

Refined intersection homology on non-Witt spaces

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We investigate a generalization to non-Witt stratified spaces of the intersection homology theory of Goresky–MacPherson. The second-named author has described the self-dual sheaves compatible with intersection homology, and the other authors have described a generalization of Cheeger’s L^2 de Rham cohomology. In this paper we first extend both of these cohomology theories by describing all sheaf complexes in the derived category of constructible sheaves that are compatible with middle perversity intersection cohomology, though not necessarily self-dual. Our main result is that this refined intersection cohomology theory coincides with the analytic de Rham theory on Thom–Mather stratified spaces. The word “refined” is motivated by the fact that the definition of this

cohomology theory depends on the choice of an additional structure (mezzo-perversity) which is automatically zero in the case of a Witt space.

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The ordinary (say, singular) homology of a singular space generally does not satisfy Poincaré duality. Remarkably, Goresky and MacPherson showed that in the case of a singular stratified space there is an adapted homology theory, intersection homology, that satisfies a generalized form of Poincaré duality [16]. On “Witt spaces” [26] the middle perversity intersection homology groups satisfy Poincaré duality and thus one obtains a signature, which is a bordism invariant. Working independently, Cheeger discovered that the de Rham cohomology of the L^2 -differential forms with respect to a certain adapted metric on the regular part of a Witt space also satisfies Poincaré duality [15]. In fact, Cheeger was able to show that his analytic theory is dual to the topological theory of Goresky–MacPherson.

A stratified pseudomanifold is Witt if, for each stratum of odd codimension, the lower middle perversity middle-dimensional rational intersection homology group of the link vanishes. For many purposes in topology and geometry, the class of Witt spaces is too restrictive. For example, Zucker’s reductive Borel–Serre compactification [27] of a locally symmetric space is generally not a Witt space. Applying this process of compactification to a Hilbert modular surface, one adds circle strata whose links are 2-tori, see [8]. Since the tori have nonzero middle homology, the compactification is not a Witt space. Non-Witt spaces are also necessary as a cycle reservoir to geometrically construct generalized homology theories (satisfying the Eilenberg–Steenrod axioms except the dimension axiom) with prescribed properties or desired coefficients, see [20]. Non-Witt spaces have for instance been used by Minatta in [23] to construct signature homology, a homology theory whose coefficients are \mathbb{Z} in nonnegative degrees divisible by 4, given by the signature, and 0 otherwise. (In fact, one of the earliest uses of Witt spaces was as a geometric cycle theory for connective KO -homology at odd primes.) Orbit spaces of smooth group actions on smooth manifolds are also generally non-Witt spaces. For example, consider the action of S^1 on $S^5 \subset \mathbb{C}^3$ such that the orbit space is $S^5/S^1 = \mathbb{CP}^2$, complex projective space. The action extends radially to an action on D^6 in such a way that the center of the disc becomes a fixed point. Then D^6/S^1 is the cone on \mathbb{CP}^2 and thus not Witt.

If a stratified space is not Witt then there are two middle perversity intersection sheaf complexes, one for the lower middle perversity \overline{m} and the other for the upper middle perversity \overline{n} , and the canonical morphism between them is not a quasi-isomorphism. In terms of the sheaf theoretic formulation of intersection homology in [17], this means that none of the Deligne sheaves $\mathbf{IC}_{\overline{p}}^\bullet$ is self-dual. A theory of self-dual sheaves on non-Witt spaces, compatible with intersection homology, has been developed by the second-named author in [4]. These sheaves “sit in between”

the two middle perversity intersection sheaves, are built from Lagrangian structures along the strata of odd codimension, and provide cohomology theories on non-Witt spaces satisfying Poincaré duality. Not every non-Witt space carries a self-dual sheaf compatible with intersection homology, and spaces that do are here termed *L-spaces*. In [5] it is shown that the signature of a perverse self-dual sheaf on an *L-space* is a bordism-invariant and does not depend on the choice of the self-dual sheaf. More generally, it is proved that the characteristic *L*-classes of such sheaves are independent of choices, so that *L*-spaces have well-defined *L*-classes, generalizing the Hirzebruch *L*-classes of smooth manifolds and the Goresky–MacPherson *L*-classes of Witt spaces to spaces that do not necessarily satisfy the Witt condition. As a historical aside, much of the research on intersection homology, Witt spaces and ultimately *L*-spaces was motivated by an early question of D. Sullivan: In 1970, he had posed the problem of finding classes of singular spaces that possess a bordism invariant signature. (It is clear that no such signature can exist on the class of all oriented stratified singular spaces.) It is precisely the class of *L*-spaces that has been used by Minatta [23] as a cycle theory for signature homology. In [8], Kulkarni and the second-named author have shown that the reductive Borel–Serre compactifications of Hilbert modular surfaces are *L*-spaces.

Cheeger’s L^2 de Rham cohomology theory has been extended to non-Witt spaces by some of the authors, [2], extending [14] and [1]. The failure of the Witt condition means analytically that one must impose a boundary condition at the singular strata in order that the exterior derivative and the Hodge de Rham operator have closed domains. These boundary conditions refine the middle perversities of Goresky–MacPherson and were termed *mezzoperversities* in [2]. Every analytic mezzoperversity has a dual mezzoperversity and the intersection pairing induces a generalized Poincaré duality, refining that of intersection homology. Not every non-Witt space carries a self-dual mezzoperversity, and smoothly stratified spaces that do are termed Cheeger spaces in [2].

Theorem 5.7 is the main result of this paper and states that the L^2 cohomology associated to a self-dual mezzoperversity and the cohomology of a certain self-dual sheaf coincide. (Note, however, that as discussed below, the former theory requires a Thom–Mather stratification, while the latter only requires a topologically stratified space.) More generally, since the analytic cohomology is defined for mezzoperversities that are not self-dual, we provide an extension of the theory of [4] to non-self-dual sheaves that are compatible with intersection homology.

Specifically, in Sec. 1 we define a category $RP(\hat{X})$ associated to a topological pseudomanifold \hat{X} which captures the candidate sheaves for refining intersection homology on non-Witt spaces. If \hat{X} is Witt, then $RP(\hat{X})$ consists up to isomorphism of a single complex of sheaves

$$\hat{X} \text{ Witt} \Rightarrow RP(\hat{X}) = \{\mathbf{IC}_{\bar{n}}^{\bullet} \cong \mathbf{IC}_{\bar{m}}^{\bullet}\},$$

while on a general pseudomanifold $RP(\hat{X})$ contains $\mathbf{IC}_{\bar{n}}^{\bullet}$ and $\mathbf{IC}_{\bar{m}}^{\bullet}$ and possibly many other sheaf complexes. In particular, the category $SD(\hat{X})$ of self-dual sheaves

of [4] is a full subcategory of the category of refined middle perversity sheaves,

$$SD(\widehat{X}) \subseteq RP(\widehat{X}).$$

Any complex \mathbf{S}^\bullet in $RP(\widehat{X})$ will have unique maps

$$\mathbf{IC}_{\overline{m}}^\bullet \rightarrow \mathbf{S}^\bullet \rightarrow \mathbf{IC}_{\overline{n}}^\bullet$$

compatible with the normalizations on the regular part of \widehat{X} .

In Sec. 2, we introduce the concept of a *topological mezzoperversity*, consisting of a compatible choice of locally constant sheaves at each stratum where the Witt condition does not hold. So this concept is empty in the Witt case. Given a topological mezzoperversity \mathcal{L} , we will use a modified Deligne truncation operation, which is essentially Banagl's truncation functor ([4]) in order to construct by an involved inductive process an object in $RP(\widehat{X})$. Conversely we will show (Theorem 2.2) that every object of $RP(\widehat{X})$ is (up to isomorphism) of this form for a unique topological mezzoperversity \mathcal{L} .

In Sec. 3, we will define the “dual topological mezzoperversity” and show that the corresponding homology groups satisfy a global duality. If a topological mezzoperversity is self-dual, then we show in Proposition 4.3 that it is equivalent to a Lagrangian structure in the sense of [4] and the resulting object in $D(\widehat{X})$ is the corresponding self-dual sheaf. We also mention (in Sec. 4) two interesting examples.

Finally, in Sec. 5 we restrict to Thom–Mather stratified spaces and show that the L^2 de Rham cohomology groups from [2] are canonically isomorphic to refined intersection cohomology groups. Indeed starting from any *analytic* mezzoperversity, as we will now refer to the mezzoperversities from [2], we obtain a sheaf complex in $RP(\widehat{X})$ by sheafifying the analytic domains associated to the mezzoperversity. The topological mezzoperversity of this sheaf complex is equivalent to the given analytic mezzoperversity, and we show that every topological mezzoperversity corresponds to an analytic one in this way. Since the global sections of the sheafifications turn out to agree with the global analytic domain of the mezzoperversity, the L^2 de Rham cohomology coincides with refined intersection cohomology (Theorem 5.7). Recall that some of the authors proved in Sec. 3.5 of [3] that these L^2 de Rham cohomology groups are stratified homotopy invariant, therefore so are these refined intersection cohomology groups. It is important to observe that certain applications of the stratified viewpoint require cohomological invariants that are not topologically invariant but are sensitive to the particular stratification chosen. This principle has been applied successfully in high-dimensional knot theory: a submanifold embedded in an ambient manifold may be regarded as a stratum. Invariants which are fine enough to detect the stratum may then yield knot invariants, see [7] and [12] on nonlocally flat PL embeddings. Note that for PL embeddings, strata arise due to nonlocal flatness of the embedding. (Example: $S^2 \hookrightarrow S^4$ obtained by suspending a nontrivial knot $S^1 \hookrightarrow S^3$.) If refined intersection cohomology groups are not topologically invariant but capable of detecting embedding properties, then they might have future applications in high-dimensional knot theory.

0. Notation

Note that this notation is different from that employed in [1, 2].

