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A Higher Atiyah–Patodi–Singer Index Theorem for the Signature Operator on Galois Coverings

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Abstract. Let (N, g) be a closed Riemannian manifold of dimension $2m - 1$ and let $\Gamma \to \widetilde{N} \to N$ be a Galois covering of N. We assume that Γ is of polynomial growth with respect to a word metric and that $\Delta_{\widetilde{N}}$ is L^2-invertible in degree m. By employing spectral sections with a *symmetry property* with respect to the $\star$-Hodge operator, we define the higher eta invariant associated with the *signature operator* on $\widetilde{N}$, thus extending previous work of Lott. If $\pi_1(M) \to \widetilde{M} \to M$ is the universal cover of a compact orientable even-dimensional manifold *with boundary* ($\partial M = N$) then, under the above invertibility assumption on $\Delta_{\partial\widetilde{M}}$, and always employing *symmetric* spectral sections, we define a *canonical* Atiyah–Patodi–Singer index class, in $K_0(C_r^*(\Gamma))$, for the signature operator of $\widetilde{M}$. Using the higher APS index theory developed in [6], we express the Chern character of this index class in terms of a local integral and of the higher eta invariant defined above, thus establishing a higher APS index theorem for the signature operator on Galois coverings. We expect the notion of a symmetric spectral section for the signature operator to have wider implications in higher index theory for signatures operators.

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0. Introduction

Let (N, g) be a compact closed Riemannian manifold and let D be the signature operator on N. We let $\Gamma = \pi_1(N)$ be the fundamental group of N and consider $C_r^*(\Gamma)$, the reduced C^*-algebra of the group. In this paper, unless otherwise stated, we shall assume Γ to be of polynomial growth with respect to a word metric.

Beside the numerical index of D, an integer, it is important to consider the *higher index* of D, a class in $K_*(C_r^*(\Gamma))$. In order to define the latter we introduce the flat bundle of C^*-algebras

$$\mathcal{V} = \widetilde{N} \times_\Gamma C_r^*(\Gamma)$$

and consider the signature operator with values in $\mathcal{V}$, denoted $\mathcal{D}$. This is an elliptic differential operator in the sense of Mishenko and Fomenko [17] and has a well-defined index class: $\mathrm{Ind}(\mathcal{D}^+) \in K_0(C_r^*(\Gamma))$ if $\dim N$ is even (so that $\mathcal{D}$ is $\mathbb{Z}_2$-graded odd) and $\mathrm{Ind}(\mathcal{D}) \in K_1(C_r^*(\Gamma))$ if $\dim N$ is odd. The Chern character of the higher signature index class can be computed through the Connes–Moscovici higher index theorem, see [4] and, for a heat-kernel proof, see [9].

The aim of this paper is to establish a parallel result for manifolds with boundary; we are thus interested in proving a higher Atiyah–Patodi–Singer index theorem *for the signature operator* on Galois coverings.

In order to understand the problems involved in the passage to manifolds with boundary, we concentrate on the Abelian case and treat the closed case first. Thus, we let N be a closed oriented even-dimensional manifold with $\pi_1(N) = \mathbb{Z}^k$. In this case, the higher index class $\mathrm{Ind}(\mathcal{D}^+)$ has, following Lustzig [13], a geometric description. We consider the dual of $\pi_1(N)$ consisting of all irreducible representation of $\pi_1(N)$; in this particular case, we obtain the k-dimensional torus $T^k \equiv \widehat{\mathbb{Z}^k} = \mathrm{Hom}(\mathbb{Z}^k, U(1))$. Each $\theta \in T^k$ defines a flat unitary line bundle L_θ over N. Considering the associated twisted signature operator D_θ, we obtain a family of generalized odd $\mathbb{Z}_2$-graded Dirac operators $\mathcal{D} = (\mathrm{D}_\theta)_{\theta \in T^k}$ and thus, according to Atiyah and Singer, an index class $\mathrm{Ind}(\mathcal{D}^+) \in K^0(T^k)$. We shall call the family $\mathcal{D} = (\mathrm{D}_\theta)_{\theta \in T^k}$ the *Lusztig family* associated to N. Notice that by Fourier transform $C^0(T^k) \cong C_r^*(\mathbb{Z}^k)$ so that $K^0(T^k) \equiv K_0(C^0(T^k)) \cong K_0(C_r^*(\mathbb{Z}^k))$ and it is not difficult to see that, under this isomorphism, the Lustzig index class and the Mishenko–Fomenko index class correspond. The higher index theorem, i.e. a formula for $\mathrm{Ch}(\mathrm{Ind}(\mathcal{D}^+)) \in H^*(T^k, \mathbb{C})$, has been established by Lustzig using the Atiyah–Singer family index theorem [2].

Let now M be an oriented compact even-dimensional manifold *with boundary*. We fix a Riemannian metric g on M which is a product near the boundary. We consider the associated Levi-Civita connection ∇^M and the signature operator D. The boundary signature operator is denoted, as usual, by D_0. We further assume, just to simplify the exposition, that $\pi_1(M) = \mathbb{Z}^k$.

Let $\mathcal{D} = (\mathrm{D}_\theta)_{\theta \in T^k}$ be the Lusztig family associated to M. In trying to define a higher signature index class, we are confronted with a rather fundamental problem. In order to define a *smooth* family of Fredholm operators out of $\mathcal{D}$, we are obliged to consider a spectral section $\mathcal{P} = (P_\theta)$ for the boundary family $\mathcal{D}_0 = (\mathrm{D}_{0,\theta})_{\theta \in T^k}$ (see [15] and, for a survey, [18]). The introduction of spectral sections is unavoidable here, the problem being that the family of Atiyah–Patodi–Singer spectral projection $(\Pi(\theta))_{\geq}$ is *not* smooth in $\theta \in T^k$. Once a spectral section has been chosen, we can define an index bundle $\mathrm{Ind}(\mathcal{D}^+, \mathcal{P}) \in K^0(T^k)$. However, different choices of spectral sections produce, in general, distinct index classes; more precisely given, two spectral sections $\mathcal{P}$, $\mathcal{Q}$, we have the relative index theorem [15]:

$$\mathrm{Ind}(\mathcal{D}^+, \mathcal{P}) - \mathrm{Ind}(\mathcal{D}^+, \mathcal{Q}) = [\mathcal{Q} - \mathcal{P}] \ \text{ in } \ K^0(T^k),$$

with $[\mathcal{Q}-\mathcal{P}]$ equal to the index class of the Fredholm family $(P_\theta Q_\theta)$: range$(Q_\theta) \to$ range(P_θ).

Because of the geometric and topological significance of the signature operator it is natural to ask whether there exists a *special class* of spectral sections having all the same index class in $K^0(T^k)$ and, moreover, with the lower index equal to the signature of M. For this, it suffices to fix a subset S of the space of all spectral sections with the property that

$$\forall\, \mathcal{P}, \mathcal{Q} \in S; \quad [\mathcal{P} - \mathcal{Q}] = 0 \in K^0(T^k). \tag{0.1}$$

We will show (see Proposition 1.3 and Definition 3.2) that, thanks to the particular structure of the signature operator, it is possible to fix such a class of spectral sections under the following assumption:

(H1) The boundary signature Laplacian on $\partial \widetilde{M}$ is L^2-invertible in degree $\dim M/2$.

Equivalently:

(H1') The family of boundary signature Laplacians $(\Delta_{0,\theta})$ is invertible in degree $\dim M/2$.

These special spectral sections will be called *symmetric*. Any choice of symmetric spectral section $\mathcal{P}$ defines a *canonical* index class independent of $\mathcal{P}$: $\mathrm{Ind}(\mathcal{D}^+, \mathcal{P}) \equiv \mathrm{Ind}(\mathcal{D}^+) \in K^0(T^k)$.

