $k(\tilde{m}, \tilde{m}')\delta f_j^{(i)}(\tilde{m}, \tilde{m}')$ which is by definition $k(\tilde{m}, \tilde{m}')(f_j^{(i)}(\tilde{m}') - f_j^{(i)}(\tilde{m}))$; ω_Γ is as usual given by $\omega_\Gamma(k) = \int_F \operatorname{Tr}_{\tilde{m}} k(\tilde{m}, \tilde{m})$, with F a fundamental domain for the Γ-action.

Pass now to manifolds with boundary and associated manifolds with cylindrical ends. Consider the small subalgebras $J_c(\tilde{V})$, $A_c(\tilde{V})$, $B_c(\partial\tilde{V} \times \mathbb{R})$ appearing in the (small) Wiener-Hopf extension constructed in Subection 4.2 (just take T =point there). We write briefly J_c, A_c, B_c and $0 \to J_c \to A_c \xrightarrow{\pi_c} B_c \to 0$. We adopt the notation of the previous sections. Given ϕ as above, we can clearly define an *absolute* cyclic p-cocycle τ_ϕ on J_c. Next, define the $(p+1)$-linear functional ψ^r_ϕ on A_c by replacing the integral in ω_Γ with Melrose' regularized integral. Consider next the cyclic p-cochain on A_c, call it $\tau^r_\phi(k_0, \dots, k_p)$, defined by

$$\frac{1}{p+1} \left(\psi^r_\phi(k_0, k_1, \dots, k_p) + \psi^r_\phi(k_1, \dots, k_p, k_0) + \cdots + \psi^r_\phi(k_p, k_0, \dots, k_{p-1}) \right) .$$

Finally, introduce the new derivation $\delta^{(i)}_{p+1}(\ell) := [\chi^0, \ell]$ with χ^0 the function on $\partial\tilde{V} \times \mathbb{R}$ induced by the characteristic function of $(-\infty, 0]$. Then the eta cocycle associated to τ_ϕ is given by

$$\sigma_\phi(\ell_0, \dots, \ell_{p+1}) = \frac{1}{(p+1)!} \sum_{\alpha \in \mathfrak{S}_{p+1}} \sum_i \operatorname{sign}(\alpha) \omega_\Gamma(\ell_0\, \delta^{(i)}_{\alpha(1)} \ell_1 \cdots \delta^{(i)}_{\alpha(p+1)} \ell_{p+1}) \tag{7.13}$$

It should be possible to prove, using the techniques of this paper, that this is a cyclic $(p+1)$-cocycle for B_c and that $(\tau^r_\phi, \sigma_\phi)$ is a *relative* cyclic p-cocycle for the pair (A_c, B_c). σ_ϕ is, by definition, the *eta cocycle* corresponding to τ_ϕ.

Proceeding exactly as above, thus introducing suitable smooth algebras, extending the cyclic cocycles, smoothing out the index classes and equating the absolute pairing $\langle \operatorname{Ind}(\tilde{D}), [\tau_\phi] \rangle$ with the relative pairing $\langle \operatorname{Ind}(\tilde{D}, \tilde{D}^\partial), [\tau^r_\phi, \sigma_\phi] \rangle$ one should obtain a higher (Atiyah-Patodi-Singer)-(Connes-Moscovici) index formula, with boundary correction term given in terms of

$$\int_0^\infty \sigma_\phi([\dot{p}_t, p_t], p_t, \ldots, p_t)dt \quad \text{with} \quad p_t := e_{t\tilde{D}^{\rm cyl}}$$

A full treatment of the general theory on foliated bundles, together with this important particular case will be treated elsewhere.

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