Let \widehat{X} be an n -dimensional pseudomanifold with a fixed topological stratification

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{n-2} \subset X_n = \widehat{X}$$

such that each X_k is a closed subset of \widehat{X} , and the k th stratum

$$Y_k = X_k \setminus X_{k-1}$$

is either empty or a manifold of dimension k . We assume that $Y_n = \widehat{X} \setminus X_{n-2}$ is dense. We will often work with the increasing sequence of open sets

$$\mathcal{U}_k = \widehat{X} \setminus X_{n-k}$$

for which we have

$$\mathcal{U}_{k+1} = \mathcal{U}_k \cup Y_{n-k},$$

and we make frequent use of the inclusions of these two subsets:

$$\mathcal{U}_k \xhookrightarrow{i_k} \mathcal{U}_{k+1} \xleftarrow{j_k} Y_{n-k}.$$

The singular and regular parts of \widehat{X} are X_{n-2} and \mathcal{U}_2 , respectively. Each point $x \in X_k \setminus X_{k-1}$ has distinguished neighborhoods in X homeomorphic to the product of a Euclidean ball of dimension k and an open cone

$$\mathbb{B}^k \times C^\circ(Z)$$

for some compact stratified space Z of dimension $n - k - 1$, called the link of Y_k at x . Moreover the stratifications of Z and \widehat{X} are compatible through this homeomorphism. So the link of a link is a link and locally \widehat{X} is an iterated bundle of cones.

We denote the category of sheaves of \mathbb{R} -vector spaces on \widehat{X} by $\mathrm{Sh}(\widehat{X})$, the category of constructible differential graded sheaves (i.e. complexes \mathbf{A}^\bullet such that each $\mathbf{A}^\bullet|_{Y_k}$ has locally constant cohomology sheaves and has finitely generated stalk cohomology) by $\mathrm{Sh}^\bullet(\widehat{X})$, and the derived category of bounded constructible complexes by $D(\widehat{X})$. We will denote the derived sheaf of an object \mathbf{A}^\bullet in $D(\widehat{X})$, i.e. the sheaf associated to the presheaf $(\mathcal{U} \mapsto \mathrm{H}^\bullet(\mathbf{A}^\bullet(\mathcal{U})))$, by

$$\mathrm{H}^\bullet(\mathbf{A}^\bullet) \in \mathrm{Sh}^\bullet(\widehat{X})$$

and its global hypercohomology, i.e. the cohomology of the global sections of an injective resolution, by $\mathbb{H}^\bullet(\widehat{X}; \mathbf{A}^\bullet)$. We denote the intersection sheaf complex associated to the lower middle perversity \overline{m} of Goresky–MacPherson by $\mathbf{IC}_{\overline{m}}^\bullet$, and similarly for the upper middle perversity \overline{n} by $\mathbf{IC}_{\overline{n}}^\bullet$. For a review of these concepts we refer to [6, 10, 17].

When we study differential forms, we will assume that the stratification is “smooth” as described below. In the Introduction we distinguish between *topological* mezzoperversities and *analytic* mezzoperversities. In the text we will reserve this

distinction until the final section, before that we develop the notion of topological mezzoperversity and refer to it as simply a mezzoperversity.

1. Refined Middle Perversity Sheaves

We define the objects of interest in the derived category, following [4, 17]. In this section, we work with topologically stratified pseudomanifolds and sheaves of \mathbb{R} -vector spaces. Let $j_x : \{x\} \hookrightarrow \widehat{X}$ denote the inclusion of a point $x \in \widehat{X}$.

Definition 1.1. Let \widehat{X} be a stratified orientable pseudomanifold and \mathbf{S}^\bullet a constructible bounded complex of sheaves of real vector spaces on \widehat{X} . We say that \mathbf{S}^\bullet satisfies the axioms $[RP]$, or is a *refined middle perversity complex of sheaves*, provided:

(RP1) Normalization: There is an isomorphism of the restriction of \mathbf{S}^\bullet to the regular part \mathcal{U}_2 of \widehat{X} and the constant rank 1 sheaf over \mathcal{U}_2 ,

$$\nu^{\mathbf{S}} : \mathbb{R}_{\mathcal{U}_2} \xrightarrow{\cong} \mathbf{S}^\bullet|_{\mathcal{U}_2}$$

(RP2) Lower bound: $H^\ell(j_x^* \mathbf{S}^\bullet) = 0$ for any $x \in \widehat{X}$ and $\ell < 0$.

(RP3) \overline{n} -stalk vanishing: $H^\ell(j_x^* \mathbf{S}^\bullet) = 0$ for any $x \in \mathcal{U}_{k+1} \setminus \mathcal{U}_2$ and $\ell > \overline{n}(k)$.

(RP4) \overline{m} -costalk vanishing: $H^\ell(j_x^! \mathbf{S}^\bullet) = 0$ for any $x \in Y_{n-k}$ and $\ell \leq \overline{m}(k) + n - k + 1$.

We denote by $RP(\widehat{X})$ the full subcategory of $D(\widehat{X})$ whose objects satisfy the axioms $[RP]$.

Remark 1.2. (1) Our normalization convention is that of Borel [10] and not that of [17]. Thus for example axiom (RP1) involves the constant rank-one sheaf over \mathcal{U}_2 as a complex concentrated in degree zero, whereas in [17] this degree is shifted by $n = \dim X$.

(2) The word “refined” in the definition above of $RP(\widehat{X})$ is motivated by the fact that these axioms will allow the existence of an additional structure, namely a mezzo-perversity, which of course is zero in the case of a Witt space. See Sec. 2.

(3) For simplicity we work with orientable \widehat{X} and the constant sheaf over the regular part of \widehat{X} . With essentially no change one can allow a locally constant system over general \widehat{X} . For example, the canonical normalization over a non-orientable \widehat{X} is to use the orientation sheaf over \mathcal{U}_2 .

Remark 1.3. The category $RP(\widehat{X})$ is closely related to the category $EP(\widehat{X})$ of equiperverse sheaves defined in [6, §9.3.1]. Indeed, the stalk and costalk vanishing conditions demanded in the respective definitions agree on strata of odd codimension. For EP , cohomology stalks on a stratum of even codimension k must vanish above $\overline{n}(k) + 1$, while for RP they already have to vanish above $\overline{n}(k)$. Similarly, there is a discrepancy of one degree for the costalk vanishing conditions. Hence $RP(\widehat{X})$ is always a full subcategory of $EP(\widehat{X})$, and they coincide if \widehat{X} has only

singular strata of odd codimension. If \widehat{X} has only strata of even codimension, then $EP(\widehat{X})$ equals $\overline{m}\mathcal{P}(X)$, the category of middle perverse sheaves on \widehat{X} , while $RP(\widehat{X})$ contains up to isomorphism only $\mathbf{IC}_{\overline{m}}^\bullet$.

As in [17], the axiom (RP4) is equivalent to knowing that, at each $x \in \mathcal{U}_{k+1} \setminus \mathcal{U}_k$, the attaching maps

$$\mathbf{H}^j(\mathbf{S}_x^\bullet) \rightarrow \mathbf{H}^j(Ri_{k*}i_k^*\mathbf{S}^\bullet)_x \quad (1.1)$$

are isomorphisms for all $j \leq \overline{m}(k)$.

Remark 1.4. One could also define refined sheaves for other perversities, but as our examples correspond to middle perversity, we do not explore this here.

Let us immediately note two important properties of the axioms.

Proposition 1.5. (i) If $\mathbf{S}^\bullet \in D(\widehat{X})$ satisfies axioms [RP], and \mathcal{DS}^\bullet is its Verdier dual, then $(\mathcal{DS}^\bullet)[-n]$ satisfies the axioms [RP].
 (ii) [Cappell–Shaneson] If \mathbf{A}^\bullet and \mathbf{B}^\bullet are in $RP(\widehat{X})$, the restriction to \mathcal{U}_2 induces an injective map

$$\mathrm{Hom}_{D(\widehat{X})}(\mathbf{A}^\bullet, \mathbf{B}^\bullet) \rightarrow \mathrm{Hom}_{D(\mathcal{U}_2)}(\mathbf{A}^\bullet|_{\mathcal{U}_2}, \mathbf{B}^\bullet|_{\mathcal{U}_2}).$$

Proof. These follow directly from the axioms: For constructible complexes of sheaves satisfying (RP1) and (RP2), it is well known [17, §5.3] that Verdier duality interchanges conditions (RP3) and (RP4), as \overline{m} and \overline{n} are complementary perversities. In [13, (1.3)], (ii) is shown to follow from (RP3) and (RP4). \square

Example 1. Let $\widehat{X} = M$ be an n -dimensional oriented smooth manifold (without boundary). Let $\mathbf{S}^\bullet = \mathbb{R}_M$ be the constant real sheaf in degree 0 on M . Then \mathbf{S}^\bullet is an object of $RP(M)$. (Axioms (RP3) and (RP4) are vacuously satisfied.) If \mathcal{A} is any local coefficient system (locally constant sheaf) in degree 0 on an oriented manifold M , then its Verdier dual is given by $\mathcal{DA} = \mathcal{A}^*[n]$, where \mathcal{A}^* is the linear dual local system with stalks $\mathcal{A}_x^* = \mathrm{Hom}(\mathcal{A}_x, \mathbb{R})$, $x \in M$. In particular, the Verdier dual of a local system on a manifold is again a local system, but there is a degree shift. Applying this to \mathbf{S}^\bullet , we obtain the self-duality isomorphism

$$(\mathcal{DS}^\bullet)[-n] = \mathbb{R}_M^* \cong \mathbb{R}_M = \mathbf{S}^\bullet,$$

where the isomorphism $\mathbb{R}_M^* \cong \mathbb{R}_M$ is given by the canonical multiplication $\mathbb{R}_M \otimes \mathbb{R}_M \rightarrow \mathbb{R}_M$. This shows in particular that $\mathcal{DS}^\bullet[-n]$ is again an object of $RP(M)$.