We can see the appearance of symmetric spectral sections in yet another way. Let us go back to a closed manifold N with $\pi_1(N) = \mathbb{Z}^k$ and let us try to define the Bismut–Cheeger *eta form* [3] associated to the Lustzig's family $\mathcal{D}$ (this eta form is nothing but the higher eta invariant of [10] when the group Γ is commutative). For simplicity, we assume that N is odd-dimensional. Since the operators are not invertible, the integral defining the Bismut–Cheeger eta form

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{STr}_{\mathrm{Cl}(1)} \left(\frac{\mathrm{d}}{\mathrm{d}s} \mathbb{B}_s \right) \exp(-\mathbb{B}_s^2) \mathrm{d}s \tag{0.2}$$

(with $\mathbb{B}_s$ the rescaled Bismut superconnection) is not known to be convergent. Let us now assume the analogue of (H1), namely that $\Delta_{\widetilde{N}}$ is L^2-invertible in degree $(\dim N + 1)/2$: the K^1-index class of the Lustzig's family is then zero and, according to [15], we can consider a spectral section $\mathcal{P}$ for $\mathcal{D}$. This gives, in turn, a $\mathcal{P}$-eta form $\widehat{\eta}_{\mathcal{P}} \in \Omega^*(T^k)/\mathrm{d}\Omega^*(T^k)$. We can see all these operations as a way of regularizing the a-priori divergent integral (0.2).

Using the *jump formula* for eta forms [15, sect. 16], we see that two different regularizations are related by the formula

$$\widehat{\eta}_{\mathcal{P}} - \widehat{\eta}_{\mathcal{Q}} = \mathrm{Ch}([\mathcal{P} - \mathcal{Q}])$$

and it is once again clear that if $\mathcal{P}$ and $\mathcal{Q}$ are *symmetric* then the difference is zero and we obtain a *canonical* higher eta form in $\Omega^*(T^k)/\mathrm{d}\Omega^*(T^k)$; we denote this form by $\widetilde{\eta}$.

Going back to a manifold with the boundary satisfying assumption (H1), we see that we can now define a higher eta invariant $\widetilde{\eta}_\partial$ associated to the boundary of the universal cover of M, $\widetilde{\eta}_\partial \in \Omega^*(T^k)/\mathrm{d}\Omega^*(T^k)$; our higher signature index theorem expresses the Chern character of the canonical index class as the difference of the usual local integral and $\widetilde{\eta}_\partial/2$.

Condition (H1) appears for the first time in [10]; as remarked there, it is a *homotopy invariant condition*. The use of symmetric spectral sections makes rigorous an heuristic argument in [10] used in order to regularize Equation (0.2). It should be also remarked that assumption (H1) can be regarded as the analytic analogue of the *antisimple condition* of Weinberger, see [20]. For a different regularization of the higher eta invariant for the signature operator, see also the recent preprint [12].

In the noncommutative case, we proceed analogously, using the APS index theory developed in [5, 6] in place of the one in [15]. Thus, if M is a compact orientable manifold with boundary satisfying assumption (H1), we can define a canonical index class $\mathrm{Ind}(\mathcal{D}^+) \in K_0(C_r^*(\Gamma))$, $\Gamma = \pi_1(M)$. If, moreover, $\pi_1(M)$ is of polynomial growth, then we can define the higher eta invariant of the boundary signature operator and prove a higher APS signature index formula. Notice that the assumption that $\Delta_{\partial\widetilde{M}}$ is L^2-invertible *in each degree* would be an unreasonable one, see [11].

The right-hand side of this formula allows for the introduction of the higher signatures $\sigma(M, \partial M; [c])$, $[c] \in H^*(\Gamma, \mathbb{C})$, of a pair $(M, \partial M)$ satisfying assumption (H1). We can also assume that $\pi_1(M) = F \times \Gamma$, F finite, Γ of polynomial growth, then fix a representation $\rho\colon F \to U(\ell)$ and consider *twisted* higher signatures $\sigma_\rho(M, \partial M; [c]), [c] \in H^*(\Gamma, \mathbb{C})$. One can conjecture that under the analogue of assumption (H1), these (twisted) higher signatures are oriented homotopy invariants of the pair $(M, \partial M)$. The conjecture appears for the first time in [10, conjecture 2]. Our improvement with respect to [10] is twofold: first the higher eta invariant for the signature operator is now correctly defined; second, the higher signatures are now expressed through the Chern character of a higher index class. This means that in order to prove the homotopy invariance of the higher signatures, it now suffices to show the homotopy invariance of the canonical signature index class $\mathrm{Ind}(\mathcal{D}^+) \in K_0(C_r^*(\Gamma))$. See also the recent preprint [12] for a rigorous treatment of the conjecture of [10].

For a positive answer to this conjecture in a special case (but with Γ allowed to be Gromov-hyperbolic), we refer the reader to [12].

The results of this paper have been announced in [11]; they have been circulating as a Preprint IHES/M/98/40.

1. Riemannian Fibrations, Signature Operators and Symmetric Spectral Sections

We consider a smooth fibration $\phi\colon X \to B$ of closed oriented odd-dimensional Riemannian manifolds. The generic point of B will be denoted by θ; each fibre $X_\theta \equiv \phi^{-1}(\theta)$ is thus assumed to be diffeomorphic, through an orientation preserving diffeomorphism, to a fixed closed oriented manifold Z of dimenion $2m-1$.

We denote by $g_{X/B}$ the smooth family of Riemannian metrics in the vertical direction. As Lusztig [13, sect. 3], we assume the existence of a Hermitian vector bundle E over X with the additional structure of being *flat* in the fibre direction, thus the restriction of E to each fiber X_θ defines a flat bundle denoted E_θ.

These data define in a natural way a family of twisted odd signature operators $\mathcal{D} = (\mathrm{D}_\theta)_{\theta\in B}$ with $\mathrm{D}_\theta\colon \Omega^*(X_\theta, E_\theta) \to \Omega^*(X_\theta, E_\theta)$,

$$\mathrm{D}_\theta(\phi) = (\sqrt{-1})^m(-1)^{p+1}(\epsilon\star d - d\star)\phi$$

with $\epsilon = 1$ if $\phi \in \Omega^{2p}(X_\theta, E_\theta) \equiv \Omega^{2p}_\theta$ and $\epsilon = -1$ if $\phi \in \Omega^{2p-1}(X_\theta, E_\theta) \equiv \Omega^{2p-1}_\theta$.

We know that each D_θ sends forms of even/odd degree into forms of even/odd degree. Moreover, D_θ commutes with the isomorphism: $\Theta = (-1)^p\star$ on both Ω^{2p}_θ and Ω^{2p-1}_θ. Thus $\mathrm{D}_\theta = \mathrm{D}^{\mathrm{even}}_\theta \oplus \mathrm{D}^{\mathrm{odd}}_\theta$ with $\mathrm{D}^{\mathrm{odd}}_\theta = \Theta\mathrm{D}^{\mathrm{even}}_\theta\Theta$. For each fixed $\theta \in B$ the Hodge theorem implies the following orthogonal decomposition of the space of differential forms on X_θ with values in E_θ:

$$\begin{aligned}\Omega^*_\theta \;=\;& \Omega^0_\theta \oplus \Omega^1_\theta \oplus \cdots \mathrm{d}\Omega^{m-2}_\theta \oplus \mathrm{H}^{m-1}_\theta \\ & \oplus \mathbf{d}^*\boldsymbol{\Omega}^m_\theta \oplus \mathbf{d}\boldsymbol{\Omega}^{m-1}_\theta \oplus H^m_\theta \oplus \mathrm{d}^*\Omega^{m+1}_\theta \oplus \cdots \oplus \Omega^{2m-1}_\theta.\end{aligned}$$

Consider for each fixed $\theta \in B$ the subspace of Ω^*_θ given by $V_\theta = \mathrm{d}^*\Omega^m_\theta \oplus \mathrm{d}\Omega^{m-1}_\theta$. This space is *invariant* under D_θ. Moreover, the first summand $V^{\mathrm{odd}}_\theta = \mathrm{d}^*\Omega^m_\theta$ is invariant for $\mathrm{D}^{\mathrm{odd}}_\theta$ and the second summand $V^{\mathrm{ev}}_\theta = \mathrm{d}\Omega^{m-1}_\theta$ is invariant for $\mathrm{D}^{\mathrm{even}}_\theta$. We denote the orthocomplement of V_θ in Ω^*_θ by W_θ,

$$W_\theta = \Omega^0_\theta \oplus \Omega^1_\theta \oplus \cdots \mathrm{d}\Omega^{m-2}_\theta \oplus \mathrm{H}^{m-1}_\theta \oplus \mathrm{H}^m_\theta \oplus \mathrm{d}^*\Omega^{m+1}_\theta \oplus \cdots \oplus \Omega^{2m-1}_\theta$$

and we have $W_\theta = W^{\mathrm{odd}}_\theta \oplus W^{\mathrm{ev}}_\theta$.

