# Dirac index classes and the noncommutative spectral flow 

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Received 19 June 2001; accepted 28 June 2002

Communicated by R. Melrose


#### Abstract

We present a detailed proof of the existence-theorem for noncommutative spectral sections (see the noncommutative spectral flow, unpublished preprint, 1997). We apply this result to various index-theoretic situations, extending to the noncommutative context results of BoossWojciechowski, Melrose-Piazza and Dai-Zhang. In particular, we prove a variational formula, in $K_{*}\left(C_{r}^{*}(\Gamma)\right)$, for the index classes associated to 1-parameter family of Dirac operators on a $\Gamma$-covering with boundary; this formula involves a noncommutative spectral flow for the boundary family. Next, we establish an additivity result, in $K_{*}\left(C_{r}^{*}(\Gamma)\right)$, for the index class defined by a Dirac-type operator associated to a closed manifold $M$ and a map $r: M \rightarrow B \Gamma$ when we assume that $M$ is the union along a hypersurface $F$ of two manifolds with boundary $M=M_{+} \cup_{F} M_{-}$. Finally, we prove a defect formula for the signature-index classes of two cut-and-paste equivalent pairs $\left(M_{1}, r_{1}: M_{1} \rightarrow B \Gamma\right)$ and $\left(M_{2}, r_{2}: M_{2} \rightarrow B \Gamma\right)$, where


$$
M_{1}=M_{+} \cup_{\left(F, \phi_{1}\right)} M_{-}, \quad M_{2}=M_{+} \cup_{\left(F, \phi_{2}\right)} M_{-}
$$

and $\phi_{j} \in \operatorname{Diff}(F)$. The formula involves the noncommutative spectral flow of a suitable 1parameter family of twisted signature operators on $F$. We give applications to the problem of cut-and-paste invariance of Novikov's higher signatures on closed oriented manifolds.
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Keywords: Index classes; Noncommutative spectral sections; Boundary value problems; Gluing formulae; Cut-and-paste invariance

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## 1. Introduction and main results

The Connes-Moscovici higher index theorem on Galois coverings [6] can be seen as a family-index theorem but with a noncommutative parameter space given by the reduced $C^{*}$-algebra of the covering group [5]. It is natural to ask to what extent results established for families of Dirac operators parametrized by a topological space $X$ can be carried over to such a noncommutative context. The past 20 years of noncommutative geometry show that such generalizations have, besides their own beauty, deep and interesting geometric applications. See Alain Connes' book [5] for a wide and fascinating treatment of many of these generalizations.

We consider the following results in family-index theory:

- on a closed manifold $N$, the existence of spectral section for a family of Dirac operators with trivial index class [25,26];
- on a manifold with boundary $M$, the definition of the Atiyah-Patodi-Singer index class and of the $b$-index class associated to the choice of a spectral section for the boundary family [25,26];
- the equality of these two index classes [25,26];
- the relative index theorem, giving a formula for the difference of two index classes associated to two different choices of spectral sections [25,26];
- on a closed manifold $N=M_{+} \cup_{F} M_{-}$, union along a hypersurface $F$ of two manifolds with boundary, the splitting formula for the index class associated to a family of Dirac operators [7];
- the notion of higher spectral flow and its relationships with family index theory [8].

For an informal presentation of some of these results we refer the reader to the surveys $[23,28]$. We shall also consider the following result in ordinary (i.e. numerical) index theory:

- Let $M_{1}=M_{+} \cup_{\left(F, \phi_{1}\right)} M_{-}$and $M_{2}=M_{+} \cup_{\left(F, \phi_{2}\right)} M_{-}, \phi_{j} \in \operatorname{Diff}(F), \partial M_{+}=F=$ $-\partial M_{-}$be two even-dimensional manifolds that are cut-and-paste equivalent; let $D_{1}, D_{2}$ be two cut-and-paste equivalent operators of Dirac-type. Then the difference ind $D_{2}^{+}$- ind $D_{1}^{+}$is equal to the spectral flow of a natural family of operators on $F$ [3].

The main goal of this paper is to establish noncommutative generalizations of the above results and give geometric applications to the problem of cut-and-paste invariance of Novikov's higher signatures on a closed oriented manifold.

The first result, i.e. the existence of noncommutative spectral section, was claimed in the odd-dimensional case by Wu in the preprint [30]. Unfortunately Wu's preprint was never published; moreover, as pointed out to us by Michel Hilsum, the proof given by Wu contained one unjustified step. For these reasons we felt it was necessary to present a full proof of the existence of noncommutative spectral sections; the missing step turned out to be nontrivial and one section of this paper is
entirely devoted to a rigorous proof of this particular result. Notice that there was much more in Wu's preprint than just the existence-theorem for noncommutative spectral sections. Of course we cannot report here on Wu's further results and we strongly feel that it is a pity that the interesting Mathematics established in [30] will most likely never be published.

Some of our papers on higher index theory freely use the existence-theorem stated in Wu's unpublished preprint as well as the consequent definition, given there, of noncommutative spectral flow (this definition is an immediate generalization of the family case treated in [8]). In these articles of ours, [16,18,19], some of the above generalizations are presented. More precisely:

- The definition, in the even-dimensional case, of the $b$-index class associated to a Dirac operator on a Galois covering with boundary once a spectral section for the boundary operator is chosen, is given in [16]. (Notice that our main interest in [16] was in proving an index formula, in noncommutative topological de Rham homology, for the Chern character of such an index class.)
- A formula for the Chern character of the noncommutative spectral flow in terms of higher eta invariants is also proved in [16].
- The gluing formula for index classes is already stated in [19].
- The existence-theorem for spectral section on even-dimensional closed manifolds is proved in [18] assuming the theorem in the odd-dimensional case.
- Finally, always in [18], we prove the relative index theorem for even-dimensional Galois coverings with boundary.

In writing this article we were also motivated by the wish of completing the picture that has emerged so far from these articles. We shall be more specific in each particular section.

The paper is organized as follows. In Section 2 we recall the notion of noncommutative spectral section and prove the fundamental result that a noncommutative spectral section for a Dirac-type operator exists if and only if the index class defined by this operator is trivial in $K$-Theory. We also recall the notion of difference class and of noncommutative spectral flow. In Section 3 we recall from $[16,31]$ the definition of $b$-index class and APS-index class associated to a Dirac-type operator on an even-dimensional Galois covering with boundary; these classes depend on the choice of a noncommutative spectral section for the boundary operator. We then prove the equality of this two index classes. We end Section 3 by extending these results to odd dimensions, using suspension as in [26]. In Section 4, following ideas of Bunke and Dai-Zhang, we prove a gluing formula for index classes. In Sections 5 and 6 we discuss applications to the problem of cut-and-paste invariance for higher signatures. Let us briefly explain what is the problem and what are the techniques that we shall be using.

Recall from [11] that two manifolds $M_{1}, M_{2}$ are SK-equivalent if

$$
\begin{equation*}
M_{1}=M_{+} \cup_{\left(F, \phi_{1}\right)} M_{-}, \quad M_{2}=M_{+} \cup_{\left(F, \phi_{2}\right)} M_{-} \tag{1}
\end{equation*}
$$

with $\partial M_{+}=F=-\partial M_{-}$and $\phi_{j} \in \operatorname{Diff}(F)$. In words, $M_{1}$ and $M_{2}$ are obtained by gluing two manifolds with boundary but the gluing diffeomorphisms are different.

Incidentally, SK stands for Schneiden $=$ cutting and Kleben $=$ pasting.
The signature of a manifold is a SK-invariant $\sigma\left(M_{1}\right)=\sigma\left(M_{2}\right)$ for $M_{1}, M_{2}$ as above. An analytic proof of this fact is given in [3]. The argument given there is a consequence of a more general formula concerning the numerical indices of two Dirac-type operators obtained one from the other by a cut-and-paste construction; the formula expresses the difference of the numerical indices in terms of the spectral flow of a suitable 1-parameter family of operators on $F$. For the particular case of the signature operator this spectral flow is defined by a 1-parameter family $\left\{D_{F}(\theta)\right\}_{\theta \in S^{1}}$, of odd signature operators acting on the fibers of the mapping torus $M\left(F, \phi_{2}^{-1} \circ \phi_{1}\right)$ associated with $\phi_{1}$ and $\phi_{2}$ and parametrized by a path of metrics. Because of the cohomological significance of the zero-eigenvalue of the signature operator, this spectral flow is equal to zero. Since the signature of a manifold is equal to the index of the signature-operator, we obtain finally

$$
\begin{equation*}
\sigma\left(M_{2}\right)-\sigma\left(M_{1}\right)=\operatorname{ind} D_{2}-\operatorname{ind} D_{1}=\operatorname{sf}\left(\left\{D_{F}(\theta)\right\}_{\theta \in S^{1}}\right)=0 . \tag{2}
\end{equation*}
$$

In Section 6 we shall be concerned with the higher analog of this argument.
We thus consider a pair $(M, r: M \rightarrow B \Gamma)$, with $M$ a closed oriented manifold and $r$ a continuous map into the classifying space of a discrete group $\Gamma$. To such a pair one can attach a set of numbers, the Novikov's higher signatures $\sigma(M, r ;[c])$ defined by

$$
\sigma(M, r ;[c])=\left\langle L(M) \cup r^{*}[c],[M]\right\rangle
$$

with $[c] \in H^{*}(B \Gamma, \mathbb{Q})=H^{*}(\Gamma, \mathbb{Q})$. Two pairs

$$
\left(M_{1}, r_{1}: M_{1} \rightarrow B \Gamma\right) \quad \text { and } \quad\left(M_{2}, r_{2}: M_{2} \rightarrow B \Gamma\right)
$$

are said to be SK-equivalent if $M_{1}$ and $M_{2}$ satisfy (1) and $\left(r_{1}\right)_{\mid M_{+}} \simeq\left(r_{2}\right)_{\mid M_{+}}$, $\left(r_{1}\right)_{\mid M_{-}} \simeq\left(r_{2}\right)_{\mid M_{-}}$, where $\simeq$ means homotopy equivalence. One can ask whether two SK-equivalent pairs $\left(M_{1}, r_{1}\right),\left(M_{2}, r_{2}\right)$ have the same higher signatures. It was observed by Lott [22, Remark 4.1], that results in [11,27] can be reinterpreted as giving examples where the answer to this question is negative; one can then try to find conditions on $\Gamma$ and $F$ so as to ensure that the higher signatures are indeed SKinvariants. The first result in this direction is given by Leichtnam-Lott-Piazza [12], as a corollary of the higher APS-index formula proved there and the higher index formula proved in [20] (see [12, Corollary 0.4$]$ ). The condition on the group $\Gamma$ is that it be Gromov hyperbolic or virtually nilpotent. The condition on $F$ is recalled in detail in Sections 6.2 and 6.3: it is an homotopy-invariant condition of the pair $\left(F, r_{\mid F}\right)$ and can be briefly formulated as a gap condition, in middle degree, for the spectrum of the differential form Laplacian on the $\Gamma$-covering associated to $\left(F, r_{\mid F}\right)$. This condition was first introduced by Lott [21], where a regularization of the a priori divergent integral defining the higher eta invariant for the signature operator was proposed. These ideas were further pursued in the articles [12,18,22].

A different treatment of the cut-and-paste result proved in [12] was subsequently given in [13]. In that paper techniques from algebraic surgery are used; the condition on $F$ remains the same but the group $\Gamma$ must only satisfy the property that the Baum-Connes map be rationally injective.

In Section 6 we shall reobtain these results using a higher version of the result of Booss-Wojciechowski, i.e. a higher analog of (2). We briefly describe the result. Let us denote by $\mathscr{D}_{M_{j}}^{\text {sign, } r_{j}}, j=1,2$, the Mishchenko-Fomenko signature operators associated to the two SK-equivalent pairs $\left(M_{j}, r_{j}\right)$ and a choice of metric $g_{j}$ on $M_{j}$; then for the corresponding index classes, in $K_{*}\left(C_{r}^{*}(\Gamma)\right)$, the following defect formula holds:

$$
\begin{equation*}
\text { Ind } \mathscr{D}_{M_{2}}^{\text {sign, } r_{2}}-\operatorname{Ind} \mathscr{D}_{M_{1}}^{\text {sign, } r_{1}}=\operatorname{sf}\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right) . \tag{3}
\end{equation*}
$$

On the right-hand side of the above formula the noncommutative spectral flow of a suitable 1-parameter family of $t$ wisted signature operators acting on the fibers of the mapping torus $M\left(F, \phi_{2}^{-1} \circ \phi_{1}\right)$ appears. Formula (3) is a consequence of two fundamental results:

- the gluing formula;
- a variational formula for the index classes, in $K_{*}\left(C_{r}^{*}(\Gamma)\right)$, associated to a 1-parameter family of Dirac operators on a $\Gamma$-covering with boundary.

It is the latter formula that involves a noncommutative spectral flow. The variational formula is proved in Section 5; there we also establish the equality of the noncommutative spectral flow for a family $\{\mathscr{D}(u)\}_{u \in[0,1]}$ with the index class associated to a suitable boundary value problem for the operator $\partial / \partial u+$ $\{\mathscr{D}(u)\}_{u \in[0,1]}$ on the cylinder.

Notice that the noncommutative spectral flow is a class in $K_{*}\left(C_{r}^{*}(\Gamma)\right)$; as such it should be more precisely called a higher noncommutative spectral flow. Numerical (i.e. lower) spectral flows in the noncommutative context have already been considered, for example, in the work of Perera, Kaminker and Phillips; see [10] and references therein.

It is important to notice that, in contrast with the numerical case treated in (2), the noncommutative spectral flow appearing in (3) will in general be different from 0 , as it is associated to a family of twisted signature operators, with the twisting bundle depending on the parameter $\theta$. By employing symmetric spectral section, as in [17], we shall show that under the cited assumption on $F$ this noncommutative spectral flow is automatically zero. This fact, together with the assumption that the Baum-Connes assembly map is rationally injective, allows one to reobtain the results of [13].

Notice that in the even-dimensional case a recent preprint [9] of Hilsum sharpens further these results by slightly weakening the assumptions on $F$. We refer the reader to his paper for the detailed statement of this weaker assumption and also for a somewhat different treatment of boundary value problems in the noncommutative
context, more in the spirit of [3]. It would be interesting to know whether the higher APS-index formula proved in [16] can be established for these more general boundary value problems (one advantage of spectral section being that they are specific enough for local index-theory arguments to be carried over).

Notice also that the cut-and-paste results in the odd-dimensional case are somewhat more involved than in the even-dimensional case and cannot be obtained by simply crossing with $S^{1}$. This phenomenon is already present in $[12,13]$ and will be further clarified in the present paper.

## 2. Proof of the existence of noncommutative spectral sections

We begin this section with two functional analytic results that will be needed in the proof of the existence-theorem. The definition of noncommutative spectral section and the statement of the existence-theorem are given in Section 2.4.

### 2.1. Very full projections and a lifting theorem

We consider in this section a unital $C^{*}$-algebra $A$ and denote by $H_{A}=l^{2}(A)=$ $l^{2}(\mathbb{N}) \otimes_{\mathbb{C}} A$ the standard $A$-Hilbert module. We consider $\mathscr{B}_{A}$ the algebra of $A$-linear continuous adjointable operators from $H_{A}$ to itself and denote by $\mathscr{K}_{A}$ the subalgebra of $\mathscr{B}_{A}$ of all the operators $T$ such that both $T$ and $T^{*}$ are $A$-compact. We then have the following exact sequence:

$$
0 \rightarrow \mathscr{K}_{A} \rightarrow \mathscr{B}_{A} \rightarrow \mathscr{C}_{A}=\mathscr{B}_{A} / \mathscr{K}_{A} \rightarrow 0,
$$

where $\mathscr{C}_{A}$ denotes the (generalized) Calkin algebra. We recall that (see [1]) the $K$ theory groups $K_{*}\left(\mathscr{B}_{A}\right)$ vanish and that $K_{*}\left(\mathscr{K}_{A}\right) \simeq K_{*}(A)$. The standard six terms exact sequence in $K$-theory gives the following isomorphism $\delta$ called the index map

$$
\begin{equation*}
\delta: K_{0}\left(\mathscr{C}_{A}\right) \rightarrow K_{1}\left(\mathscr{K}_{A}\right) \simeq K_{1}(A) . \tag{4}
\end{equation*}
$$

We shall now recall what a very full projection is, see [1, Section 6.11]. The relevance of this notion in our problem has been pointed out to us by Ralf Meyer.

Definition 1. A projection $p\left(=p^{*}\right)$ of a unital $C^{*}$-algebra $A$ is said to be very full if there exists an isometry $u \in A\left(u^{*} u=1=1_{A}\right)$ such that $u u^{*} \leqslant p$. In other words, $p$ contains a projection which is equivalent to 1 .

The following theorem and the precise structure of its proof has been suggested to us by Ralf Meyer.

Theorem 1 (Ralf Meyer). Let $p$ be a very full projection of $\mathscr{C}_{A}$ such that the $K$-theory class $[p]$ of $p$ in $K_{0}\left(\mathscr{C}_{A}\right)$ is zero (i.e. the index $\delta(p)$ is zero) and $1-p$ is very full. Then
there exists a projection $P \in \mathscr{B}_{A}$ such that $\pi(P)=p$ where $\pi: \mathscr{B}_{A} \rightarrow \mathscr{C}_{A}=\mathscr{B}_{A} / \mathscr{K}_{A}$ is the canonical projection.

Proof. The proof of the theorem relies on the next two propositions.
Proposition 1. Let $p$ be a very full projection of $\mathscr{C}_{A}$. Then there exists an isometry $W \in \mathscr{B}_{A}$ such that $\pi(W) \pi\left(W^{*}\right) \leqslant p$.

Proof. By assumption there exists $W_{1} \in \mathscr{B}_{A}$ such that $\pi\left(W_{1}\right)$ is an isometry of $\mathscr{C}_{A}$ satisfying $\pi\left(W_{1}\right) \pi\left(W_{1}\right)^{*} \leqslant p$ and $W_{1}^{*} W_{1}-\mathrm{Id} \in \mathscr{K}_{A}$. Recall that $l^{2}(A)=\oplus_{n \geqslant 0}^{\perp} A e_{n}$; for any $N \in \mathbb{N}$, we denote by $P_{N} \in \mathscr{B}_{A}$ the projection from $l^{2}(A)$ onto $\operatorname{Im} P_{N}=$ $\oplus_{n \geqslant N}^{\perp} A e_{n}$. Then there exists $N \in \mathbb{N}$ such that $\left\|P_{N}\left(W_{1}^{*} W_{1}-\mathrm{Id}\right) P_{N}\right\| \leqslant \frac{1}{4}$ where $\|\cdot\|$ denotes the $C^{*}$-norm of $\mathscr{B}_{A}$.

Now we consider and fix a unitary isomorphism $\chi: l^{2}(A) \rightarrow \operatorname{Im} P_{N} \subset l^{2}(A)$.
We have

$$
\begin{equation*}
\left\|\chi^{-1} P_{N}\left(W_{1}^{*} W_{1}-\mathrm{Id}\right) P_{N} \chi\right\| \leqslant \frac{1}{4} \tag{5}
\end{equation*}
$$

we identify $P_{N} \chi$ with an element of $\mathscr{B}_{A}$ so that $\left(P_{N} \chi\right)^{*}=\chi^{-1} P_{N}$. We then set $W_{2}=$ $W_{1} P_{N} \chi \in \mathscr{B}_{A}$ so that $W_{2}^{*}=\chi^{-1} P_{N} W_{1}^{*}$. From (5) we deduce that

$$
\left\|W_{2}^{*} W_{2}-\mathrm{Id}\right\|=\left\|\chi^{-1} P_{N}\left(W_{1}^{*} W_{1}-\mathrm{Id}\right) P_{N} \chi\right\| \leqslant \frac{1}{4} .
$$

So $W_{2}^{*} W_{2} \in \mathscr{B}_{A}$ is invertible and $\operatorname{Im} W_{2}^{*}=l^{2}(A)$. Therefore $W_{2}^{*}$ admits a polar decomposition $W_{2}^{*}=V \sqrt{W_{2} W_{2}^{*}}, \operatorname{Im} W_{2}=\operatorname{Im} W_{2} W_{2}^{*}$ is closed in $l^{2}(A)$, ker $W_{2}^{*}=$ $\operatorname{ker} W_{2} W_{2}^{*}$ and

$$
\begin{equation*}
l^{2}(A)=\operatorname{Im} W_{2} W_{2}^{*} \oplus^{\perp} \operatorname{ker} W_{2} W_{2}^{*} \tag{6}
\end{equation*}
$$

(see [29, Theorem 15.3.8] for details). Moreover, $V^{*} V$ (resp. $V V^{*}$ ) is the orthogonal projection onto $\operatorname{Im} W_{2} W_{2}^{*}$ (resp. $\operatorname{Im} W_{2}^{*} W_{2}=l^{2}(A)$ ), so $V V^{*}=\mathrm{Id}$. From the definition of $W_{2}$ we get $W_{2} W_{2}^{*}=W_{1} W_{1}^{*}+W_{1}\left(P_{N}-\mathrm{Id}\right) W_{1}^{*}$ and thus $\pi\left(W_{2} W_{2}^{*}\right)=$ $\pi\left(W_{1} W_{1}^{*}\right)$. Since by hypothesis $\pi\left(W_{2} W_{2}^{*}\right)=\pi\left(W_{1} W_{1}^{*}\right) \leqslant p$, we obtain the proposition as a consequence of the following lemma by taking $W=V^{*}$.

Lemma 1. We have $\pi\left(V^{*} V\right)=\pi\left(W_{2} W_{2}^{*}\right)$.
Proof. Using decomposition (6) and the fact that $\operatorname{Im} W_{2} W_{2}^{*}$ is closed and defines a $A$-Hilbert module, one sees that $W_{2} W_{2}^{*}$ induces a (bi-)continuous bijection

$$
W_{2} W_{2}^{*}: \operatorname{Im} W_{2} W_{2}^{*} \rightarrow \operatorname{Im} W_{2} W_{2}^{*}
$$

still denoted $W_{2} W_{2}^{*}$. We denote by $S$ its inverse; by extending $S$ by 0 on ker $W_{2} W_{2}^{*}$, we get an element of $\mathscr{B}_{A}$ still denoted $S$. Thus, by definition of the polar decomposition $W_{2}^{*}=V \sqrt{W_{2} W_{2}^{*}}$, we have $W_{2} W_{2}^{*} S=V^{*} V$. Now we consider the
universal representation of the Calkin algebra $j: \mathscr{C}_{A} \rightarrow B\left(H_{\mathscr{C}_{A}}\right)$. By applying $j \circ \pi$ to the previous identity we get $j\left(\pi\left(W_{2} W_{2}^{*}\right)\right) j(\pi(S))=j\left(\pi\left(V^{*} V\right)\right)$; so the range of the projection $j\left(\pi\left(V^{*} V\right)\right)$ is contained in the range of the projection $j\left(\pi\left(W_{2} W_{2}^{*}\right)\right)$. Moreover, from the identity $W_{2} W_{2}^{*}=V^{*} V W_{2} W_{2}^{*}=W_{2} W_{2}^{*} V^{*} V$, we get that the range of $j\left(\pi\left(W_{2} W_{2}^{*}\right)\right)$ is contained in the one of $j\left(\pi\left(V^{*} V\right)\right)$. Since $j$ is injective, we see that $\pi\left(V^{*} V\right)=\pi\left(W_{2} W_{2}^{*}\right)$ which proves the lemma.

Proposition 2. Let $p$ be a very full projection of $\mathscr{C}_{A}$ such that $1-p$ is also very full. Then there exists a projection $p^{\prime \prime}$ of $\mathscr{C}_{A}$ such that for any $k, l \in \mathbb{N}^{*}$, one can find an A-linear bicontinuous isomorphism:

$$
\chi_{k, l}: l^{2}(A) \rightarrow\left(\Pi_{i=1}^{k} l^{2}(A)\right) \oplus l^{2}(A) \oplus\left(\Pi_{j=1}^{l} l^{2}(A)\right)=\mathscr{H}_{k, l}
$$

such that

$$
\begin{equation*}
p=\chi_{k, l}^{-1}\left(1_{k} \oplus p^{\prime \prime} \oplus 0_{l}\right) \chi_{k, l}, \tag{7}
\end{equation*}
$$

where $1_{k} \oplus p^{\prime \prime} \oplus 0_{l}$ denotes the obvious diagonal-by-blocks element of the Calkin algebra $\mathscr{C}_{A}(k, 1, l)$ associated with $\mathscr{H}_{k, l}$ and $u \rightarrow \chi_{k, l}^{-1} u \chi_{k, l}$ induces an isomorphism between $\mathscr{C}_{A}$ and $\mathscr{C}_{A}(k, 1, l)$.

Proof. We shall treat (only) the case $k=l=1$ from which the general case follows immediately.

By considering a decomposition $l^{2}(A) \simeq l^{2}(A) \oplus l^{2}(A)$, we choose and fix a projection $\Pi_{1} \in \mathscr{B}_{A}$ such that ker $\Pi_{1}$ and $\operatorname{Im} \Pi_{1}$ are both isomorphic to $l^{2}(A)$. According to Proposition 1, we fix an isometry $W \in \mathscr{B}_{A}$ such that $\pi(W) \pi\left(W^{*}\right) \leqslant p$. We set $\widehat{W}=W \Pi_{1}$, so $\widehat{W}^{*}=\Pi_{1} W^{*}$. Since $W^{*} W=$ Id we get

$$
\widehat{W}^{*} \widehat{W}=\Pi_{1}, \quad \widehat{W} \widehat{W}^{*}=W \Pi_{1} W^{*}, \quad W W^{*} \widehat{W} \widehat{W}^{*}=\widehat{W} \widehat{W}^{*} W W^{*}=\widehat{W} \widehat{W}^{*}
$$

From this, we deduce that $\widehat{W} \widehat{W}^{*} \leqslant W W^{*}$. So $W W^{*}-\widehat{W} \widehat{W}^{*}=W\left(\operatorname{Id}-\Pi_{1}\right) W^{*}$ is a projection

$$
\operatorname{Im} W W^{*}=\operatorname{Im} \widehat{W} \widehat{W}^{*} \oplus \perp \operatorname{Im}\left(W W^{*}-\widehat{W} \widehat{W}^{*}\right)
$$

and we get

$$
\begin{equation*}
l^{2}(A)=\operatorname{Im} \widehat{W} \widehat{W}^{*} \oplus^{\perp} \operatorname{Im}\left(W W^{*}-\widehat{W} \widehat{W}^{*}\right) \oplus^{\perp} \operatorname{ker} W W^{*} \tag{8}
\end{equation*}
$$

Since the map

$$
\begin{gathered}
\operatorname{Im} \widehat{W}^{*} \widehat{W}=\operatorname{Im} \Pi_{1} \simeq l^{2}(A) \rightarrow \operatorname{Im} \widehat{W} \widehat{W}^{*} \\
y \rightarrow \widehat{W}(y)
\end{gathered}
$$

induces an isomorphism, one sees that

$$
\begin{equation*}
\operatorname{Im} \widehat{W} \widehat{W}^{*} \simeq l^{2}(A) \tag{9}
\end{equation*}
$$

Moreover, since $W W^{*}-\widehat{W}^{*} \widehat{W}=W\left(\operatorname{Id}-\Pi_{1}\right)\left(W\left(\operatorname{Id}-\Pi_{1}\right)\right)^{*}$, one proves in the same way that $\operatorname{Im}\left(W W^{*}-\widehat{W}^{*} \widehat{W}\right) \simeq l^{2}(A)$. Thanks to Kasparov's stabilization theorem, we then have

$$
\begin{equation*}
\operatorname{Im}\left(W W^{*}-\widehat{W}^{*} \widehat{W}\right) \oplus^{\perp} \operatorname{ker} W W^{*} \simeq l^{2}(A) \tag{10}
\end{equation*}
$$

Now we go back to the proof of the proposition. Since we have $p \pi\left(W W^{*}\right)=$ $\pi\left(W W^{*}\right) p=\pi\left(W W^{*}\right)$ (i.e. $p \geqslant \pi\left(W W^{*}\right)$ ) and $\pi\left(W W^{*}\right) \geqslant \pi\left(\widehat{W} \widehat{W}^{*}\right)$, we see that $p \geqslant \pi\left(\widehat{W} \widehat{W}^{*}\right)$ and $p \geqslant \pi\left(\left(W W^{*}-\widehat{W} \widehat{W}^{*}\right)\right.$. Using decomposition (8), we can identify $\mathscr{B}_{A}$ as a set of (3,3)-matrices; accordingly, we then see that $p\left(\in \mathscr{C}_{A}\right)$ is of the following diagonal form by blocks

$$
\begin{equation*}
p=1 \oplus 1 \oplus q_{1} . \tag{11}
\end{equation*}
$$

We set $1 \oplus q_{1}=p^{\prime}, p^{\prime}$ is a projection of the Calkin algebra associated with the lefthand side of (10). Since $1-p$ is very full, $1-p^{\prime}$ is also very full. Now we apply to $1-p^{\prime}$ and to $\operatorname{Im}\left(W W^{*}-\widehat{W}^{*} \widehat{W}\right) \oplus^{\perp}$ ker $W W^{*}$ the analog of decomposition (8) Repeating for $1-p^{\prime}$ the arguments that have been used for $p$, i.e. (9)-(11), we obtain the following decomposition:

$$
\begin{equation*}
\operatorname{Im}\left(W W^{*}-\widehat{W}^{*} \widehat{W}\right) \oplus^{\perp} \operatorname{ker} W W^{*} \simeq l^{2}(A) \oplus l^{2}(A) \tag{12}
\end{equation*}
$$

With respect to this decomposition we can write $1-p^{\prime}=q_{2} \oplus 1$ (diagonal by blocks) where $q_{2}$ is a projection in $\mathscr{C}_{A}$. So, from (8) and (12), we get a decomposition $l^{2}(A) \simeq l^{2}(A) \oplus l^{2}(A) \oplus l^{2}(A)$ such that

$$
p=1 \oplus\left(1-q_{2}\right) \oplus 0
$$

One then gets the proposition by setting $p^{\prime \prime}=1-q_{2}$.
Proof of Theorem 1 Continuation. We use the projection $p^{\prime \prime} \in \mathscr{C}_{A}$ associated to $p$ as in Proposition 2.