Let us recall Deligne’s construction of the sheaf complexes $\mathbf{IC}_{\overline{p}}^\bullet$ in $D(\widehat{X})$. For any perversity \overline{p} on \widehat{X} and $k = 2, \dots, n$ define

$$E_k^{\overline{p}} : D(\mathcal{U}_k) \rightarrow D(\mathcal{U}_{k+1}), \quad E_k^{\overline{p}}\mathbf{A}^\bullet = \tau_{\leq \overline{p}(k)} Ri_{k*}\mathbf{A}^\bullet,$$

and then set

$$\mathbf{IC}_{\overline{p}}^\bullet = E_n^{\overline{p}} \cdots E_2^{\overline{p}} \mathbb{R}_{\mathcal{U}_2}.$$

The sheaf complexes $\mathbf{IC}_{\overline{m}}^\bullet$ and $\mathbf{IC}_{\overline{n}}^\bullet$ both satisfy the axioms [RP], and we will now show that any sheaf that satisfies [RP] is closely related to these sheaves.

First, on the regular part \mathcal{U}_2 of \widehat{X} , any sheaf $\mathbf{S}^\bullet \in RP(\widehat{X})$ has natural maps

$$\mathbf{IC}_{\overline{m}}^\bullet|_{\mathcal{U}_2} \rightarrow \mathbf{S}^\bullet|_{\mathcal{U}_2} \rightarrow \mathbf{IC}_{\overline{n}}^\bullet|_{\mathcal{U}_2} \quad (1.2)$$

since by (RP1) each of these complexes has an isomorphism to the constant sheaf $\mathbb{R}_{\mathcal{U}_2}$.

Proposition 1.6. (cf. [4, §2.2]) *For any $\mathbf{S}^\bullet \in RP(\widehat{X})$ there is a unique pair of morphisms*

$$\mathbf{IC}_{\overline{m}}^\bullet \xrightarrow{\alpha} \mathbf{S}^\bullet \xrightarrow{\beta} \mathbf{IC}_{\overline{n}}^\bullet$$

extending (1.2).

Moreover, on $\mathcal{U}_{k+1} = \widehat{X} \setminus X_{n-k-1}$, these morphisms factor

$$\begin{array}{ccccc} i_{k+1}^* \mathbf{IC}_{\overline{m}}^\bullet & \xrightarrow{\alpha|_{\mathcal{U}_{k+1}}} & i_{k+1}^* \mathbf{S}^\bullet & \xrightarrow{\beta|_{\mathcal{U}_{k+1}}} & i_{k+1}^* \mathbf{IC}_{\overline{n}}^\bullet \\ & \searrow a' & \nearrow a & \searrow b & \nearrow b' \\ & E_k^{\overline{m}} i_k^* \mathbf{S}^\bullet & & E_k^{\overline{n}} i_k^* \mathbf{S}^\bullet & \end{array} \quad (1.3)$$

through two standard extensions $E_k^{\overline{m}} i_k^* \mathbf{S}^\bullet$ and $E_k^{\overline{n}} i_k^* \mathbf{S}^\bullet$ of $i_k^* \mathbf{S}^\bullet$.

Proof. Uniqueness follows from Proposition 1.5(ii), so we only need to establish existence.

Assume inductively that we have found morphisms

$$i_k^* \mathbf{IC}_{\overline{m}}^\bullet \rightarrow i_k^* \mathbf{S}^\bullet \rightarrow i_k^* \mathbf{IC}_{\overline{n}}^\bullet \quad (1.4)$$

extending (1.2). Applying the functors $E_k^{\overline{p}}$ to (1.4) we obtain morphisms

$$a' : i_{k+1}^* \mathbf{IC}_{\overline{m}}^\bullet = E_k^{\overline{m}}(i_k^* \mathbf{IC}_{\overline{m}}^\bullet) \rightarrow E_k^{\overline{m}}(i_k^* \mathbf{S}^\bullet),$$

$$b' : E_k^{\overline{n}}(i_k^* \mathbf{S}^\bullet) \rightarrow E_k^{\overline{n}}(i_k^* \mathbf{IC}_{\overline{n}}^\bullet) = i_{k+1}^* \mathbf{IC}_{\overline{n}}^\bullet$$

in $D(\mathcal{U}_{k+1})$. Adjunction $i_{k+1}^* \mathbf{S}^\bullet \rightarrow Ri_{k*} i_k^* \mathbf{S}^\bullet$ is the identity on \mathcal{U}_k and induces maps

$$\tau_{\leq \overline{p}(k)} i_{k+1}^* \mathbf{S}^\bullet \rightarrow E_k^{\overline{p}}(i_k^* \mathbf{S}^\bullet) \text{ for } \overline{p} = \overline{m} \text{ or } \overline{n}.$$

The axiom (RP4) (in the form (1.1)) implies that for \overline{m} this morphism is a quasi-isomorphism, and axiom (RP3) implies that

$$i_{k+1}^* \mathbf{S}^\bullet \cong \tau_{\leq \overline{n}(k)} i_{k+1}^* \mathbf{S}^\bullet$$

and so all together we have morphisms

$$\begin{aligned} i_{k+1}^* \mathbf{IC}_{\overline{m}}^\bullet &\xrightarrow{a'} E_k^{\overline{m}}(i_k^* \mathbf{S}^\bullet) \cong \tau_{\leq \overline{m}(k)} i_{k+1}^* \mathbf{S}^\bullet \xrightarrow{a} i_{k+1}^* \mathbf{S}^\bullet \\ &\cong \tau_{\leq \overline{n}(k)} i_{k+1}^* \mathbf{S}^\bullet \xrightarrow{b} E_k^{\overline{n}}(i_k^* \mathbf{S}^\bullet) \xrightarrow{b'} i_{k+1}^* \mathbf{IC}_{\overline{n}}^\bullet \end{aligned}$$

extending (1.2) as required. \square

Let $\mathbf{C}^\bullet(a)$, $\mathbf{C}^\bullet(b)$ and $\mathbf{C}^\bullet(ba)$ be sheaf complexes completing a, b and ba , respectively, to distinguished triangles. Using these triangles we are able to conclude the following:

Proposition 1.7. *For any $x \in \mathcal{U}_{k+1} \setminus \mathcal{U}_k$, the cohomology of the complex $\mathbf{C}^\bullet(ba)_x$ is concentrated in degree $\bar{n}(k)$, and the maps a and b in (1.3) induce an injective map*

$$\mathbf{H}^{\bar{n}(k)}(\mathbf{S}_x^\bullet) \rightarrow \mathbf{H}^{\bar{n}(k)}(\mathbf{C}^\bullet(ba)_x).$$

Proof. Let us analyze a and b for odd k . The distinguished triangles

$$\begin{array}{ccccc} E_k^{\bar{m}} i_k^* \mathbf{S}^\bullet & \xrightarrow{a} & i_{k+1}^* \mathbf{S}^\bullet, & i_{k+1}^* \mathbf{S}^\bullet & \xrightarrow{b} & E_k^{\bar{n}} i_k^* \mathbf{S}^\bullet, \\ & \swarrow +1 & \searrow & \swarrow +1 & \searrow & \\ & \mathbf{C}^\bullet(a) & & \mathbf{C}^\bullet(b) & & \\ & & E_k^{\bar{m}} i_k^* \mathbf{S}^\bullet \xrightarrow{ba} E_k^{\bar{n}} i_k^* \mathbf{S}^\bullet & & & \\ & & \swarrow +1 & \searrow & & \\ & & \mathbf{C}^\bullet(ba) & & & \end{array}$$

fit together into a diagram

$$\begin{array}{ccccc} & & ba & & \\ & \curvearrowright & & \curvearrowleft & \\ E_k^{\bar{m}} i_k^* \mathbf{S}^\bullet & \xrightarrow{a} & i_{k+1}^* \mathbf{S}^\bullet & \xrightarrow{b} & E_k^{\bar{n}} i_k^* \mathbf{S}^\bullet \\ & \swarrow +1 & \searrow & \swarrow +1 & \searrow \\ & \mathbf{C}^\bullet(a) & & \mathbf{C}^\bullet(b) & \\ & & \mathbf{C}^\bullet(ba) & & \end{array}$$

which by the “octahedral axiom” we can complete to the octahedral diagram

$$\begin{array}{ccccc} & & ba & & \\ & \curvearrowright & & \curvearrowleft & \\ E_k^{\bar{m}} i_k^* \mathbf{S}^\bullet & \xrightarrow{a} & i_{k+1}^* \mathbf{S}^\bullet & \xrightarrow{b} & E_k^{\bar{n}} i_k^* \mathbf{S}^\bullet \\ & \swarrow +1 & \searrow & \swarrow +1 & \searrow \\ & \mathbf{C}^\bullet(a) & \xrightarrow{+1} & \mathbf{C}^\bullet(b) & \\ & \swarrow +1 & \searrow & \swarrow +1 & \searrow \\ & \mathbf{C}^\bullet(ba) & & \mathbf{C}^\bullet(ba) & \end{array} \quad (1.5)$$

Taking the stalks of the cohomology sheaves at a point $x \in \mathcal{U}_{k+1} \setminus \mathcal{U}_k$, we obtain the following diagram

$$\begin{array}{ccccc}
 & & \mathbf{H}^j(ba) & & \\
 & \nearrow & & \searrow & \\
 \mathbf{H}^j(E_k^{\overline{m}} i_k^* \mathbf{S}^\bullet)_x & \xrightarrow{\mathbf{H}^j(a)} & \mathbf{H}^j(\mathbf{S}_x^\bullet) & \xrightarrow{\mathbf{H}^j(b)} & \mathbf{H}^j(E_k^{\overline{n}} i_k^* \mathbf{S}^\bullet)_x \\
 & \nwarrow & \nwarrow & \nwarrow & \\
 & & \mathbf{H}^j(\mathbf{C}^\bullet(a)_x) & \xleftarrow{+1} & \mathbf{H}^j(\mathbf{C}^\bullet(b)_x) \\
 & \nwarrow & & \nearrow & \\
 & & \mathbf{H}^j(\mathbf{C}^\bullet(ba)_x) & &
 \end{array}$$

(Note: Dotted arrows in the original image connect the top row to the bottom row, and the bottom row to the bottom-most row, with labels +1.)

where we have used dotted arrows for maps into the corresponding \mathbf{H}^{j+1} . From (1.1) we know that the map $\mathbf{H}^j(b)$ is an isomorphism for $j \leq \overline{m}(k)$, and by construction $\mathbf{H}^j(ba)$ is the identity map for these j , hence $\mathbf{H}^j(a)$ is also an isomorphism and

$$\mathbf{H}^j(\mathbf{C}^\bullet(ba)_x) = \mathbf{H}^j(\mathbf{C}^\bullet(a)_x) = 0 \quad \text{for all } j \leq \overline{m}(k),$$

$$\mathbf{H}^j(\mathbf{C}^\bullet(b)_x) = 0 \quad \text{for all } j < \overline{m}(k).$$