Finally, we denote by C_θ the restriction of D_θ to V_θ and by G_θ the restriction of D_θ to W_θ:

$$C_\theta \equiv \mathrm{D}_\theta|_{V_\theta};\quad G_\theta \equiv \mathrm{D}_\theta|_{W_\theta};\quad \mathrm{D}_\theta = \begin{pmatrix} C_\theta & 0 \\ 0 & G_\theta\end{pmatrix}.$$

Notice that $G_\theta = G^{\mathrm{odd}}_\theta \oplus G^{\mathrm{ev}}_\theta$. We now define

$$\begin{aligned}\Omega^<_\theta \;&=\; \Omega^0_\theta \oplus \Omega^1_\theta \oplus \cdots \Omega^{m-2}_\theta \oplus \mathrm{d}\Omega^{m-2}_\theta,\\ \Omega^>_\theta \;&=\; \mathrm{d}^*\Omega^{m+1}_\theta \oplus \Omega^{m+1}_\theta \cdots \oplus \Omega^{2m-1}_\theta.\end{aligned}$$

As usual, these two subspaces decompose in even and odd forms. Moreover, $W_\theta = \Omega^<_\theta \oplus H^{m-1}_\theta \oplus H^m_\theta \oplus \Omega^>_\theta$.

We now make the following assumption

(H1) *The family of twisted signature Laplacians Δ_θ is invertible in degree m.*

Thus there exists a *smooth* family of pseudodifferential operators $F = (F_\theta)_{\theta\in B}$, $F_\theta \in \Psi^{-2}(X_\theta, \Lambda^m(X_\theta) \otimes E_\theta)$, such that $(\Delta_\theta)^{[m]}F_\theta = F_\theta(\Delta_\theta)^{[m]} = \mathrm{Id}_{\Omega^m_\theta}$.

Notice that assumption (H1) also implies the invertibility of the family of twisted Laplacians in degree $m-1$. In particular, $H^m_\theta = H^{m-1}_\theta = \{0\}$ for each $\theta \in B$ so that $W_\theta = \Omega^<_\theta \oplus \Omega^>_\theta$ and thus $\Omega^*_\theta = V_\theta \oplus (\Omega^<_\theta \oplus \Omega^>_\theta)$. Following [10, 19] we now define an involution α_θ on W_θ:

$$\alpha_\theta = \mathrm{Id} \ \text{ on } \ \Omega^<_\theta, \quad \alpha_\theta = -\mathrm{Id} \ \text{ on } \ \Omega^>_\theta.$$

It is immediate from the structure of D_θ that

$$G_\theta \circ \alpha_\theta + \alpha_\theta \circ G_\theta = 0.$$

In other words, α_θ gives a grading to W_θ, $W^+_\theta = \Omega^<_\theta$, $W^-_\theta = \Omega^>_\theta$ and the family $\mathcal{G} = (G_\theta)_{\theta\in B}$ is *odd* with respect to such a grading. It is important to notice that, because of assumption (H1), the decomposition $\Omega^*_\theta = V_\theta \oplus W_\theta$, the operators C_θ, G_θ and the involution α_θ, all depend smoothly upon $\theta \in B$.

PROPOSITION 1.1. *If the family $\mathcal{D} = (D_\theta)_\theta$ satisfies assumption* (H1)*, then its K^1-index class vanishes:* $\mathrm{Ind}(\mathcal{D}) = 0$ *in* $K^1(B)$.

Proof. It suffices to show that the family $\mathcal{G}$ has zero K^1-index. However, this is clear since the family $\mathcal{G}$ is homotopic, through self-adjoint Fredholm families to

$$\begin{pmatrix} \mathrm{Id} & \mathcal{G}^- \\ \mathcal{G}^+ & -\mathrm{Id} \end{pmatrix}$$

which is invertible. □

According to proposition 1 in [15], there exists a spectral section $\mathcal{P} = (P_\theta)_{\theta\in B}$ for $\mathcal{D}$. Thus, $\mathcal{P}$ is a *smooth* family $(P_\theta)_{\theta\in B}$ of self-adjoint projections with $P_\theta \in \Psi^0(X_\theta; \Lambda^*(X_\theta) \otimes E_\theta)$ and satisfying the following property:

$$\exists\, R \in \mathbb{R} \text{ such that: } \mathrm{D}_\theta u = \lambda u \Rightarrow \begin{cases} P_\theta u = u & \text{if } \lambda > R \\ P_\theta u = 0 & \text{if } \lambda < -R. \end{cases} \tag{1.1}$$

Since D_θ decomposes diagonally with respect to the decomposition $V_\theta \oplus W_\theta$, we can choose a spectral section which is also diagonal. Namely, both $\mathcal{C} = (C_\theta)$ and $\mathcal{G} = (G_\theta)$ are self-adjoint families with $\mathcal{C}$ invertible by construction; the K^1-index class of both families is zero and we can choose a spectral section for each of them. This will produce a diagonal spectral section for the family $\mathcal{D}$. In fact, we can choose a diagonal spectral section

$$\mathcal{P} = (P_\theta = \Pi_{\geq,\theta} + P'_\theta)_{\theta\in B},$$

where for each θ, $\Pi_{\geq,\theta}$ is the APS spectral projection for C_θ and P'_θ is a spectral section for G_θ. We will set $\Pi_\geq = (\Pi_{\geq,\theta})_{\theta\in B}$ and $\mathcal{P}' = (P'_\theta)_{\theta\in B}$ so that $\mathcal{P} = \Pi_\geq + \mathcal{P}'$. If, in addition,

$$P'_\theta \circ \alpha + \alpha \circ P'_\theta = \alpha \tag{1.2}$$

on W_θ, then we shall call such a spectral section *symmetric*. A symmetric spectral section is thus a spectral section which is diagonal with respect to the splitting $\Omega_\theta = V_\theta \oplus W_\theta$, it is equal to the APS spectral projection on V_θ *and* it is a Cl(1) spectral section on W_θ with respect to the involution α, see [16].

PROPOSITION 1.2. *If the family of twisted signature operators* $\mathcal{D} = (\mathrm{D}_\theta)$ *satisfies assumption* (H1), *then it admits a symmetric spectral section* $\mathcal{P}$ *defined by a smooth family* $(P_\theta)_\theta$ *as above.*

PROPOSITION 1.3. *If* $\mathcal{P}$ *and* $\mathcal{Q}$ *are two symmetric spectral sections for* $\mathcal{D} = (\mathrm{D}_\theta)$, *then*

$$[\mathcal{P} - \mathcal{Q}] = 0 \text{ in } K^0(B) \otimes \mathbb{Q}.$$

PROPOSITION 1.4. *Under the above assumptions, there is a well-defined eta form* $\widetilde{\eta} \in \Omega^*(B)/\mathrm{d}\Omega^*(B)$ *associated to the family* $\mathcal{D}$.

Proofs. In order to prove Proposition 1.2, we must show, according to [16, proposition 2], that the odd $\mathbb{Z}_2$-graded family $\mathcal{G} = (G_\theta)$ has zero K^0-index class. To this end, we recall the map Θ: $\Omega^*_\theta \to \Omega^*_\theta$ equal to $(-1)^p\star$ on Ω^{2p-1}_θ and Ω^{2p}_θ. Then we have

$$\Theta V_\theta = V_\theta, \quad \Theta\Omega^<_\theta = \Omega^>_\theta, \quad \Theta\Omega^>_\theta = \Omega^<_\theta, \quad \mathrm{D}_\theta\Theta = \Theta\mathrm{D}_\theta.$$

Thus $G^-_\theta = \Theta G^+_\theta\Theta$. Consider now the family ΘG^+_θ: $\Omega^<_\theta \to \Omega^<_\theta$. This is a *self-adjoint* family; thus the K^0-index class is equal to zero. On the other hand, the latter index class is precisely $\mathrm{Ind}(\mathcal{G}^+)$ since Θ defines a family of isomorphisms. Thus, $\mathrm{Ind}(\mathcal{G}^+) = 0$ in $K^0(B)$ as required. Notice that the space W_θ is not the space of sections of a vector bundle. However, an inspection of the proof of the existence of spectral sections in [15, 16] shows that this is not a problem.