Lemma 2. The $K$-theory class of 1 in $K_{0}\left(\mathscr{C}_{A}\right)$ is zero.
Proof. We recall that the index map (4) $\delta$ is an isomorphism and is also the connecting map associated with the six terms exact sequence in $K$-theory. Since the projection 1 in the Calkin algebra $\mathscr{C}_{A}$ is the image of the projection defined by Id in $\mathscr{B}_{A}$, we have $\delta([1])=0$. Since $\delta$ is an isomorphism, the lemma is proved.

Proof of Theorem 1 Conclusion. Since $[p]$ is (by assumption) zero, the previous lemma then shows that $\left[p^{\prime \prime}\right]=0$. Thus there exists $k, l^{\prime} \in \mathbb{N}^{*}$ and a unitary element
$u \in U\left(M_{k+1+l^{\prime}}\left(\mathscr{C}_{A}\right)\right)$ such that

$$
u\left(1_{k} \oplus 0_{1} \oplus 0_{l^{\prime}}\right) u^{-1}=1_{k} \oplus p^{\prime \prime} \oplus 0_{l^{\prime}}
$$

We set $l=k+2 l^{\prime}+1$ and $z=\operatorname{diag}\left(u, u^{-1}\right)$. We then have

$$
z\left(1_{k} \oplus 0_{1} \oplus 0_{l}\right) z^{-1}=1_{k} \oplus p^{\prime \prime} \oplus 0_{l}
$$

Since $z$ belongs to the connected component of 1 in $U\left(M_{k+1+l}\left(\mathscr{C}_{A}\right)\right)$ (see [1]), we can find $Z$ in $U\left(M_{k+1+l}\left(\mathscr{B}_{A}\right)\right)$ such that $\pi(Z)=z$. Now we apply Proposition 2 with $k$ and $l=k+2 l^{\prime}+1$ and we set

$$
\begin{equation*}
P=\chi_{k, l}^{-1} Z\left(\operatorname{Id}_{k} \oplus 0_{1} \oplus 0_{l}\right) Z^{-1} \chi_{k, l} \in \mathscr{B}_{A} . \tag{13}
\end{equation*}
$$

We thus get a projection $P$ of $\mathscr{B}_{A}$; using the two equalities (7) and (13) and the fact that $\pi(Z)=z$, one sees that $\pi(P)=p$ which proves Theorem 1 .

### 2.2. Spectral cuts and very full projections

We now consider a finitely generated group $\Gamma$; we shall apply the above results in a particular case to be described below, where the $C^{*}$-algebra $A$ will be equal to $C_{r}^{*}(\Gamma)$, the reduced $C^{*}$-algebra associated to the group $\Gamma$.

It should be remarked, however, that all arguments leading to the main result in this section and to the existence-theorem in the next section, can be easily generalized to any Dirac-type operator acting on the section of a $A$-vector bundle in the sense of Mishchenko-Fomenko, with $A$ any unital $C^{*}$-algebra.

We shall briefly denote by $\Lambda$ the reduced $C^{*}$-algebra $C_{r}^{*}(\Gamma)$.
We next consider a smooth compact riemannian manifold $M$ of dimension $2 m+$ 1 , a continuous map $f: M \rightarrow B \Gamma$ and a complex hermitian clifford module $E \rightarrow M$ endowed with a unitary clifford connection. Let $\widetilde{M} \rightarrow M$ be the $\Gamma$-normal cover of $M$ associated with $f: M \rightarrow B \Gamma$. Let $\mathscr{V}_{f}=\Lambda \times_{\Gamma} \widetilde{M} \rightarrow M$ be the $\Lambda$-flat bundle associated to these data. Then we denote by $\mathscr{D}$ the $\Lambda$-Dirac-type operator acting on $C^{\infty}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$, the set of smooth section of the bundle $E \otimes_{\mathbb{C}} \mathscr{V}_{f} \rightarrow M$. In fact $\mathscr{D}$ defines a self-adjoint unbounded regular operator acting on the $\Lambda$-Hilbert module $L_{\Lambda}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$, the completion of $C^{\infty}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ with respect to the 1 hermitian scalar product $\langle;\rangle$ associated to the above data. Of course, $L_{\Lambda}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ is isomorphic to $l^{2}(\Lambda)$, and the algebra of continuous $\Lambda$-linear adjointable operators $B L_{A}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ is isomorphic to $\mathscr{B}_{A}$. We recall that for any $F \in C^{0}([-\infty,+\infty], \mathbb{C}), F(\mathscr{D})$ is well defined as an element of $B L_{A}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$. Moreover, if $K L_{A}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ denotes the ideal of
adjointable $\Lambda$-compact operators then the following Calkin algebra:

$$
C_{\Lambda}(E)=\frac{B L_{\Lambda}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)}{K L_{\Lambda}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)}
$$

is isomorphic to $\mathscr{C}_{\Lambda}$.
Now we recall the following:
Definition 2. A smooth map $\chi \in C^{\infty}(\mathbb{R},[0,1])$ is said to be a spectral cut if there exists two reals $a<b$ such that for any $t<a, \chi(t)=0$ and for any $t>b, \chi(t)=1$.

Theorem 2. Let $\chi$ be any spectral cut, then $\chi(\mathscr{D})$ and $\mathrm{Id}-\chi(\mathscr{D})$ define two very full projections of the Calkin algebra $C_{A}(E)$.

Proof. We will treat only the case of $\chi(\mathscr{D})$, the case of $\mathrm{Id}-\chi(\mathscr{D})$ being similar. We shall need the following.

Technical Lemma. Let $\chi$ be a spectral cut. There exists an orthonormal sequence $\left(e_{k}\right)_{k \in \mathbb{N}}$ of elements of $L_{\Lambda}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ such that $\forall k, l \in \mathbb{N},\left\langle e_{k} ; e_{l}\right\rangle=\delta_{k, l} l_{\Gamma}$ and $\chi(\mathscr{D})\left(e_{k}\right)=e_{k}$.

Granting this result, we shall now prove Theorem 2 . We denote by $F \subset L_{\Lambda}^{2}\left(M, E \otimes \mathbb{C}^{V_{f}}\right)$ the closure of $\oplus_{k \in \mathbb{N}}^{\perp} \Lambda e_{k}$. Thus, $F$ is $\Lambda$-Hilbert module contained in the range of $\chi(\mathscr{D})$. We denote by $p_{F}=\sum_{k \in \mathbb{N}}\left\langle. ; e_{k}\right\rangle e_{k}$ the orthogonal projection onto $F, p_{F}$ belongs to $B_{\Lambda} L_{\Lambda}^{2}\left(M, E \otimes \mathbb{C} \mathscr{V}_{f}\right)$. We fix a $\Lambda$-unitary isomorphism $\theta: L_{\Lambda}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right) \rightarrow F$ and denote by $j$ the canonical injection $j: F \rightarrow L_{A}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$. Then $j \circ \theta$ defines an element of $B_{A} L_{A}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ whose adjoint is given by $(j \circ \theta)^{*}=\theta^{*} \circ p_{F}$. Then one checks easily that

$$
\left.\theta^{*} \circ p_{F} \circ j \circ \theta=\operatorname{Id}_{L_{A}^{2}(M, E} \otimes_{\subset} \mathscr{r}_{f}\right), \quad j \circ \theta \circ \theta^{*} \circ p_{F}=p_{F}
$$

But, according to the Technical Lemma we have

$$
\chi(\mathscr{D}) \circ p_{F}=p_{F}=\left(\chi(\mathscr{D}) \circ p_{F}\right)^{*}=p_{F} \circ \chi(\mathscr{D})
$$

therefore we deduce that, in the Calkin algebra, the projection $\chi(\mathscr{D})$ contains the very full projection $p_{F}$. Theorem 2 is proved.

### 2.3. Proof of the Technical Lemma

This section can be skipped at first reading. Recall that the principal symbol of $D$ (acting on section of $E$ ) is $\sigma(D)(x ; \xi)=\sqrt{-1} c(\xi)$ where $c(\xi)$ denotes the Clifford multiplication by $\xi \in T_{x}^{*} M$; thus the principal symbol of $\mathscr{D}$ is $\sigma(\mathscr{D})(x ; \xi)=$ $\sqrt{-1} c(\xi) \otimes_{\mathbb{C}} \mathrm{Id}_{\mathscr{r}_{f}} . \quad$ Let $\quad x \in M$ and $\xi \in S_{x}^{*} M \quad$ (the unit cotangent sphere)
then $(\sqrt{-1} c(\xi))^{2}=\mathrm{Id}$ and

$$
\begin{equation*}
E_{x}=\operatorname{ker}(\sqrt{-1} c(\xi)-\mathrm{Id}) \oplus^{\perp} \operatorname{ker}(\sqrt{-1} c(\xi)+\mathrm{Id}) \tag{14}
\end{equation*}
$$

In other words, the two bundles $\operatorname{ker}(\sqrt{-1} c(\xi) \pm \mathrm{Id}) \rightarrow S_{x}^{*} M$ define a direct orthogonal sum decomposition of the trivial pulled back bundle $\pi^{*}\left(E_{x}\right) \rightarrow S_{x}^{*} M$ where $\pi$ : $S_{x}^{*} M \rightarrow\{x\}$ denotes the canonical projection. We start with an elementary result.

Lemma 3. Let $\xi \in S_{x}^{*} M$, we then have:
(1) For any spectral cut $\chi$, the principal symbol $\sigma(\chi(D))(x ; \xi)$ is the orthogonal projection of $\pi^{*}\left(E_{x}\right)$ onto $\operatorname{ker}(\sqrt{-1} c(\xi)-\mathrm{Id})$.
(2)

$$
\begin{aligned}
\operatorname{dim} \operatorname{ker}(\sqrt{-1} c(\xi)-\mathrm{Id}) & =\operatorname{dim} \operatorname{ker}(\sqrt{-1} c(\xi)+\mathrm{Id}) \\
& =\frac{1}{2} \operatorname{dim} E_{x} \geqslant 2^{m-1}
\end{aligned}
$$

Proof. (1) We denote by $|D|^{-1}$ the operator defined as the inverse of $D$ on Im $D \in L^{2}(M, E)$ and by zero on ker $D$. Identity (14) shows that the principal symbol $\sigma\left(\operatorname{Id}+\frac{D}{|D|}\right)(x, \xi)$ is the orthogonal projection of $\pi^{*}\left(E_{x}\right)$ onto $\operatorname{ker}(\sqrt{-1} c(\xi)-\mathrm{Id})$. Since $\chi(D)-\left(\operatorname{Id}+\frac{D}{|D|}\right)$ is a smoothing operator, we see that $\chi(D)$ and $\operatorname{Id}+\frac{D}{|D|}$ have the same principal symbol, (1) is thus proved.
(2) This result is implicitly proved in the literature; we give the proof for the sake of completeness. By writing the clifford module $E_{x}$ as a direct sum of irreducible clifford modules, one can reduce oneself to the case where $E_{x}$ itself is irreducible. Recall then that the Clifford algebra $\mathrm{Cl}\left(\mathbb{C}^{2 m+1}\right)$ is isomorphic to the algebra $M_{2^{m}}(\mathbb{C}) \oplus M_{2^{m}}(\mathbb{C})$ so that $\mathrm{Cl}\left(\mathbb{C}^{2 m+1}\right)$ admits to inequivalent complex irreducible modules $\mathscr{S}^{+}, \mathscr{S}^{-}$of the same dimension $2^{m}$, thus $E_{x}$ is one of the $\mathscr{S}^{ \pm}$and $\operatorname{dim} E_{x}=$ $2^{m}$. Moreover, there exists an algebra isomorphism $\theta$ :

$$
\theta: \mathrm{Cl}\left(\mathbb{C}^{2 m+2}\right) \rightarrow \operatorname{End}_{\mathbb{C}}\left(\mathscr{S}^{+} \oplus \mathscr{S}^{-}\right)
$$

As usual, the $\mathrm{Cl}\left(\mathbb{C}^{2 m+1}\right)$ is identified to the even part $\mathrm{Cl}^{0}\left(\mathbb{C}^{2 m+2}\right)$ of the Clifford algebra $\mathrm{Cl}\left(\mathbb{C}^{2 m+2}\right)$. By the isomorphism induced by the map $\xi \in \mathbb{C}^{2 m+1} \rightarrow e_{2 m+2} \cdot \xi$ where $e_{2 m+2}$ is the last vector of the canonical basis of $\mathbb{R}^{2 m+2}$. For any $\xi \in \mathbb{C}^{2 m+1}$, we set $c(\xi)=\theta\left(e_{2 m+2} \cdot \xi\right)$, the operator $c(\xi)$ sends $\mathscr{S}^{ \pm}$into itself and its restriction to $E_{x}$ (identified to one of the $\mathscr{S}^{ \pm}$) coincides with the Clifford multiplication (also denoted $c(\xi)$ above) induced by the principal symbol $\sigma(\mathscr{D})(x ; \xi)$ of $\mathscr{D}$. So we have to prove that for any $\xi \in S^{2 m}=\left\{\eta \in \mathbb{R}^{2 m+1} /\|\eta\|=1\right\}$ :

$$
\operatorname{dim} \operatorname{ker}(\sqrt{-1} c(\xi)-\mathrm{Id})_{\mid \mathscr{S}^{ \pm}}=\operatorname{dim} \operatorname{ker}(\sqrt{-1} c(\xi)+\mathrm{Id})_{\mid \mathscr{Y}^{ \pm}}
$$

Since $m \geqslant 1$ (and thus $2 m+2>2$ ), Definition 3.20 and Proposition 3.21 of [2] show that

$$
\begin{equation*}
\operatorname{Tr} \sqrt{-1} c(\xi)_{\mid \mathscr{S}^{+}}=\operatorname{Tr} \sqrt{-1} c(\xi)_{\mid \mathscr{S}^{-}} \tag{15}
\end{equation*}
$$

Now we consider the Chirality operator $\beta=\left(\sqrt{-1}^{m+1}\right) e_{1} \ldots e_{2 m+1}$; in this formula $\left(e_{1}, \ldots, e_{2 m+1}\right)$ is the canonical oriented basis of $\mathbb{R}^{2 m+1}$. Let $\xi \in S^{2 m}$, recall (see [2, pp. 109-110]) that $\beta \cdot e_{2 m+2} \cdot \xi=-e_{2 m+2} \cdot \xi \cdot \beta$ and $\theta(\beta) c(\xi)=-c(\xi) \theta(\beta)$.

Since $\theta(\beta)$ sends $\mathscr{S}^{ \pm}$onto $\mathscr{S}^{\mp}$ we then get

$$
\begin{equation*}
\operatorname{ker}(\sqrt{-1} c(\xi) \pm \mathrm{Id})_{\mid \mathscr{S}^{ \pm}} \simeq \operatorname{ker}(\sqrt{-1} c(\xi) \mp \mathrm{Id})_{\mid \mathscr{S}^{\mp}} \tag{16}
\end{equation*}
$$

Moreover, since $(\sqrt{-1} c(\xi))^{2}=$ Id we have
$\operatorname{Tr} \sqrt{-1} c(\xi)_{\mid \mathscr{S} \pm}=\operatorname{dim} \operatorname{ker}(\sqrt{-1} c(\xi)-\mathrm{Id})_{\mid \mathscr{\mathscr { L } ^ { \pm }}}-\operatorname{dim} \operatorname{ker}(\sqrt{-1} c(\xi)+\mathrm{Id})_{\mid \mathscr{\mathscr { L }} \pm}$.
Using the three equations (15)-(17) we get

$$
\operatorname{dim} \operatorname{ker}(\sqrt{-1} c(\xi)-\mathrm{Id})_{\left.\right|_{\mathscr{S}^{ \pm}}}=\operatorname{dim} \operatorname{ker}(\sqrt{-1} c(\xi)+\mathrm{Id})_{\left.\right|_{\mathscr{S}^{ \pm}}}
$$

which proves the lemma.
Lemma 4. Let $\chi$ be a spectral cut. There exists a strictly increasing sequence of real numbers $\left(a_{k}\right)_{k \in \mathbb{N}}$ such that $\chi$ is equal to 1 on $] a_{0}-1,+\infty\left[, \lim a_{k}=+\infty\right.$ and for any $k \in \mathbb{N}$, one can find $\phi_{k} \in C_{\text {comp }}^{\infty}(] a_{k}, a_{k+1}[, \mathbb{R})$ and $u_{k} \in L_{\Lambda}^{2}\left(M, E \otimes \mathbb{C}_{f}\right)$ such that $\left\langle\phi_{k}(\mathscr{D})\left(u_{k}\right) ; \phi_{k}(\mathscr{D})\left(u_{k}\right)\right\rangle$ is invertible in $\Lambda$.

Proof. We fix a spectral cut $\chi_{1}$ whose support is contained in $\left[a_{0}+1,+\infty[\right.$, thus $\chi$ is equal to 1 on an open neighborhood of the support of $\chi_{1}$. We are going to show the existence of $v \in L_{A}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ such that $\left\langle\chi_{1}(\mathscr{D})(v) ; \chi_{1}(\mathscr{D})(v)\right\rangle$ is invertible in $\Lambda$. Then one can find $\psi \in C_{\text {comp }}^{\infty}(\mathbb{R},[0,1])$ (with $\psi \equiv 1$ on a big enough interval) such that $\left\langle\psi(\mathscr{D}) \chi_{1}(\mathscr{D})(v) ; \psi(\mathscr{D}) \chi_{1}(\mathscr{D})(v)\right\rangle$ is invertible in $\Lambda$. This will imply the lemma in the following way: we fix $a_{1}>a_{0}$ such that the support of $\phi_{0}=\psi \chi_{1}$ is contained in $] a_{0}, a_{1}\left[\right.$, then, to construct $\phi_{1}$, we proceed as above, replacing $a_{0}$ by $a_{1}$. Thus inductively, we construct the $a_{k}$ and the $\phi_{k}$ satisfying the requirements of the lemma.

Let $U$ be a (small) nonempty open subset of $M$ over which the bundles $E$ and $\mathscr{V}_{f}$ are trivial and for which there exists a diffeomorphism $\Psi: U \rightarrow B\left(0, r_{0}\right)$ onto an open ball $B\left(0, r_{0}\right)$ of $\mathbb{R}^{n}$ where $n=2 m+1$ is the dimension of $M$. For each $\left.\left.r \in\right] 0, r_{0}\right]$, we shall set $U_{r}=\Psi^{-1}(B(0, r)) \subset U$. Until the end of the proof of Lemma 4 we identify each $\left.\left.U_{r}(r \in] 0, r_{0}\right]\right)$ with the open ball $B(0, r)$ of $\mathbb{R}^{n}$ and $E_{\mid U_{r}}$ with $U_{r} \times \mathbb{C}^{N}$. Let $D$ denote the Dirac operator acting on the sections of $E$. Let $a \in C^{\infty}(M,[0,1])$ with support contained in a small chart and such that $a \equiv 1$ on a neighborhood of $\bar{U}$. Since the principal symbol $\sigma\left(a \chi_{1}^{2}(\mathscr{D})\right)(x ; \xi)$ of $a \chi_{1}^{2}(\mathscr{D})$ is equal to
$\sigma\left(a \chi_{1}^{2}(D)\right)(x ; \xi) \otimes \mathrm{Id}$, we can find a $\Lambda$-pseudo-differential operator $\mathscr{R}$ of order-1 acting on the section of $E \otimes_{\mathbb{C}} \mathscr{V}_{f}$ such that for any $u \in C_{c}^{\infty}(U ; E)$ one has

$$
a \chi_{1}^{2}(\mathscr{D})\left(u 1_{\Gamma}\right)=a \chi_{1}^{2}(D)(u) 1_{\Gamma}+\mathscr{R}\left(u 1_{\Gamma}\right) .
$$

We then observe that $\left\langle a \chi_{1}^{2}(\mathscr{D})\left(u 1_{\Gamma}\right) ; u 1_{\Gamma}\right\rangle$ is equal to

$$
\begin{equation*}
\left\langle\chi_{1}(\mathscr{D})\left(u 1_{\Gamma}\right) ; \chi_{1}(\mathscr{D})\left(u 1_{\Gamma}\right)\right\rangle=\left\|\chi_{1}(D)(u)\right\|_{L^{2}}^{2} 1_{\Gamma}+\left\langle\mathscr{R}\left(u 1_{\Gamma}\right) ; u 1_{\Gamma}\right\rangle, \tag{18}
\end{equation*}
$$

where $1_{\Gamma}$ denotes the neutral element of $\Gamma$, so that $u 1_{\Gamma}$ is indeed a section of $E \otimes_{\mathbb{C}} \mathscr{V}_{f}$. The idea of the rest of the proof of Lemma 4 consists in finding a $u \in C_{c}^{\infty}(U ; E)$ such that $\left\|\chi_{1}(D)(u)\right\|_{L^{2}}^{2}=1$ and $\left\|\left\langle\mathscr{R}\left(u 1_{\Gamma}\right) ; u 1_{\Gamma}\right\rangle\right\|_{\Lambda}<1$, where $\|\cdot\| \|_{\Lambda}$ is the $C^{*}$-norm of $\Lambda$.

For any $(x ; \xi) \in T^{*} U \backslash 0$ we shall denote by $p(x ; \xi)$ the projection of $E_{x}$ onto $\operatorname{ker}\left(\sqrt{-1} c\left(\frac{\xi}{|\xi|_{x}}\right)-\mathrm{Id}\right) \quad$ along $\quad \operatorname{ker}\left(\sqrt{-1} c\left(\frac{\xi}{|\xi|_{x}}\right)+\mathrm{Id}\right)$. Let $\left.\left.r \in\right] 0, r_{0} / 4\right] \quad$ and $\alpha \in C_{c}^{\infty}(U ;[0,1])$ which is equal to 1 on $U_{2 r}$; the choice of $r$ will be made small enough according to the next sublemma. Then there exists a pseudo-differential operator $T$ of order -1 such that for any $u \in C_{c}^{\infty}\left(U_{r} ; E\right)$ one has

$$
\begin{equation*}
\chi_{1}(D)(u)(x)=\alpha(x) \frac{1}{(2 \pi)^{n}} \int_{|\xi| \geqslant 1} \exp (\sqrt{-1} x \cdot \xi) p(x ; \xi) \hat{u}(\xi) d \xi+T(u)(x) \tag{19}
\end{equation*}
$$

We fix a point $\left(0 ; \xi_{0}\right) \in S^{*} U_{r}$ (where 0 is the origin of $\mathbb{R}^{n}$ ) and a small contractible open neighborhood $C\left(\xi_{0}\right)$ of $\xi_{0}$ in $S_{0}^{*} U_{r}$. We still denote by $C\left(\xi_{0}\right)$ the open conic neighborhood of $\xi_{0}$ in $T_{0}^{*} U_{r} \backslash 0$ induced from $C\left(\xi_{0}\right)$ by $\mathbb{R}^{*+}$-homogeneity. Since the bundle defined by the range of $p(0 ; \xi)$ is trivial over $C\left(\xi_{0}\right)$, we may find $v \in S\left(\mathbb{R}^{n} ; \mathbb{C}^{N}\right)$ such that $\int_{\mathbb{R}^{n}}|\hat{v}(\xi)|^{2} d \xi=(2 \pi)^{n}$ and

$$
\begin{gathered}
\text { supp } \hat{v} \subset C\left(\xi_{0}\right) \cap\left\{\xi \in \mathbb{R}^{n} /|\xi| \geqslant 1 / 2\right\}, \\
\forall \xi \in C\left(\xi_{0}\right), p(0 ; \xi)(\widehat{v}(\xi))=\widehat{v}(\xi)
\end{gathered}
$$

Let $\beta \in C_{c}^{\infty}\left(U_{r} ;[0,1]\right)$ be such that $\beta \equiv 1$ on an open neighborhood of $0 \in \mathbb{R}^{n}$. We then set for any real $R \geqslant 2, u_{R}(x)=\beta(x) R^{\frac{n}{2}} v(R x)$. Lemma 4 is an easy consequence of the following:

Sublemma 1. If $r>0$ (and thus also supp $\alpha$ ) is small enough then one has:
(1) $\lim _{R \rightarrow+\infty}\left\|u_{R}\right\|_{L^{2}}=1$ and $\lim _{R \rightarrow+\infty}\left\|u_{R}\right\|_{H^{-1}}=0$.
(2) $\varliminf_{R \rightarrow+\infty}\left\|\chi_{1}(D)\left(u_{R}\right)\right\|_{L^{2}}>2 / 3$.

Proof of Lemma 4 admitting Sublemma 1. With the notations of Eq. (18), let $C$ be the operator norm of $\mathscr{R} \in B_{\Lambda}\left(H_{\Lambda}^{1}\left(M, E \otimes \mathscr{V}_{f}\right) ; L_{\Lambda}^{2}\left(M, E \otimes \mathscr{V}_{f}\right)\right)$. According to Lemma 1 we choose $r>0$ and $R>2$ such that, setting $u=\frac{u_{R}}{\left\|\chi_{1}(D)\left(u_{R}\right)\right\|_{L^{2}}}$, we have $\|u\|_{L^{2}} \leqslant 3$ and
$\|u\|_{H^{-1}} \leqslant \frac{1}{10 C+10}$. Then, using the Cauchy-Schwartz inequality one has immediately

$$
\left\|\left\langle\mathscr{R}\left(u 1_{\Gamma}\right) ; u 1_{\Gamma}\right\rangle\right\|_{\Lambda}<1
$$

where $\left\|\|_{\Lambda}\right.$ denotes the $C^{*}$-norm of $\Lambda$. Next, using Eq. (18), one checks easily that $\left\langle\chi_{1}(\mathscr{D})\left(u 1_{\Gamma}\right) ; \chi_{1}(\mathscr{D})\left(u 1_{\Gamma}\right)\right\rangle$ is invertible in $\Lambda$ which proves Lemma 4.