If $j > \overline{n}(k)$, then $\mathbf{H}^j(E_k^{\overline{m}} i_k^* \mathbf{S}^\bullet)_x$ and $\mathbf{H}^j(\mathbf{S}_x^\bullet)$ both vanish and thus $\mathbf{H}^j(\mathbf{C}^\bullet(a)_x) = 0$ for these j . We conclude that $\mathbf{C}^\bullet(a)$ is concentrated in a single degree, namely $\overline{n}(k)$. Moreover, if $x \in \mathcal{U}_k$, then $\mathbf{H}^j(\mathbf{C}^\bullet(a)_x) = 0$ for all j , since $i_k^* a$ is the identity map. So $\mathbf{C}^\bullet(a)$ is supported over the manifold $\mathcal{U}_{k+1} \setminus \mathcal{U}_k$. Summarizing, $\mathbf{C}^\bullet(a)$ has the form

$$\mathbf{C}^\bullet(a) = j_{k*} \mathcal{A}[-\overline{n}(k)],$$

where \mathcal{A} is a locally constant sheaf on the manifold $M = Y_{n-k}$. Note that as j_k is a closed inclusion, the functor j_{k*} is just extension by zero and we have $j_{k*} = j_k!$.

We shall now show that $\mathbf{C}^\bullet(b)$ is also concentrated in the degree $\overline{n}(k)$. We prove this following [4, §2.4]. As pointed out in Proposition 1.5(i), the Verdier dual $\mathcal{D}(\mathbf{S}^\bullet)[-n]$ is also in $RP(\widehat{X})$. Thus we have a distinguished triangle

$$\begin{array}{ccc}
 i_{k+1}^* \mathcal{D}(\mathbf{S}^\bullet)[-n] & \xrightarrow{b'} & E_k^{\overline{n}} i_k^* \mathcal{D}(\mathbf{S}^\bullet)[-n] \\
 & \nwarrow & \nwarrow \\
 & \mathbf{C}^\bullet(b') &
 \end{array}$$

(Note: Dotted arrows in the original image connect the top row to the bottom row, with labels +1.)

which dualizes to

$$\begin{array}{ccc}
 E_k^{\overline{m}} i_k^* \mathbf{S}^\bullet & \xrightarrow{\mathcal{D}(b')[-n]} & i_{k+1}^* \mathbf{S}^\bullet \\
 & \nwarrow & \nwarrow \\
 & \mathcal{D}(\mathbf{C}^\bullet(b'))[-n+1] &
 \end{array}$$

(Note: Dotted arrows in the original image connect the top row to the bottom row, with labels +1.)

By Proposition 1.5(ii), we know that $\mathcal{D}(b')[-n] = a$ and so we have an isomorphism of triangles

$$\begin{array}{ccccccc} E_k^{\overline{m}} i_k^* \mathbf{S}^\bullet & \xrightarrow{\mathcal{D}(b')[-n]} & i_{k+1}^* \mathbf{S}^\bullet & \longrightarrow & \mathcal{D}(\mathbf{C}^\bullet(b'))[-n+1] & \xrightarrow{+1} & \dots \\ \downarrow = & & \downarrow = & & \downarrow \cong & & \\ E_k^{\overline{m}} i_k^* \mathbf{S}^\bullet & \xrightarrow{a} & i_{k+1}^* \mathbf{S}^\bullet & \longrightarrow & \mathbf{C}^\bullet(a) & \xrightarrow{+1} & \dots \end{array}$$

which shows that $\mathcal{D}(\mathbf{C}^\bullet(b'))[-n+1] \cong \mathbf{C}^\bullet(a)$. In the following calculation, we distinguish in our notation between the Verdier dualizing functor $\mathcal{D}_{\widehat{X}} = \mathcal{D}$ on \widehat{X} and the dualizing functor \mathcal{D}_M on the manifold $M = Y_{n-k}$. We have

$$\begin{aligned} \mathbf{C}^\bullet(b') &\cong (\mathcal{D}_{\widehat{X}} \mathbf{C}^\bullet(a))[-n+1] \\ &\cong (\mathcal{D}_{\widehat{X}}(j_{k*} \mathcal{A}[-\overline{n}(k)]))[-n+1] \\ &\cong (\mathcal{D}_{\widehat{X}}(j_{k*} \mathcal{A}))[\overline{n}(k) - n + 1] \\ &\cong j_{k!}(\mathcal{D}_M \mathcal{A})[\overline{n}(k) - n + 1] \\ &\cong j_{k*} \mathcal{A}^*[n - k][\overline{n}(k) - n + 1] \\ &\cong j_{k*} \mathcal{A}^*[\overline{n}(k) - k + 1] \\ &\cong j_{k*} \mathcal{A}^*[-\overline{n}(k)]. \end{aligned}$$

(Recall that for odd k , $\overline{n}(k) = (k-1)/2$.) This shows that $\mathbf{C}^\bullet(b')$ is concentrated in degree $\overline{n}(k)$. Since every object of $RP(\widehat{X})$ is of the form $\mathcal{D}\mathbf{S}^\bullet[-n]$ for some $\mathbf{S}^\bullet \in \text{Ob } RP(\widehat{X})$, this also shows that $\mathbf{C}^\bullet(b)$ is concentrated in degree $\overline{n}(k)$.

If k is odd, then $\overline{n}(k) = \overline{m}(k) + 1$ and, using the information above, the diagram for $\overline{n}(k)$ collapses to a pair of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{H}^{\overline{n}(k)}(\mathbf{S}_x^\bullet) & \xrightarrow{\mathbf{H}^{\overline{n}(k)}(b)} & \mathbf{H}^{\overline{n}(k)}(E_k^{\overline{m}} i_k^* \mathbf{S}^\bullet)_x & \longrightarrow & \mathbf{H}^{\overline{n}(k)}(\mathbf{C}^\bullet(b)_x) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow = \\ 0 & \longrightarrow & \mathbf{H}^{\overline{n}(k)}(\mathbf{C}^\bullet(a)_x) & \longrightarrow & \mathbf{H}^{\overline{n}(k)}(\mathbf{C}^\bullet(ba)_x) & \longrightarrow & \mathbf{H}^{\overline{n}(k)}(\mathbf{C}^\bullet(b)_x) \longrightarrow 0 \end{array}$$

and so we have in particular an injective map $\mathbf{H}^{\overline{n}(k)}(\mathbf{S}_x^\bullet) \cong \mathbf{H}^{\overline{n}(k)}(\mathbf{C}^\bullet(a)_x) \rightarrow \mathbf{H}^{\overline{n}(k)}(\mathbf{C}^\bullet(ba)_x)$. \square

One interpretation of this proposition is that $i_{k+1}^* \mathbf{S}^\bullet$ differs from the two canonical extensions of $i_k^* \mathbf{S}^\bullet$ to \mathcal{U}_{k+1} , namely $E_k^{\overline{m}} i_k^* \mathbf{S}^\bullet$ and $E_k^{\overline{n}} i_k^* \mathbf{S}^\bullet$, by the choice of a subsheaf $\mathbf{H}^{\overline{n}(k)}(\mathbf{S}_x^\bullet)$ of $\mathbf{H}^{\overline{n}(k)}(\mathbf{C}^\bullet(ba)_x)$. Indeed, it is possible to assemble $i_k^* \mathbf{S}^\bullet$ and the sheaf

$$\mathbf{W}_S(Y_{n-k}) \in \text{Sh}(Y_{n-k}), \quad \mathbf{W}_S(Y_{n-k}) = \mathbf{H}^{\overline{n}(k)}(\mathbf{S}^\bullet)|_{Y_{n-k}} \quad (1.6)$$

to an object over \mathcal{U}_{k+1} satisfying $[RP]$, quasi-isomorphic to $\mathbf{S}^\bullet|_{\mathcal{U}_{k+1}}$.

To carry out this assembly, let us recall the modified truncation functor from [4, §5]: Given a constructible complex of sheaves \mathbf{A}^\bullet on a stratified pseudomanifold M , and an injective sheaf map

$$\phi : \mathbf{E} \rightarrow \mathbf{H}^p(\mathbf{A}^\bullet),$$

we use the quotient map

$$\pi : \ker d^p \rightarrow \mathbf{H}^p(\mathbf{A}^\bullet)$$

to define a new (constructible) complex of sheaves

$$\begin{aligned} \tau_{\leq p}(\mathbf{A}^\bullet, \mathbf{E}) &\in \mathrm{Sh}^\bullet(M), \\ (\tau_{\leq p}(\mathbf{A}^\bullet, \mathbf{E}))^j &= \begin{cases} \mathbf{A}^j & \text{if } j < p, \\ \pi^{-1}(\phi(\mathbf{E})) & \text{if } j = p, \\ 0 & \text{if } j > p. \end{cases} \end{aligned}$$

Notice that this new complex satisfies

$$\mathbf{H}^j(\tau_{\leq p}(\mathbf{A}^\bullet, \mathbf{E})) = \begin{cases} \mathbf{H}^j(\mathbf{A}^\bullet) & \text{if } j < p, \\ \phi(\mathbf{E}) & \text{if } j = p, \\ 0 & \text{if } j > p. \end{cases} \quad (1.7)$$

It is an important fact that this modified truncation defines a functor. Let $\mathrm{Sh}^\bullet(M)$ denote the category of complexes of sheaves on M , and $MS(M)$ the associated category of morphisms. Let $\mathrm{Sh}^\bullet(M) \rtimes MS(M)$ denote the twisted product category whose objects are pairs

$$(\mathbf{A}^\bullet, \mathbf{E} \xrightarrow{\phi} \mathbf{H}^p(\mathbf{A}^\bullet))$$

with ϕ injective, and whose morphisms are pairs

$$(f, h) \in \mathrm{Hom}((\mathbf{A}^\bullet, \mathbf{E} \xrightarrow{\phi} \mathbf{H}^p(\mathbf{A}^\bullet)), (\mathbf{B}^\bullet, \mathbf{F} \xrightarrow{\psi} \mathbf{H}^p(\mathbf{B}^\bullet)))$$

with $f : \mathbf{A}^\bullet \rightarrow \mathbf{B}^\bullet$ a sheaf complex morphism and $h : \mathbf{E} \rightarrow \mathbf{F}$ a sheaf morphism such that

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{\phi} & \mathbf{H}^p(\mathbf{A}^\bullet) \\ \downarrow h & & \downarrow \mathbf{H}^p(f) \\ \mathbf{F} & \xrightarrow{\psi} & \mathbf{H}^p(\mathbf{B}^\bullet) \end{array}$$

commutes. In a similar fashion, we define $D(M) \rtimes MS(M)$ starting from the derived category of bounded constructible sheaf complexes on M . We know, from [4, Theorem 2.5], that modified truncation defines a covariant functor

$$\tau_{\leq p}(\cdot, \cdot) : \mathrm{Sh}^\bullet(M) \rtimes MS(M) \rightarrow \mathrm{Sh}^\bullet(M)$$

which descends to the derived category

$$\tau_{\leq p}(\cdot, \cdot) : D(M) \rtimes MS(M) \rightarrow D(M).$$

We can now carry out the assembly referred to above.