In order to prove Proposition 1.3, we consider two symmetric spectral sections $\mathcal{P}$, $\mathcal{Q}$. With respect to the decomposition $\Omega^*_\theta = V_\theta \oplus W_\theta$ we write $\mathcal{P} = \Pi_\geq + \mathcal{P}'$ and $\mathcal{Q} = \Pi_\geq + \mathcal{Q}'$ as before the statement of Proposition 1.2. Then we obtain

$$\begin{aligned}[\mathcal{P} - \mathcal{Q}] &= [\Pi_\geq + \mathcal{P}' - \Pi_\geq - \mathcal{Q}'] = [\mathrm{Id} - \alpha\mathcal{P}'\alpha - \mathrm{Id} + \alpha\mathcal{Q}'\alpha]\\ &= [\alpha\mathcal{Q}'\alpha - \alpha\mathcal{P}'\alpha] = [\mathcal{Q}' - \mathcal{P}'] = [\mathcal{Q} - \mathcal{P}] = -[\mathcal{P} - \mathcal{Q}].\end{aligned}$$

Thus $2[\mathcal{P} - \mathcal{Q}] = 0$ in $K^0(B)$; Proposition 1.3 is proved.

Proposition 1.4 follows at once from Propositions 1.3, 1.2 and the jump formula for eta forms proved in [15, proposition 17]. □

Notice that if B is a torus, $B = T^k$, then $K^0(T^k)$ has no torsion and $[\mathcal{P} - \mathcal{Q}] = 0$ in $K^0(T^k)$ for two symmetric spectral sections.

Remark. For simplicity, we have proved the existence of symmetric spectral sections and, thus, of a canonical eta form, under assumption (H1). Suppose, more generally, that the following holds:

(H2) *The space H^m_θ of twisted harmonic forms in degree m is of constant dimension in $\theta \in \widehat{\mathbb{Z}^k}$.*

Under this weaker assumption we can still prove Proposition 1.1 and give the notion of symmetric spectral section: for this it is enough to consider the new decomposition

$$\Omega^*_\theta = V_\theta \oplus \Omega^<_\theta \oplus \Omega^>_\theta \text{ with } V_\theta = H^{m-1}_\theta \oplus \mathrm{d}^*\Omega^m_\theta \oplus \mathrm{d}\Omega^{m-1}_\theta \oplus H^m_\theta.$$

It is clear that Propositions 1.2, 1.3 and 1.4 still hold.

2. $\mathbb{Z}^k$-Galois Coverings and Higher Eta Forms

We shall now specialize the above general picture to the Lustzig family. We consider a closed oriented $(2m-1)$-dimensional compact manifold N and a $\mathbb{Z}^k$-Galois covering $\mathbb{Z}^k \to \widetilde{N} \to N$. (At some point, we shall take N to be the boundary of an even-dimensional manifold M with fundamental group $\mathbb{Z}^k$ and $\widetilde{N} = \partial\widetilde{M}$, $\widetilde{M}$ being the universal cover of M.)

We denote by ν a classifying map ν: $N \to B\mathbb{Z}^k$ and follow the notations of section 3 of [10]. The space $B\mathbb{Z}^k$ is a k-dimensional torus. It is the dual torus to $T^k = \widehat{\mathbb{Z}^k} = \mathrm{Hom}(\mathbb{Z}^k, U(1))$. On the product $(T^k)^* \times T^k$, there is a canonical Hermitian line bundle H with a canonical Hermitian connection ∇^H. The bundle H is flat when restricted to any fibre of the projection $(T^k)^* \times T^k \to T^k$, see [13]. Using the map $\nu \times \mathrm{Id}$: $N \times T^k \to (T^k)^* \times T^k$, we obtain a line bundle E_0 on $N \times T^k$ with a natural Hermitian (pulled-back) connection ∇^{E_0}. We are now within the framework of the previous section, namely we have a fibration of odd-dimensional closed manifolds ϕ: $N \times T^k \to T^k$, with fibres diffeomorphic to a fixed $(2m-1)$-dimensional manifold, a Hermitian line bundle E_0 over the total space with a flat structure in the fibre directions.

Let $\mathcal{D} = (\mathrm{D}_\theta)_{\theta \in T^k}$ the associated family of twisted odd-signature operators. We assume that $\mathcal{D}$ satisfies assumption (H1).

DEFINITION 2.1. If the family $\mathcal{D}$ satisfies assumption (H1) we define the higher eta invariant for the signature operator of the covering $\mathbb{Z}^k \to \widetilde{N} \to N$,

$\widetilde{\eta} \in \Omega^*(T^k)/\mathrm{d}\Omega^*(T^k)$, as the $\mathcal{P}$-eta form $\widehat{\eta}_P$ associated to one (and therefore any) *symmetric* spectral section $\mathcal{P}$.

It follows from Proposition 1.4 that this definition is well posed. For the definition of the $\mathcal{P}$-eta form we refer the reader to [15]. Notice that one can extend Definition 2.1 to the case in which $\mathcal{D}$ only satisfies assumption (H2).

3. Manifolds with Boundary

Let M be a $2m$-dimensional oriented manifold with boundary. We fix a b-metric g_M on M [14] which is a product near the boundary. We denote by ∇^M the associated Levi-Civita connection and by D the signature operator of M. Assume now that $\pi_1(M) = \mathbb{Z}^k$. Let $\nu_M\colon M \to B\mathbb{Z}^k$ be a classifying map for M. We identify $B\mathbb{Z}^k$ with the Albanese variety of M, $\mathrm{Alb}(M) = H_1(M,\mathbb{R})/H_1(M,\mathbb{Z})_{\mathrm{modTor}}$. As in [10], we choose ν_M to be constant in the normal direction near the boundary and in the same homotopy class of the Albanese map. The dual torus to $\mathrm{Alb}(M)$ is the Picard variety $\mathrm{Pic}(M)$. As in the previous section, we denote by H and ∇^H the canonical Hermitian line bundle of $\mathrm{Alb}(M) \times \mathrm{Pic}(M)$ and its canonical partially flat connection. We denote by E_0 the pulled-back bundle $(\nu_M \times \mathrm{Id})^* H$ on $M \times \mathrm{Pic}(M)$; we endow this bundle with the pulled-back connection ∇^{E_0}. In this way, we obtain a family of b-differential operators on M parametrized by $\mathrm{Pic}(M)$, $\mathcal{D} = (\mathrm{D}_\theta)_{\theta\in\mathrm{Pic}(M)}$, with D_θ equal to the signature operator with values in the flat line bundle $E_0|_{M\times\theta}$. We shall call this family the *Lusztig family of M* and pose $\mathrm{Pic}(M) = T^k$ and $B\mathbb{Z}^k = (T^k)^*$.

We denote by $\mathcal{D}_0 = (\mathrm{D}_{0,\theta})_{\theta\in T^k}$ the boundary family of $\mathcal{D}$. It is important to make the identifications used in [15, 16] explicit. It suffices to specify these identifications for the signature operator on M: we identify ${}^b\Lambda^+(M)|_{\partial M}$ with $\Lambda^*(\partial M)$ through the map $M^+ : {}^b\Lambda^+(M)|_{\partial M} \to \Lambda^*(\partial M)$, $M^+ \equiv (\tau^+)^{-1}$, with

$$\tau^+\colon \Lambda^*(\partial M) \to {}^b\Lambda^+(M)|_{\partial M} \quad \tau^+(\alpha_\partial) = \alpha_\partial + \mathrm{cl}(\omega_M)\alpha_\partial,$$

ω_M being the chirality operator of M defined by the b-metric g_M: $\mathrm{cl}(\omega_M)^2 = \mathrm{Id}$. We then define $M^-\colon {}^b\Lambda^-(M)|_{\partial M} \to \Lambda^*(\partial M)$ to be $M^+ \circ \mathrm{cl}(\sqrt{-1}\,\mathrm{d}x/x)$. The following lemma gives a concrete expression of M^+ and we leave the easy proof to the reader.

LEMMA 3.1. *Let $\omega_{\partial M}$ be the chirality operator of ∂M. Let*

$$\alpha = \alpha^0 + \frac{\mathrm{d}x}{x} \wedge \alpha^1 \in C^\infty(\partial M \times [0,1];\ {}^b\Lambda^*_\theta(M))$$

be a b-differential form where α^0, α^1 do not involve $\mathrm{d}x/x$. Then $\mathrm{cl}(\omega_M)(\alpha) = \alpha$ if and only if $\alpha^0 = -\,\mathrm{cl}(\omega_{\partial M})(\alpha^1)$. Moreover, if $\mathrm{cl}(\omega_M)(\alpha) = \alpha$ then $M^+(\alpha) = \alpha^0$.