Proof of Sublemma 1. (1) is easy and left to the reader. Let us sketch the proof of (2). According to (1) and Eq. (19) we have to consider the following term:

$$
\begin{equation*}
\alpha(x) \frac{1}{(2 \pi)^{n}} \int_{|\xi| \geqslant 1} \exp (\sqrt{-1} x \cdot \xi) p(x ; \xi) \widehat{u_{R}}(\xi) d \xi \tag{20}
\end{equation*}
$$

Then, by writing $p(x ; \xi)-p(0 ; \xi)=\sum_{j=1}^{n} x_{j} g_{j}(x ; \xi)$, we see that at the expense of shrinking $r$ (and thus supp $\alpha$ ), the $L^{2}$-operator norm of the pseudo-differential operator $\sum_{j=1}^{n} x_{j} g_{j}\left(x ; D_{x}\right)$ will be small so that we may replace the term (20) by

$$
\begin{equation*}
\alpha(x) \frac{1}{(2 \pi)^{n}} \int_{|\xi| \geqslant 1} \exp (\sqrt{-1} x . \xi) p(0 ; \xi) \widehat{u_{R}}(\xi) d \xi \tag{21}
\end{equation*}
$$

Now recall that $1-\alpha(x)$ is identically zero on a neighborhood of supp $\beta$. Then, since the operator with the following Schwartz kernel

$$
(1-\alpha(x)) \frac{1}{(2 \pi)^{n}} \int_{|\xi| \geqslant 1} \int_{\mathbb{R}^{n}} \exp (\sqrt{-1} x . \xi) p(0 ; \xi) \beta(y) d \xi d y
$$

is smoothing we may replace (thanks to (1)) the term (20) by

$$
v_{R}(x)=\frac{1}{(2 \pi)^{n}} \int_{|\xi| \geqslant 1} \exp (\sqrt{-1} x . \xi) p(0 ; \xi) \widehat{u_{R}}(\xi) d \xi
$$

But, using standard analysis and all the properties of the function $v$, one checks that $\lim _{R \rightarrow+\infty}\left\|v_{R}\right\|_{L^{2}\left(\mathbb{R}^{n}\right)}=1$, which proves Sublemma 1.

Recall that our goal in this section is to provide a proof of the following.
Technical Lemma. Let $\chi$ be a spectral cut. There exists an orthonormal sequence $\left(e_{k}\right)_{k \in \mathbb{N}}$ of elements of $L_{A}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ such that $\forall k, l \in \mathbb{N},\left\langle e_{k} ; e_{l}\right\rangle=\delta_{k, l} 1_{\Gamma}$ and $\chi(\mathscr{D})\left(e_{k}\right)=e_{k}$.

However, by setting for each $k \in \mathbb{N}$ (and with the notation of Lemma 4):

$$
e_{k}=\left(\left\langle\phi_{k}(\mathscr{D})\left(u_{k}\right), \phi_{k}(\mathscr{D})\left(u_{k}\right)\right\rangle\right)^{-1 / 2} \quad \phi_{k}(\mathscr{D})\left(u_{k}\right)
$$

one deduces immediately this result from Lemma 4.

### 2.4. Existence of spectral sections in the odd-dimensional case

We begin by recalling the definition of noncommutative spectral section. This definition is motivated by the family (i.e. commutative) case treated in Melrose-Piazza [25]; the definition of noncommutative spectral section was given in the unpublished paper [30] and recalled in [31] (which is a published paper).

Definition $3(\mathrm{Wu})$. A projection $\mathscr{P} \in B L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ is said to be a noncommutative spectral section for $\mathscr{D}$ if there exist two spectral cuts $\chi_{1}, \chi_{2}$ such that $\chi_{2} \equiv 1$ on a neighborhood of the support of $\chi_{1}$, and $\operatorname{Im} \chi_{1}(\mathscr{D}) \subset \operatorname{Im} \mathscr{P} \subset \operatorname{Im} \chi_{2}(\mathscr{D})$.

We recall that $\mathscr{D}$ defines an index class Ind $(\mathscr{D}) \in K_{1}(\Lambda)$; one way to define this class is to apply isomorphism (4) to the projection, in the Calkin algebra $C_{\Lambda}(E)$, defined by $\frac{1}{2}(F+1)$, with $F=\left(1+\mathscr{D}^{2}\right)^{-\frac{1}{2}} \mathscr{D}$. The following theorem is stated by Wu in [30,31].

Theorem 3. There exists a spectral section for $\mathscr{D}$ if and only if the index class Ind (D) vanishes in $K_{1}(\Lambda)$.

Remark. Although we state the theorem for the particular $C^{*}$-algebra $\Lambda=C_{r}^{*}(\Gamma)$, it should be clear that the proof easily extends to any Dirac-type operator acting on the sections of a $A$-bundle, as in the work of Mishchenko-Fomenko, with $A$ any unital $C^{*}$-algebra.

Proof of Theorem 3. First we remark that for any smooth spectral cut $\chi$, we have Ind $(\mathscr{D})=\delta([\chi(\mathscr{D})])$; indeed $\frac{1}{2}(F+1)=\chi(\mathscr{D})$ in the Calkin algebra.

We start by proving that if a spectral section $\mathscr{P}$ exists, then $\delta([\chi(\mathscr{D})]=0$. Let us fix a spectral cut $\chi$. By the very definition of spectral section, $\mathscr{P}$ and $\chi(\mathscr{D})$ induce the same element in the Calkin algebra $C_{\Lambda}(E)$. So the $K$-theory class $[\chi(\mathscr{D})] \in K_{0}\left(C_{\Lambda}(E)\right)$ is the image of $[\mathscr{P}] \in K_{0}\left(B_{A} L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)\right)$ by the map induced by the projection $\pi$ : $B_{A} L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right) \rightarrow C_{\Lambda}(E)$. Since $K_{0}\left(B_{\Lambda} L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)\right)=\{0\}$, we see immediately that $[\chi(\mathscr{D})]=0$ and thus that $\delta([\chi(\mathscr{D})]=0$.

We shall now prove the converse. First of all we state the most important consequence of our results in the preceding three sections. Let us fix a spectral cut $\chi$. From Theorems 1 and 2 we infer that there exists a projection $\mathscr{2}$ in $B_{A} L_{\Lambda}^{2}\left(M ; E \otimes \mathbb{C} \mathscr{V}_{f}\right)$ such that $\mathscr{Q}-\chi(\mathscr{D})$ is a $\Lambda$-compact operator. As pointed out to us by Michel Hilsum, the existence of such a projection $\mathscr{2}$ was claimed without explanation in [30]. Equipped with this fundamental result, we can now finish the proof.

All the arguments we give from here until the end of the this section are due to F. Wu.
First we have the following lemma, which is a noncommutative generalization of Lemma 1 of [25].

Lemma $5(\mathrm{Wu})$. Let $\mathscr{E}$ be a $\Lambda$-Hilbert module isomorphic to $l^{2}(\Lambda)$. Let $\mathscr{Q}=\mathscr{Q}^{2}=$ $\mathscr{Q}^{*} \in \mathscr{B}_{A}(\mathscr{E})$ be a projection and $\mathscr{J} \in \mathscr{B}_{A}(\mathscr{E})$ such that $\|\mathscr{Q}-\mathscr{J}\| \leqslant \frac{1}{2}$. Then there is a function $f$, holomorphic in a neighborhood of the spectrum of $\left|\mathscr{2} \mathscr{J}^{*}\right|$, such that $\mathscr{P}=f\left(\mathscr{J} \mathscr{V} \mathscr{J}^{*}\right) \in \mathscr{B}_{\Lambda}(\mathscr{E})$ is the orthogonal projection onto $\mathscr{J} \mathscr{Z}(\mathscr{E})$. Furthermore, if $\mathscr{2}-\mathscr{J} \in \mathscr{K}_{\Lambda}(\mathscr{E})$, then $\mathscr{P}-\mathscr{2} \in \mathscr{K}_{\Lambda}(\mathscr{E})$ too.

Proof. For any $x \in \mathscr{Q}(\mathscr{E})$ one has $x=\mathscr{2}(x)=\mathscr{J}(x)+(\mathscr{2}-\mathscr{F})(x)$. Hence we have

$$
\forall x \in \mathscr{Q}(\mathscr{E}), \quad\|\mathscr{J}(x)\| \geqslant \frac{1}{2}\|x\| .
$$

From this inequality, one deduces easily that $\mathscr{J} \mathscr{Q}$ has closed range. Hence $(\mathscr{J} \mathscr{Q})^{*}=$ $2 \mathscr{J}^{*}$ has also closed range and Theorem 15.3 .8 of [29] shows that

$$
\begin{gathered}
\mathscr{J} \mathscr{Q}(\mathscr{E})=\operatorname{ker}\left(\mathscr{2} \mathscr{J}^{*}\right)^{\perp}=\operatorname{ker}\left(\left|\mathscr{Q} \mathscr{J}^{*}\right|\right)^{\perp}=\left|\mathscr{V} \mathscr{J}^{*}\right|(\mathscr{E}), \\
\mathscr{E}=\operatorname{ker}\left(\left|\mathscr{Q} \mathscr{J}^{*}\right|\right) \oplus \mathscr{J} \mathscr{Q}(\mathscr{E})=\operatorname{ker}\left(\left|\mathscr{Q} \mathscr{J}^{*}\right|^{2}\right) \oplus \mathscr{J} \mathscr{Q}(\mathscr{E})=\operatorname{ker}\left(\mathscr{J} \mathscr{Q} \mathscr{J}^{*}\right) \oplus \mathscr{J} \mathscr{Q}(\mathscr{E}) .
\end{gathered}
$$

Therefore, 0 is an isolated point in the spectrum of $\left|\mathscr{Q} \mathscr{J}^{*}\right|$ and hence of $\left|\mathscr{Q} \mathscr{J}^{*}\right|^{2}=$ $\mathscr{J} \mathscr{\mathscr { J }} \mathscr{J}^{*}$. Let $f(\lambda)=1$ for $\lambda$ in a neighborhood of $\operatorname{Spec}\left(\left|\mathscr{2} \mathscr{J}^{*}\right|^{2}\right) \backslash\{0\}$ and $f(\lambda)=0$ for $\lambda$ near 0 . Then $f$ is holomorphic in a neighborhood of $\operatorname{Spec}\left(\left|\mathscr{2} \mathscr{J}^{*}\right|^{2}\right)$ and

$$
\mathscr{P}=f\left(\left|\mathscr{Q} \mathscr{J}^{*}\right|^{2}\right)=f\left(\mathscr{J} \mathscr{Q} \mathscr{J}^{*}\right)
$$

is the orthogonal projection onto $\mathscr{J} \mathscr{(}(\mathscr{E})$.
Now we show that if in addition $\mathscr{Q}-\mathscr{J} \in \mathscr{K}_{\Lambda}(\mathscr{E})$ then $\mathscr{P}-\mathscr{Q} \in \mathscr{K}_{\Lambda}(\mathscr{E})$. To this end, we first observe that, since $\mathscr{P}(\mathscr{E})=\mathscr{J} \mathscr{Z}(\mathscr{E})$, we have $\mathscr{P}(\mathscr{J} \mathscr{Q})=\mathscr{J} \mathscr{Q}$. Since $\pi(\mathscr{J} \mathscr{Q})=\pi(\mathscr{Q})$ in the Calkin algebra $\mathscr{C}_{\Lambda}(\mathscr{E})$, this implies that

$$
\pi(\mathscr{P}) \pi(\mathscr{Q})=\pi(\mathscr{P}) \pi(\mathscr{\mathscr { V }})=\pi(\mathscr{P} \mathscr{J} \mathscr{Q})=\pi(\mathscr{J} \mathscr{Q})=\pi(\mathscr{Q}) \in \mathscr{C}_{\Lambda}(\mathscr{E}) .
$$

Taking adjoints on both sides, noticing that both $\pi(\mathscr{P})$ and $\pi(\mathscr{2})$ are self-adjoint elements in $\mathscr{C}_{\Lambda}(\mathscr{E})$, we get

$$
\begin{equation*}
\pi(\mathscr{Q})=\pi(\mathscr{Q}) \pi(\mathscr{P}) . \tag{22}
\end{equation*}
$$

On the other hand, since $\mathscr{I}: Q(\mathscr{E}) \rightarrow \mathscr{P}(\mathscr{E})$ is an isomorphism, $\mathscr{J}^{-1} \mathscr{P} \in \mathscr{B}_{\Lambda}(\mathscr{E})$ is welldefined, satisfying $\mathscr{J}\left(\mathscr{J}^{-1} \mathscr{P}\right)=\mathscr{P}$. Passing to $\mathscr{C}_{\Lambda}(\mathscr{E})$ and noticing that $\pi(\mathscr{Q}) \pi(\mathscr{J})=$ $\pi(\mathscr{J}) \pi(\mathscr{Q})$, we obtain $\pi(\mathscr{Q}) \pi(\mathscr{P})=\pi(\mathscr{P})$. From Eq. (22) we get $\pi(\mathscr{P})=\pi(\mathscr{Q})$ in $\mathscr{C}_{\Lambda}(\mathscr{E})$ which proves the lemma.

Proof of Theorem 3 Conclusion. Let $\varphi \in C_{c}^{\infty}(\mathbb{R} ;[0,1])$ such that $\varphi(t)=1$ for $t \in[-1,1]$ and $\varphi(t)=0$ for $|t| \geqslant 2$. We set $\varphi_{N}(t)=\varphi(t / N)$ for $N \in \mathbb{N}^{*}$. One checks easily that as $N \rightarrow+\infty$ :

$$
\varphi_{N}(\mathscr{D})(\chi(\mathscr{D})-\mathscr{Q}) \rightarrow \chi(\mathscr{D})-\mathscr{2}
$$

in the norm of $\mathscr{K}_{\Lambda}(\mathscr{E})$. Thus there is $N_{0}$ such that

$$
\|(\mathrm{Id}-\mathscr{2})-\mathscr{J}\| \leqslant \frac{1}{2}
$$

where

$$
\mathscr{J}=\mathrm{Id}-\chi(\mathscr{D})-\varphi_{N_{0}}(\mathscr{D})(\chi(\mathscr{D})-Q) .
$$

Applying Lemma 5 to $\mathrm{Id}-\mathscr{2}$ instead of $\mathscr{Q}$ we see that there is a projection $\mathscr{P}_{1}=\mathscr{P}_{1}^{*}=\mathscr{P}_{1}^{2}$ in $\mathscr{B}_{\Lambda}(\mathscr{E})$ such that $(\mathrm{Id}-\mathscr{Q})-\left(\mathrm{Id}-\mathscr{P}_{1}\right)=\in \mathscr{K}_{\Lambda}(\mathscr{E})$ and

$$
\left(\mathrm{Id}-\mathscr{P}_{1}\right)(\mathscr{E})=\mathscr{J}(\mathrm{Id}-\mathscr{Q})(\mathscr{E})
$$

Choose another spectral cut $\chi_{0}$ such that

$$
\chi_{0}(t)=0 \quad \text { on } \operatorname{Supp}(1-\chi) \cup\left[-2 N_{0}, 2 N_{0}\right]
$$

Then we have $\left(1-\chi_{0}\right)(1-\chi)=(1-\chi)$ and $\left(1-\chi_{0}\right) \varphi_{N_{0}}=\varphi_{N_{0}}$. Thus we have

$$
\left(\operatorname{Id}-\chi_{0}(\mathscr{D})\right)(\operatorname{Id}-\chi(\mathscr{D}))=(\operatorname{Id}-\chi(\mathscr{D})), \quad\left(\operatorname{Id}-\chi_{0}(\mathscr{D})\right) \varphi_{N_{0}}(\mathscr{D})=\varphi_{N_{0}}(\mathscr{D}) .
$$

This implies that $\left(\operatorname{Id}-\chi_{0}(\mathscr{D})\right) J=J$. Hence we have

$$
\begin{aligned}
& \left(\operatorname{Id}-\mathscr{P}_{1}\right)\left(L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)\right) \\
& \quad=\operatorname{Im}\left(\operatorname{Id}-\chi_{0}(\mathscr{D})\right) J(\operatorname{Id}-\mathscr{Q}) \subset\left(\operatorname{Id}-\chi_{0}(\mathscr{D})\right)\left(L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)\right) .
\end{aligned}
$$

Let $\chi_{1}$ be a spectral cut such that $\chi_{0}(t)=1$ on a neighborhood of $\operatorname{Supp}\left(\chi_{1}\right)$. Then we get, for any $x \in L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ :

$$
\chi_{1}(\mathscr{D})\left(\mathscr{P}_{1}(x)\right)=\chi_{1}(\mathscr{D})(x)-\chi_{1}(\mathscr{D})\left(\left(\operatorname{Id}-\mathscr{P}_{1}\right)(x)\right) .
$$

Since $\operatorname{Im}\left(\operatorname{Id}-\mathscr{P}_{1}\right) \subset\left(\operatorname{Id}-\chi_{0}(\mathscr{D})\right)\left(L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)\right)$, there is $y \in L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ such that $\left(\operatorname{Id}-\mathscr{P}_{1}\right)(x)=\left(\operatorname{Id}-\chi_{0}(\mathscr{D})\right)(y)$. Hence

$$
\chi_{1}(\mathscr{D})\left(\left(\operatorname{Id}-\mathscr{P}_{1}\right)(x)\right)=\chi_{1}(\mathscr{D})\left(\operatorname{Id}-\chi_{0}(\mathscr{D})\right)(y)=0 .
$$

Therefore $\chi_{1}(\mathscr{D}) \mathscr{P}_{1}=\chi_{1}(\mathscr{D})$. Taking the adjoint shows also $P_{1} \chi_{1}(\mathscr{D})=\chi_{1}(\mathscr{D})$. In particular, we have found a spectral cut $\chi_{1}$ and a projection $\mathscr{P}_{1}$ such that $\operatorname{Im} \chi_{1}(\mathscr{D}) \subset \operatorname{Im}\left(\mathscr{P}_{1}\right)$.

We now proceed to modify the projection $\mathscr{P}_{1}$ to a spectral section of $\mathscr{D}$. Let $\psi$ be a spectral cut such that $\psi(t)=1$ on $\left[-1,+\infty\left[\right.\right.$, we set $\psi_{N}(t)=\psi(t / N)$. Since $\chi_{1}(\mathscr{D})-$ $\mathscr{P}_{1}$ is a $\Lambda$-compact operator, one checks easily that

$$
\psi_{N}(\mathscr{D})\left(\chi_{1}(\mathscr{D})-\mathscr{P}_{1}\right) \rightarrow \chi_{1}(\mathscr{D})-\mathscr{P}_{1} .
$$

Thus there exists $N_{0}>0$ such that $\psi_{N_{0}}(t) \chi_{1}(t)=\chi_{1}(t)$ and

$$
\left\|\mathscr{P}_{1}-\chi_{1}(\mathscr{D})+\psi_{N_{0}}(\mathscr{D})\left(\chi_{1}(\mathscr{D})-\mathscr{P}_{1}\right)\right\|=\left\|\mathscr{P}_{1}-\psi_{N_{0}}(\mathscr{D}) \mathscr{P}_{1}\right\|<\frac{1}{2} .
$$

Applying Lemma 5, we obtain a projection $\mathscr{P}=\mathscr{P}^{2}=\mathscr{P}^{*}$ in $B_{\Lambda} L_{\Lambda}^{2}\left(M ; E \otimes \mathbb{C} \mathscr{V}_{f}\right)$ such that

$$
\operatorname{Im}(\mathscr{P})=\psi_{N_{0}}(\mathscr{D}) P_{1}\left(L_{A}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)\right)
$$

Let $\chi_{2}$ be a smooth spectral cut such that $\chi_{2}(t) \psi_{N_{0}}(t)=\psi_{N_{0}}(t)$. Then $\chi_{2}(\mathscr{D}) \psi_{N_{0}}(\mathscr{D}) \mathscr{P}_{1}=\psi_{N_{0}}(\mathscr{D}) \mathscr{P}_{1}$ and we have

$$
\mathscr{P}\left(L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)\right)=\operatorname{Im}\left(\chi_{2}(\mathscr{D}) \psi_{N_{0}}(\mathscr{D}) \mathscr{P}_{1}\right) \subset \operatorname{Im} \chi_{2}(\mathscr{D}) .
$$

On the other hand, since $\mathscr{P}_{1} \chi_{1}(\mathscr{D})=\chi_{1}(\mathscr{D})$, we have

$$
\chi_{1}(\mathscr{D})=\psi_{N_{0}}(\mathscr{D}) \chi_{1}(\mathscr{D})=\psi_{N_{0}}(\mathscr{D}) \mathscr{P}_{1} \chi_{1}(\mathscr{D}) .
$$

This implies that

$$
\operatorname{Im} \chi_{1}(\mathscr{D})=\operatorname{Im}\left(\psi_{N_{0}}(\mathscr{D}) \mathscr{P}_{1} \chi_{1}(\mathscr{D})\right) \subset \operatorname{Im}\left(\psi_{N_{0}}(\mathscr{D}) \mathscr{P}_{1}\right)=\operatorname{Im} \mathscr{P} .
$$

Therefore, $\operatorname{Im} \chi_{1}(\mathscr{D}) \subset \operatorname{Im} \mathscr{P} \subset \operatorname{Im} \chi_{2}(\mathscr{D})$ and $\mathscr{P}$ is a spectral section of $\mathscr{D}$. Theorem 3 is proved.

### 2.5. Difference classes and the noncommutative spectral flow

We recall the notion of difference class associated to two spectral section. The definition is already given in $[30,31]$ and it is an easy and natural extension to the noncommutative setting of the definition given in [25]. We shall need the following proposition of Wu whose proof (left to the reader) uses the same technique as in the proof of Theorem 3.

Proposition 3. If there exists a spectral section for $\mathscr{D}$, then for any spectral cut $\chi_{1}$, there is a spectral cut $\chi_{2}$ such that $\chi_{1} \chi_{2}=\chi_{1}$, and a spectral section $\mathscr{R}$ satisfying $\operatorname{Im} \chi_{1}(\mathscr{D}) \subset \operatorname{Im} \mathscr{R} \subset \operatorname{Im} \chi_{2}(\mathscr{D})$. Similarly, for any spectral cut $\chi_{2}$, there is a spectral cut $\chi_{1}$ such that $\chi_{1} \chi_{2}=\chi_{1}$, and a spectral section $\mathscr{2}$ satisfying $\operatorname{Im} \chi_{1}(\mathscr{D}) \subset \operatorname{Im} \mathscr{Q} \subset \operatorname{Im} \chi_{2}(\mathscr{D})$.

This proposition implies immediately the following:
Corollary 1. Let $\mathscr{P}_{1}, \mathscr{P}_{2}$ be two spectral section of $\mathscr{D}$. Then there exist spectral section $\mathscr{R}, \mathscr{2}$ such that for $i=1,2$

$$
\mathscr{P}_{i} \mathscr{R}=\mathscr{R}_{\mathscr{P}_{i}}=\mathscr{P}_{i} \quad \text { and } \quad \mathscr{P}_{i} \mathscr{Q}=\mathscr{Q} \mathscr{P}_{i}=\mathscr{Q} .
$$

Definition 4. Let $\mathscr{P}_{1}$ and $\mathscr{P}_{2}$ be two spectral sections of $\mathscr{D}$. Then there exists a class $\left[\mathscr{P}_{1}-\mathscr{P}_{2}\right] \in K_{0}(\Lambda)$, called the difference class, which is defined as follows: choose a spectral section $\mathscr{Q}$ of $\mathscr{D}$ satisfying $\mathscr{P}_{i} \mathscr{Q}=\mathscr{Q}=\mathscr{Q}_{\mathscr{P}_{i}}$ for $i \in\{1,2\}$, such a spectral section $\mathscr{Q}$ exists by Corollary 1. Then $\mathscr{P}_{1}-\mathscr{Q}$ and $\mathscr{P}_{2}-\mathscr{Q}$ are projections in $\mathscr{K}_{A}\left(L_{A}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)\right)$ and $\left[\mathscr{P}_{1}-\mathscr{P}_{2}\right]:=\left[\mathscr{P}_{1}-2\right]-\left[\mathscr{P}_{2}-\mathscr{Q}\right]$ is well defined as a class in $K_{0}\left(\mathscr{K}_{\Lambda}\left(L_{\Lambda}^{2}\left(M ; E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)\right)\right)=K_{0}(\Lambda)$.

Remark. The definition is well posed, independent of the choice of 2 . The proof of this fact proceeds, mutatis mutandis, as in [25, p. 127].

Now we recall Wu's definition of noncommutative spectral flow; this is an extension to the noncommutative context of the notion of higher spectral flow given by Dai and Zhang in [8].

Let $\left(f_{u}\right)_{u \in[0,1]}$ be a continuous family of maps $f_{u}: M \rightarrow B \Gamma$. Let $\left(g_{u}\right)_{u \in[0,1]},\left(h_{u}\right)_{u \in[0,1]}$ and $\left(\nabla^{E, u}\right)_{u \in[0,1]}$ be continuous families of respectively riemannian metrics on $M$, hermitian metrics on $E$ and Clifford unitary connections on $\left(E, h_{u}\right)$ with respect to $g_{u}$ for each $u \in[0,1]$. Let $\left(\mathscr{D}_{u}\right)_{u \in[0,1]}$ be the associated continuous family of Dirac-type operators, where each $\mathscr{D}_{u}$ acts on $L_{A}^{2}\left(M ; E \otimes \mathbb{C} \mathscr{V}_{f_{u}}\right)$. Recall that there exists an isomorphism

$$
\mathscr{U}: K_{1}\left(C^{0}([0,1] ; \mathbb{C}) \otimes \Lambda\right) \simeq K_{1}(\Lambda),
$$

which is implemented by the evaluation map $f(\cdot) \otimes \lambda \rightarrow f(0) \lambda$. Now, assume that the index $\delta\left(\left[\chi\left(\mathscr{D}_{0}\right)\right]\right)$ vanishes in $K_{1}(\Lambda)$, then using the above isomorphism $\mathscr{U}$, one gets that $\delta\left(\left[\chi\left(\left(\mathscr{D}_{u}\right)_{u \in[0,1]}\right)\right]\right)$ vanishes in $K_{1}\left(C^{0}([0,1]) \otimes \Lambda\right)$.

Summarizing the family $\left(\mathscr{D}_{u}\right)_{u \in[0,1]}$ admits a (total) spectral section $\mathscr{P}=\left(\mathscr{P}_{u}\right)_{u \in[0,1]}$.

Definition 5. If $\mathscr{L}_{0}$ (resp. $\mathscr{Q}_{1}$ ) is a spectral section associated with $\mathscr{D}_{0}$ (resp. $\mathscr{D}_{1}$ ) then the noncommutative spectral flow $\operatorname{sf}\left(\left(\mathscr{D}_{u}\right)_{u \in[0,1]} ; \mathscr{Q}_{0}, \mathscr{V}_{1}\right)$ from $\left(\mathscr{D}_{0}, \mathscr{L}_{0}\right)$ to $\left(\mathscr{D}_{1}, \mathscr{V}_{1}\right)$ through $\left(\mathscr{D}_{u}\right)_{u \in[0,1]}$ is the $K_{0}(\Lambda)$-class:

$$
\operatorname{sf}\left(\left(\mathscr{D}_{u}\right)_{u \in[0,1]} ; \mathscr{V}_{0}, \mathscr{Q}_{1}\right)=\left[\mathscr{Q}_{1}-\mathscr{P}_{1}\right]-\left[\mathscr{Q}_{0}-\mathscr{P}_{0}\right] \in K_{0}(\Lambda) .
$$

This definition does not depend on the particular choice of the total spectral section $\mathscr{P}=(\mathscr{P})_{u \in[0,1]}$.

If the family is periodic (i.e. $\mathscr{D}_{1}=\mathscr{D}_{0}$ ) and if we take $\mathscr{V}_{1}=\mathscr{Q}_{0}$ then the spectral flow $\operatorname{sf}\left(\left(\mathscr{D}_{u}\right)_{u \in[0,1]} ; \mathscr{Q}_{0}, \mathscr{Q}_{0}\right)$ does not depend on the choice of $\mathscr{Q}_{1}=\mathscr{Q}_{0}$ and defines a $K$ theory class which is intrinsically associated to the given periodic family; we shall denote this class by $\operatorname{sf}\left(\left(\mathscr{D}_{u}\right)_{u \in S^{1}}\right)$.

More generally we can consider a periodic family of operators $\left(\mathscr{D}_{u}\right)$ as above but acting on the fibers of a fiber bundle $P \rightarrow S^{1}$ with fibers diffeomorphic to our manifold $M$. Also in this case there is a well-defined noncommutative spectral flow
sf $\left(\left(\mathscr{D}_{u}\right)_{u \in S^{1}}\right) \in K_{0}(\Lambda)$. We shall encounter an example of this more general situation in Section 6.