Proposition 1.8. (cf. [10, Lemma 2.4]) *Let $\mathbf{S}^\bullet \in D(\mathcal{U}_{k+1})$ satisfy the axioms $[RP]$ on \mathcal{U}_{k+1} , let $\mathbf{W}_S(Y_{n-k})$ be the sheaf from (1.6), and let*

$$Ei_k^* \mathbf{S}^\bullet = \begin{cases} \tau_{\leq \overline{n}(k)} Ri_{k*} i_k^* \mathbf{S}^\bullet & \text{if } k \text{ even} \\ \tau_{\leq \overline{n}(k)} (Ri_{k*} i_k^* \mathbf{S}^\bullet, \mathbf{W}_S(Y_{n-k})) & \text{if } k \text{ odd} \end{cases}$$

then $\mathbf{S}^\bullet \cong Ei_k^* \mathbf{S}^\bullet$.

Proof. We must distinguish two cases: The case k even and the case k odd. Suppose first that k is even. By axiom $(RP4)$, phrased as (1.1), the adjunction map $\mathbf{S}^\bullet \xrightarrow{f} Ri_{k*} i_k^* \mathbf{S}^\bullet$ induces an isomorphism

$$\tau_{\leq \overline{n}(k)} \mathbf{S}^\bullet \xrightarrow{f} \tau_{\leq \overline{n}(k)} Ri_{k*} i_k^* \mathbf{S}^\bullet,$$

since we have $\overline{n}(k) = \overline{m}(k)$ when k is even. The canonical map $\tau_{\leq \overline{n}(k)} \mathbf{S}^\bullet \rightarrow \mathbf{S}^\bullet$ is an isomorphism by axiom $(RP3)$. Putting these two isomorphisms together, we obtain an isomorphism $\mathbf{S}^\bullet \cong \tau_{\leq \overline{n}(k)} Ri_{k*} i_k^* \mathbf{S}^\bullet = Ei_k^* \mathbf{S}^\bullet$ as claimed.

Suppose that k is odd. The adjunction map f participates in a commutative diagram

$$\begin{array}{ccc} \mathbf{W}_S(Y_{n-k}) & \xrightarrow{\text{id}} & \mathbf{H}^{\overline{n}(k)}(\mathbf{S}^\bullet) \\ \downarrow \text{id} & & \downarrow \mathbf{H}^{\overline{n}(k)}(f) \\ \mathbf{H}^{\overline{n}(k)}(\mathbf{S}^\bullet) & \xrightarrow{\mathbf{H}^{\overline{n}(k)}(f)} & \mathbf{H}^{\overline{n}(k)}(Ri_{k*} i_k^* \mathbf{S}^\bullet) \end{array}$$

which by functoriality of $\tau_{\leq \overline{n}(k)}(\cdot, \cdot)$ induces a map

$$\tau_{\leq \overline{n}(k)}(\mathbf{S}^\bullet, \mathbf{W}_S(Y_{n-k})) \rightarrow Ei_k^* \mathbf{S}^\bullet.$$

This map is an isomorphism: In degrees $\leq \overline{m}(k)$ it is a cohomology isomorphism by $(RP4)$ in the form (1.1); in degree $\overline{n}(k)$ it induces the map

$$\mathbf{H}^{\overline{n}(k)}(f) : \mathbf{H}^{\overline{n}(k)}(\mathbf{S}^\bullet) \rightarrow \mathbf{H}^{\overline{n}(k)}(f)(\mathbf{H}^{\overline{n}(k)}(\mathbf{S}^\bullet)),$$

which is also an isomorphism. The canonical map

$$\tau_{\leq \overline{n}(k)}(\mathbf{S}^\bullet, \mathbf{W}_S(Y_{n-k})) = \tau_{\leq \overline{n}(k)} \mathbf{S}^\bullet \rightarrow \mathbf{S}^\bullet$$

is an isomorphism by $(RP3)$. Consequently,

$$\mathbf{S}^\bullet \cong \tau_{\leq \overline{n}(k)}(\mathbf{S}^\bullet, \mathbf{W}_S(Y_{n-k})) \cong Ei_k^* \mathbf{S}^\bullet,$$

as required. \square

2. Mezzoperversities

We begin with describing, by an involved iterative process, the analogue of Deligne's construction in the category $[RP]$. Recall from [17] that given a perversity, one can

inductively construct the intersection sheaf complex in $D(\mathcal{U}_k)$ for each k , obtaining finally an object in $D(\widehat{X})$. To construct an object in $[RP]$ one needs more information than just a classical perversity. We examine the analogous procedure for constructing an object in $[RP]$ inductively over \mathcal{U}_k .

All sheaves in $[RP]$ are isomorphic to the constant sheaf $\mathcal{P}_2 = \mathbb{R}$ over \mathcal{U}_2 and to $\mathcal{P}_3 = \tau_{\leq \overline{n}(2)} Ri_{2*} \mathbb{R}$ over \mathcal{U}_3 . First we choose $\mathbf{W}(Y_{n-3})$ a subsheaf of $\mathbf{H}^{\overline{n}(3)}(Ri_{3*} \mathcal{P}_3)$ over Y_{n-3} then we set

$$\mathcal{L}_4 = \{\mathbf{W}(Y_{n-3})\}, \quad \mathcal{P}_4(\mathcal{L}_4) = \tau_{\leq \overline{n}(3)}(Ri_{3*} \mathcal{P}_3, \mathbf{W}(Y_{n-3})) \text{ over } \mathcal{U}_4.$$

We can continue inductively in this way to construct an element $\mathcal{P}(\mathcal{L})$ of $D(\widehat{X})$. Indeed, if we have constructed \mathcal{L}_k and $\mathcal{P}_k(\mathcal{L}_k)$ over \mathcal{U}_k , then we extend our construction to \mathcal{U}_{k+1} in the following way. If k is even, set:

$$\mathcal{L}_{k+1} = \mathcal{L}_k = \{\mathbf{W}(Y_{n-3}), \dots, \mathbf{W}(Y_{n-(k-1)})\}, \quad \mathcal{P}_{k+1}(\mathcal{L}_{k+1}) = \tau_{\leq \overline{n}(k)} Ri_{k*} \mathcal{P}_k(\mathcal{L}_k).$$

If k is odd, choose

$$\mathbf{W}(Y_{n-k}) \text{ a subsheaf of } \mathbf{H}^{\overline{n}(k)}(Ri_{k*} \mathcal{P}_k(\mathcal{L}_k))$$

and then set

$$\begin{aligned} \mathcal{L}_{k+1} &= \mathcal{L}_k \cup \{\mathbf{W}(Y_{n-k})\} \\ &= \{\mathbf{W}(Y_{n-3}), \dots, \mathbf{W}(Y_{n-k})\}, \\ \mathcal{P}_{k+1}(\mathcal{L}_{k+1}) &= \tau_{\leq \overline{n}(k)}(Ri_{k*} \mathcal{P}_k(\mathcal{L}_k), \mathbf{W}(Y_{n-k})). \end{aligned}$$

Definition 2.1. We refer to the sequence of sheaves

$$\mathbf{W}(Y_{n-k}) \in \text{Sh}(Y_{n-k})$$

together with the injective sheaf maps

$$\mathbf{W}(Y_{n-k}) \rightarrow \mathbf{H}^{\overline{n}(k)}(Ri_{k*} \mathcal{P}_k(\mathcal{L}_k))$$

as a (topological) *mezzoperversity* \mathcal{L} and to the resulting sheaf complex $\mathcal{P}(\mathcal{L})$ as the *Deligne sheaf* associated to the mezzoperversity.

As anticipated, every $\mathcal{P}(\mathcal{L})$ is an element of $[RP]$. Indeed, it clearly satisfies $(RP1)$ – $(RP3)$, and from (1.7) we know that the map

$$\mathcal{P}_k(\mathcal{L}) \rightarrow Ri_{k*} \mathcal{P}_{k-1}(\mathcal{L})$$

is a quasi-isomorphism for degrees $\leq \overline{m}(k)$. The commutative diagram

$$\begin{array}{ccc} \mathcal{P}_k(\mathcal{L}) & \xrightarrow{\text{adj.}} & Ri_{k*} i_k^* \mathcal{P}_k(\mathcal{L}) = Ri_{k*} \mathcal{P}_{k-1}(\mathcal{L}) \\ & \searrow \text{q. iso } \leq \overline{m}(k) & \swarrow \cong \\ & Ri_{k*} \mathcal{P}_{k-1}(\mathcal{L}) & \end{array}$$

where we have used that $i_k^* \mathcal{P}_k(\mathcal{L}) = \mathcal{P}_{k-1}(\mathcal{L})$, shows that the attaching maps

$$\mathbf{H}^j(\mathcal{P}_k(\mathcal{L})_x) \rightarrow \mathbf{H}^j(Ri_{k*} i_k^* \mathcal{P}_k(\mathcal{L}))_x$$

are isomorphisms for all $j \leq \overline{n}(k)$. We next prove that conversely every element of $[RP]$ is isomorphic to a Deligne sheaf.