With these identifications, the boundary family $\mathcal{D}_0 = (\mathrm{D}_{0,\theta})_{\theta \in T^k}$ becomes a family of twisted signature operators of the type described in Section 1. We now make the following assumption on the *boundary* family:

(H1) The family of boundary signature Laplacians $\Delta_{0,\theta}$, $\theta \in \mathrm{Pic}(M)$, is smoothly invertible in degree $m = \dim M/2$.

Equivalently, we could assume that

(H1') The signature Laplacian on $\partial\widetilde{M}$ is L^2-invertible in degree m.

Thanks to Proposition 1.2, we can thus fix a *symmetric* spectral section $\mathcal{P} = (P_\theta)$ for $\mathcal{D}_0$.

DEFINITION 3.2. Let M be an orientable manifold with boundary with $\pi_1(M) = \mathbb{Z}^k$ and satisfying assumption (H1). The higher index class of M, in $K^0(\mathrm{Pic}(M)) \equiv K^0(\widehat{\mathbb{Z}^k})$, is the index class associated to the generalized Atiyah–Patodi–Singer boundary value problem $\mathrm{Ind}(\mathcal{D}^+, \mathcal{P})$ fixed by one (and therefore any) symmetric spectral section $\mathcal{P}$.

That this is a well-posed definition follows immediately from Proposition 1.3 and from the relative index theorem of [15] which states that $\mathrm{Ind}(\mathcal{D}^+, \mathcal{P}) - \mathrm{Ind}(\mathcal{D}^+, \mathcal{Q}) = [\mathcal{Q} - \mathcal{P}]$. We denote the higher index class of M by $\mathrm{Ind}(\mathcal{D}^+)$. Notice that assumption (H1) also implies the existence of a higher eta invariant $\widetilde{\eta}_\partial \in \Omega^*(T^k)/\mathrm{d}\Omega^*(T^k)$.

Making use of the APS family index theorem proved in [15], we can now state the following *higher APS index theorem for the signature operator*:

THEOREM 3.3. *Let M be a manifold with boundary with fundamental group equal to $\mathbb{Z}^k$ and satisfying assumption* (H1'). *For the Chern character of the canonical index class of M the following formula holds:*

$$\mathrm{Ch}(\mathrm{Ind}(\mathcal{D}^+)) = \int_M L(M, \nabla^M) \wedge \mathrm{e}^{-(\nabla^{E_0})^2} - \frac{1}{2}\widetilde{\eta}_\partial \quad \text{in } H^{2*}_{\mathrm{dR}}(\mathrm{Pic}(M)) \equiv H^{2*}_{\mathrm{dR}}(\widehat{\mathbb{Z}^k}). \tag{3.1}$$

The curvature of the line bundle E_0 can be explicitly computed, see [10, 13].

Remark. The higher index class $\mathrm{Ind}(\mathcal{D}^+)$ can also be defined under the weaker assumption considered at the end of Section 2, namely:

(H2) *The signature Laplacian on $\partial\widetilde{M}$ acting on forms of degree $m = \dim M/2$ has a gap at* 0.

There is a corresponding higher APS signature index theorem. We leave the precise statement and the proof to the reader.

4. Noncommutative Symmetric Spectral Sections

We now pass to the noncommutative case. We first assume that (N, g) is a closed $(2m-1)$-dimensional compact orientable Riemannian manifold and that $\Gamma \to \widetilde{N} \to N$ is a Galois covering of N. We endow $\widetilde{N}$ with the lifted metric $\tilde{g}$. We consider the (odd) signature operators D on N and $\widetilde{\mathrm{D}}$ on $\widetilde{N}$. We denote by $\mathcal{D}$ the signature operator with values in the flat $C_r^*(\Gamma)$-bundle $\mathcal{V} = \widetilde{N} \times_\Gamma C_r^*(\Gamma)$; thus

$$\mathcal{D}\colon \Omega^*(N, \mathcal{V}) \equiv C^\infty(N, \mathcal{V} \otimes \Lambda^*(N)) \to C^\infty(N, \mathcal{V} \otimes \Lambda^* N) \equiv \Omega^*(N, \mathcal{V}).$$

Notice that $\mathcal{D} = \pm\sqrt{-1}(\epsilon \star \mathrm{d}_{\mathcal{V}} - \mathrm{d}_{\mathcal{V}}\star)$ with $\mathrm{d}_{\mathcal{V}}$ the exterior differentiation with values in $\mathcal{V}$. The space $\Omega^*(N, \mathcal{V})$ has a natural structure of pre-Hilbert $C_r^*(\Gamma)$-module; we denote by $\Omega^*_{L^2}(N, \mathcal{V})$ the associated Hilbert module.

We now make our fundamental assumption:

(H1) *The signature Laplacian $\Delta_{\widetilde{N}}$ is L^2-invertible in degree m.*

Remark. Notice that the assumption that $\Delta_{\widetilde{N}}$ is L^2-invertible *in each degree* would be an unreasonable one, see [11]. It is precisely this remark that makes the treatment of higher index theory for the signature operator more complicated. Notice, finally, that assumption (H1) implies the L^2-invertibility of $\Delta_{\widetilde{N}}$ in degree $(m-1)$.

We then have the following proposition:

PROPOSITION 4.1. *Under assumption* (H1) *there exist orthogonal decompositions:*

$$\begin{aligned} \Omega^m(N, \mathcal{V}) &= \mathrm{d}_{\mathcal{V}}\Omega^{m-1}(N, \mathcal{V}) \oplus \mathrm{d}^*_{\mathcal{V}}\Omega^{m+1}(N, \mathcal{V}), \\ \Omega^m_{L^2}(N, \mathcal{V}) &= \overline{\mathrm{d}_{\mathcal{V}}\Omega^{m-1}(N, \mathcal{V})} \oplus \overline{\mathrm{d}^*_{\mathcal{V}}\Omega^{m+1}(N, \mathcal{V})}. \end{aligned} \tag{4.1}$$

Proof. Let us first prove the second decomposition. Assumption (H1) implies that $\operatorname{Im} \Delta^{[m]}_{\mathcal{V}} = \Omega^m_{L^2}(N, \mathcal{V})$. Moreover, we certainly have

$$\operatorname{Im} \Delta^{[m]}_{\mathcal{V}} \subset \overline{\mathrm{d}_{\mathcal{V}}\Omega^{m-1}(N, \mathcal{V})} \oplus \overline{d^*_{\mathcal{V}}\Omega^{m+1}(N, \mathcal{V})};$$

since the two terms in the above right-hand side are orthogonal, we immediately obtain the second decomposition in Equation (4.1). Now we get the left-hand side formula by first observing that $\Delta^m_{\mathcal{V}}$ is an elliptic pseudo-differential operator inducing an isomorphism on $\Omega^m(N, \mathcal{V})$ and then proceeding as above. □

A similar decomposition holds for $\Omega^{m-1}(N,\mathcal{V})$. Thanks to the above proposition, we can consider the orthogonal decomposition $\Omega^*(N,\mathcal{V}) = V \oplus W$ with (we abbreviate $\Omega^*(N,\mathcal{V}) \equiv \Omega^*_{\mathcal{V}}$)

$$\begin{aligned} V &= \mathrm{d}^*_{\mathcal{V}}\Omega^m_{\mathcal{V}} \oplus \mathrm{d}_{\mathcal{V}}\Omega^{m-1}_{\mathcal{V}}, \quad W = \Omega^<_{\mathcal{V}} \oplus \Omega^>_{\mathcal{V}},\\ \Omega^<_{\mathcal{V}} &= \Omega^0_{\mathcal{V}} \oplus \Omega^1_{\mathcal{V}} \oplus \cdots \Omega^{m-2}_{\mathcal{V}} \oplus \mathrm{d}_{\mathcal{V}}\Omega^{m-2}_{\mathcal{V}},\\ \Omega^>_{\mathcal{V}} &= \mathrm{d}^*_{\mathcal{V}}\Omega^{m+1}_{\mathcal{V}} \oplus \Omega^{m+1}_{\mathcal{V}} \cdots \oplus \Omega^{2m-1}_{\mathcal{V}} \end{aligned}$$

We still denote by α the natural involution on W equal to the identity on $\Omega^<_{\mathcal{V}}$ and minus the identity on $\Omega^>_{\mathcal{V}}$. The operator $\mathcal{D}$ splits diagonally with respect to the decomposition $\Omega^*(N,\mathcal{V}) = V \oplus W$. Moreover, its restriction to W anticommutes with α.