Remark. 1. The spectral flow is additive with respect to the composition of path; this means that if $(\mathscr{A}(t))_{t \in[a, b]}$ is a 1-parameter family of Dirac-type operators as in Definition 5 and if $(\mathscr{B}(t))_{t \in[b, c]}$ is a second such family such that $\mathscr{B}(b)=\mathscr{A}(b)$, then, for the family $(\mathscr{D}(t))_{t \in[a, c]}:=(\mathscr{A}(t)) \cup(\mathscr{B}(t))$ the following additivity formula holds
$\operatorname{sf}\left((\mathscr{D}(t)) ; \mathscr{P}_{a}, \mathscr{P}_{c}\right)=\operatorname{sf}\left((\mathscr{A}(t)) ; \mathscr{P}_{a}, \mathscr{P}_{b}\right)+\operatorname{sf}\left(\left(\mathscr{B}^{(t)}\right) ; \mathscr{P}_{b}, \mathscr{P}_{c}\right) \quad$ in $K_{0}\left(C_{r}^{*}(\Gamma)\right)$.
This formula follows immediately from the definition.
2. For a motivation behind Definition 5 we refer the reader to the original work of Dai-Zhang [8].
3. The definition of noncommutative spectral flow, together with some noncommutative extensions of results of Dai-Zhang, have been already published in [16]. In particular, a formula for the Chern character of the noncommutative spectral flow was proved there (Theorem 5.3); the formula involves the higher eta invariant.
4. The definition of noncommutative spectral flow can be given for any 1parameter family of Dirac-type operators, not necessarily associated to a variation of classifying map, of the metrics and of the connection. In fact, the definition can be easily extended to the more general situation of Dirac-type operators on $A$-bundles, as in the work of Mishchenko-Fomenko.

### 2.6. Existence of spectral sections in the even-dimensional case

Now we consider a smooth closed riemannian manifold $M$ of dimension $2 m$, a continuous map $f: M \rightarrow B \Gamma$ and a complex hermitian $\mathbb{Z}_{2}$-graded Clifford module $E=E^{+} \oplus E^{-} \rightarrow M$ endowed with a unitary Clifford connection. Let $\widetilde{M} \rightarrow M$ be the $\Gamma$-normal cover of $M$ associated with $f: M \rightarrow B \Gamma$. Let $\mathscr{V}_{f}=$ $\Lambda \times_{\Gamma} \widetilde{M} \rightarrow M$ be the $\Lambda$-flat bundle associated to these data. Then we denote by $\mathscr{D}$ the $\Lambda$-Dirac-type operator acting on $C^{\infty}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ associated to these data. In fact,

$$
\mathscr{D}=\left(\begin{array}{cc}
0 & \mathscr{D}^{-} \\
\mathscr{D}^{+} & 0
\end{array}\right)
$$

defines an odd self-adjoint unbounded regular operator acting on the $\Lambda$-Hilbert module $L_{A}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$.

Let $\tau$ denotes the grading of $L_{A}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)$ induced by the grading automorphism $\tau_{E}$ defining the splitting $E=E^{+} \oplus E^{-}$We clearly have $\mathscr{D} \tau=-\tau \mathscr{D}$.

Definition 6. A $\mathrm{Cl}(1)$-spectral section for $\mathscr{D}$ is a spectral section $\mathscr{P}$ with the additional property that

$$
\mathscr{P}_{\tau}+\tau \mathscr{P}=\tau .
$$

Lemma 4.3 of [18] shows the following:
Theorem 4. $\mathscr{D}$ admits a $\mathrm{Cl}(1)$-spectral section $\mathscr{P} \in B_{\Lambda}\left(L_{\Lambda}^{2}\left(M, E \otimes_{\mathbb{C}} \mathscr{V}_{f}\right)\right)$ if and only if the index of $\mathscr{D}^{+}$is zero in $K_{0}(\Lambda)$.

## 3. Manifolds with boundary, spectral sections and index classes

## 3.1. $A P S$ and b-index classes in the even-dimensional-case

Let $(M, g)$ be a smooth riemannian manifold of dimension $n=2 m$ with boundary, $E=E^{+} \oplus E^{-}$a $\mathbb{Z}_{2}$-graded hermitian Clifford module endowed with a unitary Clifford connection $\nabla$. We assume that on a collar neighborhood $(\simeq[0,1] \times \partial M)$ of $\partial M, g$ and $\nabla$ have a product structure. We consider also a finitely generated group $\Gamma$, a classifying map $f: M \rightarrow B \Gamma$ and denote by $\Lambda$ the reduced $C^{*}$-algebra of $\Gamma$. Let $\widetilde{M} \rightarrow M$ be the associated $\Gamma$-normal cover. We consider $\mathscr{V}_{f}=\Lambda \times{ }_{\Gamma} \widetilde{M}$, a flat bundle over $M$. These data define a $\mathbb{Z}_{2}$-graded $\Lambda$-linear Dirac type operator $\mathscr{D}$ acting on $C^{\infty}\left(M, \mathscr{V}_{f} \otimes_{\mathbb{C}} E\right)$ which, in a collar neighborhood $(\simeq[0,1] \times \partial M)$, may be written as:

$$
\mathscr{D}=\left(\begin{array}{ll}
0 & \mathscr{D}^{-} \\
\mathscr{D}^{+} & 0
\end{array}\right)
$$

where $\mathscr{D}^{+}=\sigma_{\mathscr{D}^{+}}(d x)\left(\frac{\partial}{\partial x}+\mathscr{D}_{0}\right), \mathscr{D}_{0}$ is the induced boundary operator acting on the sections of $\left(\mathscr{V}_{f} \otimes_{\mathbb{C}} E^{+}\right)_{\mid \partial M}$ and $\mathscr{D}^{-}=\sigma^{-}\left(\frac{\partial}{\partial x}+\sigma^{+} \mathscr{D}_{0} \sigma^{-}\right)$where we have set $\sigma^{ \pm}=\sigma_{\mathscr{P}^{ \pm}}(d x)$. We now consider the $\Lambda$-Hilbert modules $L_{\Lambda}^{2}\left(M, \mathscr{V}_{f} \otimes E\right)$ and $L_{\Lambda}^{2}\left(\partial M,\left(\mathscr{V}_{f} \otimes E^{ \pm}\right)_{\mid \partial M}\right)$ defined by the above data. Then the operator $\mathscr{D}_{0}$ defines a self-adjoint regular unbounded $\Lambda$-linear operator acting on $L_{\Lambda}^{2}\left(\partial M,\left(\mathscr{V}_{f} \otimes E\right)_{\mid \partial M}\right)$. By the cobordism invariance of the index class (see, for example, [16, Proposition 2.3]) we know that Ind $\mathscr{D}_{0}$ is zero in $K_{1}(\Lambda)$; thus Theorem 3 shows that $\mathscr{D}_{0}$ admits spectral section $\mathscr{P} \in B L_{\Lambda}^{2}\left(\partial M,\left(\mathscr{V}_{f} \otimes E^{+}\right)_{\mid \partial M}\right)$, moreover (see [16, Theorem 2.6 1]), one may assume that $\mathscr{P}$ is a $\Lambda$-pseudo-differential operator of order zero. Then we define an odd operator acting on $L_{\Lambda}^{2}\left(\partial M,\left(\mathscr{V}_{f} \otimes E\right)_{\mid \partial M}\right)$ by

$$
B_{\mathscr{P}}=\left(\begin{array}{cc}
0 & (\mathrm{Id}-\mathscr{P}) \sigma^{-} \\
\sigma^{-} \mathscr{P} & 0
\end{array}\right) .
$$

Next we introduce the domain $\operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}\right)$ of $\mathscr{D}$ associated with the global APSboundary condition defined by $\mathscr{P}$ :

$$
\operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}\right)=\left\{\xi \in H_{A}^{1}\left(M, \mathscr{V}_{f} \otimes E\right) / B_{\mathscr{P}}\left(\xi_{\mid \partial M}\right)=0\right\}
$$

and will denote by $\mathscr{D}_{\mathscr{P}}$ the restriction of $\mathscr{D}$ to $\operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}\right)$. In a similar and obvious way one defines $\mathscr{D} \frac{ \pm}{\mathscr{P}}$. We also set

$$
\operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}^{2}\right)=\left\{\xi \in \operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}\right) / \mathscr{D}(\xi) \in \operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}\right)\right\}
$$

Then Wu has shown in [31, p. 174], that the heat operator $H(s)=e^{-s \mathscr{P}_{\mathscr{P}}^{+} \mathscr{P}_{\mathcal{P}}^{-}}$is well defined for $s$ real $>0$ as a $\Lambda$-compact operator sending $L_{\Lambda}^{2}\left(M, \mathscr{V}_{f} \otimes E\right)$ into $\operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}^{2}\right)$ and that $\mathrm{Id}+\mathscr{D}_{\mathscr{P}}^{2}$ is invertible with an inverse being $\Lambda$-compact. Thus $\mathscr{D}_{\mathscr{P}}^{-} \int_{0}^{+\infty} H(s) d s$ is a parametrix for $\mathscr{D}_{\mathscr{P}}^{+}$and $\mathscr{D}_{\mathscr{P}}^{+}$is indeed a $\Lambda$-Fredholm operator from $\operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}^{+}\right)$to $L_{A}^{2}\left(M, \mathscr{V}_{f} \otimes E^{-}\right)$; we will denote by $\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}^{+}, \mathscr{P}\right) \in K_{0}(\Lambda)$ the associated higher (APS) index class.

Now we are going to recall the definition of the $b$-higher index class (see $[14,16]$ ) associated to $\mathscr{D}^{+}$. For background material on the $b$-calculus we refer the reader to the basic reference [24]. According to [31,16] there exists a $\Lambda$-compact self-adjoint operator $\mathscr{A}_{\mathscr{P}}^{0} \in K L_{\Lambda}^{2}\left(\partial M,\left(\mathscr{V}_{f} \otimes E\right)_{\mid \partial M}\right)$ such that $\mathscr{D}_{0}^{+}+\mathscr{A}_{\mathscr{P}}^{0}$ is invertible and

$$
\mathscr{P}=\frac{1}{2}\left(\mathrm{Id}+\frac{\mathscr{D}_{0}^{+}+\mathscr{A}_{\mathscr{P}}^{0}}{\left|\mathscr{D}_{0}^{+}+\mathscr{A}_{\mathscr{P}}^{0}\right|}\right) .
$$

It is proved in [16, Proposition 2.10] that the operator $\mathscr{A}_{\mathscr{P}}^{0}$ can be chosen to be a $\Lambda$ smoothing operator.

We add to $M$ a cylinder $[0,1] \times \partial M$ and still denote by $M$ the extended manifold. Then we extend the metric $g$ to a $b$-metric having a product structure $\left(\frac{d x}{x}\right)^{2}+g_{\partial M}(y)$ on $[0,1 / 2] \times \partial M$ (this amounts to add an infinite cylinder). Then we denote by ${ }^{b} \mathscr{D}$ the $b-\Lambda$-operator associated to $\mathscr{D}$ acting on $L_{b, \Lambda}^{2}\left(M, \mathscr{V}_{f} \otimes E\right)$ (see [14, Section 11]). Let $\rho \in C_{c}^{\infty}\left(\mathbb{R}, \mathbb{R}^{+}\right)$be a nonnegative even smooth test function such that $\int_{\mathbb{R}} \rho(x) d x=1$. We set $\rho_{\varepsilon}(x)=\frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right)$ and then consider the Fourier transform of $\rho_{\varepsilon}$ :

$$
\widehat{\rho}_{\varepsilon}(z)=\int_{\mathbb{R}} e^{-i t z} \rho\left(\frac{t}{\varepsilon}\right) \varepsilon^{-1} d t
$$

Then there exists an operator $\mathscr{A}_{\mathscr{P}} \in \Psi_{b, 1}^{-\infty}\left(M, \mathscr{V}_{f} \otimes E^{+}, \mathscr{V}_{f} \otimes E^{-}\right)$such that the indicial family of ${ }^{b} \mathscr{D}^{+}+\mathscr{A}_{\mathscr{P}}$ is given by

$$
I\left({ }^{b} \mathscr{D}^{+}+\mathscr{A}_{\mathscr{P}}, z\right)=\mathscr{D}_{0}+i z+\widehat{\rho}_{\varepsilon}(z) \mathscr{A}_{\mathscr{P}}^{0}
$$

and is invertible for any $z \in \mathbb{R}$ (see [25,16, Lemma 6.1]). So ${ }^{b} \mathscr{D}^{+}+\mathscr{A}_{\mathscr{P}}$ belongs to the $b-\Lambda$-calculus and is a $\Lambda$-Fredholm operator from $H_{b, \Lambda}^{1}\left(M, \mathscr{V}_{f} \otimes E^{+}\right)$to
$L_{b, A}^{2}\left(M, \mathscr{V}_{f} \otimes E^{-}\right)$. We shall denote by $\operatorname{Ind}_{b}\left({ }^{b} \mathscr{D}^{+}, \mathscr{P}\right) \in K_{0}(\Lambda)$ the associated higher index class.

Theorem 5. With the above notations, the following equality holds:

$$
\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}^{+}, \mathscr{P}\right)=\operatorname{Ind}_{b}\left({ }^{b} \mathscr{D}^{+}, \mathscr{P}\right)
$$

Proof. Let $\varphi \in C^{\infty}([0,1] ;[0,1])$, such that $\varphi(x)=0$ for $\frac{3}{4} \leqslant x \leqslant 1$ and $\varphi(x)=1$ for $0 \leqslant x \leqslant \frac{1}{2}$. For each $t \in[0,1]$, one defines two perturbed Dirac-type $\Lambda$-linear operators by setting: $\quad \mathscr{D}^{+}(t)=\sigma^{+}\left(\frac{\partial}{\partial x}+\mathscr{D}_{0}+t \varphi(x) \mathscr{A}_{\mathscr{P}}^{0}\right) \quad$ and: $\quad \mathscr{D}^{-}(t)=\sigma^{-}\left(\frac{\partial}{\partial x}+\sigma^{+}\left(\mathscr{D}_{0}+\right.\right.$ $\left.\left.t \varphi(x) \mathscr{A}_{\mathscr{P}}^{0}\right) \sigma^{-}\right)$. We then set

$$
\mathscr{D}(t)=\left(\begin{array}{cc}
0 & \mathscr{D}^{-}(t) \\
\mathscr{D}^{+}(t) & 0
\end{array}\right) .
$$

Wu has constructed (see [31]) for each $s>0$ a heat operator $e^{-s \mathscr{Q}^{2}(1)}$ which sends $L_{\Lambda}^{2}\left(M, \mathscr{V}_{f} \otimes E\right)$ into $\operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}^{2}\right)$. Observe that for each $t \in[0,1], \mathscr{D}^{2}(1)-\mathscr{D}^{2}(t)$ is a smoothing operator. Thus, using Duhamel expansion's formula one then defines for each $s>0$ the heat operator $e^{-s \mathscr{O}^{2}(t)}$ which sends $L_{A}^{2}\left(M, \mathscr{V}_{f} \otimes E\right)$ into $\operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}^{2}\right)$. Then,

$$
\mathscr{D}^{-}(t) \int_{0}^{+\infty} e^{-s \mathscr{D}^{+}(t) \mathscr{D}^{-}(t)} d s
$$

defines a parametrix for $\mathscr{D}^{+}(t)$. So the $\left(\mathscr{D}^{+}(t)\right)_{t \in[0,1]}$ define a continuously family of $\Lambda$-Fredholm operators from $H_{\Lambda}^{1}\left(M, \mathscr{V}_{f} \otimes E^{+}\right)$to $L_{\Lambda}^{2}\left(M, \mathscr{V}_{f} \otimes E^{-}\right)$. Thus for any $t \in[0,1], \operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}(t)^{+}, \mathscr{P}\right)=\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}(1)^{+}, \mathscr{P}\right)$. Now, following closely the arguments of Section 10 of [12], one easily checks that $\mathscr{D}(1)^{+}$induces a $\Lambda$-Fredholm operator, denoted ${ }^{\text {cyl }} \mathscr{D}(1)^{+}$from $H_{b, A}^{1}\left(M, \mathscr{V}_{f} \otimes E^{+}\right)$to $L_{b, A}^{2}\left(M, \mathscr{V}_{f} \otimes E^{-}\right)$and that:

$$
\begin{gathered}
\operatorname{Ind}\left({ }^{\mathrm{cyl}} \mathscr{D}(1)^{+}\right)=\operatorname{Ind}_{b}\left({ }^{b} \mathscr{D}^{+}, \mathscr{P}\right) \in K_{0}(\Lambda), \\
\operatorname{Ind}\left({ }^{\mathrm{cyl}} \mathscr{D}(1)^{+}\right)=\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}(1)^{+}, \mathscr{P}\right) .
\end{gathered}
$$

Since we have seen that $\operatorname{Ind}^{\operatorname{APS}}\left(\mathscr{D}(1)^{+}, \mathscr{P}\right)=\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}(0)^{+}, \mathscr{P}\right)$ the theorem is proved.

By combining the previous theorem and Proposition 6.1 of [18], i.e. the relative index theorem for $b$-index classes, one gets at once the following relative index theorem for APS-index classes:

Theorem 6. With the above notations, let $\mathscr{P}, \mathscr{P}^{\prime}$ be two spectral section for $\mathscr{D}_{0}$. Then one has

$$
\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}^{+}, \mathscr{P}\right)-\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}^{+}, \mathscr{P}^{\prime}\right)=\left[\mathscr{P}^{\prime}-\mathscr{P}\right] \in K_{0}(\Lambda) .
$$

## 3.2. $A P S$ - and b-index classes in the odd-dimensional case

We shall now address the odd-dimensional case. Let $(M, g)$ be a smooth riemannian manifold with boundary of dimension $n=2 m+1, E$ a hermitian Clifford module endowed with a unitary Clifford connection $\nabla$. We assume that in a collar neighborhood of $\partial M,(\simeq[0,1] \times \partial M=\{(x, y)\}), g$ and $\nabla$ have a product structure. We fix a Clifford action $\mathrm{Cl}_{\partial}$ of $T^{*} \partial M$ on $E_{0}=E_{\mid \partial M}$ by setting for any $\eta \in T^{*} \partial M, \mathrm{Cl}_{\partial}(\eta)=\mathrm{Cl}(d x) \mathrm{Cl}(\eta)$ where Cl denotes the Clifford action of $T^{*} M$ on $E$. We set $\tau=\sqrt{-1} \mathrm{Cl}(d x)$; since $\tau^{2}=\mathrm{Id}, E_{0}=E_{0}^{+} \oplus E_{0}^{-}$is $\mathbb{Z}_{2}$-graded with $E_{0}^{ \pm}=$ $\operatorname{ker}(\tau- \pm \mathrm{Id})$.

We consider a finitely generated group $\Gamma$, and a classifying map $f: M \rightarrow B \Gamma$. These data define as before a $\Lambda$-linear Dirac type operator $\mathscr{D}$ acting on $C^{\infty}\left(M, \mathscr{V}_{f} \otimes_{\mathbb{C}} E\right)$ which, in a collar neighborhood $(\simeq[0,1] \times \partial M)$, may be written as:

$$
\mathscr{D}=\sigma_{\mathscr{D}}(d x)\left(\frac{\partial}{\partial x}+\mathscr{D}_{0}\right),
$$

where $\mathscr{D}_{0}$ is the induced boundary $\mathbb{Z}_{2}$-graded operator acting on the section of $\left(\mathscr{V}_{f} \otimes_{\mathbb{C}} E\right)_{\mid \partial M}$. The operator $\mathscr{D}_{0}$ is a self-adjoint regular unbounded $\Lambda$-linear operator acting on $L_{\Lambda}^{2}\left(\partial M,\left(\mathscr{V}_{f} \otimes E\right)_{\mid \partial M}\right)$; we know that Ind $\mathscr{D}_{0}$ is zero in $K_{0}(\Lambda)$, so Theorem 4 shows that $\mathscr{D}_{0}$ admits a $\mathrm{Cl}(1)$-spectral section $\mathscr{P} \in B L_{\Lambda}^{2}(\partial M$, $\left.\left(\mathscr{V}_{f} \otimes E\right)_{\mid \partial M}\right)$; by definition we have $\tau \mathscr{P}+\mathscr{P}_{\tau}=\tau$. One may assume as in the even case that $\mathscr{P}$ is a $\Lambda$-pseudo-differential operator of order zero. We fix such a $\mathrm{Cl}(1)$ spectral section $\mathscr{P}$ in the rest of this section. We can find (proceeding as in the evendimensional case) a self-adjoint odd $\Lambda$-compact operator $\mathscr{A}_{\mathscr{P}}^{0}$ acting on $L_{A}^{2}\left(\partial M,\left(\mathscr{V}_{f} \otimes E\right)_{\mid \partial M}\right)$ such that $\mathscr{D}_{0}+\mathscr{A}_{\mathscr{P}}^{0}$ is invertible and

$$
\begin{equation*}
\mathscr{P}=\frac{1}{2}\left(\operatorname{Id}+\frac{\mathscr{D}_{0}+\mathscr{A}_{\mathscr{P}}^{0}}{\left|\mathscr{D}_{0}+\mathscr{A}_{\mathscr{P}}^{0}\right|}\right) . \tag{24}
\end{equation*}
$$

The fact that $\mathscr{A}_{\mathscr{P}}^{0}$ be odd is implied by the condition $\tau \mathscr{P}+\mathscr{P} \tau=\tau$, as in [26].
We denote by ${ }^{b} \mathscr{D}$ the $b$-operator induced by $\mathscr{D}$. Proceeding as in [25, Lemma 9] and in [16], one checks the existence of a self-adjoint operator $\mathscr{A}_{\mathscr{P}} \in \Psi_{b, 1}^{-\infty}$ $\left(M, \mathscr{V}_{f} \otimes E, \mathscr{V}_{f} \otimes E\right)$ with indicial family is given by

$$
\begin{equation*}
I\left({ }^{b} \mathscr{D}+\mathscr{A}, z\right)=\mathscr{D}_{0}+i z+\widehat{\rho}_{\varepsilon}(z) \mathscr{A}_{\mathscr{P}}^{0} . \tag{25}
\end{equation*}
$$

This indicial family is invertible for any $z \in \mathbb{R}$.

Proposition 4. The operator ${ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}$ is self-adjoint regular on $L_{b, \Lambda}^{2}\left(M, \mathscr{V}_{f} \otimes E\right)$.
Proof. We follow [31, p. 174]. Proceeding as in [17, p. 328] and as in [25, Proposition 8] one constructs for any $s>0$ a heat operator $H(s)=e^{-s\left(b^{( } \mathscr{D}+\mathscr{A} \mathscr{P}\right)^{2}}$ such that $R=$ $\int_{0}^{+\infty} e^{-s} H(s) d s$ sends $L_{b, 4}^{2}\left(M, \mathscr{V}_{f} \otimes E\right)$ into $H_{b, \Lambda}^{2}\left(M, \mathscr{V}_{f} \otimes E\right)$ and (Id $+\left({ }^{b} \mathscr{D}+\right.$ $\left.\left.\mathscr{A}_{\mathscr{P}}\right)^{2}\right) \circ R=\mathrm{Id}$. Thus $\left(\operatorname{Id}+\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)^{2}\right)$ is surjective. Moreover, for any $\xi, \xi^{\prime} \in H_{b, \Lambda}^{1}$ $\left(M, \mathscr{V}_{f} \otimes E\right)$ one has

$$
\left\langle\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)\left(\xi_{1}\right) ; \xi_{2}\right\rangle=\left\langle\xi_{1} ;\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)\left(\xi_{2}\right)\right\rangle \in \Lambda,
$$

which completes the proof of the proposition.
Proceeding along the same lines, one proves the following:
Proposition 5. Let $\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}$ be the operator induced by $\mathscr{D}$ defined on the domain $\operatorname{dom}\left(\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}\right)=\left\{\xi \in H_{b, \Lambda}^{1}\left(M, \mathscr{V}_{f} \otimes E\right) / \mathscr{P}\left(\xi_{\mid \partial M}\right)=0\right\}$. Then $\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}$ is a self-adjoint unbounded regular operator on $L_{A}^{2}\left(M, \mathscr{V}_{f} \otimes E\right)$.

Definition 7. One defines two suspended families indexed by $t \in[0, \pi]$ by the following two formulae:
(1) $\left(\Sigma\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)\right)_{t}:=\sqrt{\mathrm{Id}}+\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)^{2} \cos t+\sqrt{-1}\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right) \sin t$.

The family $\Sigma\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)=\left(\Sigma\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)\right)_{t \in[0, \pi]}$ defines a $\left(C_{c}^{0}(] 0, \pi[) \otimes \Lambda\right)$-linear operator acting on $C_{c}^{0}(] 0, \pi[) \otimes H_{b, \Lambda}^{1}\left(M, \mathscr{V}_{f} \otimes E\right)$.
(2) $\left(\Sigma\left(\mathscr{D}_{P}^{\mathrm{APS}}\right)\right)_{t}:=\sqrt{\mathrm{Id}+\left(\mathscr{D}_{P}^{\mathrm{APS}}\right)^{2}} \cos t+\sqrt{-1} \mathscr{D}_{P}^{\mathrm{APS}} \sin t$.

The easy proof of the next proposition is left to the reader.
Proposition 6. (1) The index class $\operatorname{Ind}\left(\Sigma\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)\right)$ is well defined in $K_{0}\left(C_{c}^{0}(] 0, \pi[) \otimes \Lambda\right) \simeq K_{1}(\Lambda)$.
(2) The index class $\operatorname{Ind}\left(\Sigma\left(\mathscr{D}_{P}^{\mathrm{APS}}\right)\right)$ is well defined in $K_{0}\left(C_{c}^{0}(] 0, \pi[) \otimes \Lambda\right) \simeq K_{1}(\Lambda)$.

We can use suspension in order to extend to the odd-dimensional case Definition 4. We follow Proposition 4 of [26].

Definition 8. Consider two $\mathrm{Cl}(1)$-spectral section $P_{1}, P_{2}$ of $\mathscr{D}_{0}$, let $\mathscr{A}_{P_{j}}^{0}(j \in\{1,2\})$ be a $\mathbb{Z}_{2}$-graded self-adjoint $\Lambda$-compact operator acting on $L_{\Lambda}^{2}\left(\partial M ;\left(\mathscr{V}_{f} \otimes E\right)_{\mid \partial M}\right)$ such that for each $j \in \in\{1,2\}, \mathscr{D}_{0}+\mathscr{A}_{P_{j}}^{0}$ is invertible and

$$
P_{j}=\frac{1}{2}\left(\operatorname{Id}+\frac{\mathscr{D}_{0}+\mathscr{A}_{P_{j}}^{0}}{\left|\mathscr{D}_{0}+\mathscr{A}_{P_{j}}^{0}\right|}\right)
$$

Then for any $t \in[0, \pi]$,

$$
\mathscr{D}_{0}^{j}(t)=\tau\left(\mathrm{Id}+\left(\mathscr{D}_{0}+\mathscr{A}_{P_{j}}^{0}\right)^{2}\right)^{1 / 2} \cos t+\left(\mathscr{D}_{0}+\mathscr{A}_{P_{j}}^{0}\right) \sin t
$$

is invertible, where $\tau=\sqrt{-1} \mathrm{Cl}(d x)$ is the involution of $E_{\partial M}$. Let $P_{j}(t)$ denote the projection onto the positive part of $\mathscr{D}_{0}^{j}(t)$. According to Corollary 1 , we consider a continuous family $(Q(t))_{t \in[0, \pi]}$ of projections in $B_{A} L_{A}^{2}\left(\partial M ;\left(\mathscr{V}_{f} \otimes E\right)_{\mid \partial M}\right)$ such that for any $t \in[0, \pi]$ and $j \in\{1,2\}, P_{j}(t) Q(t)=Q(t) P_{j}(t)=Q(t)$. Then

$$
\left[P_{1}-P_{2}\right]=\left[\left(P_{1}(t)-P_{2}(t)\right)_{t}\right]=\left[\left(P_{1}(t)-Q(t)\right)_{t}\right]-\left[\left(P_{2}(t)-Q(t)\right)_{t}\right]
$$

is a well-defined class in $K_{0}\left(C_{c}^{0}(] 0, \pi[) \otimes \Lambda\right) \simeq K_{1}(\Lambda)$ which does not depend on the choice of $\mathscr{A}_{P_{j}}^{0}(j \in\{1,2\})$ and $(Q(t))_{t \in[0, \pi]}$.