Theorem 2.2. *If \mathbf{S}^\bullet is an object in $RP(\widehat{X})$ and $\nu_2 : \mathbb{R}_{\mathcal{U}_2} \xrightarrow{\cong} \mathbf{S}^\bullet|_{\mathcal{U}_2}$ is the normalization isomorphism from (RP1), then one can construct a mezzoperversity \mathcal{L} with Deligne sheaf $\mathcal{P}(\mathcal{L})$ and an isomorphism*

$$\nu : \mathcal{P}(\mathcal{L}) \rightarrow \mathbf{S}^\bullet$$

extending ν_2 to \widehat{X} .

Proof. The construction of \mathcal{L} is given inductively. Over \mathcal{U}_2 there is nothing to show. Over \mathcal{U}_3 we have the isomorphism

$$\nu_3 = (\tau_{\leq \overline{n}(2)} Ri_{2*})(\nu_2) : \tau_{\leq \overline{n}(2)} Ri_{2*} \mathcal{P}_2 = \mathcal{P}_3 \rightarrow \tau_{\leq \overline{n}(2)} Ri_{2*} \mathbf{S}^\bullet|_{\mathcal{U}_2} \cong \mathbf{S}^\bullet|_{\mathcal{U}_3},$$

so the statement holds over \mathcal{U}_3 . Over \mathcal{U}_4 we know from Proposition 1.8 that

$$\mathbf{S}^\bullet|_{\mathcal{U}_4} \cong \tau_{\leq \overline{n}(3)}(Ri_{3*} i_3^* \mathbf{S}^\bullet, \mathbf{W}_\mathbf{S}(Y_{n-3}))$$

where $\mathbf{W}_\mathbf{S}(Y_{n-3}) = \mathbf{H}^{\overline{n}(3)}(\mathbf{S}^\bullet)|_{Y_{n-3}}$. Since the map

$$\mathbf{H}^{\overline{n}(3)}(Ri_{3*} \mathcal{P}_3) \xrightarrow{\mathbf{H}^{\overline{n}(3)}(Ri_{3*} \nu_3)} \mathbf{H}^{\overline{n}(3)}(Ri_{3*} \mathbf{S}^\bullet|_{\mathcal{U}_3})$$

is an isomorphism, we can define an injective sheaf map ψ by the diagram

$$\begin{array}{ccc} \mathbf{W}_\mathbf{S}(Y_{n-3}) & \xrightarrow{\phi} & \mathbf{H}^{\overline{n}(3)}(Ri_{3*} \mathbf{S}^\bullet|_{\mathcal{U}_3}) \\ \text{id} \downarrow & & \uparrow \cong \mathbf{H}^{\overline{n}(3)}(Ri_{3*} \nu_3) \\ \mathbf{W}_\mathbf{S}(Y_{n-3}) & \xrightarrow{\psi} & \mathbf{H}^{\overline{n}(3)}(Ri_{3*} \mathcal{P}_3) \end{array}$$

Here, ϕ is the map induced on $\mathbf{H}^{\overline{n}(3)}$ by the adjunction $\mathbf{S}^\bullet|_{\mathcal{U}_4} \rightarrow Ri_{3*} i_3^* \mathbf{S}^\bullet|_{\mathcal{U}_4}$. This diagram shows that the pair $(Ri_{3*} \nu_3^{-1}, \text{id})$ is an isomorphism in the category $D(\mathcal{U}_4) \rtimes MS(\mathcal{U}_4)$, so after applying the functor $\tau_{\leq \overline{n}(3)}(\cdot, \cdot)$ we obtain an isomorphism

$$\begin{aligned} \nu_4 : \tau_{\leq \overline{n}(3)}(Ri_{3*} \mathcal{P}_3, \mathbf{W}_\mathbf{S}(Y_{n-3})) \\ = \mathcal{P}_4(\mathcal{L}_4) \rightarrow \tau_{\leq \overline{n}(3)}(Ri_{3*} \mathbf{S}^\bullet|_{\mathcal{U}_3}, \mathbf{W}_\mathbf{S}(Y_{n-3})) \cong \mathbf{S}^\bullet|_{\mathcal{U}_4}, \end{aligned}$$

where \mathcal{L}_4 is $\mathbf{W}_\mathbf{S}(Y_{n-3})$ together with the injective sheaf map ψ . Thus the statement holds over \mathcal{U}_4 . One now continues to proceed as above to show that the statement holds over all of \widehat{X} . \square

Definition 2.3. We refer to the mezzoperversity \mathcal{L} in this theorem as *the mezzoperversity of \mathbf{S}^\bullet* and denote it $\mathcal{L}_\mathbf{S}$.

3. Global Duality

In this section we work with \widehat{X} a compact oriented topologically stratified pseudo-manifold and with sheaves of \mathbb{R} -vector spaces.

In Proposition 1.5(i), we pointed out that if $\mathbf{S}^\bullet \in D(\widehat{X})$ satisfies axioms $[RP]$, and \mathcal{DS}^\bullet is its Verdier dual, then $(\mathcal{DS}^\bullet)[-n]$ satisfies the axioms $[RP]$ as well. If \mathcal{L} is the mezzoperversity of \mathbf{S}^\bullet we refer to the mezzoperversity of $(\mathcal{DS}^\bullet)[-n]$ as the *dual mezzoperversity* of \mathcal{L} and denote it \mathcal{DL} .

Global duality for refined intersection homology groups is an immediate consequence upon taking hypercohomology. Indeed, for any $\mathbf{S}^\bullet \in D(\widehat{X})$, we have (see, e.g. [17, §1.12], [6, §4.4])

$$\mathbb{H}^j(\widehat{X}; \mathbf{S}^\bullet) \cong \mathrm{Hom}(\mathbb{H}^{n-j}(\widehat{X}; \mathcal{DS}^\bullet[-n]), \mathbb{R})$$

and hence a non-degenerate bilinear pairing

$$\mathbb{H}^j(\widehat{X}; \mathbf{S}^\bullet) \otimes \mathbb{H}^{n-j}(\widehat{X}; \mathcal{DS}^\bullet[-n]) \rightarrow \mathbb{R}.$$

In terms of mezzoperversities, this says that there is a natural non-degenerate pairing

$$\mathbb{H}^j(\widehat{X}; \mathcal{P}(\mathcal{L})) \otimes \mathbb{H}^{n-j}(\widehat{X}; \mathcal{P}(\mathcal{DL})) \rightarrow \mathbb{R}.$$

The category $RP(\widehat{X})$ always contains the sheaf complexes \mathbf{IC}_m^\bullet and \mathbf{IC}_n^\bullet , corresponding to the zero mezzoperversity and the mezzoperversity whose maps $\mathbf{W}(Y_{n-k}) \rightarrow \mathbf{H}^{\overline{n}(k)}(Ri_{k*}\mathcal{P}_k(\mathcal{L}_k))$ are isomorphisms. These mezzoperversities are dual and the pairing above is the pairing described by Goresky–MacPherson [17, §5]. In the next section, we give two further examples where \mathcal{L} and \mathcal{DL} can be identified.

4. Example: Self-Dual Sheaves

In this section we work with oriented n -dimensional topologically stratified pseudo-manifolds \widehat{X} and sheaves of \mathbb{R} -vector spaces. As pointed out in [4, §1.9], one could more generally work with vector spaces over any field of characteristic not equal to two.

Theorem 4.1. *If \mathbf{S}^\bullet is a self-dual sheaf in $SD(\widehat{X})$ in the sense of [4] and \mathcal{L} is its Lagrangian structure, then $\mathbf{S}^\bullet[-n]$ satisfies axioms $[RP]$ and its mezzoperversity \mathcal{L}_S is naturally identified with \mathcal{L} .*

Proof. The category $SD(\widehat{X})$ is defined in [4] by four axioms $(SD1)$ – $(SD4)$. Up to a degree shift by the dimension of \widehat{X} , the first three axioms coincide with $(RP1)$ – $(RP3)$ and by [4, Lemma 2.1] elements of $SD(\widehat{X})$ also satisfy $(RP4)$. Finally, one checks easily that both the Lagrangian structure and the mezzoperversity coincide with the cohomology sheaves of \mathbf{S}^\bullet in degree $\overline{n}(k)$ over the strata of codimension k for odd k (together with the requisite monomorphisms). Therefore, the theorem is proved. \square

Note that there are topological restrictions to finding objects in the category $SD(\widehat{X})$, for instance the signature of the links must vanish.

Definition 4.2. A *topological stratified pseudomanifold* \widehat{X} is an *L-space* if $SD(\widehat{X}) \neq \emptyset$.

Example 2. The suspension $\widehat{X} = \Sigma T^2$ of the 2-torus T^2 is an *L-space*, as the signature of T^2 vanishes. The suspension $\widehat{X} = \Sigma \mathbb{CP}^2$ of the complex projective space \mathbb{CP}^2 is not an *L-space*, as \mathbb{CP}^2 has signature 1. Note that both of these examples are indeed non-Witt spaces.

In the next section we show that the mezzoperversities in the analytic theory of [2] coincide with the topological mezzoperversities described above. In this setting of smoothly stratified spaces, a *Cheeger space* is one admitting a self-dual analytic mezzoperversity. Thus in the smooth setting, a self-dual mezzoperversity is equivalent to a Lagrangian structure in the sense of [4], in particular we have

Proposition 4.3. A smoothly stratified pseudomanifold is a Cheeger space iff it is an *L-space*.

Example 3. Banagl-Kulkarni's work ([8]) shows that the reductive Borel–Serre compactification of a Hilbert modular surface is a Cheeger space.

Notice though that an *L-space* is a topologically stratified space, so the strata need not even be smoothable manifolds. Thus *L-spaces* constitute a larger class of spaces than Cheeger spaces.

5. Hodge/De Rham Cohomology of *ie*-Metrics

In this section we assume that all sheaves are real vector spaces and that the stratification is smooth.