Recall that the notion of spectral section has been extended to the noncommutative context by Wu [21]. Proceeding as in Section 1, we see that under assumption (H1), the operator $\mathcal{D}$ has a trivial index class in $K_1(C^*_r(\Gamma))$. Thus, according to theorem 2.2 in [21] (and its sharpening in [10, th. 2.6]), $\mathcal{D}$ admits a spectral section $\mathcal{P} \in \Psi^0_{C^*_r(\Gamma)}(N, \Lambda^*(N) \otimes \mathcal{V})$.

DEFINITION 4.2. The spectral section $\mathcal{P}$ is *symmetric* if $\mathcal{P}$ is diagonal with respect to the splitting $\Omega^*(N,\mathcal{V}) = V \oplus W$ and

$$\mathcal{P}|_V = \Pi_{\geq}, \quad \mathcal{P}|_W \circ \alpha + \alpha \circ \mathcal{P}|_W = \alpha.$$

In the commutative case, we remarked that this last condition simply means that $\mathcal{P}|_W$ is a Cl(1)-spectral section for $\mathrm{D}|_W$, the latter operator being $\mathbb{Z}_2$-graded odd with respect to the grading given by α. The existence of Cl(1)-spectral sections for odd $\mathbb{Z}_2$-graded operators was left open by Wu. Thus, we need to prove the following lemma.

LEMMA 4.3. *Let A be a unital C^*-algebra and let $\mathcal{H}$ be a $\mathbb{Z}_2$-graded full Hilbert module for A. We denote by α the grading on $\mathcal{H}$. Let $\mathcal{D}$ be an odd, self-adjoint, densely defined, A-linear, unbounded regular operator. If the K_0-index class of $\mathcal{D}^+$ is trivial in $K_0(A)$, then there exists a* Cl(1)*-spectral section $\mathcal{P}$ for $\mathcal{D}$, i.e. a spectral section with the property that*

$$\mathcal{P} \circ \alpha + \alpha \circ \mathcal{P} = \alpha. \tag{4.2}$$

Proof. Observe that since $\mathcal{D}$ is odd and self-adjoint, $\mathcal{D}$ has trivial K_1-index. Thus, there certainly exists a spectral section $\mathcal{P}'$ for $\mathcal{D}$. The basic remark to be made is that, given a free submodule H^n of $\mathcal{H}$, we can find a $R > 0$ such that if ϕ is a compactly supported smooth function with values in [0, 1] and equal to one on $[-R, R]$, then $\phi(\mathcal{D})(\mathcal{H})$ contains a free sub-module isomorphic to H^n. Using proposition 2.5 and corollary 2.6 in [21] together with the above remark, it is easy to see that for any free submodule H^n of $\mathcal{H}$ there exists a new spectral section

$\mathcal{Q}'$ with the following two properties: $\mathcal{Q}' \circ \mathcal{P}' = \mathcal{P}' \circ \mathcal{Q}' = \mathcal{P}'$ and the range of $\mathcal{Q}' - \mathcal{P}'$ contains a free sub-module isomorphic to H^n. The proof proceeds now as in [16]. We omit the details. □

PROPOSITION 4.4. *If the signature Laplacian on $\widetilde{N}$ satisfies assumption* (H1) *then $\mathcal{D}$ admits a symmetric spectral section $\mathcal{P}$. Moreover, if $\mathcal{Q}$ is a second symmetric spectral section, then*

$$[\mathcal{P} - \mathcal{Q}] = 0 \text{ in } K_0(C_r^*(\Gamma)) \otimes \mathbb{C}. \tag{4.3}$$

Proof. We proceed as in the commutative case, see the proof of Proposition 1.2: the existence of Cl(1)-spectral sections for odd $\mathbb{Z}_2$-graded operators with vanishing K_0-index is provided by Lemma 4.3 above. □

Remark. If we relax assumption (H1) and only assume that

(H2) *The signature Laplacian $\Delta_{\widetilde{N}}$ has a gap in middle degree,*

then we can consider the finitely generated projective modules of harmonic forms $\mathcal{H}^m \subset \Omega^m_{\mathcal{V}}$, $\mathcal{H}^{m-1} \subset \Omega^{m-1}_{\mathcal{V}}$. Using once again the Mishenko–Fomenko calculus, we obtain a Hodge decomposition

$$\begin{aligned} \Omega^m(N, \mathcal{V}) &= \mathcal{H}^m \oplus \mathrm{d}_{\mathcal{V}}\Omega^{m-1}(N, \mathcal{V}) \oplus \mathrm{d}^*_{\mathcal{V}}\Omega^{m+1}(N, \mathcal{V}), \\ \Omega^m_{L^2}(N, \mathcal{V}) &= \mathcal{H}^m \oplus \overline{\mathrm{d}_{\mathcal{V}}\Omega^{m-1}(N, \mathcal{V})} \oplus \overline{\mathrm{d}^*_{\mathcal{V}}\Omega^{m+1}(N, \mathcal{V})} \end{aligned}$$

and similarly for $\Omega^{m-1}(N, \mathcal{V})$, $\Omega^{m-1}_{L^2}(N, \mathcal{V})$. Using this Hodge decomposition we can extend the notion of a symmetric spectral section as in Section 1 and prove the analogue of Proposition 4.4.

5. Higher Eta Invariants and Higher ρ-Invariants for Signature Operators

Let (N, g) and $\Gamma \to \widetilde{N} \to N$ be as in the previous section. Recall that if Γ is of polynomial growth with respect to a word metric, then we can consider the dense subalgebra $\mathcal{B}^\infty \subset C_r^*(\Gamma)$ of rapidly decreasing functions in $C_r^*(\Gamma)$ and $\mathcal{V}^\infty = \widetilde{N} \times_\Gamma \mathcal{B}^\infty$.

PROPOSITION 5.1. *Let assumption* (H1) *hold and let the group Γ be of polynomial growth. Then we can always choose a symmetric $\mathcal{B}^\infty$-spectral section, i.e. a symmetric spectral section in $\Psi^0_{\mathcal{B}^\infty}(N, \mathcal{V}^\infty \otimes \Lambda^*(N))$.*

Proof. The proof follows immediatly from theorem 2.6 of [6] and Proposition 4.4. □

We can now give the following fundamental definition:

DEFINITION 5.2. If assumption (H1) holds and Γ is of polynomial growth, we define the higher eta invariant $\widetilde{\eta}$ of the signature operator $\mathcal{D}$ associated to $\widetilde{N} \to N$ as the $\mathcal{P}$-higher eta invariant $\widehat{\eta}_{\mathcal{P}} \in \overline{\overline{\widehat{\Omega}}}_*(\mathcal{B}^\infty)/d\overline{\overline{\widehat{\Omega}}}_*(\mathcal{B}^\infty)$ associated to one (and therefore any) symmetric $\mathcal{B}^\infty$-spectral section $\mathcal{P}$ for $\mathcal{D}$.

It follows from Equation (4.3) and formula (5.1) in [6] that this definition is well posed. For the definition of $\mathcal{P}$-higher eta invariant, we refer the reader to [6]. The space $\overline{\overline{\widehat{\Omega}}}_*(\mathcal{B}^\infty)$ is the space of noncommutative differential forms with rapidly decreasing coefficients modulo the closure of the space of graded commutators.

We shall now investigate the variational properties of $\widetilde{\eta} \equiv \widehat{\eta}_{\mathcal{P}}$. The higher eta invariant just defined depends on several choices: Lott's connection ∇ depends on a function h on $\widetilde{N}$; the signature operator involves the metric and, of course, we had to choose a trivializing operator $\widetilde{A}^0_{\mathcal{P}}$ associated with the symmetric spectral section $\mathcal{P}$. That the higher eta invariant is independent of the choice of the particular trivializing operator is proved in [6]. Thus, we consider the variation of $\widetilde{\eta}$ with respect to the function h and to the metric g. Recall that assumption (H1) is *independent* of the metric g (it is a homotopy invariant condition).