Remark. In fact the previous definition of difference class $\left[P_{1}-P_{2}\right]$ holds equally well when one replaces $\partial M$ (resp. $f: \partial M \rightarrow B \Gamma$ ) by a closed manifold $F$ (resp. a map $f^{\prime}: F \rightarrow B \Gamma$ ) which is not necessarily the boundary of a manifold.

There is a third index class that one can consider. Following [26], we are going to associate a Dirac suspension to the $\Lambda$-operator ${ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}$. For any $\theta \in \mathbb{R} / 2 \pi \mathbb{Z}$, we denote by $L_{\theta}$ the line bundle over $\mathbb{R} / \pi \mathbb{Z}$ obtained by identifying $(0, v)$ with $(\pi, \exp (-\sqrt{-1} \theta) v), v \in \mathbb{C}$. Notice that the first circle has length $2 \pi$ whereas the second circle has length $\pi$. We then denote by $\mathscr{L}$ the bundle of $C^{0}(\mathbb{R} / \pi \mathbb{Z})$-modules over $\mathbb{R} / 2 \pi \mathbb{Z}$ whose fiber over $\theta \in \mathbb{R} / 2 \pi \mathbb{Z}$ is given by $\mathscr{L}_{\theta}=C^{0}\left(\mathbb{R} / \pi \mathbb{Z} ; L_{\theta}\right)$ (recall that the $\mathscr{L}_{\theta}$ are topologically trivial). We set

$$
S \Lambda=C^{0}(\mathbb{R} / \pi \mathbb{Z}) \otimes \Lambda
$$

Proposition 7. One can define a Dirac suspension of ${ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}$ which is a $\mathbb{Z}_{2}$-graded Sイ-Fredholm operator, ${ }^{b} \mathscr{D}_{\mathscr{P}}^{\Sigma}$ :
$H_{b, S \Lambda}^{1}\left(\mathbb{R} / 2 \pi \mathbb{Z} \times M ; \mathscr{L} \otimes_{\mathbb{C}} \mathscr{V}_{f} \otimes E \otimes \mathbb{C}^{2}\right) \rightarrow L_{b, S \Lambda}^{2}\left(\mathbb{R} / 2 \pi \mathbb{Z} \times M ; \mathscr{L} \otimes \mathbb{C} \mathscr{V}_{f} \otimes E \otimes \mathbb{C}^{2}\right)$,
which thus has a well-defined index class Ind $\left({ }^{b} \mathscr{D}_{\mathscr{P}}^{\Sigma}\right) \in K_{0}(S \Lambda)$.
Proof. We follow Section 5 of [26] and explain the modifications that are needed in the present noncommutative context. We introduce the following operator acting on $C^{\infty}(\mathbb{R} / 2 \pi \mathbb{Z}, \mathscr{L}):$

$$
\partial^{L}=-\sqrt{-1} \frac{\partial}{\partial \theta}+\sqrt{-1} \frac{t-\frac{1}{2}}{\pi}
$$

Then, for each $\theta \in \mathbb{R} / 2 \pi \mathbb{Z}$ and $n \in \mathbb{Z}$, we consider the section $e_{n}(t, \theta)$ of $L_{\theta}$ defined by $e_{n}(t, \theta)=e^{\sqrt{-1} n \theta}$ if $t \in\left[0, \pi\left[\right.\right.$ and $e_{n}(t, \theta)=e^{\sqrt{-1}(n-1) \theta}$ if $t=\pi$. Next, we fix $\varepsilon>0$
and $\delta>0$ two small (enough) reals and a function $f_{\varepsilon, \delta} \in C_{c}^{\infty}([0, \pi] \times \mathbb{R})$ which is equal to zero on $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$ and equal to one on $[\delta, \pi-\delta] \times\left[\frac{-1}{\varepsilon}, \frac{1}{\varepsilon}\right]$. Then we introduce the $C^{0}(\mathbb{R} / \pi \mathbb{Z})$-linear operator $R_{\varepsilon, \delta}$ acting on $C^{\infty}(\mathbb{R} / 2 \pi \mathbb{Z}, \mathscr{L})$ and defined by the following Schwartz kernel:

$$
K\left(R_{\varepsilon, \delta}\right)\left(t, \theta, t^{\prime}, \theta^{\prime}\right)=\sum_{n \in \mathbb{Z}} f_{\varepsilon, \delta}\left(t, n+\frac{(t-1 / 2)}{\pi}\right) \delta_{t=t^{\prime}} \otimes\left(e_{n}(t, \theta) \otimes \overline{e_{n}\left(t, \theta^{\prime}\right)}\right) .
$$

Then the required Dirac suspension ${ }^{b} \mathscr{D}_{P}^{\Sigma}$ is defined to be

$$
\left(\begin{array}{cc}
0 & \partial^{L} \otimes \mathrm{Id}+\mathrm{Id} \otimes \sqrt{-1} \mathscr{D}+R_{\varepsilon, \delta} \otimes \sqrt{-1} \mathscr{A} \\
\partial^{L} \otimes \mathrm{Id}-\mathrm{Id} \otimes \sqrt{-1} \mathscr{D}-R_{\varepsilon, \delta} \otimes \sqrt{-1} \mathscr{A} & 0
\end{array}\right)
$$

This operator is the sum of a Dirac $b-S \Lambda$-operator and a $b-S \Lambda$ operator of order $-\infty$; moreover its indicial family is invertible for $z \in \mathbb{R}$; it follows from [16] that the operator is $S \Lambda$-Fredholm. The proposition is proved.

In the next theorem we relate the three index classes introduced above; we also state the relative index theorem in the odd-dimensional case. We devote the next subsection to its proof.

Theorem 7. With the notation introduced above we have:
(1) $\operatorname{Ind}\left(\Sigma\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)\right)=\operatorname{Ind}\left(\Sigma\left(\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}\right)\right)$.
(2) Let $j: K_{0}\left(C_{c}^{0}(] 0, \pi[) \otimes \Lambda\right) \rightarrow K_{0}(S \Lambda)$ be the map induced by the inclusion $] 0, \pi[\rightarrow \mathbb{R} / \pi \mathbb{Z}$. Then

$$
j\left(\operatorname{Ind}\left(\Sigma\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)\right)\right)=\operatorname{Ind}\left({ }^{b} \mathscr{D}_{\mathscr{P}}^{\Sigma}\right) \in K_{0}(S \Lambda) .
$$

(3) Let $\mathscr{P}_{1}, \mathscr{P}_{2}$ be two $\mathrm{Cl}(1)$-spectral section for $\mathscr{D}_{0}$, then

$$
\operatorname{Ind}\left(\Sigma\left(\mathscr{D}_{\mathscr{P}_{1}}^{\mathrm{APS}}\right)\right)-\operatorname{Ind}\left(\Sigma\left(\mathscr{D}_{\mathscr{P}_{2}}^{\mathrm{APS}}\right)\right)=\left[\mathscr{P}_{2}-\mathscr{P}_{1}\right] \in K_{1}(\Lambda) .
$$

We can also give the odd-dimensional version of Definition 5 thus introducing the noncommutative spectral flow for an odd-dimensional manifold $F$ endowed with a $\mathbb{Z}_{2}$-graded hermitian Clifford module $E$.

Let $\left(f_{u}\right)_{u \in[0,1]}$ be a continuous family of maps $f_{u}: F \rightarrow B \Gamma$. Let $\left(g_{u}\right)_{u \in[0,1]},\left(h_{u}\right)_{u \in[0,1]}$, and $\left(\nabla^{E, u}\right)_{u \in[0,1]}$ be continuous families of respectively riemannian metrics on $F$, hermitian metrics on $E$, and Clifford unitary connections on ( $E, h_{u}$ ) with respect to $g_{u}$ for each $u \in[0,1]$. Let $\left(\mathscr{D}_{u}\right)_{u \in[0,1]}$ be the associated continuous family of odd Diractype operators (as above) where each $\mathscr{D}_{u}$ acts on $L_{\Lambda}^{2}\left(M ; E \otimes \mathbb{C} \mathscr{V}_{f_{u}}\right)$. Assume that the
index Ind $\mathscr{D}_{0}^{+}$vanishes in $K_{0}(\Lambda)$. Then Ind $\left(\mathscr{D}_{u}^{+}\right)_{u \in[0,1]}$ vanishes in $K_{0}\left(C^{0}([0,1]) \otimes \Lambda\right)$ and thus $\left(\mathscr{D}_{u}\right)_{u \in[0,1]}$ admits a $\mathrm{Cl}(1)$-spectral section $\left(\mathscr{P}_{u}\right)_{u \in[0,1]}$.

Definition 9. If $\mathscr{Q}_{0}$ (resp. $\mathscr{Q}_{1}$ ) is a $\mathrm{Cl}(1)$-spectral section associated with $\mathscr{D}_{0}$ (resp. $\mathscr{D}_{1}$ ) then the noncommutative spectral flow $\operatorname{sf}\left(\left(\mathscr{D}_{u}\right)_{u \in[0,1]} ; \mathscr{D}_{0}, \mathscr{Q}_{1}\right)$ from $\left(\mathscr{D}_{0}, \mathscr{L}_{0}\right)$ to $\left(\mathscr{D}_{1}, \mathscr{V}_{1}\right)$ through $\left(\mathscr{D}_{u}\right)_{u \in[0,1]}$ is the $K_{1}(\Lambda)$-class:

$$
\operatorname{sf}\left(\left(\mathscr{D}_{u}\right)_{u \in[0,1]} ; \mathscr{V}_{0}, \mathscr{Q}_{1}\right)=\left[\mathscr{Q}_{1}-\mathscr{P}_{1}\right]-\left[\mathscr{Q}_{0}-\mathscr{P}_{0}\right] \in K_{1}(\Lambda) .
$$

This definition does not depend on the particular choice of the spectral section $\left(\mathscr{P}_{u}\right)_{u \in[0,1]}$. If moreover the family is periodic (i.e. $\mathscr{D}_{0}=\mathscr{D}_{1}$ ) and if we take $\mathscr{Q}_{0}=\mathscr{Q}_{1}$ then $\operatorname{sf}\left(\left(\mathscr{D}_{u}\right)_{u \in[0,1]} ; \mathscr{V}_{0}, \mathscr{V}_{0}\right)$ does not depend on the choice of $\mathscr{L}_{0}=\mathscr{V}_{1}$.

Remark. As in the even case, the definition of noncommutative spectral flow can be given for any 1-parameter family of Dirac-type operators.

### 3.3. Proof of Theorem 7

This section can be skipped at first reading. We shall adapt to the noncommutative context the arguments in [26]. This proceeds in a rather straightforward way; in fact, the suspension-arguments do not distinguish very much between the commutative and noncommutative context.
(1) Our goal is to show that $\operatorname{Ind}\left(\Sigma\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)\right)=\operatorname{Ind}\left(\Sigma\left(\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}\right)\right)$. We adapt to the noncommutative setting the proof of Proposition 5 of [26]; we shall also employ arguments used in a similar proposition proved in [12, Theorem 10.1]. Let $\varphi \in C^{\infty}([0,1] ;[0,1])$ such that $\varphi(x)=1$ for $0 \leqslant x \leqslant 1 / 2$ and $\varphi(x)=0$ for $3 / 4 \leqslant x$. By considering the family of operators

$$
\mathscr{D}(t)=\sigma_{\mathscr{D}}\left(\frac{\partial}{\partial x}+\mathscr{D}_{0}+t \varphi(x) \mathscr{A}_{\mathscr{P}}^{0}\right)
$$

and using an homotopy argument one can replace, in the definition of $\mathscr{D}_{\mathscr{P}}^{\text {APS }}, \mathscr{D}$ by $\mathscr{D}(1)$. We shall denote by $\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}(1)$ the associated self-adjoint $\Lambda$-Fredholm operator. Then let $\mu>0$ such that $[-\mu, \mu] \cap \operatorname{spec}\left(\mathscr{D}_{0}+\mathscr{A}_{\mathscr{P}}^{0}\right)=\emptyset$. We observe that we do not change the index class if, in Definition 7, we take instead of $\left(\left(\Sigma\left(\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}\right)\right)_{t}\right)$ the family: $\left(\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}(1)+s \sqrt{-1} \mathrm{Id}\right)_{s \in[-\mu / 2, \mu / 2]}$. Now, for any $s \in[-\mu / 2, \mu / 2]$ we set

$$
\mathscr{P}_{s \sqrt{-1}}=\frac{1}{2}\left(\operatorname{Id}+\frac{\mathscr{D}_{0}+\mathscr{A}_{\mathscr{P}}^{0}+s \sqrt{-1} \tau}{\left|\mathscr{D}_{0}+\mathscr{A}_{\mathscr{P}}^{0}+s \sqrt{-1} \tau\right|}\right) .
$$

Proceeding as in [26, p. 303], using the family $\left(\mathscr{P}_{t s \sqrt{-1}}\right)_{t \in[0,1]}$ and an homotopy argument, one checks easily the following:

Lemma 6. Ind $\left(\Sigma\left(\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}\right)\right)$ coincides with the index class defined by the suspended family (parametrized by $s \in[-\mu / 2, \mu / 2]$ ):

$$
\begin{gathered}
\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}(1)+s \sqrt{-1} \mathrm{Id}: \operatorname{Dom}_{s} \rightarrow L_{A}^{2}\left(M, \mathscr{V}_{f} \otimes E\right), \\
\operatorname{Dom}_{s}=\left\{u \in H_{A}^{1}\left(M, \mathscr{V}_{f} \otimes E\right) / \mathscr{P}_{s \sqrt{-1}}\left(\xi_{\mid \partial M}\right)=0\right\} .
\end{gathered}
$$

Now we extend $M$ to a manifold $M_{e}$, with cylindrical end and the boundary defining functions related by $x=\exp (\tilde{x})$. Let us fix $\chi \in C^{\infty}(\mathbb{R}, \mathbb{R})$ such that $\chi(x)=1$ for $x<1$ and $\chi(x) \in C^{\infty}\left(M_{e}\right)$ has support in the collar neighborhood of the boundary. Following closely [26, p. 303], we embed $\operatorname{Dom}_{s}$ into $H_{b, A}^{1}\left(M_{e}, \mathscr{V}_{f} \otimes E\right)$ and obtain the decomposition

$$
\operatorname{Dom}_{s} \oplus G_{s}=H_{b, \Lambda}^{1}\left(M_{e}, \mathscr{V}_{f} \otimes E\right)
$$

where

$$
G_{s}=\left\{u \in H_{b, \Lambda}^{1}\left(M_{e}, \mathscr{V}_{f} \otimes E\right) / \mathscr{P}_{s \sqrt{-1}}\left(u_{\mid x=1}\right)=u_{\mid x=1} \text { and } \forall x>1 u=\chi(x) u_{\mid x=1}\right\} .
$$

Moreover, $\mathscr{D}(1)$ induces an operator, denoted ${ }^{\text {cyl }} \mathscr{D}_{\mathscr{P}}$ from $H_{b, \Lambda}^{1}\left(M_{e}, \mathscr{V}_{f} \otimes E\right)$ to $L_{b, A}^{2}\left(M_{e}, \mathscr{V}_{f} \otimes E\right)$. Then, proceeding as in [26, pp. 303-304], one checks easily that for $s \in]-\mu, \mu\left[,^{\text {cyl }} \mathscr{D}_{\mathscr{P}}-\sqrt{-1} s\right.$ Id induces an invertible operator from $G_{s}$ onto its range denoted $H_{s}$ and that

$$
L_{b, A}^{2}\left(M_{e}, \mathscr{V}_{f} \otimes E\right)=L^{2}\left(M, \mathscr{V}_{f} \otimes E\right) \oplus H_{s}
$$

From this and Lemma 6 , we conclude that $\operatorname{Ind}\left(\Sigma\left(\mathscr{D}_{\mathscr{P}}^{\mathrm{APS}}\right)\right)=\operatorname{Ind}\left(\Sigma\left({ }^{\text {cyl }} \mathscr{D}_{\mathscr{P}}\right)\right)$. Lastly, an homotopy argument (see [12, Section 10]) shows that $\operatorname{Ind}\left(\Sigma\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)\right)=$ Ind $\left(\Sigma\left({ }^{\text {cyl }} \mathscr{D}_{\mathscr{P}}\right)\right)$ which completes the proof of (1).
(2) Let $j: K_{0}\left(C_{c}^{0}(] 0, \pi[) \otimes \Lambda\right) \rightarrow K_{0}(S \Lambda)$ be the map induced by the inclusion $] 0, \pi\left[\rightarrow \mathbb{R} / \pi \mathbb{Z}\right.$. Our goal is to prove that $j\left(\operatorname{Ind}\left(\Sigma\left({ }^{b} \mathscr{D}+\mathscr{A}_{\mathscr{P}}\right)\right)\right)=\operatorname{Ind}\left({ }^{b} \mathscr{D}_{\mathscr{P}}^{\Sigma}\right) \in K_{0}(S \Lambda)$. We observe that $L_{b, S \Lambda}^{2}\left(\mathbb{R} / 2 \pi \mathbb{Z} \times M ; \mathscr{L} \otimes_{\mathbb{C}} \mathscr{V}_{f} \otimes E \otimes \mathbb{C}^{2}\right)$ is isomorphic to the field of continuous $\Lambda$-Hilbert modules over $\mathbb{R} / \pi \mathbb{Z}$ given by $\oplus_{n \in \mathbb{Z}} \mathbb{C} e_{n}(t,.) \otimes$ $L_{b, 4}^{2}\left(M ; \mathscr{V}_{f} \otimes E \otimes \mathbb{C}^{2}\right)$. Now an easy translation in the noncommutative setting of the proof of the Proposition 6 of [26, pp. 308-309] allows to finish the proof of (2).
(3) We shall now prove the relative index theorem in the odd-dimensional case $\operatorname{Ind}\left(\Sigma\left(\mathscr{D}_{\mathscr{P}_{1}}^{\mathrm{APS}}\right)\right)-\operatorname{Ind}\left(\Sigma\left(\mathscr{D}_{\mathscr{P}_{2}}^{\mathrm{APS}}\right)\right)=\left[\mathscr{P}_{2}-\mathscr{P}_{1}\right] \in K_{1}(\Lambda)$. We follow the proof of Proposition 11 of [26] and explain the modifications allowing to extend it to our noncommutative setting. We fix three real numbers, $a<0, b>1$ and $\varepsilon>0$. We consider a function $\psi_{\varepsilon}(t, \lambda) \in C^{\infty}([0, \pi) \times \mathbb{R})$ which vanishes on the complement of the open rectangle $] 0, \pi[\times] a, a+b \pi[$ and is equal to one on the rectangle $[\varepsilon, \pi-\varepsilon] \times$ $[a+\varepsilon, a+b \pi-\varepsilon]$. Next we introduce the $C^{0}(\mathbb{R} / \pi \mathbb{Z})$-linear operator $R_{\varepsilon}$ acting
on $C^{\infty}(\mathbb{R} / 2 \pi \mathbb{Z}, \mathscr{L})$ and defined by the following Schwartz kernel:

$$
K\left(R_{\varepsilon}\left(t, \theta, t^{\prime}, \theta^{\prime}\right)\right)=\sum_{n \in \mathbb{Z}} \psi_{\varepsilon}\left(t, n+\frac{(t-1 / 2)}{\pi}\right) \delta_{t=t^{\prime}} \otimes\left(e_{n}(t, \theta) \otimes \overline{e_{n}\left(t, \theta^{\prime}\right)}\right)
$$

We fix two operators $\mathscr{A}_{\mathscr{P}_{j}}^{0}(j \in\{1,2\})$ as in Definition 8 and we associate to them two operators $\mathscr{A}_{\mathscr{P}_{j}}^{0}$ as in (25). Then, we consider the Dirac suspension ${ }^{b} \mathscr{D}_{\mathscr{P}_{j}}^{\Sigma}$ defined by

$$
\left(\begin{array}{cc}
0 & \partial^{L} \otimes \mathrm{Id}+\mathrm{Id} \otimes \sqrt{-1} \mathscr{D}+R_{\varepsilon} \otimes \sqrt{-1} \mathscr{A}_{\mathscr{P}_{j}} \\
\partial^{L} \otimes \mathrm{Id}-\mathrm{Id} \otimes \sqrt{-1} \mathscr{D}-R_{\varepsilon} \otimes \sqrt{-1} \mathscr{A}_{\mathscr{P}_{j}} & 0
\end{array}\right) .
$$

Next we observe that $L_{b, S \Lambda}^{2}\left(\mathbb{R} / 2 \pi \mathbb{Z} \times \partial M ; \mathscr{L} \otimes \mathbb{C} \mathscr{V}_{f} \otimes E_{\mid \partial M}\right)$ is isomorphic to the field of continuous $\Lambda$-Hilbert modules over $\mathbb{R} / \pi \mathbb{Z}$ given by $V_{0}(t) \oplus V(t)$ where

$$
\begin{gathered}
V_{0}(t)=\mathbb{C} e_{0}(t, .) \otimes L_{\Lambda}^{2}\left(\partial M ; \mathscr{V}_{f} \otimes E_{\mid \partial M}\right), \\
V(t)=\underset{n \in \mathbb{Z}\{0\}}{\oplus} \mathbb{C} e_{n}(t, .) \otimes L_{\Lambda}^{2}\left(\partial M ; \mathscr{V}_{f} \otimes E_{\mid \partial M}\right),
\end{gathered}
$$

where we have written $\mathscr{V}_{f}$ instead of $\left(\mathscr{V}_{f}\right)_{\mid \partial M}$.
Now we consider the two following families of operators:

$$
\begin{align*}
& \left(\mathrm{Id}+\left(\mathscr{D}_{0}+\mathscr{A}_{\mathscr{P}_{j}}^{0}\right)^{2}\right)^{1 / 2} \tau \cos t+\left(\mathscr{D}_{0}+\mathscr{A}_{\mathscr{P}_{j}}^{0}\right) \sin t  \tag{26}\\
& \left(\operatorname{Id}+\left(\mathscr{D}_{0}+\psi_{\varepsilon}\left(t, \frac{(t-1 / 2)}{\pi}\right) \mathscr{A}_{\mathscr{P}_{j}}^{0}\right)^{2}\right)^{1 / 2} \tau \cos t \\
& \quad+\left(\mathscr{D}_{0}+\psi_{\varepsilon}\left(t, \frac{(t-1 / 2)}{\pi}\right) \mathscr{A}_{\mathscr{P}_{j}}^{0}\right) \sin t \tag{27}
\end{align*}
$$

where $j \in\{1,2\}$. Observe that for each $t \in(0, \pi]$, the operators of (26) and (27) are invertible, then by considering the projections onto their positive part, one defines the class $\left[\mathscr{P}_{2}^{\varepsilon}-\mathscr{P}_{1}^{\varepsilon}\right] \in K_{1}(\Lambda)$ exactly as in Definition 8. Proceeding as in [26, pp. 316317], one checks that

$$
\left[\mathscr{P}_{2}^{\varepsilon}-\mathscr{P}_{1}^{\varepsilon}\right]=\operatorname{Ind}\left(\Sigma\left(\mathscr{D}_{\mathscr{P}_{1}}^{\mathrm{APS}}\right)\right)-\operatorname{Ind}\left(\Sigma\left(\mathscr{D}_{\mathscr{P}_{2}}^{\mathrm{APS}}\right)\right) .
$$

Since the operators in (26) and (27) are invertible and coincide for at least one real $t \in] 1 / 2, \pi]$, we have $\left[\mathscr{P}_{2}^{\varepsilon}-\mathscr{P}_{1}^{\varepsilon}\right]=\left[\mathscr{P}_{2}-\mathscr{P}_{1}\right]$.

The theorem is now completely proved.

## 4. A splitting formula for index classes

### 4.1. The even-dimensional case

In this section we shall establish a splitting (or gluing) formula for index classes; this result is stated without proof in [19]. Our first task consists in computing higher APS-index classes for the cylinder in the even-dimensional case.

Let $N$ be a closed manifold of dimension $2 m+1, f: N \rightarrow B \Gamma$ a continuous map and $E \rightarrow N$ be a Clifford hermitian module over $N$ endowed with a unitary Clifford connection. Let $\mathscr{D}_{N}$ be the associated $\Lambda$-Dirac-type operator acting on $L_{A}^{2}\left(N, \mathscr{V}_{f} \otimes E\right)$. We fix a constant map $c:[-1,1] \rightarrow B \Gamma$. Let $\mathscr{D}$ be the associated $\mathbb{Z}_{2}$-graded Dirac-type operator acting on $L_{\Lambda}^{2}\left([-1,1] \times N, \mathscr{V}_{f \times c} \otimes E \otimes \mathbb{C}^{2}\right)$, with boundary operator equal to $\mathscr{D}_{N}$ at $\{-1\} \times N$ and $-\mathscr{D}_{N}$ at $\{1\} \times N$ (notice that the normals to the two components of the boundary are inward pointing at $-1 \times N$ and outward pointing at $\{1\} \times N$. We fix a spectral section $\mathscr{P}$ at $\{-1\} \times N$ for the boundary operator outward pointing at $\{-1\} \times N$ and consider the spectral section Id $-\mathscr{P}$ at $\{1\} \times N$.

## Lemma 7.

$$
\operatorname{Ind}^{\mathrm{APS}}(\mathscr{D}, \mathscr{P}, \operatorname{Id}-\mathscr{P})=0 \in K_{0}(\Lambda)
$$

Proof. Let $\mathscr{A}_{\mathscr{P}}^{0}$ be a self-adjoint $\Lambda$-compact operator such that $\mathscr{D}_{N}+\mathscr{A}_{\mathscr{P}}^{0}$ is invertible and $\mathscr{P}$ is the projection onto to the positive part of $\mathscr{D}_{N}+\mathscr{A}_{\mathscr{P}}^{0}$. We consider the operator ${ }^{\text {cyl }} \mathscr{D}_{\mathscr{P}}$ defined by

$$
\left(\begin{array}{cc}
0 & \mathrm{cyl} \mathscr{D}_{\mathscr{P}}^{-} \\
\mathrm{cyl} \mathscr{D}_{\mathscr{P}}^{+} & 0
\end{array}\right)
$$

where ${ }^{\text {cyl }} \mathscr{D}_{\mathscr{P}}^{+}=\mathrm{Cl}\left(\frac{d x}{1-x^{2}}\right)\left(\left(1-x^{2}\right) \frac{\partial}{\partial x}+\mathscr{D}_{N}+\mathscr{A}_{\mathscr{P}}^{0}\right)$ and ${ }^{\text {cyl }} \mathscr{D}_{\mathscr{P}}^{-}$is the adjoint of ${ }^{\text {cyl }} \mathscr{D}_{\mathscr{P}}^{+}$. The boundary operator of ${ }^{\text {cyl }} \mathscr{D}_{\mathscr{P}}^{+}$at $\{ \pm 1\} \times N$ is $\mp\left(\mathscr{D}_{N}+\mathscr{A}_{\mathscr{P}}^{0}\right)$. Then, proceeding as in Section 10 of [12], one checks that ${ }^{\text {cyl }} \mathscr{D}_{\mathscr{P}}$ is $\Lambda$-Fredholm from $H_{b, 4}^{1}([-1,1] \times$ $\left.N, \mathscr{V}_{f \times c} \otimes E \otimes \mathbb{C}^{2}\right)$ to $L_{b, A}^{2}\left([-1,1] \times N, \mathscr{V}_{f \times c} \otimes E \otimes \mathbb{C}^{2}\right)$ and that $\operatorname{Ind}^{\mathrm{APS}}(\mathscr{D}, \mathscr{P}, \mathrm{Id}-$ $\mathscr{P})=\operatorname{Ind}\left({ }^{\mathrm{cyl}} \mathscr{D}_{\mathscr{P}}^{+}\right)$. But using Mellin transform on the cylinder one sees that ${ }^{\mathrm{cyl}} \mathscr{D}_{\mathscr{P}}^{2}$ is invertible, because $\xi^{2}+\left(\mathscr{D}_{N}+\mathscr{A}_{\mathscr{P}}^{0}\right)^{2}$ is invertible on $L^{2}(\mathbb{R}, d \xi) \otimes L_{A}^{2}\left(N, \mathscr{V}_{f} \otimes E\right)$. So the index class is zero, which proves the lemma.