A smooth stratification is the nomenclature introduced in [1]. It coincides with the notion of *espace stratifié* in [11], controlled pseudomanifold [24], and Thom–Mather stratified space [22], [25], see also [19]. Mather showed [22, §8] that any Whitney stratified subset of a smooth space, such as real and complex analytic varieties, admits Thom–Mather control data, hence a smooth stratification. In [1] it is shown that this class of spaces is equivalent to manifolds with corners endowed with *iterated fibration structures*, a class first defined by Richard Melrose.

Definition 5.1. A manifold with corners \widetilde{X} has an *iterated fibration structure* if every boundary hypersurface H is the total space of a fibration

$$\phi_H : H \rightarrow Y_H$$

whose base Y_H is itself a manifold with corners, and at each intersection $H_1 \cap H_2$ we have a commutative diagram

$$\begin{array}{ccc} H_1 \cap H_2 & \xrightarrow{\phi_{H_1}} & \phi_{H_1}(H_1 \cap H_2) \\ & \searrow \phi_{H_2} & \swarrow \phi_{H_1 H_2} \\ & Y_{H_2} & \end{array}$$

where all arrows are fibrations.

Given a manifold with corners \tilde{X} with an iterated fibration structure, we obtain a (smoothly) stratified pseudomanifold by collapsing the fibrations in an appropriate order. Conversely, given a smoothly stratified pseudomanifold \hat{X} , radial blow-ups of the strata in an appropriate order results in a manifold with corners with an iterated fibration structure, called the resolution of \hat{X} . There is a natural map

$$\beta : \tilde{X} \rightarrow \hat{X}$$

that identifies the interior of \tilde{X} with the regular part of \hat{X} , either of which we denote X . For details, we refer the reader to [1].

There is a natural class of functions on \tilde{X} respecting the iterated fibration structure,

$$\mathcal{C}_{\Phi}^{\infty}(\tilde{X}) = \{f \in \mathcal{C}^{\infty}(\tilde{X}) : i_H^* f = \phi_H^* f_{Y_H}\},$$

where $i_H : H \hookrightarrow \tilde{X}$ is the inclusion of the hyperplane H and $f_{Y_H} \in \mathcal{C}^{\infty}(Y_H)$. We view this as a natural replacement for the smooth functions on \hat{X} , so we define

$$\mathcal{C}^{\infty}(\hat{X}) = \{f \in \mathcal{C}^0(\hat{X}) : \beta^* f|_X \in \mathcal{C}_{\Phi}^{\infty}(\tilde{X})|_X\}. \quad (5.1)$$

Lemma 5.2. *Given any open cover of a smoothly stratified space, \hat{X} , there is a subordinate partition of unity in $\mathcal{C}^{\infty}(\hat{X})$.*

Proof. We follow the presentation of [21, Chap. 2] for smooth manifolds. We start by defining the analogue of a “regular cover”.

Since \hat{X} has finitely many strata, each of which is paracompact, \hat{X} is itself paracompact. (Let us assume, for simplicity of notation, that we have picked radial functions r at each stratum so that $\{r < 3\}$ is a (Thom–Mather) tubular neighborhood.) Let us briefly say that an open cover $\{\mathcal{V}_{\alpha} : \alpha \in \mathcal{A}\}$ of \hat{X} is *esteemed* if it is a countable, locally finite cover by precompact open sets such that: if $\mathcal{V}_{\alpha} \subseteq \mathcal{U}_2$ then there is a coordinate chart

$$\phi_{\alpha} : \mathcal{V}_{\alpha} \rightarrow \mathbb{R}^n$$

such that $\phi_{\alpha}(\mathcal{V}_{\alpha}) = \mathbb{B}_3(0)$, and we set $h(\alpha) = n$; if \mathcal{V}_{α} is not contained in \mathcal{U}_2 then there is a singular stratum $Y_{h(\alpha)}$, a point $q_{\alpha} \in Y_{h(\alpha)}$, and a distinguished neighborhood chart

$$\phi_{\alpha} : \mathcal{V}_{\alpha} \rightarrow \mathbb{R}^{h(\alpha)} \times [0, 3)_r \times Z_{q_{\alpha}}$$

such that $\phi_\alpha(\mathcal{V}_\alpha) = \mathbb{B}_3(0) \times [0, 3)_r \times Z_{q_\alpha}$; finally, we demand that the collection $\{\tilde{\mathcal{V}}_\alpha : \alpha \in \mathcal{A}\}$ given by

$$\tilde{\mathcal{V}}_\alpha = \begin{cases} \phi_\alpha^{-1}(\mathbb{B}_1(0)) & \text{if } \mathcal{U}_\alpha \subseteq X \\ \phi_\alpha^{-1}(\mathbb{B}_1(0) \times [0, 1)_r \times Z_{q_\alpha}) & \text{otherwise} \end{cases}$$

still covers \hat{X} . Thus essentially a cover is esteemed if it is made up of distinguished open sets.

If \hat{X} is smooth then an esteemed cover is just a regular cover, and every open cover has a regular refinement [21, Proposition 2.24]. Using this fact at each stratum, it is easy to see that any open cover of \hat{X} has an esteemed refinement. It is also easy to show that an esteemed cover has a subordinate partition of unity in $\mathcal{C}^\infty(\hat{X})$: For each $k < n$, let $\rho_k : \mathbb{R}^k \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be a smooth non-negative function equal to one on $\mathbb{B}_1(0) \times [0, 1)$ and supported in $\mathbb{B}_3(0) \times [0, 3)$, also let $\rho_n : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth non-negative function equal to one on $\mathbb{B}_1(0)$ and supported in $\mathbb{B}_3(0)$. For each α let

$$\bar{\psi}_\alpha : \mathcal{V}_\alpha \rightarrow \mathbb{R}, \quad \bar{\psi}_\alpha = \rho_{h(\alpha)} \circ \phi_\alpha,$$

where we let $\rho_{h(\alpha)}$ act on $\mathbb{B}_3(0) \times [0, 3)_r \times Z_{q_\alpha}$ by disregarding the last coordinate. Finally, the functions

$$\psi_\alpha = \frac{\bar{\psi}_\alpha}{\sum_{\alpha \in \mathcal{A}} \bar{\psi}_\alpha}$$

are a partition of unity in $\mathcal{C}^\infty(\hat{X})$ subordinate to $\{\mathcal{V}_\alpha : \alpha \in \mathcal{A}\}$. \square

There is a class of metrics that reflects the conic structure of a stratified space near a singular stratum, the *iterated incomplete edge, or iie, metrics* (also known as iterated wedge metrics). First consider \hat{X} with a total of two strata, say with filtration $Y \subseteq \hat{X}$, and let x be a *boundary defining function* for Y meaning a non-negative function $x \in \mathcal{C}^\infty(\hat{X})$ such that $Y = x^{-1}(0)$ and $|dx|$ is bounded away from zero as $x \rightarrow 0$. An iie metric on \hat{X} is a Riemannian metric on $\hat{X}^{\text{reg}} = \mathcal{U}_2$ that in a tubular neighborhood of Y has the form

$$dx^2 + x^2 g_Z + \phi^* g_Y, \tag{5.2}$$

where g_Y is a Riemannian metric on Y and g_Z restricts to the fibers over Y to be a Riemannian metric on the link. An iie metric on a general stratified space \hat{X} is defined, inductively over the depth of \hat{X} , as a Riemannian metric on \hat{X}^{reg} that near each singular stratum has the form (5.2) where now g_Z is itself an iie metric on the stratified space Z . As an iie metric is defined on the regular part of \hat{X} , we can equally well consider it as a metric on the interior of \hat{X} . Any smoothly stratified space can be endowed with an iie metric. (In our constructions below we will assume that we are working with a *rigid, suitably scaled* iie metric, these metrics also exist on any stratified space as shown in [1] to which we refer for the definitions and details.)

On a singular manifold it is often useful to replace the (co)tangent bundle with a bundle adapted to the geometry. For a stratified space with an iie metric, g , a natural choice is the iie-cotangent bundle, ${}^{\text{iie}}T^*X$, defined over \tilde{X} as those one-forms whose restriction to each boundary hypersurface vanishes on vertical vector fields. Thus sections of ${}^{\text{iie}}T^*X$ are locally spanned by

$$dx, \quad dy, \quad x dz.$$

We can also identify ${}^{\text{iie}}T^*X$ as the bundle whose sections are one-forms on \tilde{X} with bounded pointwise length with respect to g . This makes it easy to see that the metric g induces a bundle metric on ${}^{\text{iie}}T^*X$. An advantage of ${}^{\text{iie}}T^*X$ over $T^*\tilde{X}$ is that forms like $x dz$ that vanish at $x = 0$ as sections of $T^*\tilde{X}$ do not vanish at $x = 0$ as sections of ${}^{\text{iie}}T^*X$, which better reflects the structure of the space. The metric on ${}^{\text{iie}}T^*X$ allows us to define the space of L^2 iie-differential forms,

$$L^2(\tilde{X}; \Lambda^*({}^{\text{iie}}T^*X)) = L^2(\hat{X}^{\text{reg}}; \Lambda^*T^*\hat{X}^{\text{reg}}),$$

where the equality comes from identifying an L^2 -form on \tilde{X} with its restriction to the interior of \tilde{X} .