Consider a 1-parameter family of input informations, with parameter $r \in [0, 1]$. We choose symmetric spectral sections $\mathcal{Q}_0$ for $\mathcal{D}(0)$ and $\mathcal{Q}_1$ for $\mathcal{D}(1)$. Since $\{\mathcal{D}(r)\}$ has trivial index class in $K_1(C[0,1] \otimes \mathcal{B}^\infty)$, we can choose a spectral section $\mathcal{P} = \{\mathcal{P}_r\}$ associated to $\{\mathcal{D}(r)\}$. We can and we shall choose $\mathcal{P}_r$ symmetric for each r. By definition of higher spectral flow [6, 21] and by Proposition 4.4, we have that the higher spectral flow $\mathrm{sf}(\{\mathcal{D}(r)\}; \mathcal{Q}_0, \mathcal{Q}_1)$ from $(\mathcal{D}_0, \mathcal{Q}_0)$ to $(\mathcal{D}_1, \mathcal{Q}_1)$ is *zero* in $K_0(C^*_r(\Gamma)) \otimes \mathbb{C}$. Recall now theorem 5.3 of [6]:

$$\widehat{\eta}(\mathcal{D}_1, \mathcal{Q}_1) - \widehat{\eta}(\mathcal{D}_0, \mathcal{Q}_0) = 2\,\mathrm{Ch}(\mathrm{sf}(\{\mathcal{D}(r)\}; \mathcal{Q}_0, \mathcal{Q}_1)) - \int_0^1 a_0(r)\,\mathrm{d}r \tag{5.1}$$

with a_0 *local* and more precisely given by the regularized limit, as $s \downarrow 0$, of

$$\int_0^1 \frac{2}{\sqrt{\pi}} \mathrm{STR}_{\mathrm{Cl}(1)} \left[\frac{\mathrm{d}}{\mathrm{d}r} (\mathbb{B}_s(r)) \exp(-\mathbb{B}_s^2(r)) \right]$$

with

$$\mathbb{B}_s(r) = \Upsilon \nabla(r) + \sigma s (\widetilde{\mathrm{D}}(r) + \phi(s) \widetilde{A}^0_{\mathcal{P}_r}), \quad r \in [0, 1],$$

($\phi(s)$ being a smooth function which is zero for $s < 1$ and one for $s > 2$). Since $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are symmetric, the left-hand side of Equation (5.1) is equal to the difference of the higher eta invariants of $\mathcal{D}_1$ and $\mathcal{D}_0$ and we can thus conclude that this difference is *local*. It is important to point out that this local expression involves Lott's bi-form ω (see [6, prop. 27]). Thus, if we decompose $\overline{\overline{\widehat{\Omega}}}_*(\mathcal{B}^\infty)$ into a direct sum of subcomplexes labeled by the conjugacy classes of Γ, we see that the difference of two higher eta invariants defined in terms of two different input data is concentrated in the subcomplex labeled by $\langle e \rangle$, the conjugacy class of the identity.

Recall [10] that given $x \in \Gamma$ it is possible to define a cochain complex C_x^k with $H_x^k \cong H^k(N_{\langle x\rangle})$; here $N_{<x>}$ is the quotient $\{x\}\backslash C_x$, with $\{x\}$ equal to the cyclic group generated by x and C_x the centralizer of x in Γ; the cyclic cohomology of $\mathbb{C}\Gamma$ can be expressed in terms of $H^k(N_{\langle x\rangle})$. Given $c \in Z_x^k$, there is a cyclic cocycle $\tau_c \in ZC^k(\mathbb{C}\Gamma)$ defined as

$$\tau_c(\gamma_0, \ldots, \gamma_k) = \begin{cases} 0 & \text{if } \gamma_0 \cdots \gamma_k \notin \langle x\rangle \\ c(g, g\gamma_0, \ldots, g\gamma_0 \ldots \gamma_{k-1}) & \text{if } \gamma_0 \cdots \gamma_k = g^{-1}xg. \end{cases} \tag{5.2}$$

We have thus proved

PROPOSITION 5.3. *Let τ_c be the cyclic cocycle given by Equation (5.2). We assume that N satisfies* (H1) *and that Γ is of polynomial growth. Suppose that τ_c extends as a cyclic cocycle of $\mathcal{B}^\infty$. If $x \neq e$, then the pairing $\langle \widetilde{\eta}, \tau_c\rangle$ is independent of h and of the Riemannian metric g on N.*

Let $\widetilde{\eta} = \oplus_{\langle x\rangle \in \langle\Gamma\rangle} \widetilde{\eta}(\langle x\rangle)$ the expression of the higher eta invariant in terms of the subcomplexes of $\overline{\widehat{\Omega}}_*(\mathcal{B}^\infty)$ labeled by the conjugacy classes of Γ. Following what Lott has done in the invertible case [6], we can now define *the higher ρ-invariant for a signature operator* satisfying assumption (H1) as $\widetilde{\rho} = \oplus_{\langle x\rangle \neq \langle e\rangle} \widetilde{\eta}(\langle x\rangle)$. This is a closed noncommutative form. Moreover, because of the above remarks (and the properties of Lott's bi-form) its class in $\overline{H}_*(\mathcal{B}^\infty)$ is a *metric invariant*. It would be very interesting to express this class as the Chern character of a secondary signature class in $K_0(\mathcal{B}^\infty)$. Notice that the numbers appearing in Proposition 5.3 are precisely the pairing of $\widetilde{\rho}$ with τ_c.

6. A Higher APS Index Theorem for the Signature Operator

Let M be a $2m$-dimensional compact orientable manifold with boundary. We fix an exact b-metric g on M [14] and consider the lifted metric $\tilde{g}$ on the universal cover $\Gamma \equiv \pi_1(M) \to \widetilde{M} \to M$. We denote by ∇^M the Levi-Civita connection associated to g.

We denote by D, $\widetilde{\mathrm{D}}$ and $\mathcal{D}$ the signature operators on M, $\widetilde{M}$ and on M with values in the flat bundle $\mathcal{V}$ defined by $\widetilde{M}$ and $C_r^*(\Gamma)$. We denote by D_0, $\widetilde{\mathrm{D}_0}$ and $\mathcal{D}_0$ the associated boundary operators. Notice that the boundary-covering $\Gamma \to \partial\widetilde{M} \to M$ and the operator

$$\mathcal{D}_0\colon C^\infty(\partial M, \mathcal{V}|_{\partial M} \otimes \Lambda^*(\partial M)) = \Omega^*(\partial M, \mathcal{V}|_{\partial M}) \to \Omega^*(\partial M, \mathcal{V}|_{\partial M})$$

are of the type considered in the previous section.

We now make the assumption

(H1) *The boundary signature Laplacian $\Delta_{\partial\widetilde{M}}$ is L^2-invertible in degree m.*

In order to prove that this assumption implies the existence of a *canonical* index class, as in the commutative case (see Definition 3.2), we need to extend to the noncommutative context the relative index theorem proved in [15]. Since the index class $\mathrm{Ind}(\mathcal{D}^+, \mathcal{P})$ is defined through the trivializing perturbation $\mathcal{A}^0_{\mathcal{P}}$ (on ∂M) and the associated regularizing operator $\mathcal{D}^+ + \mathcal{A}^+_{\mathcal{P}}$ (on M), this needs an extra argument.

Although we only state it for the signature operator, it will be clear that the proof applies to any Dirac-type operator associated to an exact b-metric.

PROPOSITION 6.1. *Let $\mathcal{D}$ be the signature operator with values in $\mathcal{V}$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two spectral sections for the boundary operator $\mathcal{D}_0$, then*

$$\mathrm{Ind}(\mathcal{D}^+, \mathcal{P}_2) - \mathrm{Ind}(\mathcal{D}^+, \mathcal{P}_1) = [\mathcal{P}_1 - \mathcal{P}_2]. \tag{6.1}$$

Proof. We choose $\mathcal{P}_1$ and $\mathcal{P}_2$ as in the proof of proposition 16 in [15], using the results of Wu [21]. We obtain a family of trivializing operators $\mathcal{D}_0(r)$ on ∂M, $r \in [-1, 1]$, together with the corresponding regularizing operators $\mathcal{D}(r)$ on M. See [MP 1] for the details. By construction, we have

$$\mathrm{Ind}(\mathcal{D}^+, \mathcal{P}_2) = \mathrm{Ind}(\mathcal{D}(-1)^+) \quad \mathrm{Ind}(\mathcal{D}^+, \mathcal{P}_1) = \mathrm{Ind}(\mathcal{D}(1)^+).$$