Lemma 8. We use the notations of the previous lemma. Let $\mathscr{P}$ and $\mathscr{2}$ be two spectral section for $\mathscr{D}_{N}$ at $\{-1\} \times N$. Then

$$
\operatorname{Ind}^{\mathrm{APS}}(\mathscr{D}, \mathscr{P}, \operatorname{Id}-\mathscr{Q})=[\mathscr{Q}-\mathscr{P}] \in K_{0}(\Lambda) .
$$

Proof. Theorem 6 implies that

$$
\operatorname{Ind}^{\mathrm{APS}}(\mathscr{D}, \mathscr{P}, \operatorname{Id}-\mathscr{Q})-\operatorname{Ind}^{\mathrm{APS}}(\mathscr{D}, \mathscr{P}, \operatorname{Id}-\mathscr{P})=[\mathscr{Q}-\mathscr{P}] .
$$

Since Lemma 7 implies that $\operatorname{Ind}^{\text {APS }}(\mathscr{D}, \mathscr{P}$, Id $-\mathscr{P})=0$ one gets immediately the lemma.

Now we are going to prove the gluing formula for index classes; before stating it, we introduce suitable notations. Let $(M, g)$ be a smooth closed riemannian manifold dimension $n=2 m, E=E^{+} \oplus E^{-}$a $\mathbb{Z}_{2}$-graded hermitian Clifford module endowed with a unitary Clifford connection $\nabla$. We consider also a finitely generated group $\Gamma$, a classifying map $f: M \rightarrow B \Gamma$ and denote by $\Lambda$ the reduced $C^{*}$-algebra of $\Gamma$. Let $\widetilde{M} \rightarrow M$ be the associated $\Gamma$-normal cover. We consider the following flat bundle over $M$ :

$$
\mathscr{V}_{f}=\Lambda \times_{\Gamma} \tilde{M}
$$

These data define a $\mathbb{Z}_{2}$-graded $\Lambda$-linear Dirac-type operator $\mathscr{D}$ acting on $L_{\Lambda}^{2}\left(M, \mathscr{V}_{f} \otimes E\right)$. This operator has a well-defined index class $\operatorname{Ind}\left(\mathscr{D}^{+}\right) \in K_{0}(\Lambda)$. Let $F$ be a closed cutting hypersurface of $M$ such that $M=M_{+} \cup M_{-}$where $M_{ \pm}$are two manifolds whose (common) boundary is $F$. We assume that all these data have a product structure near $F$. Let $\mathscr{P}$ and $\mathscr{Q}$ be two spectral section for the boundary operator of the operator induced by the restriction $\mathscr{D}_{\mid M_{+}}$of $\mathscr{D}$ to $M_{+}$; observe that Id $-\mathscr{2}$ is a spectral section for the boundary operator of $\mathscr{D}_{\mid M_{-}}$.

Theorem 8. $\operatorname{Ind}\left(\mathscr{D}^{+}\right)=\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{\mid M_{+}}^{+}, \mathscr{P}\right)+\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{\mid M_{-}}^{+}, \operatorname{Id}-\mathscr{Q}\right)+[\mathscr{P}-\mathscr{Q}]$.
Proof. One just has to adapt to the noncommutative setting (and the evendimensional case) the proof of Theorem 3.1 of [7]. Thus following an idea of Bunke [4], one considers a unitary map $W=a+b+c-d$ from

$$
L_{\Lambda}^{2}\left(M, \mathscr{V}_{f} \otimes E\right) \oplus L_{A}^{2}\left([-1,1] \times F,\left(\mathscr{V}_{f} \otimes E\right)_{[-1,1] \times F}\right)
$$

to

$$
\begin{aligned}
& L_{\Lambda}^{2}\left(M_{+} \cup[0,1] \times F,\left(\mathscr{V}_{f} \otimes E\right)_{M_{+} \cup[0,1] \times F}\right) \oplus L_{\Lambda}^{2}\left(M_{-} \cup[-1,0]\right. \\
& \left.\quad \times F,\left(\mathscr{V}_{f} \otimes E\right)_{M_{-} \cup[-1,0] \times F}\right),
\end{aligned}
$$

where $a, b, c$, and $d$ are defined as in [7, p. 315]. Then one finishes the proof as in [7] by using Lemma 8 instead of Theorem 3.3 of [7]. We omit the (easy) details.

Remark. In this even-dimensional case, but for more general boundary value problems, Hilsum has also established the above gluing result, with a different proof.

See [9, Theorem 9.2]. Notice that our formula

$$
\operatorname{Ind}\left(\mathscr{D}^{+}\right)=\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{\mid M_{+}}^{+}, \mathscr{P}\right)+\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{\mid M_{-}}^{+}, \operatorname{Id}-\mathscr{P}\right)
$$

is a direct consequence of the APS-theory on Galois coverings developed in [16,31] and of the ideas of Bunke (and Dai-Zhang). This much has been known to us since the appearance of [7] and of [15], but the lack of geometric applications of such a formula has held us from publishing the result until now; the applications we shall give, in Section 6, to the cut-and-paste invariance of higher signatures seem to be a good enough motivation to finally write down the (easy) proof.

### 4.2. The odd-dimensional case

Our next task consists in computing higher APS-index classes for the cylinder in the odd-dimensional case. Let $N$ be a closed manifold of dimension $2 m, f: N \rightarrow B \Gamma$ a continuous map and $E \rightarrow N$ be a $\mathbb{Z}_{2}$-graded Clifford hermitian module over $N$ endowed with a unitary Clifford connection. Let $\mathscr{D}_{N}$ be the associated $\Lambda$-Dirac-type operator acting on $L_{\Lambda}^{2}\left(N, \mathscr{V}_{f} \otimes E\right)$. We fix a constant map $c:[-1,1] \rightarrow B \Gamma$. Let $\mathscr{D}$ be the associated $\mathbb{Z}_{2}$-graded Dirac-type operator acting on $L_{\Lambda}^{2}\left([-1,1] \times N, \mathscr{V}_{f \times c} \otimes\right.$ $E \otimes \mathbb{C}^{2}$ ), with boundary operator equal to $\pm \mathscr{D}_{N}$ at $\{\mp 1\} \times N$. We fix a $\mathrm{Cl}(1)$ spectral section $\mathscr{P}$ at $\{-1\} \times N$ and consider the $\mathrm{Cl}(1)$-spectral section Id $-\mathscr{P}$ at $\{1\} \times N$.

Notation. In order to simplify the notation we denote by $\operatorname{Ind}^{\mathrm{APS}}(\mathscr{D}, \mathscr{P})$ the $K_{1}$-index class appearing in the statement of Theorem 7.

The proof of the following lemma is basically the same as the one of Lemma 7 and will be omitted.

## Lemma 9.

$$
\operatorname{Ind}^{\mathrm{APS}}(\mathscr{D}, \mathscr{P}, \operatorname{Id}-\mathscr{P})=0 \in K_{0}\left(C_{c}^{0}(] 0, \pi[) \otimes \Lambda\right) \simeq K_{1}(\Lambda)
$$

Lemma 10. We use the notations of the previous lemma. Let $\mathscr{P}$ and $\mathscr{2}$ be two $\mathrm{Cl}(1)-$ spectral section for $\mathscr{D}_{N}$ at $\{-1\} \times N$. Then

$$
\operatorname{Ind}^{\mathrm{APS}}(\mathscr{D}, \mathscr{P}, \operatorname{Id}-\mathscr{2})=[\mathscr{Q}-\mathscr{P}] \in K_{1}(\Lambda)
$$

Proof. Theorem 7.3 implies that

$$
\operatorname{Ind}^{\mathrm{APS}}(\mathscr{D}, \mathscr{P}, \operatorname{Id}-\mathscr{Q})-\operatorname{Ind}^{\mathrm{APS}}(\mathscr{D}, \mathscr{P}, \operatorname{Id}-\mathscr{P})=[\mathscr{Q}-\mathscr{P}] .
$$

Since Lemma 9 implies that $\operatorname{Ind}^{\text {APS }}(\mathscr{D}, \mathrm{Id}-\mathscr{P}, \mathscr{P})=0$ one gets immediately the lemma.

Let $(M, g)$ be a smooth closed riemannian manifold dimension $n=2 m+1, E$ a hermitian Clifford module endowed with a unitary Clifford connection $\nabla$. As in the even case we consider a finitely generated group $\Gamma$, a classifying map $f: M \rightarrow B \Gamma$ and denote by $\Lambda$ the reduced $C^{*}$-algebra of $\Gamma$ and by

$$
\mathscr{V}_{f}=\Lambda \times_{\Gamma} \widetilde{M}
$$

the associated flat bundle. These data define a $\Lambda$-linear unbounded self-adjoint Dirac-type operator $\mathscr{D}$ acting on $L_{\Lambda}^{2}\left(M,\left(\mathscr{V}_{f} \otimes E\right)\right)$. This operator has a well-defined index class $\operatorname{Ind}(\mathscr{D}) \in K_{1}(\Lambda)$ which is defined by suspension. Let $F$ be a closed cutting hypersurface of $M$ such that $M=M_{+} \cup M_{-}$where $M_{ \pm}$are two manifolds whose $F$ has (common) boundary is $F$. We assume that all these data have a product structure near $F$. Let $\mathscr{P}$ and $\mathscr{2}$ be two $\mathrm{Cl}(1)$-spectral section for the boundary operator of the operator induced by the restriction $\mathscr{D}_{\mid M_{+}}$of $\mathscr{D}$ to $M_{+}$. Then we have

Theorem 9. Ind $(\mathscr{D})=\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{\mid M_{+}}, \mathscr{P}\right)+\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{\mid M_{-}}, \operatorname{Id}-\mathscr{2}\right)+[\mathscr{P}-\mathscr{2}]$.
Proof. Once again the proof of Theorem 3.1 of [7] (which follows an idea of Bunke [4]) extends without problems; we simply have to use Lemma 10 instead of Theorem 3.3 of [7]. We omit the details.

## 5. Index classes and the noncommutative spectral flow

### 5.1. A variational formula for APS-index classes

Let $X$ be an even-dimensional riemannian manifold with boundary and let $E$ be a $\mathbb{Z}_{2}$-graded complex vector bundle over $X$. We assume that there exists a 1-parameter family of riemannian metrics $\left\{g_{u}\right\}_{u \in[1,2]}$ on $X$ and a 1-parameter family of hermitian metrics $\left\{h_{u}^{E}\right\}_{u \in[1,2]}$ on $E$ so that $\left(E, h_{u}\right)$ is a unitary $\mathbb{Z}_{2}$-graded Clifford module for $\left(X, g_{u}\right)$. We also assume that there is a 1-parameter family of connections $\nabla^{E, u}$ that are unitary with respect to $h_{u}^{E}$ and Clifford with respect to the Levi-Civita connection associated to $g_{u}$. We assume that these data depend continuously on $u \in[1,2]$. Let $D_{u}$ be the Dirac operator associated to $\left(g_{u}, h_{u}^{E}, \nabla^{E, u}\right)$. Let $f_{u}: X \rightarrow B \Gamma, u \in[1,2]$ be a 1-parameter family of continuous maps. Let $\left\{\mathscr{D}_{u}\right\}_{u \in[1,2]}$, $\mathscr{D}_{u}: \mathscr{C}^{\infty}\left(X, E \otimes \mathscr{V}_{u}\right) \rightarrow \mathscr{C}^{\infty}\left(X, E \otimes \mathscr{V}_{u}\right)$, be the resulting family of $C_{r}^{*}(\Gamma)$-linear operators.

Proposition 8. Let us denote by $\left\{\left(\mathscr{D}_{u}\right)_{0}\right\}$ the family of boundary operators associated to $\left\{\mathscr{D}_{u}, u \in[1,2]\right\}$. We fix noncommutative spectral section $\mathscr{P}_{1}, \mathscr{P}_{2}$ for $\left(\mathscr{D}_{1}\right)_{0}$
and $\left(\mathscr{D}_{2}\right)_{0}$, respectively. Then

$$
\begin{equation*}
\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{2}^{+}, \mathscr{P}_{2}\right)-\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{1}^{+}, \mathscr{P}_{1}\right)=\operatorname{sf}\left(\left\{\left(\mathscr{D}_{u}\right)_{0}\right\} ; \mathscr{P}_{2}, \mathscr{P}_{1}\right) \quad \text { in } K_{0}\left(C_{r}^{*}(\Gamma)\right) . \tag{28}
\end{equation*}
$$

Proof. We are going to extend to the noncommutative setting the proof of Theorem 5.2 of [8]. Let $(\mathscr{2}(u))_{u \in[1,2]}$ be a total spectral section for the family $\left\{\left(\mathscr{D}_{u}\right)_{0}, u \in[1,2]\right\}$. Then we get a continuous family of elliptic boundary value problems ( $\left.\mathscr{D}_{u}, \mathscr{Q}(u)\right)$, $1 \leqslant u \leqslant 2$. By the homotopy invariance of the $K$-theoretic index class we get

$$
\operatorname{Ind}\left(\mathscr{D}_{1}^{+}, \mathscr{Q}(1)\right)=\operatorname{Ind}\left(\left(\mathscr{D}_{2}^{+}, \mathscr{2}(2)\right) .\right.
$$

By the relative index Theorem 6 we get

$$
\operatorname{Ind}\left(\mathscr{D}_{1}^{+}, \mathscr{2}(1)\right)=\operatorname{Ind}\left(\mathscr{D}_{1}^{+}, \mathscr{P}_{1}\right)+\left[\mathscr{P}_{1}-\mathscr{2}(1)\right]
$$

and

$$
\operatorname{Ind}\left(\mathscr{D}_{2}^{+}, \mathscr{Q}(2)\right)=\operatorname{Ind}\left(\mathscr{D}_{2}^{+}, \mathscr{P}_{2}\right)+\left[\mathscr{P}_{2}-\mathscr{2}(2)\right] .
$$

The proposition is then an immediate consequence of the definition of the higher spectral flow and of the three previous equalities.

Remark. 1. A similar result holds for any 1-parameter family of Dirac-type operators, not necessarily defined by a variation of the geometric data.
2. One can immediately extend the previous proposition to the odd-dimensional case; in such a case we need to fix $\mathrm{Cl}(1)$-spectral section.

Notation. In order to treat simultaneously the odd- and even-dimensional case, from now on we shall not write the superscript + in the even-dimensional case (see formula (28)); thus both in the even- and in odd-dimensional case we shall simply write $\operatorname{Ind}^{\text {APS }}(\mathscr{D}, \mathscr{P})$ for the index class associated to $\mathscr{D}$ and the choice of a spectral section for the boundary operator. We shall implicitly choose $\mathrm{Cl}(1)$-spectral section in the odd-dimensional case.

### 5.2. Index classes on the cylinder and the noncommutative spectral flow

It is well known that for a smooth 1-parameter family $\{D(u)\}_{u \in[0,1]}$ of Dirac operators on a closed odd-dimensional manifold $N$, there is a formula equating the spectral flow of $\{D(u)\}_{u \in[0,1]}$ to the index of the operator $\partial / \partial u+D(u)$ on the cylinder $[0,1] \times N$ with APS-boundary conditions. In this section we shall extend this result to the higher case. Thus let $\{\mathscr{D}(u)\}_{u \in[0,1]}$ be a smooth family of $\Lambda$-linear Dirac-type operators as in Section 2.5; without loss of generality we may assume that this family is constant near $u=0$ and 1 . We first treat the case in which $N$ is odd dimensional. We fix reference spectral section $\mathscr{2}_{0}$ and $\mathscr{Q}_{1}$ for $\mathscr{D}(0)$ and $\mathscr{D}(1)$
respectively; we can consider the noncommutative spectral flow $\operatorname{sf}(\{D(u)\}$; $\left.\mathscr{Q}_{0}, \mathscr{Q}_{1}\right) \in K_{0}(\Lambda)$ and the index class, associated to the cylinder $[0,1] \times N$,

$$
\operatorname{Ind}^{\operatorname{APS}}\left(\frac{\partial}{\partial u}+\mathscr{D}(u) ; \mathscr{Q}_{0}, \operatorname{Id}-\mathscr{Q}_{1}\right) \in K_{0}(\Lambda)
$$

Theorem 10. The following formula holds in $K_{0}(\Lambda)$ :

$$
\begin{equation*}
\operatorname{Ind}^{\mathrm{APS}}\left(\frac{\partial}{\partial u}+\mathscr{D}(u) ; \mathscr{D}_{0}, \operatorname{Id}-\mathscr{Q}_{1}\right)=\operatorname{sf}\left(\{D(u)\} ; \mathscr{Q}_{0}, \mathscr{Q}_{1}\right) . \tag{29}
\end{equation*}
$$

Proof. Consider a total spectral section $\left(\mathscr{P}_{u}\right)_{u \in[0,1]}$ for our family $\{\mathscr{D}(u)\}_{u \in[0,1]}$. We attach a cylinder $[-1,0] \times N$ on the left of our cylinder $[0,1] \times N$; we also attach a cylinder $[1,2] \times N$ on the right and consider the differential operator $\partial / \partial u+\widehat{\mathscr{D}}(u)$ on $[-1,2] \times N$ with $\widehat{\mathscr{D}}(u)=\mathscr{D}(0)$ for $u \in[-1,0], \widehat{\mathscr{D}}(u)=\mathscr{D}(u)$ for $u \in[0,1]$ and $\widehat{\mathscr{D}}(u)=$ $\mathscr{D}(1)$ for $u \in[1,2]$. Consider the operator $\partial / \partial u+\widehat{\mathscr{D}}(u)$ with APS-boundary condition $\mathscr{2}_{0}$ at $u=-1$ and APS-boundary condition Id $-\mathscr{Q}_{1}$ at $u=2$. From the gluing formula of Theorem 8 we know that

$$
\begin{aligned}
& \operatorname{Ind}^{\mathrm{APS}}\left(\partial / \partial u+\widehat{\mathscr{D}}(u) ; \mathscr{Q}_{0}, \mathrm{Id}-\mathscr{2}_{1}\right) \\
& \quad=\operatorname{Ind}^{\mathrm{APS}}\left(\partial / \partial u+\mathscr{D}(0) ; \mathscr{Q}_{0}, \operatorname{Id}-\mathscr{P}_{0}\right)+\operatorname{Ind}^{\mathrm{APS}}\left(\partial / \partial u+\mathscr{D}(u) ; \mathscr{P}_{0}, \operatorname{Id}-\mathscr{P}_{1}\right) \\
& \quad+\operatorname{Ind}^{\mathrm{APS}}\left(\partial / \partial u+\mathscr{D}(1) ; \mathscr{P}_{1}, \operatorname{Id}-\mathscr{Q}_{1}\right) ;
\end{aligned}
$$

for the first and third summand on the right-hand side we can use Lemma 8, obtaining

$$
\begin{aligned}
& \operatorname{Ind}^{\mathrm{APS}}\left(\partial / \partial u+\widehat{\mathscr{D}}(u) ; \mathscr{Q}_{0}, \mathrm{Id}-\mathscr{2}_{1}\right) \\
& \quad=\left[\mathscr{P}_{0}-\mathscr{Q}_{0}\right]+\operatorname{Ind}^{\mathrm{APS}}\left(\partial / \partial u+\mathscr{D}(u) ; \mathscr{P}_{0}, \mathrm{Id}-\mathscr{P}_{1}\right)+\left[\mathscr{\mathscr { V }}_{1}-\mathscr{P}_{1}\right] .
\end{aligned}
$$

Since $\operatorname{sf}\left(\{D(u)\} ; \mathscr{Q}_{0}, \mathscr{Q}_{1}\right)=\left[\mathscr{Q}_{1}-\mathscr{P}_{1}\right]-\left[\mathscr{Q}_{0}-\mathscr{P}_{0}\right]$, we are left with the task of proving that $\operatorname{Ind}^{\mathrm{APS}}\left(\partial / \partial u+\mathscr{D}(u) ; \mathscr{P}_{0}, \mathrm{Id}-\mathscr{P}_{1}\right)=0$. By the equality of the APS-index and of the $b$-index (see Section 3.1) it suffices to show that

$$
\operatorname{Ind}_{b}\left(u(1-u) \partial / \partial u+\mathscr{D}(u) ; \mathscr{P}_{0}, \operatorname{Id}-\mathscr{P}_{1}\right)=0 .
$$

We know that the latter index class is associated to a perturbation

$$
u(1-u) \partial / \partial u+\mathscr{D}(u)+\mathscr{A}_{\mathscr{P}_{0}}^{\text {left }}+\mathscr{A}_{\mathscr{P}_{1}}^{\text {right }} \in \Psi_{b, A}^{1} .
$$

We also know that $\operatorname{Ind}^{\mathrm{APS}}\left(\partial / \partial u+\mathscr{D}(0) ; \mathscr{P}_{0}, \operatorname{Id}-\mathscr{P}_{0}\right)=0$. Thus

$$
\operatorname{Ind}_{b}\left(u(1-u) \partial / \partial u+\mathscr{D}(0) ; \mathscr{P}_{0}, \operatorname{Id}-\mathscr{P}_{0}\right)=0
$$

which means that the operator $u(1-u) \partial / \partial u+\mathscr{D}(0)+\mathscr{A}_{\mathscr{P}_{0}}^{\text {left }}+\mathscr{A}_{\mathscr{P}_{0}}^{\text {right }}$ has zero index. We only need to show that the $b$-pseudo-differential operators

$$
u(1-u) \partial / \partial u+\mathscr{D}(u)+\mathscr{A}_{\mathscr{P}_{0}}^{\text {left }}+\mathscr{A}_{\mathscr{P}_{1}}^{\text {right }} \quad u(1-u) \partial / \partial u+\mathscr{D}(0)+\mathscr{A}_{\mathscr{P}_{0}}^{\text {left }}+A_{\mathscr{P}_{0}}^{\text {right }}
$$

are homotopic through $\Lambda$-Fredholm operators. Consider the 1 -parameter family of $b$-pseudo-differential operators

$$
B(s)=u(1-u) \partial / \partial u+\mathscr{D}(s u)+\mathscr{A}_{\mathscr{P}_{0}}^{\text {left }}+\mathscr{A}_{\mathscr{P}_{s}}^{\text {right }} s \in[0,1],
$$

where we recall that $\left(\mathscr{P}_{s}\right)_{s \in[0,1]}$ is the chosen total spectral section. Then $B(s)$ is a 1-parameter continuous family of $b$-pseudo-differential operators connecting
$u(1-u) \partial / \partial u+\mathscr{D}(0)+\mathscr{A}_{\mathscr{P}_{0}}^{\text {left }}+\mathscr{A}_{\mathscr{P}_{0}}^{\text {right }} \quad$ and $\quad u(1-u) \partial / \partial u+\mathscr{D}(u)+\mathscr{A}_{\mathscr{P}_{0}}^{\text {left }}+\mathscr{A}_{\mathscr{P}_{1}}^{\text {right }}$.
Each $B(s)$ is clearly $b$-elliptic (i.e. the $b$-principal symbol is invertible outside the 0 section of the compressed cotangent bundle). We only need to show that the indicial family $I(B(s), \lambda)$ is invertible for each $s \in[0,1]$ and for each $\lambda \in \mathbb{R}$. The indicial family of $B(s)$ on the left boundary is fixed and invertible, being equal to $(\mathscr{D}(0))_{0}+i \lambda+$ $\widehat{\rho}_{\varepsilon}(\lambda) \mathscr{A}_{\mathscr{P}_{0}}^{0}$. Thus we concentrate on the right boundary; there the indicial family is equal to

$$
i \lambda-(\mathscr{D}(s))_{0}-\widehat{\rho}_{\varepsilon}(\lambda) \mathscr{A}_{\mathscr{P}}^{0} .
$$

This is invertible (for a fixed $s$ ) for every $\lambda \in \mathbb{R}$ if and only if it is invertible in $\lambda=0$; but this is clear, as $\widehat{\rho}_{\varepsilon}(0)=1$ and $\mathscr{A}_{\mathscr{P}_{s}}^{0}$ is, by construction, the perturbation that makes $(\mathscr{D}(s))_{0}$ invertible. Summarizing $B(s)$ is a family of (symbolically) elliptic $b$ -pseudo-differential operators with invertible indicial family $I(B(s), \lambda), \lambda \in \mathbb{R}$, for each $s \in[0,1]$. By the $b-\Lambda$-calculus these operators are $\Lambda$-Fredholm acting on $H_{b, \Lambda}^{1}$. The theorem is proved.

Remark. A similar result holds for even-dimensional closed manifolds $N$.

## 6. Cut and paste invariance for higher signatures

### 6.1. A defect formula

Let $\left(M_{i}, r_{i}\right)(1 \leqslant i \leqslant 2)$ be a closed oriented smooth manifold endowed with a continuous map $r_{i}: M_{i} \rightarrow B \Gamma$.

Definition 10 (Karras et al. [11]). We say that $\left(M_{1}, r_{1}\right)$ and ( $M_{2}, r_{2}$ ) are SKequivalent if there exist decompositions

$$
M_{1}=M_{+} \cup_{\left(F, \phi_{1}\right)} M_{-}, \quad M_{2}=M_{+} \cup_{\left(F, \phi_{2}\right)} M_{-}
$$

with

$$
\partial M_{+}=F=\partial M_{-}, \quad \phi_{i} \in \operatorname{Diff}^{+}(F)
$$

and such that $\left(r_{1}\right)_{\mid M_{ \pm}} \simeq\left(r_{2}\right)_{\mid M_{ \pm}}(\simeq$ meaning homotopy equivalence $)$.
Notice that $F$ is, by assumption, an embedded hypersurface in $M_{1}$ and in $M_{2}$ which is endowed with the orientation induced by the one of $M_{+}$and the inward normal vector to $\partial M_{+}$. The orientation-preserving diffeomorphism $\phi$ is thought as going from $\partial M_{+}$to $\partial M_{-}$; by construction $\left(r_{i}\right)_{\mid \partial M_{+}}=\left(r_{i}\right)_{\mid \partial M_{-}}{ }^{\circ} \phi$. Notice finally that the orientations of $T M_{+}$and $T M_{-}$are opposite over the boundary $F$. For simplicity we assume that $F$ is two-sided (i.e. the normal bundle to $F$ is trivial).

Let $\left(M_{i}, r_{i}\right)(1 \leqslant i \leqslant 2)$ be as in Definition 10 . We denote by $\widetilde{M}_{i}$ the $\Gamma$-coverings defined by $r_{i}$. Thus $\widetilde{M}_{i}:=\left(r_{i}\right)^{*} E \Gamma$. We also let $\tilde{F}_{i} \rightarrow F$ be the covering associated with $\left(r_{i}\right)_{\mid F}: F \rightarrow B \Gamma$. We consider the flat bundles $\mathscr{V}_{i}=\widetilde{M}_{i} \times{ }_{\Gamma} \Lambda$ over $M_{i}$. We now introduce riemannian metrics $g_{i}$ on $M_{i}$ and endow the coverings $\widetilde{M}_{i}$ with the lifted metrics. We denote by $\mathscr{D}_{M_{i}}^{\text {sign, } r_{i}}$ the associated signature operator on $M_{i}$ with coefficients in $\mathscr{V}_{i}$. The restriction of this operator to $M_{ \pm}$will be denoted by $\mathscr{D}_{M_{ \pm}}^{\text {sign, } r_{i}}$; we use the precise notation $\mathscr{D}_{\partial M_{ \pm}}^{\text {sign, } r_{i}}$ for the associated boundary operators.

Our goal in this section is to express the difference

$$
\text { Ind } \mathscr{D}_{M_{2}}^{\text {sign, } r_{2}}-\operatorname{Ind} \mathscr{D}_{M_{1}}^{\text {sign, } r_{1}} \in K_{*}\left(C_{r}^{*}(\Gamma)\right)
$$

in terms of a noncommutative spectral flow, on $F$, naturally associated to the above geometric data.

To do so, we need an extension of the glueing formula proved in Sections 4.1 and 4.2. Let more generally $M=M_{+} \cup_{\phi} M_{-}$with $\partial M_{+}=F_{+}, \partial M_{-}=F_{-}$and $\phi$ : $F_{+} \rightarrow F_{-}$a diffeomorphism; let us fix a metric $g$ on $M$ and a map $r: M \rightarrow B \Gamma$. Giving such a metric $g$ is equivalent to giving a metric $g(+)$ on $M_{+}$and a metric $g(-)$ on $M_{-}$ such that $\phi^{*}\left(\left.g(-)\right|_{F_{-}}\right)=\left.g(+)\right|_{F_{+}}$. Similarly, the map $r$ defines by restriction maps $r_{ \pm}: M_{ \pm} \rightarrow B \Gamma$ such that $\left.\left(r_{+}\right)\right|_{F_{+}}=\left.\left(r_{-}\right)\right|_{F_{-}} \circ \phi$. For the sake of brevity we denote $\left.\left(r_{+}\right)\right|_{F_{+}}=r_{+}^{\partial}$ and $\left.\left(r_{-}\right)\right|_{F_{-}}=r_{-}^{\partial}$; we also write $\mathscr{V}_{+}^{\partial}$ for the flat bundle defined over $F_{+}$ by $r_{+}^{\partial}$ and we write $\mathscr{V}_{-}^{\partial}$ for the analogous bundle over $F_{-}$. The pull-back $\phi^{*}$ defines an isometry between the $L^{2}$-Hilbert module $L_{\Lambda}^{2}\left(F_{-}, \Lambda^{*} F_{-} \otimes \mathscr{V}_{-}^{2}\right)$ defined by $\left.g(-)\right|_{F_{-}}$ and the $L^{2}$-Hilbert module $L_{\Lambda}^{2}\left(F_{+}, \Lambda^{*} F_{+} \otimes \mathscr{V}_{+}^{\partial}\right)$ defined by $\left.g(+)\right|_{F_{+}}$. One checks easily that with our conventions

$$
\begin{equation*}
\mathscr{D}_{F_{+}}^{\text {sign, }\left(r_{+}^{\mathscr{\theta}}\right)}=-\phi^{*}\left(\mathscr{D}_{F_{-}}^{\text {sign,( }\left(r_{-}^{P}\right)}\right)\left(\phi^{*}\right)^{-1} . \tag{30}
\end{equation*}
$$

Let $\mathscr{P}$ be a spectral section for $\mathscr{D}_{F_{+}}^{\text {sign, }\left(r_{+}^{(r)}\right.}$ and consider the projection

$$
\mathscr{P}^{\phi}:=\left(\phi^{*}\right)^{-1} \mathscr{P} \phi^{*} ;
$$

then it is clear from (30) that the projection $\mathrm{Id}-\left(\phi^{*}\right)^{-1} \mathscr{P} \phi^{*}$ is a spectral section for $\mathscr{D}_{F_{-}}^{\text {sign, }\left(r_{-}^{\theta}\right)}$.

The gluing formula of Section 4 becomes in this more general setting:

$$
\begin{equation*}
\operatorname{Ind} \mathscr{D}_{M}^{\text {sign,r}}=\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{M_{+}}^{\text {sign, } r_{+}}, \mathscr{P}\right)+\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{M_{-}}^{\text {sign, } r_{-}}, 1-\mathscr{P}^{\phi}\right) . \tag{31}
\end{equation*}
$$

The proof of formula (31) proceeds exactly as the proof of the gluing formula in Section 4 (see Theorems 8, 9) once we use the following lemma. First we fix notations. Let $\mathrm{Cyl}_{\phi}:=\left([-1,0] \times F_{+}\right) \cup_{\phi}\left([0,1] \times F_{-}\right)$so that $\partial \mathrm{Cyl}_{\phi}=F_{+} \sqcup F_{-}$. Let $g_{\phi}$ any metric on $\mathrm{Cyl}_{\phi}$ such that

$$
\left.\left(g_{\phi}\right)\right|_{\partial \mathrm{Cy}_{\phi}}=\phi^{*} h \sqcup h
$$

with $h$ a metric on $F_{-}$. Let $c:[-1,1] \rightarrow B \Gamma$ be a constant map and let $r_{\phi}: \mathrm{Cyl}_{\phi} \rightarrow B \Gamma$ the map induced by $c \times r_{+}^{\partial}:[-1,0] \times F_{+} \rightarrow B \Gamma$ and $c \times r_{-}^{\partial}:[0,1] \times F_{-} \rightarrow B \Gamma$.

Lemma 11. Let $\mathscr{D}_{\mathrm{Cyl}_{\phi}}$ be the signature operator on $\mathrm{Cyl}_{\phi}$ associated to $g_{\phi}$ and $r_{\phi}$. Then for each spectral section $\mathscr{P}$ as above

$$
\operatorname{Ind}\left(\mathscr{D}_{\mathrm{Cyl}_{\phi}} ; \mathscr{P}, \operatorname{Id}-\mathscr{P}^{\phi}\right)=0 \quad \text { in } K_{*}\left(C_{r}^{*}(\Gamma)\right)
$$

Proof. The diffeomorphism $\phi$ induces in a natural way a diffeomorphism $\Phi$ : $[-1,1] \times F_{+} \rightarrow \mathrm{Cyl}_{\phi}$. On $[-1,1] \times F_{+}$we consider the pull-back metric $\Phi^{*} g_{\phi}$ and the product metric $d t^{2}+\phi^{*} h$. Let $\mathscr{D}_{[-1,1] \times F^{+}}^{\Phi}$ the signature operator on $[-1,1] \times F_{+}$ associated to the metric $\Phi^{*} g_{\phi}$ and to the map $c \times r_{+}^{\partial}$ and let $\mathscr{D}_{[-1,1] \times F^{+}}$be the signature operator associated to the product metric and to the map $c \times r_{+}^{\partial}$. We then have

$$
\begin{aligned}
& \operatorname{Ind}\left(\mathscr{D}_{\mathrm{Cyl}_{\phi}} ; \mathscr{P}, \operatorname{Id}-\mathscr{P}^{\phi}\right)=\operatorname{Ind}\left(\mathscr{D}_{[-1,1] \times F^{+}}^{\Phi} ; \mathscr{P}, \operatorname{Id}-\mathscr{P}\right) \\
& \quad=\operatorname{Ind}\left(\mathscr{D}_{[-1,1] \times F^{+}} ; \mathscr{P}, \operatorname{Id}-\mathscr{P}\right)=0 ;
\end{aligned}
$$

the first equality follows from the fact that the two boundary problems are obtained one from the other by conjugation; the second equality follows from the fact that the two metrics are equal in a neighborhood of the boundary, the third equality follows from Lemmas 7 and 9. The Lemma is proved.

We now go back to our problem of expressing in a suitable way the difference $\operatorname{Ind}\left(\mathscr{D}_{M_{1}}^{\text {sign, } r_{1}}\right)-\operatorname{Ind}\left(\mathscr{D}_{M_{2}}^{\text {sign, } r_{2}}\right)$. We perturb the metrics $g_{1}$ and $g_{2}$ so as to be product-like near $F$. Thanks to the additivity formula (31) we have

$$
\begin{equation*}
\operatorname{Ind}\left(\mathscr{D}_{M_{1}}^{\text {sign }, r_{1}}\right)=\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{M_{+}}^{\text {sign, } r_{1}}, \mathscr{P}_{1}\right)+\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{M_{-}}^{\text {sign }, r_{1}}, 1-\mathscr{P}_{1}^{\phi_{1}}\right) \tag{32}
\end{equation*}
$$

with $\mathscr{P}_{1}$ a noncommutative spectral section for the boundary operator $\mathscr{D}_{\partial M_{+}}^{\text {sign, } r_{1}}$. Similarly,

$$
\begin{equation*}
\operatorname{Ind}\left(\mathscr{D}_{M_{2}}^{\text {sign }, r_{2}}\right)=\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{M_{+}}^{\text {sign, } r_{2}}, \mathscr{P}_{2}\right)+\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{M_{-}}^{\text {sign }, r_{2}}, 1-\mathscr{P}_{2}^{\phi_{2}}\right) \tag{33}
\end{equation*}
$$

with $\mathscr{P}_{2}$ a noncommutative spectral section for $\mathscr{D}_{\partial M_{+}}^{\text {sign, } r_{2}}$. Consider $\left(g_{1}\right)_{\mid M_{+}}$and $\left(g_{2}\right)_{\mid M_{+}}$ and let $g_{+, u}$, with $u \in[1,2]$, be a path of riemannian metrics on $M_{+}$connecting them. We can choose these metrics to be product-like near the boundary of $M_{+}$. By assumption there is a continuous family of classifying maps $r_{+, u}: M_{+} \rightarrow B \Gamma$, $u \in[1,2]$, connecting $\left(r_{1}\right)_{\mid M_{+}}$and $\left(r_{2}\right)_{\mid M_{+}}$. Thus, there is a 1-parameter family of Dirac-type operators on $M_{+},\left\{\mathscr{D}_{M_{+}}^{\text {sign,u}}\right\}_{u \in[1,2]}$ with boundary family $\left\{\mathscr{D}_{\partial M_{+}}^{\text {sign }, u}\right\}_{u \in[1,2]}$. We can thus apply the variational formula in Section 5.1 and obtain

$$
\begin{align*}
& \operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{M_{+}}^{\text {sign, } r_{2}}, \mathscr{P}_{2}\right)-\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{M_{+}}^{\text {sign }, r_{1}}, \mathscr{P}_{1}\right) \\
& \quad=\operatorname{sf}\left(\left\{\mathscr{D}_{\partial M_{+}}^{\text {sign }, u}\right\} ; \mathscr{P}_{2}, \mathscr{P}_{1}\right) \quad \text { in } \quad K_{*}\left(C_{r}^{*}(\Gamma)\right) . \tag{34}
\end{align*}
$$

Notice that it does not make sense to take a path joining $g_{1}$ and $g_{2}$ (indeed, $M_{1}$ and $M_{2}$ are in general different manifolds).

We apply the same reasoning to $M_{-}$. Thus by assumption there is a continuous family of classifying maps $r_{-, u}: M_{-} \rightarrow B \Gamma, u \in[1,2]$, connecting $\left(r_{1}\right)_{\mid M_{-}}$and $\left(r_{2}\right)_{\mid M_{-}}$ and; we also choose a path of metrics $g_{-, u}, u \in[1,2]$, connecting $\left(g_{1}\right)_{\mid M_{-}}$and $\left(g_{2}\right)_{\mid M_{-}}$. We obtain a family of operators $\mathscr{D}_{M_{-}}^{\text {sign,u }}$ connecting $\mathscr{D}_{M_{-}}^{\text {sign, } r_{2}}$ and $\mathscr{D}_{M_{-}}^{\text {sign, } r_{1}}$ together with the formula

$$
\begin{align*}
& \operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{M_{-}}^{\text {sign }, r_{2}}, 1-\mathscr{P}_{2}^{\phi_{2}}\right)-\operatorname{Ind}^{\mathrm{APS}}\left(\mathscr{D}_{M_{-}}^{\text {sign, } r_{1}}, 1-\mathscr{P}_{1}^{\phi_{1}}\right) \\
& \quad=\operatorname{sf}\left(\left\{\mathscr{D}_{\partial M_{-}}^{\text {sign,u}}\right\} ; 1-\mathscr{P}_{2}^{\phi_{2}}, 1-\mathscr{P}_{1}^{\phi_{1}}\right) \tag{35}
\end{align*}
$$

Summarizing, in $K_{*}\left(C_{r}^{*}(\Gamma)\right)$ :

$$
\begin{align*}
& \text { Ind } \mathscr{D}_{M_{2}}^{\text {sign }, r_{2}}-\operatorname{Ind} \mathscr{D}_{M_{1}}^{\text {sign, } r_{1}} \\
& \quad=\operatorname{sf}\left(\left\{\mathscr{D}_{\partial M_{+}}^{\text {sign }, u}\right\} ; \mathscr{P}_{2}, \mathscr{P}_{1}\right)+\operatorname{sf}\left(\left\{\mathscr{D}_{\partial M_{-}}^{\text {sign }, u}\right\} ; 1-\mathscr{P}_{2}^{\phi_{2}}, 1-\mathscr{P}_{1}^{\phi_{1}}\right) . \tag{36}
\end{align*}
$$

Notice that $\mathscr{D}_{\partial M_{+}}^{\text {sign,1 }}$ is conjugated through $\phi_{1}^{*}$ to $-\mathscr{D}_{\partial M_{-}}^{\text {sign,1 }}$ and that $\mathscr{D}_{\partial M_{+}}^{\text {sign,2 }}$ is conjugated through $\phi_{2}^{*}$ to $-\mathscr{D}_{\partial M_{-}}^{\text {sign, } 2}$; on the other hand $\mathscr{D}_{\partial M_{+}}^{\text {sign, }}$ and $\mathscr{D}_{\partial M_{-}}^{\text {sign,u }}$ are in general unrelated for $u \in(1,2)$.

We can put together the family $\mathscr{D}_{\partial M_{+}}^{\text {sign }, u}, u \in[1,2]$ and the family $\mathscr{D}_{\partial M_{-}}^{\text {sign,u }}, u \in[1,2]$; with an harmless and obvious abuse of notation we are thus considering the family

$$
\begin{equation*}
\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}:=\left\{\mathscr{D}_{\partial M_{+}}^{\text {sign }, u}\right\}_{u \in[1,2]} \sqcup\left\{\mathscr{D}_{\partial M_{-}}^{\text {sign }, u}\right\}_{u \in[2,1]} \tag{37}
\end{equation*}
$$

which is a $S^{1}$-family acting on the fibers of the mapping torus $M\left(F, \phi_{2}^{-1} \circ \phi_{1}\right) \rightarrow S^{1}$. Recall that the mapping torus $M\left(F, \phi_{2}^{-1} \circ \phi_{1}\right)$ is obtained from $F \times[1,2] \sqcup F \times[1,2]$ by identifying $\left(\phi_{1}(x), 1\right)$ (in the first $F \times[1,2]$ ) with $(x, 1)$ (in the second $F \times[1,2]$ ) and similarly $\left(\phi_{2}(x), 2\right)$ with $(x, 2)$, for any $x \in F$.

Using (23) we can then write

$$
\begin{equation*}
\operatorname{sf}\left(\left\{\mathscr{D}_{\partial M_{+}}^{\text {sign,u }}\right\} ; \mathscr{P}_{2}, \mathscr{P}_{1}\right)+\operatorname{sf}\left(\left\{\mathscr{D}_{\partial M_{-}}^{\text {sign }, u}\right\} ; 1-\mathscr{P}_{2}^{\phi_{2}}, 1-\mathscr{P}_{1}^{\phi_{1}}\right)=\operatorname{sf}\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right. \tag{38}
\end{equation*}
$$

obtaining, finally, a proof of the following.
Theorem 11. The following formula holds in $K_{*}\left(C_{r}^{*}(\Gamma)\right)$ :

$$
\begin{equation*}
\operatorname{Ind} \mathscr{D}_{M_{2}}^{\text {sign, } r_{2}}-\operatorname{Ind} \mathscr{D}_{M_{1}}^{\text {sign, } r_{1}}=\operatorname{sf}\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right) \tag{39}
\end{equation*}
$$

where $\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}$ is the $S^{1}$-family of twisted signatures operators on $F$ defined by (37).

One important consequence of our discussion so far is the following
Corollary 2. Assume that $\Gamma$ is such that the assembly map of the Baum-Connes conjecture is rationally injective. Let $\left(M_{1}, r_{1}\right),\left(M_{2}, r_{2}\right)$ be two $S K$-equivalent pairs as in Definition 10. If, with the above notations,

$$
\operatorname{sf}\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right)=0 \quad \text { in } K_{*}\left(C_{r}^{*}(\Gamma)\right) \otimes_{\mathbb{Z}} \mathbb{Q}
$$

then the higher signatures of $\left(M_{1}, r_{1}\right)$ and $\left(M_{2}, r_{2}\right)$ are the same.
Proof. It is proven in [13] (just after Theorem 0.5) that, under the stated assumption on $\Gamma$, the equality of the two index classes imply the equality of the higher signatures. The proposition is then a consequence of Theorem 11.

### 6.2. Remark

1. Formula (39) is a higher version of formula (25.22) in [3]. In the numerical case treated there, the numerical spectral flow is also equal to the numerical index of the mapping torus defined by the diffeomorphism $\psi=\left(\phi_{2}\right)^{-1} \circ \phi_{1}$. We shall now discuss the extension of the latter result to the higher context. We consider two SKequivalent pairs $\left(M_{1}, r_{1}\right),\left(M_{2}, r_{2}\right)$ as in Definition 10 . We consider the mapping torus $M(F, \psi)$.

As explained in [11, p. 16], see also [13], the mapping torus $M(F, \psi)$ comes with a map $\hat{r}: M(F, \psi) \rightarrow B \Gamma$ such that

$$
\begin{equation*}
\left(M_{2}, r_{2}\right)-\left(M_{1}, r_{1}\right) \text { is bordant to }(M(F, \psi), \hat{r}) . \tag{40}
\end{equation*}
$$

Let us recall the argument: one considers the space $Y$ obtained from $M_{1} \times$ $[0,3] \sqcup M_{2} \times[0,3]$ and the following identifications: for $x \in F$ identify $(x, t) \in \partial M_{+} \times$ $[0,1]$ with $\left(\phi_{1}(x), t\right) \in \partial M_{\times}[0,1]$ and $(x, t) \in \partial M_{+} \times[2,3]$ with $\left(\phi_{2}(x), t\right) \in \partial M_{-}$ $\times[2,3]$. Then, after smoothing, $\partial Y=M_{1}-M_{2}-M(F, \psi)$. Moreover the two homotopies $\left(r_{j}\right)_{\mid M_{+}} \simeq\left(r_{j}\right)_{\mid M_{-}}$can be used to define a map $R: Y \rightarrow B \Gamma$, with $R_{\mid M_{j}}=r_{j}$; one then sets $\hat{r}:=R_{\mid M(F, \psi)}$ which proves (40). Using the metrics considered in the discussion preceding the statement of Theorem 11 we can endow the mapping torus $M(F, \psi)$ with a riemannian metric; we shall then denote by $\mathscr{D}_{M(F, \psi)}^{\text {sign, },}$ the signature operator with coefficients in the flat bundle associated with $\hat{r}$.

Proposition 9. One has

$$
\begin{equation*}
\text { Ind } \mathscr{D}_{M_{2}}^{\text {sign, } r_{2}}-\operatorname{Ind} \mathscr{D}_{M_{1}}^{\text {sign }, r_{1}}=\operatorname{Ind} \mathscr{D}_{M(F, \psi)}^{\text {sign, }, \hat{r}} . \tag{41}
\end{equation*}
$$

Proof. According to (40) the difference of pairs $\left[M_{1}, r_{1}\right]-\left[M_{2}, r_{2}\right]$ is bordant to $[M(F, \psi), r]$. By the cobordism invariance of the index class (see, for example, [16, Proposition 2.3], one gets immediately the proposition.

From the previous proposition and Theorem 11 we immediately obtain the following higher analog of Proposition 25.1 in [3].

Corollary 3. With the notation introduced above the following formula holds in $K_{*}\left(C_{r}^{*}(\Gamma)\right)$

$$
\begin{equation*}
\text { Ind } \mathscr{D}_{M(F, \psi)}^{\text {signn }}=\operatorname{sf}\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right) \tag{42}
\end{equation*}
$$

2. We have obtained formula (41) by exploiting two different ways for computing the difference Ind $\mathscr{D}_{M_{2}}^{\text {sign, } r_{2}}$ - Ind $\mathscr{D}_{M_{1}}^{\text {sign, } r_{1}}$. In fact, a direct proof of this fact is essentially given in [30] using the intersection product in KK-theory.
3. One can also establish formula (41) using the gluing formula for index classes. In the even-dimensional case, this is the route that is taken in [9]. See Theorem 14.1 there. The cut-and-paste results for higher signatures are then obtained in [9], always in the even-dimensional case, by giving sufficient conditions for the index class of the mapping torus to be zero. Corollary 3 shows why the approach we follow here, directly inspired by [12], and the approach in [9] are compatible.
4. For future applications, we briefly point out that the defect formula (39) is still valid if one replaces the signature operators by Dirac-type operators. In this generality one needs to give the appropriate definitions. Thus, we assume that we are given metrics $g_{1}$ on $M_{1}$ and $g_{2}$ on $M_{2}$ and Clifford bundles $E_{1}$ on $M_{1}$ and $E_{2}$ on $M_{2}$. We let $D_{1}$ and $D_{2}$ be two Dirac-type operators on these bundles. We assume that the two bundles are obtained by a clutching construction involving two Clifford bundles $E_{+} \rightarrow M_{+}, E_{-} \rightarrow M_{-}$and two bundles isomorphisms $\left(E_{+}\right)_{\mid \partial M_{+}} \simeq\left(E_{-}\right)_{\mid \partial M_{-}}$covering, respectively, $\phi_{1}$ and $\phi_{2}$. Given two classifying maps $r_{j}: M_{j} \rightarrow B \Gamma(1 \leqslant j \leqslant 2)$ as above
one defines two $\Lambda$-linear Dirac-type operators $\mathscr{D}_{M_{j}}^{r_{j}}$. Moreover, assuming that the two paths of metrics $\left\{g_{+, u}, u \in[1,2]\right\}$ and $\left\{g_{-, u}, u \in[1,2]\right\}$ considered in Section 6.1 induce (now) a Clifford action on, respectively, $E_{+}$and $E_{-}$for each $u \in[1,2]$, one may define an associated $S^{1}$-family $\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right)$ so that the defect formula (39) holds verbatim. A typical example of this situation is given when $M_{j}$ are spin manifolds and the operators are Dirac operators. We leave the details to the interested reader.

### 6.3. Vanishing spectral flow in the even-dimensional case

In this section we consider two SK-equivalent pairs $\left(M_{1}, r_{1}\right),\left(M_{2}, r_{2}\right)$ as in Definition 10 such that $\operatorname{dim} M_{1}=\operatorname{dim} M_{2}=2 m$. We shall give sufficient conditions for the noncommutative spectral flow appearing in the defect formula of Theorem 11 to be zero. Let $F$ be as in the previous subsection and consider $r=\left(r_{1}\right)_{\mid F}: F \rightarrow B \Gamma$.

Following [21] and then [12] we shall make the following:
Assumption A. Let $\Omega_{(2)}^{\ell}\left(F, \mathscr{V}_{r}\right)$ denote the $L_{\Lambda}^{2}$-completion of $\Omega^{\ell}\left(F, \mathscr{V}_{r}\right)$. We assume that the operator

$$
d: \Omega_{(2)}^{m-1}\left(F, \mathscr{V}_{r}\right) \rightarrow \Omega_{(2)}^{m}\left(F, \mathscr{V}_{r}\right)
$$

with domain equal to $\Lambda$-Sobolev space $H_{\Lambda}^{1}$, has closed image.
This assumption appeared for the first time in [21, Section 7]; there it was assumed that the $L^{2}$-spectrum of the differential form laplacian of $\tilde{F}$ (the covering defined by $r: F \rightarrow B \Gamma$ ), acting on the vector space of differential forms of degree $m$, has a full gap at zero. Assumption A (which is a homotopy-invariant assumption) is equivalent to the slightly weaker assumption that the differential form laplacian of $\tilde{F}$ has a strictly positive spectrum on $L^{2}\left(\widetilde{F}, \bigwedge^{m-1} T^{*} \tilde{F}\right) / \operatorname{ker}(d)$. We refer the reader to [12] for a proof of this fact and for examples where Assumption A is fulfilled.

We first recall (and slightly extend) the definition of symmetric spectral section [17]. Under Assumption A we have

$$
\begin{aligned}
& \Omega_{(2)}^{m}\left(F, \mathscr{V}_{r}\right)=d \Omega_{(2)}^{m-1}\left(F, \mathscr{V}_{r}\right) \oplus\left(d \Omega_{(2)}^{m-1}\left(F, \mathscr{V}_{r}\right)\right)^{\perp}, \\
& \Omega_{(2)}^{m-1}\left(F, \mathscr{V}_{r}\right)=d^{*} \Omega_{(2)}^{m}\left(F, \mathscr{V}_{r}\right) \oplus\left(d^{*} \Omega_{(2)}^{m}\left(F, \mathscr{V}_{r}\right)\right)^{\perp} .
\end{aligned}
$$

Then, we can write $\Omega^{*}\left(F, \mathscr{V}_{r}\right)=V \oplus W$ where (we abbreviate $\Omega_{\mathscr{V}}^{*}=\Omega^{*}\left(F, \mathscr{V}_{r}\right)$ ):

$$
\begin{gathered}
V=d^{*} \Omega_{\mathscr{V}}^{m}+d \Omega_{\mathscr{V}}^{m-1}, \quad W=\Omega_{\mathscr{V}}^{<} \oplus \Omega_{\mathscr{V}}^{>}, \\
\Omega_{\mathscr{V}}^{<}=\Omega_{\mathscr{V}}^{0} \oplus \cdots \oplus \Omega_{\mathscr{V}}^{m-2} \oplus\left(d^{*} \Omega_{\mathscr{V}}^{m}\right)^{\perp}, \\
\Omega_{\mathscr{V}}^{>}=\left(d \Omega_{\mathscr{V}}^{m-1}\right)^{\perp} \oplus \Omega_{\mathscr{V}}^{m+1} \oplus \cdots \oplus \Omega_{\mathscr{V}}^{2 m-1} .
\end{gathered}
$$

It is clear that $\mathscr{D}_{F}^{\text {sign,r }}$ sends $V$ (resp. $W$ ) into itself. Using Assumption A and proceeding as in the proof of Lemma 2.1 of [12], one checks easily that $\mathscr{D}_{F}^{\text {sign,r }}$ induces an invertible operator on the $L_{A}^{2}$-completion of $V$ (with domain $H^{1}$ ) and we denote by $\Pi_{>}$the projection onto the positive part. Moreover, the proof of Theorem 2 shows immediately that the restriction of $\mathscr{D}_{F}^{\text {sign,r }}$ to the $L_{A}^{2}$-completion of $W$ admits a spectral section. Notice that although $W$ is not the space of section of a $\Lambda$-bundle over $F$ the proof of Theorem 2 still applies in this case. Then, Lemma 4.3 of [17] shows that $\mathscr{D}_{F}^{\text {sign,r }}$ admits a symmetric spectral section $\mathscr{P}$ in the sense that $\mathscr{P}$ is diagonal with respect to the decomposition $\Omega^{*}\left(F, \mathscr{V}_{r}\right)=V \oplus W$ and

$$
\mathscr{P}_{\mid V}=\Pi_{>}, \quad \mathscr{P}_{\mid W^{\circ}} \alpha+\alpha_{\circ} \mathscr{P}_{\mid W}=\alpha,
$$

where $\alpha$ is the involution of $W$ equal to identity on $\Omega_{\gamma}^{<}$and to minus the identity on $\Omega_{\mathscr{V}}^{>}$. Recall from [17, Proposition 4.4] that if $\mathscr{2}$ is another symmetric spectral section then $[\mathscr{P}-\mathbb{Q}]=0$ in $K_{0}(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q}$. We immediately obtain the following.

Proposition 10. Let $\left(M_{1}, r_{1}\right),\left(M_{2}, r_{2}\right)$ be two SK-equivalent pairs as in Definition 10. We set $r:=\left(r_{1}\right)_{\mid F}$. If Assumption $A$ holds for $(F, r)$ then

$$
\operatorname{sf}\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right)=0 \quad \text { in } K_{*}\left(C_{r}^{*}(\Gamma)\right) \otimes_{\mathbb{Z}} \mathbb{Q} .
$$

Proof. In the definition of $\operatorname{sf}\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right)$, we can assume that all the spectral section involved are symmetric. Since, by definition, the noncommutative spectral flow is given in terms of difference classes associated to these symmetric spectral section, it follows by Proposition 4.4 in [17] that $\operatorname{sf}\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right)$ is zero in $K_{0}\left(C_{r}^{*}(\Gamma)\right) \otimes_{\mathbb{Z}} \mathbb{Q}$. The proposition is proved.

Remark. Notice that because of the homotopy-invariance of Assumption A we could equivalently set $r:=\left(r_{2}\right)_{\mid \partial M_{+}}$.