The family $\mathcal{D}(r)$ on M is a family of $C^*_r(\Gamma)$-Fredholm operators on unweighted b-Sobolev spaces except at $r = 0$. At $r = 0$, the boundary operator associated to $\mathcal{D}(0)$, i.e. the trivializing operator $\mathcal{D}_0(0)$, has null space which is a finitely generated projective module equal to the range of $(\mathcal{P}_1 - \mathcal{P}_2)$. This means, see [5], that the operator $\mathcal{D}(0)$ will be $C^*_r(\Gamma)$-Fredholm acting on $\pm t$-weighted b-Sobolev spaces, $t > 0$, t small. Let $\mathrm{Ind}_t(\mathcal{D}(0)^+)$ be the t-weighted index class. Then, proceeding as in the commutative case, it is easy to see that

$$\mathrm{Ind}_{(-t)}(\mathcal{D}(0)^+) - \mathrm{Ind}_t(\mathcal{D}(0)^+) = [\mathrm{null}(\mathcal{D}_0(0))] \equiv [\mathcal{P}_1 - \mathcal{P}_2]$$

(the point being here that we have different weights but *the same operator* $\mathcal{D}(0)^+$). On the other hand, a simple homotopy argument shows that

$$\mathrm{Ind}_{(-t)}(\mathcal{D}(0)^+) = \mathrm{Ind}_{(-t)}(\mathcal{D}(-1)^+), \quad \mathrm{Ind}_{(t)}(\mathcal{D}(0)^+) = \mathrm{Ind}_{(t)}(\mathcal{D}(1)^+).$$

We then have

$$\begin{aligned}
\mathrm{Ind}(\mathcal{D}^+, \mathcal{P}_2) - \mathrm{Ind}(\mathcal{D}^+, \mathcal{P}_1) &= \mathrm{Ind}(\mathcal{D}(-1)^+) - \mathrm{Ind}(\mathcal{D}(1)^+) \\
&= \mathrm{Ind}_{(-t)}(\mathcal{D}(-1)^+) - \mathrm{Ind}_{(t)}(\mathcal{D}(1)^+) \\
&= \mathrm{Ind}_{(-t)}(\mathcal{D}(0)^+) - \mathrm{Ind}_{(t)}(\mathcal{D}(0)^+) \\
&= [\mathrm{null}(\mathcal{D}_0(0))] = [\mathcal{P}_1 - \mathcal{P}_2].
\end{aligned}$$

The proposition is proved. $\square$

DEFINITION 6.2. Let M be a compact orientable even-dimensional manifold with boundary with fundamental group Γ and let assumption (H1) hold. We define the canonical signature index class $\mathrm{Ind}(\mathcal{D}^+)$, in $K_0(C_r^*(\Gamma)) \otimes \mathbb{C}$, as the Atiyah–Patodi–Singer index class associated to one (and therefore any) *symmetric* spectral section $\mathcal{P}$ for the boundary operator $\mathcal{D}_0$.

The relative index (Propisition 6.1) and Proposition 4.4 imply that this is a well-posed definition.

We can now state the higher APS index theorem *for the signature operator*:

THEOREM 6.3. *Let assumption* (H1) *hold and let* Γ *be of polynomial growth. Then*

1. *There exist a symmetric* $\mathcal{B}^\infty$*-spectral section* $\mathcal{P}$ *for the boundary operator* $\mathcal{D}_0$.
2. *The higher eta invariant* $\widetilde{\eta}_\partial \equiv \widehat{\eta}_{\mathcal{P}} \in \overline{\overline{\widehat{\Omega}}}_*(\mathcal{B}^\infty)$ *and the higher index class* $\mathrm{Ind}(\mathcal{D}^+) = \mathrm{Ind}(\mathcal{D}^+, \mathcal{P})$ *are well defined, independent of the particular choice of symmetric spectral section.*
3. *Let* $\omega_M \in \Omega^*(M) \otimes \overline{\overline{\widehat{\Omega}}}_*(\mathcal{B}^\infty)$ *the biform introduced by Lott in his heat-kernel proof of the Connes–Moscovici higher index theorem; for the Chern character of the canonical index class* $\mathrm{Ind}(\mathcal{D}^+) = K_0(\mathcal{B}^\infty) \in K_0(C_r^*(\Gamma))$ *the following higher APS index formula holds:*

$$\mathrm{Ch}(\mathrm{Ind}(\mathcal{D}^+)) = \left[\int_M L(M, \nabla^M) \wedge \omega_M - \frac{1}{2}\widetilde{\eta}_\partial\right] \text{ in } \overline{H}_*(\mathcal{B}^\infty).$$

Proof. Part (1) follows from Proposition 4.4. Part (2) has already been proved. Part (3) is a direct application of the main theorem in [6].

DEFINITION 6.4. Let M be an even-dimensional oriented manifold with boundary with fundamental group Γ of polynomial growth and satisfying assumption (H1). Let $[c] \in H^*(\Gamma, \mathbb{C})$ and let $\tau_c \in HC^*(\mathbb{C}\Gamma)$ be the cyclic cohomology class associated to the corresponding (extendable) cyclic cocycle. Following [10] we define the higher signatures of M as the complex numbers

$$\sigma(M, \partial M; [c]) \equiv \left\langle \int_M L(M, \nabla^M) \wedge \omega - \frac{1}{2}\widetilde{\eta}_{\partial M} ,\ \tau_c \right\rangle. \tag{6.2}$$

We conjecture that these numbers are homotopy invariants of the pair $(M, \partial M)$. Thanks to the above higher APS index formula this conjecture would follow from the homotopy invariance of the canonical index class $\mathrm{Ind}(\mathcal{D}^+) \in K_0(C_r^*(\Gamma))$.

Remark. So far we have treated the case in which the manifold with boundary is of even dimension. The odd dimensional case is reduced to the even-dimensional one by suspension, as in [16]. In order to contain the size of this paper, we only sketch the arguments, leaving the details to the interested reader. Notice that not all results of [16] have been extended to the noncommutative context. Thus a rigorous treatment of what follows would indeed require a few careful explanations in which these results are established.

Let M be odd-dimensional, so that $\dim(\partial M) = 2m$. One can give, in general, the notion of a symmetric Cl(1)-spectral section for a twisted signature operator $\mathcal{D}$ on a closed $2m$-dimensional manifold N satisfying assumption (H1) (i.e. $\Delta_{\widetilde{N}}$ is L^2-invertible in degree m). The definition makes use of the decomposition, analogue of that considered after Proposition 4.1, $\Omega^*(N, \mathcal{V}) = V \oplus W$ with $V = \Omega^m_{\mathcal{V}} \oplus \mathrm{d}\Omega^m_{\mathcal{V}} \oplus \mathrm{d}^*\Omega^m_{\mathcal{V}}$ and W the orthogonal complement. Using the usual natural grading α on W, we get a decomposition $W = \Omega^<_{\mathcal{V}} \oplus \Omega^>_{\mathcal{V}}$ and a Cl(1)-spectral section $\mathcal{P}$ is said to be symmetric if it is diagonal with respect to this decomposition, it is the APS spectral projection on V and satisfies condition (1.2) on W. The difference class $[\mathcal{P} - \mathcal{Q}] \in K_1(C^*_r(\Gamma)) \otimes \mathbb{C}$ is zero for any pair of symmetric Cl(1)-spectral sections. Using this result and the jump formula for the odd $\mathcal{P}$-higher eta invariant (this follows from the noncommutative analogue of lemma 6 in [16]), one can define *the* odd higher eta invariant $\widetilde{\eta}$ of a closed even-dimensional manifold N satisfying (H1) and with a fundamental group of polynomial growth; $\widetilde{\eta} \in \overline{\widehat{\Omega}}_*(\mathcal{B}^\infty)/d\overline{\widehat{\Omega}}_*(\mathcal{B}^\infty)$. We can apply these arguments to $N = \partial M$ and to the Γ-cover $\partial\widetilde{M} \to \partial M$, $\Gamma = \pi_1(M)$. The use of symmetric Cl(1)-spectral sections gives both a higher eta invariant, $\widetilde{\eta}_\partial$, and a canonical signature index class $\mathrm{Ind}(\mathcal{D}) \in K_1(C^*_r(\Gamma))$. Observing that the suspended Dirac family considered in [16] in nothing but the Lustzig family of $S^1 \times M$ and extending to the noncommutative context the computation presented in [16], one can prove, by suspension, that the Chern character of $\mathrm{Ind}(\mathcal{D})$ is equal to the noncommutative de Rham class of the usual local integral minus $\widetilde{\eta}_\partial/2$.

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