We have now obtained an analytic proof of the main result in [13].
Theorem 12. Assume that $\Gamma$ is such that the assembly map of the Baum-Connes conjecture is rationally injective. Let $\left(M_{1}, r_{1}\right)$ and $\left(M_{2}, r_{2}\right)$ be two SK-equivalent pairs as in Definition 10. We assume that Assumption A holds for $(F, r)$ with $r:=\left(r_{1}\right)_{\mid F}$. Then the higher signatures of $\left(M_{1}, r_{1}\right)$ and $\left(M_{2}, r_{2}\right)$ are the same.

The proof is a direct consequence of Proposition 10 and of Corollary 2.
Remark. The previous theorem was first proved by Leichtnam-Lott-Piazza in [12] in the case of virtually nilpotent or Gromov hyperbolic groups. It was then extended by Leichtnam-Lueck [13], using techniques from algebraic surgery, to the case where $\Gamma$ satisfies the assumption of Theorem 12.

### 6.4. Vanishing spectral flow in the odd-dimensional case

In this section we consider two SK-equivalent pairs $\left(M_{1}, r_{1}\right),\left(M_{2}, r_{2}\right)$ as in Definition 10 such that $\operatorname{dim} M_{1}=\operatorname{dim} M_{2}=2 m+1$. Let $F$ be as in the previous section and consider $r=\left(r_{1}\right)_{\mid F}: F \rightarrow B \Gamma$.

As in the even case we shall make the following assumption on $(F, r)$ :

Assumption B. The operator

$$
d: \Omega_{(2)}^{m-1}\left(F, \mathscr{V}_{r}\right) \rightarrow \Omega_{(2)}^{m}\left(F, \mathscr{V}_{r}\right)
$$

with domain equal to $\Lambda$-Sobolev space $H_{\Lambda}^{1}$, has closed range.
It is proved in [12] that Assumption B is equivalent to the hypothesis that $L^{2}$ spectrum of the differential form laplacian of $\widetilde{F}$ acting on the vector space of differential forms of degree $m$ admits a gap at zero.

Assumption B (in the slightly stronger form of requiring $L^{2}$-invertibility in middle degree) appears for the first time in Section 7 of [21].

We refer the reader to [12] for examples where Assumption B is satisfied.
Now we recall results and notations allowing to introduce the concept of symmetric $\mathrm{Cl}(1)$-spectral section. We define the rescaled flat exterior derivative $\tilde{d}: \Omega^{*}\left(F, \mathscr{V}_{r}\right) \rightarrow \Omega^{*+1}\left(F, \mathscr{V}_{r}\right)$ by setting

$$
\forall \omega \in \Omega^{|\omega|}\left(F, \mathscr{V}_{r}\right), \quad \tilde{d}(\omega)=i^{|\omega|} d \omega .
$$

Then using the Hodge duality operator $*$ one defines the involution $\tau$ :

$$
\forall \omega \in \Omega^{|\omega|}\left(F, \mathscr{V}_{r}\right), \quad \tau(\omega)=i^{-|\omega|(2 m-|\omega|)} * \omega .
$$

We refer to the Section 1 of [12] for the definition of the quadratic form $Q$ and the $\Lambda$ hermitian scalar product $\langle;\rangle$ on $\Omega^{*}\left(F, \mathscr{V}_{r}\right)$. The signature operator $\mathscr{D}_{F}^{\text {sign,r }}$ is then equal to $\tilde{d}-\tau \tilde{d} \tau$.

Let $\mathrm{H}^{m}\left(F, \mathscr{V}_{r}\right)$ denote the $m$ th cohomology group of the locally constant sheaf $\mathscr{V}_{r}$. Using Hodge duality and Assumption B one proves easily that the two following orthogonal decomposition holds:

$$
\begin{gathered}
\Omega_{(2)}^{m}\left(F, \mathscr{V}_{r}\right)=\mathrm{H}^{m}\left(F, \mathscr{V}_{r}\right) \oplus \overline{\tau \tilde{d} \tau\left(\Omega^{m+1}\left(F, \mathscr{V}_{r}\right)\right)} \oplus \overline{\tilde{d}\left(\Omega^{m-1}\left(F, \mathscr{V}_{r}\right)\right)}, \\
\Omega_{(2)}^{*}\left(F, \mathscr{V}_{r}\right)=V \oplus W \quad \text { with } \\
V=\Omega_{(2)}^{m}\left(F, \mathscr{V}_{r}\right) \oplus \mathscr{D}_{F}^{\text {sign }, r}\left(H_{A}^{1}\left(F ; \bigwedge^{m} T^{*} F \otimes \mathscr{V}_{r}\right)\right), \quad W=V^{\perp},
\end{gathered}
$$

$V$ being a closed complementable $\Lambda$-Hilbert submodule of $\Omega_{(2)}^{*}\left(F, \mathscr{V}_{r}\right)$. We observe that both $\mathscr{D}_{F}^{\text {sign }, r}$ and the grading $\tau$ send $V$ (resp. $W$ ) into itself.

We shall consider the involution $\alpha$ of $W$ such that $\alpha$ induces Id (resp. -Id) on the differential forms of degree $<m$ (resp. $>m$ ).

Proposition 11. At the expense of replacing $F$ by two disjoint copies of $F$, one can find a finitely generated projective $\Lambda$-submodule $N$ of $V$ such that:
(0) $\mathscr{D}_{F}^{\text {sign, } r}(N) \subset N, N^{+} \simeq N^{-}$.
(1) $\tau(N) \subset N$ and $\mathrm{H}^{m}\left(F, \mathscr{V}_{r}\right) \subset N$ as a $\Lambda$-submodule.
(2) $N$ admits a lagrangian $\Lambda$-submodule $L$ with respect to the quadratic form $Q$ and the orthogonal $\Lambda$-projection $P_{L}$ from $N$ onto $L$ is well defined.
(3) $V=N \oplus N^{\perp}$.

We will say that such an $L$ is a stable lagrangian of $\mathrm{H}^{m}\left(F, \mathscr{V}_{r}\right)$ (associated with $\left.N\right)$.
Proof. The index class of the signature operator on $F$ with values in $\mathscr{V}_{r}$ is zero in $K_{0}(\Lambda)$, by cobordism invariance. Notice that this index class is the sum of the index classes associated to the restrictions $\mathscr{D}_{V}, \mathscr{D}_{W}$ of the signature operator $\mathscr{D}_{F}^{\text {sign,r }}$ to $V$ and $W$, respectively. Under Assumption B, one checks easily that

$$
\alpha\left(\mathscr{D}_{W}^{+}\right)^{*} \alpha=\mathscr{D}_{W}^{+}
$$

and so 2 Ind $\mathscr{D}_{W}^{+}=0$ in $K_{0}(\Lambda)$. Replacing $F$ by the disjoint union of two copies of $F$, we may assume that Ind $\mathscr{D}_{W}^{+}=0$ in $K_{0}(\Lambda)$. Therefore, we may conclude that the index class associated to $\mathscr{D}_{F}^{\text {sign, } r}$ restricted to $V$ is also zero. Extending the proof of Proposition 2 on page 295 of [26] to the case of $\left(\mathscr{D}_{F}^{\text {sign, } r}\right)_{\mid V}$ along the lines of [18, see Lemma 4.3]), we infer that there exists a real number $R>0$ such that for any spectral section $P^{\prime}$ as in the statement of the theorem, $N=\operatorname{ker}\left(P^{\prime}+\tau P^{\prime} \tau\right)$ is a finitely generated projective submodule of $V$ containing $\mathrm{H}^{m}\left(F, \mathscr{V}_{r}\right)$ and such that
(a) $N$ is the range of a projection of $V$ and thus $N \oplus N^{\perp}=V$.
(b) $N=N^{+} \oplus N^{-}$, with $N^{ \pm}=\{\omega ; \tau \omega= \pm \omega\}$ and $N^{+}$isomorphic to $N^{-}$.

Let then $j: N^{+} \rightarrow N^{-}$be a unitary isomorphism; our lagrangian $L$ is given by

$$
\left\{\omega+j(\omega), \omega \in N^{+}\right\}
$$

The proposition is proved.
Remark. 1. Notice that, as in [26, p. 295], we have in fact proved the following more precise statement: one can find a real number $R>0$ such that for any spectral section $\mathscr{P}^{\prime}$ of $\left(\mathscr{D}_{F}^{\text {sign, } r}\right)_{\mid V}$ satisfying $\mathscr{P}^{\prime} \circ \xi\left(\left(\mathscr{D}_{F}^{\text {sign, } r}\right)_{\mid V}\right)=0$ for any $\xi \in C^{\infty}(\mathbb{R}, \mathbb{R}), \xi \equiv 0$ on
$\left[R,+\infty\left[\right.\right.$, the module $N:=\operatorname{ker}\left(\mathscr{P}^{\prime}+\tau P^{\prime} \tau\right)$ is finitely generated projective and satisfies the 4 properties of the above proposition.
2. This notion of stable lagrangian is essentially the same as the one used in Section 3 of [12].
3. In the previous proposition one may assume that $N$ is included in the range of $\psi\left(\mathscr{D}_{F}^{\text {sign,r}} \mid V\right)$ for a suitable function $\psi \in C_{\text {comp }}^{\infty}(\mathbb{R} ; \mathbb{R})$.

Now, let $L$ be a stable lagrangian of $\mathrm{H}^{m}\left(F, \mathscr{V}_{r}\right)$ (associated with $N$ ) as in Proposition 11. We observe that $\mathscr{D}_{F}^{\text {sign,r }}$ is diagonal with respect to the decomposition

$$
\Omega_{(2)}^{*}\left(F, \mathscr{V}_{r}\right)=\left(N \oplus N^{\perp}\right) \oplus W
$$

and that the restriction of $\mathscr{D}_{F}^{\text {sign }, r}$ to $N^{\perp}$ is invertible. We shall denote by $\Pi_{>}\left(N^{\perp}\right)$ the APS projection onto the positive part of the restriction of $\mathscr{D}_{F}^{\text {sign, } r}$ to $N^{\perp}$. We then may give the following:

Definition 11. Let $\mathscr{R}$ be a spectral section $\in B_{\Lambda}(W)$ for $\mathscr{D}_{F}^{\text {sign,r }} \mid W$ which is $\mathrm{Cl}(1)$ for both $\alpha$ and $\tau$. The self-adjoint projection

$$
\mathscr{P}(L, \mathscr{R})=\mathscr{P}_{L} \oplus \Pi_{>}\left(N^{\perp}\right) \oplus \mathscr{R}
$$

of $B_{A}\left(\Omega_{(2)}^{*}\left(F, \mathscr{V}_{r}\right)\right)$ is called symmetric $\mathrm{Cl}(1)$-spectral section for $\mathscr{D}_{F}^{\text {sign }, r}$.
One checks the existence of a spectral section $\mathscr{R}$ as in the above proposition by proceeding as in the proof of Lemma 4.3 of [18]. Therefore, there exists such symmetric $\mathrm{Cl}(1)$-spectral section.

Let $L$ and $L^{\prime}$ be two stable lagrangians for $\mathrm{H}^{m}\left(F, \mathscr{V}_{r}\right)$, as in Proposition 11. The proof of the proposition and the arguments of [26, p. 295] shows that if in the statement of Proposition 11 one replaces the real number $R$ by a bigger one then $N^{ \pm}$ gets replaced by $N^{ \pm} \oplus M^{ \pm}$where $\left(\mathscr{D}_{F}^{\text {sign }, r}\right)^{+}$induces an isomorphism from the finitely generated projective $C_{r}^{*}(\Gamma)$-module $M^{+}$onto the $C_{r}^{*}(\Gamma)$-module $M^{-}$. Thus, at the expense of replacing $L$ by

$$
\left\{\omega \oplus \beta+j(\omega) \oplus\left(\mathscr{D}_{F}^{\text {sign }, r}\right)^{+}(\beta),(\omega, \beta) \in N^{+} \times M^{+}\right\}
$$

(and similarly for $L^{\prime}$ ), one may assume that both $L$ and $L^{\prime}$ are associated with the same $N$.

Definition 12. Let $L, L^{\prime}$ be two lagrangians as above. Let $\mathscr{P}_{L}, \mathscr{P}_{L^{\prime}}$ be the $\Lambda$ endomorphisms of $N$ given by the orthogonal projections onto $L, L^{\prime}$ respectively. According to Definition 8 these two projections define a difference class [ $\mathscr{P}_{L}-$ $\left.\mathscr{P}_{L^{\prime}}\right] \in K_{0}\left(C_{c}^{0}(] 0, \pi[) \otimes \Lambda\right)=K_{1}(\Lambda)$; we set $\left[L-L^{\prime}\right]:=\left[\mathscr{P}_{L}-\mathscr{P}_{L^{\prime}}\right]$.

Remark. 1. The class $\left[L-L^{\prime}\right] \in K_{1}(\Lambda)$ depends only on $L$ and $L^{\prime}$.
2. One can give a different definition of the class $\left[L-L^{\prime}\right] \in K_{1}(\Lambda)$ (see [12, Section 3]: if $j, j^{\prime}$ are two unitary isomorphims $N^{+} \rightarrow N^{-}$such that $L=\left\{\omega+j(\omega) / \omega \in N^{+}\right\}$, $L^{\prime}=\left\{\omega+j^{\prime}(\omega) / \omega \in N^{+}\right\}$, one defines the class $\left[L-L^{\prime}\right]$ to be $\left[j \circ\left(j^{\prime}\right)^{-1}\right] \in K_{1}(\Lambda)$. The two definitions should be compatible through the suspension isomorphism, but we have not looked into the details.

We have the following relative result (where we are always under Assumption B):
Proposition 12. Let $L$ and $L^{\prime}$ be two stable lagrangians as in Definition 12. Let $\mathscr{P}(L, \mathscr{R})$ and $\mathscr{P}\left(L^{\prime}, \mathscr{R}^{\prime}\right)$ be two symmetric $\mathrm{Cl}(1)$-spectral section for $\mathscr{D}_{F}^{\text {sign }, r}$ as in Definition 11, then one has

$$
\left[\mathscr{P}(L, \mathscr{R})-\mathscr{P}\left(L^{\prime}, \mathscr{R}^{\prime}\right)\right]=\left[L-L^{\prime}\right] \quad \text { in } K_{1}(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q} .
$$

Proof. We set $\mathscr{P}_{1}=\mathscr{P}(L, \mathscr{R})$ and $\mathscr{P}_{2}=\mathscr{P}\left(L^{\prime}, \mathscr{R}^{\prime}\right)$ and assume that $L$ and $L^{\prime}$ are associated with (the same) $N$. We then consider the odd self-adjoint $\Lambda$-compact operator $\mathscr{A}_{\mathscr{P}_{j}}^{0}$ associated with $\mathscr{P}_{j}(j \in\{1,2\})$ as in Definition 8. Thus $\mathscr{D}_{F}^{\text {sign,r }}+\mathscr{A}_{\mathscr{P}_{j}}^{0}$ is invertible and we can moreover assume that $\mathscr{A}_{\mathscr{P}_{j}}^{0}$ is diagonal with respect to the decomposition

$$
\Omega_{(2)}^{*}\left(F, \mathscr{V}_{r}\right)=\left(N \oplus N^{\perp}\right) \oplus W
$$

and that its restriction to $W$, denoted $\mathscr{A}_{j}^{0}(W)$, is odd with respect to the grading $\alpha$. Of course we can assume that the restrictions of $\mathscr{A}_{\mathscr{P}_{j}}^{0}$ to $N^{\perp}$ coincide.

Now, for $j \in\{1,2\}$ and $t \in[0, \pi]$, we denote by $\mathscr{P}_{j}(t)$ the projection onto the positive part of

$$
\tau\left(\mathrm{Id}+\left[\mathscr{D}_{F}^{\mathrm{sin}, r} \mid W+\mathscr{A}_{j}^{0}(W)\right]^{2}\right)^{\frac{1}{2}} \cos t+\left(\mathscr{D}_{F}^{\mathrm{sign}, r} \mid W+\mathscr{A}_{j}^{0}(W)\right) \sin t .
$$

Since $\tau \alpha=-\alpha \tau$ and both $\mathscr{D}_{F}^{\text {sign,r }} \mid W$ and the $\mathscr{A}_{j}^{0}(W)$ are odd with respect to the grading $\alpha$, one checks immediately that $\alpha\left(\mathscr{P}_{1}(t)-\mathscr{P}_{2}(t) \alpha=-\left(\mathscr{P}_{1}(t)-\mathscr{P}_{2}(t)\right)\right.$ for any $t \in[0, \pi]$. Therefore

$$
\begin{equation*}
\left[\left(\mathscr{P}_{1}(t)-\mathscr{P}_{2}(t)\right)_{t \in[0, \pi]}\right]=-\left[\left(\mathscr{P}_{1}(t)-\mathscr{P}_{2}(t)\right)_{t \in[0, \pi]}\right] \quad \text { in } K_{0}\left(C_{c}^{0}(] 0, \pi[) \otimes \Lambda\right) \tag{43}
\end{equation*}
$$

and this $K$-theory class is rationally zero.
Next, we denote by $\mathscr{A}_{j}^{0}(N)$ the restriction of $\mathscr{A}_{\mathscr{P}_{j}}^{0}$ to $N$ for $j \in\{1,2\}$. For $t \in[0, \pi]$ we denote by $\mathscr{Q}_{j}(t)$ the projection onto the positive part of

$$
\tau\left(\mathrm{Id}+\left[\mathscr{D}_{F}^{\text {sign,r}} \mid N+\mathscr{A}_{j}^{0}(N)\right]^{2}\right)^{\frac{1}{2}} \cos t+\left(\mathscr{D}_{F}^{\text {sign }, r} \mid N+\mathscr{A}_{j}^{0}(N)\right) \sin t .
$$

Then by Definition 12 of the difference class one has

$$
\begin{equation*}
\left[\left(\mathscr{Q}_{1}(t)-\mathscr{Q}_{2}(t)\right)_{t \in[0, \pi]}\right]=\left[L-L^{\prime}\right] \quad \text { in } K_{0}\left(C_{c}^{0}(] 0, \pi[) \otimes \Lambda\right) \simeq K_{1}(\Lambda) . \tag{44}
\end{equation*}
$$

By combining the two Eqs. (43) and (44) and recalling Definition 8, one gets immediately the result of the proposition.

We recall that $\left(M_{1}, r_{1}\right)$ and $\left(M_{2}, r_{2}\right)$ are two SK-equivalent pairs of dimension $2 m+1$ as in Definition 10. Let us fix metrics $g_{1}$ on $M_{1}$ and $g_{2}$ on $M_{2}$; we can assume these metrics to be product-like near the hypersurface $F$. Recall that we are trying to give sufficient conditions for the noncommutative spectral flow appearing in the defect formula $\operatorname{Ind}\left(\mathscr{D}_{M_{1}}^{\text {sign, } r_{1}}\right)-\operatorname{Ind}\left(\mathscr{D}_{M_{2}}^{\text {sign, } r_{2}}\right)=\operatorname{sf}\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right)$ to be zero. We thus consider the $S^{1}$-family

$$
\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}=\left\{\mathscr{D}_{\partial M_{+}}^{\text {sign,u}}\right\}_{u \in[1,2]} \sqcup\left\{\mathscr{D}_{\partial M_{-}}^{\text {sign }, u}\right\}_{u \in[2,1]}
$$

defined on the fibers of the mapping torus $M\left(F, \phi_{2}^{-1} \circ \phi_{1}\right)$ by (37). Since the noncummutative spectral flow of $\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}$ is the sum of two terms, one coming from a variation in $\partial M_{+}$and the other coming from a variation in $\partial M_{-}$, we can look separately at each of them.

By following the proof of Proposition 11 we may construct two continuous families of spaces $N_{u}^{+}$, and $V_{u}^{+}, u \in[1,2]$ (associated with $\mathscr{D}_{\partial M_{+}}^{\text {sign }, u}$ ), we can also fix smoothly-varying families of Lagrangians $L_{u}^{+} \subset N_{u}^{+}$.

We thus consider the stable lagrangian $L_{2}^{+}$and a $\mathrm{Cl}(1)$-symmetric spectral section for $\mathscr{D}_{\partial M_{+}}^{\text {sign, } 2}$ of the type $\mathscr{P}\left(L_{2}^{+}, R_{2}^{+}\right)$; similarly we consider $L_{1}^{+}$and a $\mathrm{Cl}(1)$-symmetric spectral section for $\mathscr{D}_{\partial M_{+}}^{\text {sign, }}$ of the type $\mathscr{P}\left(L_{1}^{+}, R_{1}^{+}\right)$. Consider the $\mathrm{Cl}(1)$-symmetric spectral section Id $-\mathscr{P}\left(L_{j}^{+}, \mathscr{R}_{j}^{+}\right)^{\phi_{j}}, j=1,2$, for $\mathscr{D}_{\partial M_{-}}^{\text {sign, } j}$; then, using formula (30), one checks easily that we may write

$$
\mathrm{Id}-\mathscr{P}\left(L_{j}^{+}, \mathscr{R}_{j}^{+}\right)^{\phi_{j}}=\mathscr{P}\left(\left(\phi_{j}^{-1}\right)^{*} L_{j}^{+}, R_{j}^{-, \phi_{j}}\right),
$$

where of course $\left(\phi_{j}^{-1}\right)^{*} L_{j}^{+}$is a stable lagrangian and $R_{j}^{-, \phi_{j}}=\left(\phi_{j}^{-1}\right)^{*} \mathscr{R}_{j}^{+} \phi_{j}^{*}$. We know that

$$
\begin{aligned}
& \operatorname{Ind}\left(\mathscr{D}_{M_{2}}^{\operatorname{sign}, r_{2}}\right)-\operatorname{Ind}\left(\mathscr{D}_{M_{1}}^{\text {sign }, r_{1}}\right) \\
& \quad=\operatorname{sf}\left(\left\{\mathscr{D}_{\partial M_{+}}^{\text {sign }, u}\right\} ; \mathscr{P}\left(L_{2}^{+}, R_{2}^{+}\right), \mathscr{P}\left(L_{1}^{+}, R_{1}^{+}\right)\right) \\
& \quad+\operatorname{sf}\left(\left\{\mathscr{D}_{\partial M_{-}}^{\operatorname{sign}, u}\right\} ; \operatorname{Id}-\mathscr{P}\left(L_{2}^{+}, R_{2}^{+}\right)^{\phi_{2}}, \mathrm{Id}-\mathscr{P}\left(L_{1}^{+}, R_{1}^{+}\right)^{\phi_{1}}\right) .
\end{aligned}
$$

We set $L_{2}^{-}=\left(\phi_{2}^{-1}\right)^{*} L_{2}^{+}$; by following the proof of Proposition 11 we can construct a continuous family of stable lagrangians $L_{u}^{-}, u \in[1,2]$ extending $L_{2}^{-}$down to $u=1$; notice that we are flowing from $u=2$ to $u=1$. These stable lagrangians are thus
associated with $\mathscr{D}_{\partial M_{-}}^{\text {sign, }}$ and are constructed out of a continuous family of spaces $N_{u}^{-}$ as in Proposition 11. We denote the value at $u=1$ of this family by $L_{1}^{-}\left(\phi_{2}\right)$.

The definition of spectral flow depends on the choice of a total spectral section we choose for $\left\{\mathscr{D}_{\partial M_{+}}^{\text {sign, }}\right\}$ a total spectral section of the type $\mathscr{P}\left(L_{u}^{+}, R_{u}^{+}\right)$and we choose for $\left\{\mathscr{D}_{\partial M_{-}}^{\text {sign }, u}\right\}$ a total spectral section of the type $\mathscr{P}\left(L_{u}^{-}, R_{u}^{-}\right)$with $L_{u}^{ \pm}$as above. With these choices the first spectral flow appearing on the right-hand side of the above formula is equal to zero, whereas for the second we obtain the following simple expression:

$$
\begin{equation*}
\left[\left(\phi_{1}^{-1}\right)^{*} L_{1}^{+}-L_{1}^{-}\left(\phi_{2}\right)\right] \in K_{1}\left(C_{r}^{*}(\Gamma)\right) \otimes_{\mathbb{Z}} \mathbb{Q} ; \tag{45}
\end{equation*}
$$

here Proposition 12 has been used. Summarizing

$$
\begin{equation*}
\text { Ind } \mathscr{D}_{M_{2}}^{\text {sign }, r_{2}}-\operatorname{Ind} \mathscr{D}_{M_{1}}^{\text {sign, } r_{1}}=\left[\left(\phi_{1}^{-1}\right)^{*} L_{1}^{+}-L_{1}^{-}\left(\phi_{2}\right)\right] \in K_{1}\left(C_{r}^{*}(\Gamma)\right) \otimes_{\mathbb{Z}} \mathbb{Q} . \tag{46}
\end{equation*}
$$

We shall say that the family $\left(\mathscr{D}_{F}(\theta)\right)_{\theta \in S^{1}}$ admits an invariant stable lagrangian (see also [12, pp. 625-627]) if we can choose the above stable lagrangians so that $\left(\phi_{1}^{-1}\right)^{*} L_{1}^{+}=L_{1}^{-}\left(\phi_{2}\right)$. We can then state the main result of this subsection.

Proposition 13. Let $\left(M_{1}, r_{1}\right)$ and $\left(M_{2}, r_{2}\right)$ be two SK-equivalent pairs of dimension $2 m+1$ as in Definition 10. If Assumption B holds for $\left(F, r=\left(r_{1}\right)_{\mid F}\right)$ and if the family $\left(\mathscr{D}_{F}(\theta)\right)_{\theta} \in S^{1}$ admits an invariant stable lagrangian then

$$
\operatorname{sf}\left(\left\{\mathscr{D}_{F}(\theta)\right\}_{\theta \in S^{1}}\right)=0 \quad \text { in } K_{1}(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q} .
$$

and so

$$
\text { Ind } \mathscr{D}_{M_{2}}^{\text {sign, } r_{2}}=\operatorname{Ind} \mathscr{D}_{M_{1}}^{\text {sign }, r_{1}} \quad \text { in } K_{1}(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q} .
$$

The previous proposition implies the main result of [13]:
Theorem 13. Assume that $\Gamma$ is such that the assembly map of the Baum-Connes conjecture is rationally injective. Let $\left(M_{1}, r_{1}\right)$ and $\left(M_{2}, r_{2}\right)$ be two SK-equivalent pairs of dimension $2 m+1$ as in Definition 10. If Assumption B holds for $(F, r)$ and if the family $\left(\mathscr{D}_{F}(\theta)\right)_{\theta \in S^{1}}$ admits an invariant stable lagrangian then $\left(M_{1}, r_{1}\right)$ and $\left(M_{2}, r_{2}\right)$ have the same higher signatures.

Remark. 1. The previous theorem was first proved by Leichtnam-Lott-Piazza in [12] in the case of virtually nilpotent of Gromov hyperbolic groups. It was then extended by Leichtnam-Lueck [13], using techniques from algebraic surgery, to the case where $\Gamma$ satisfies the condition of Theorem 13.
2. As a final remark we point out that our $K$-theoretic approach to the cut-andpaste invariance of higher signatures can be extended to foliations. This is not the case either with the original approach in [12] (which uses local index formulae) or
with the algebraic-surgery approach of [13]. In fact, finding a proof of the cut-andpaste results in these papers that could be generalized to foliations was one of the motivations for proving Theorem 11. We plan to deal with the technicalities of this program in a future publication.

## Acknowledgments

We heartily thank Ralf Meyer for pointing out the relevance of the notion of very full projection and for suggesting the statement of Theorem 1 as well as the structure of its proof. We thank Michel Hilsum for alerting us of the unjustified step in Wu's preprint; we also thank Etienne Blanchard, Ulrich Bunke and Thomas Schick for helpful comments. The second author wishes to thank John Lott for interesting discussions concerning the cut-and-paste invariance of higher signatures. Finally, we thank again Michel Hilsum and Ralf Meyer for some very useful comments on an earlier version of the paper.
E. Leichtnam and P. Piazza are members of the RTN "Geometric Analysis" of the European Community. P. Piazza has received financial support from the "Shortterm mobility program" of the Consiglio Nazionale delle Ricerche, from Ministero dell'Università e della Ricerca Scientifica e Tecnologica (Cofinanziamento MURST '99) and by the Istituto Nazionale di Alta Matematica through the GNSAGA. The writing of this paper was completed in April 2001 while P. Piazza was visiting the MSRI, Berkeley, as a member of the Spectral Invariants program (from March 1st to May 12th, 2001); he thanks the institute for the invitation and partial financial support.

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