The exterior derivative on \hat{X}^{reg} extends from \hat{X}^{reg} to a differential operator on \tilde{X} acting on iie-differential forms

$$d : \mathcal{C}_c^\infty(\tilde{X}; \Lambda^j({}^{\text{iie}}T^*X)) \rightarrow \mathcal{C}_c^\infty(\tilde{X}; \Lambda^{j+1}({}^{\text{iie}}T^*X)), \quad (5.3)$$

where we will use \mathcal{C}_c^∞ to denote forms that are compactly supported in the interior of \tilde{X} or equivalently the regular part of \hat{X} . The space $\mathcal{C}_c^\infty(\tilde{X}; \Lambda^j({}^{\text{iie}}T^*X))$ is dense in $L^2(\tilde{X}; \Lambda^j({}^{\text{iie}}T^*X))$ so the operator (5.3) has two canonical extensions to a closed operator on the latter; the minimal extension of d is the operator d with domain

$$\begin{aligned} \mathcal{D}_{\min}(d) &= \{\omega \in L^2(\tilde{X}; \Lambda^*({}^{\text{iie}}T^*X)) : \exists (\omega_n) \\ &\subseteq \mathcal{C}_c^\infty(\tilde{X}; \Lambda^j({}^{\text{iie}}T^*X)) \text{ s.t. } \omega_n \rightarrow \omega \text{ and } (d\omega_n) \text{ Cauchy}\}, \end{aligned}$$

where we set $d\omega = \lim d\omega_n$, and the maximal extension of d is the operator with domain

$$\begin{aligned} \mathcal{D}_{\max}(d) &= \{\omega \in L^2(\tilde{X}; \Lambda^*({}^{\text{iie}}T^*X)) : \\ &d\omega, \text{ computed distributionally, is in } L^2(\tilde{X}; \Lambda^*({}^{\text{iie}}T^*X))\}. \end{aligned}$$

All closed extensions $(d, \mathcal{D}(d))$ of (5.3) satisfy

$$\mathcal{D}_{\min}(d) \subseteq \mathcal{D}(d) \subseteq \mathcal{D}_{\max}(d).$$

The formal adjoint of d ,

$$\delta : \mathcal{C}_c^\infty(\tilde{X}; \Lambda^j({}^{\text{iie}}T^*X)) \rightarrow \mathcal{C}_c^\infty(\tilde{X}; \Lambda^{j-1}({}^{\text{iie}}T^*X)), \quad (5.4)$$

has analogously defined domains $\mathcal{D}_{\min}(\delta)$ and $\mathcal{D}_{\max}(\delta)$ and these satisfy

$$(d, \mathcal{D}_{\min}(d))^* = (\delta, \mathcal{D}_{\max}(\delta)), \quad (d, \mathcal{D}_{\max}(d))^* = (\delta, \mathcal{D}_{\min}(\delta)).$$

We now recall the natural class of domains for d studied in [2]. Consider first a stratified space \widehat{X} with a total of two strata, with filtration $Y \subseteq \widehat{X}$, as above. Let H be the boundary hypersurface lying above Y in the resolution \widetilde{X} of \widehat{X} . If \widehat{X} is a Witt space, i.e. if Y has even codimension or the vertical (Hodge) cohomology groups^a $\mathcal{H}^{\text{mid}}(Z)$ vanish, then

$$\mathcal{D}_{\max}(d) = \mathcal{D}_{\min}(d).$$

Otherwise, given $\omega \in \mathcal{D}_{\max}(d)$, let ω_δ be its orthogonal projection off of $\ker(\delta, \mathcal{D}_{\min}(\delta))$. It is shown in *loc.cit.* that ω_δ has a (distributional) asymptotic expansion at the boundary with leading term $\alpha(\omega_\delta) + dx \wedge \beta(\omega_\delta)$ and that $\omega \in \mathcal{D}_{\min}(d)$ if and only if $\alpha(\omega_\delta) = \beta(\omega_\delta) = 0$. Thus we can define domains for d by imposing conditions on this leading term. The vertical cohomology bundle $\mathcal{H}^{\text{mid}}(H/Y) \rightarrow Y$ has a natural flat structure, which can equivalently be thought of as coming from the exterior derivative on H or from the homotopy invariance of the vertical de Rham cohomology. If $W \subseteq \mathcal{H}^{\text{mid}}(H/Y)$ is a flat subbundle, then we define a domain corresponding to *Cheeger ideal boundary conditions* from W by

$$\mathcal{D}_W(d) = \{\omega \in \mathcal{D}_{\max}(d) : \alpha(\omega_\delta) \text{ is a distributional section of } W\}.$$

In [2] it is shown that $(d, \mathcal{D}_W(d))$ is a closed operator and that filtering by differential form degree yields a Fredholm de Rham complex, for any such W .

For a general stratified space, we define domains for d inductively. If \widehat{X} is Witt, then $\mathcal{D}_{\min}(d) = \mathcal{D}_{\max}(d)$. Otherwise given a form $\omega \in \mathcal{D}_{\max}(d)$ and ω_δ its projection off of $\ker(\delta, \mathcal{D}_{\min}(\delta))$, ω has a distributional expansion at^b Y_{n-3} whose leading term we denote

$$\alpha_{n-3}(\omega_\delta) + dx \wedge \beta_{n-3}(\omega_\delta).$$

We can impose Cheeger ideal boundary conditions at Y_{n-3} by choosing a flat subbundle $W(Y_{n-3})$ of the middle-degree vertical Hodge bundle over Y_{n-3} ,

$$\mathcal{H}^{\text{mid}}(Z_{n-3}) - \mathcal{H}^{\text{mid}}(H_{n-3}/Y_{n-3}) \rightarrow Y_{n-3},$$

where H_{n-3} is the boundary hypersurface of \widetilde{X} lying above Y_{n-3} , and defining

$$\widetilde{\mathcal{L}}_3 = \{W(Y_{n-3})\},$$

$$\mathcal{D}_{\max, \widetilde{\mathcal{L}}_3}(d) = \{\omega \in \mathcal{D}_{\max}(d) : \alpha(\omega_\delta) \text{ is a distributional section of } W(Y_{n-3})\}.$$

This domain defines a closed extension of d with adjoint $(\delta, \mathcal{D}_{\min, \widetilde{\mathcal{L}}_3}(\delta))$.

The next odd codimensional stratum is Y_{n-5} and we denote the link of \widehat{X} at this stratum by Z_{n-5} . This link is itself a stratified space; it has a single singular

^a $\mathcal{H}^{\text{mid}}(Z)$ refers to space of harmonic forms on Z with respect to the induced metric, of differential form degree equal to half of the dimension of Z .

^bIf $Y_{n-3} = \emptyset$ then there is no need to impose a Cheeger type ideal boundary condition here, for simplicity of exposition we assume for the moment that all singular strata of odd codimension are non-empty and non-Witt.

stratum the link of which is Z_{n-3} , so the boundary conditions $\tilde{\mathcal{L}}_3$ imposed on the exterior derivative of \hat{X} at Y_{n-3} induce boundary conditions $\tilde{\mathcal{L}}_3(Z_{n-5})$ for the exterior derivative of Z_{n-5} . The Hodge bundle over Y_{n-5} with these boundary conditions,

$$\mathcal{H}_{\tilde{\mathcal{L}}_3(Z_{n-5})}^{\text{mid}}(Z_{n-5}) - \mathcal{H}_{\tilde{\mathcal{L}}_3(Z_{n-5})}^{\text{mid}}(H_{n-5}/Y_{n-5}) \rightarrow Y_{n-5},$$

inherits a flat structure and to impose boundary conditions at Y_{n-5} we choose a flat subbundle W_{n-5} . Unfortunately, the form ω_δ above need not have an asymptotic expansion at Y_{n-5} . However, if $\omega \in \mathcal{D}_{\max, \tilde{\mathcal{L}}_3}(d)$ and $\omega_{\delta, \tilde{\mathcal{L}}_3}$ is its orthogonal projection of $\ker(\delta, \mathcal{D}_{\min, \tilde{\mathcal{L}}_3}(\delta))$ then $\omega_{\delta, \tilde{\mathcal{L}}_3}$ has a distributional asymptotic expansion at Y_{n-5} , with leading term denoted

$$\alpha(\omega_{\delta, \tilde{\mathcal{L}}_3}) + dx \wedge \beta(\omega_{\delta, \tilde{\mathcal{L}}_3}).$$

We define

$$\begin{aligned} \tilde{\mathcal{L}}_5 &= \{W(Y_{n-3}), W(Y_{n-5})\}, \\ \mathcal{D}_{\max, \tilde{\mathcal{L}}_5}(d) &= \{\omega \in \mathcal{D}_{\max}(d) : \alpha(\omega_\delta) \text{ is a distributional section of } W(Y_{n-3}) \\ &\quad \text{over } Y_{n-3} \text{ and } \alpha(\omega_{\delta, \tilde{\mathcal{L}}_3}) \text{ is a distributional section of } W(Y_{n-5}) \\ &\quad \text{over } Y_{n-5}\}. \end{aligned}$$

We continue in this way, iteratively choosing and imposing boundary conditions at each non-Witt stratum of even codimension, until eventually we can define, with $\ell = 0$ if n is odd, and else $\ell = 1$,

$$\begin{aligned} \tilde{\mathcal{L}} &= \{W(Y_{n-3}), W(Y_{n-5}), \dots, W(Y_\ell)\}, \\ \mathcal{D}_{\tilde{\mathcal{L}}}(d) &= \{\omega \in \mathcal{D}_{\max}(d) : \alpha(\omega_\delta) \text{ is a distributional section of } W(Y_{n-3}) \text{ over } Y_{n-3}, \\ &\quad \alpha(\omega_{\delta, \tilde{\mathcal{L}}_3}) \text{ is a distributional section of } W(Y_{n-5}) \text{ over } Y_{n-5}, \dots, \\ &\quad \alpha(\omega_{\delta, \tilde{\mathcal{L}}_{n-\ell-2}}) \text{ is a distributional section of } W(Y_\ell) \text{ over } Y_\ell\}. \end{aligned}$$

We refer to these boundary conditions as *Cheeger ideal boundary conditions* corresponding to the analytic mezzoperversity $\tilde{\mathcal{L}}$. It is shown in [2] that $(d, \mathcal{D}_{\tilde{\mathcal{L}}}(d))$ is a closed operator and filtering by differential form degree yields a Fredholm de Rham complex.

A useful fact is that this domain is closed under multiplication by functions in the space $\mathcal{C}^\infty(\hat{X})$ from (5.1). Indeed, if $\omega \in \mathcal{D}_{\tilde{\mathcal{L}}}(d)$ and $f \in \mathcal{C}^\infty(\hat{X})$ then $f\omega \in L^2(\tilde{X}; \Lambda^*(\text{iie} T^*X))$ and

$$d(f\omega) = df \wedge \omega + f d\omega$$

is also in $L^2(\tilde{X}; \Lambda^*(\text{iie} T^*X))$ since df is an iie one-form; moreover it is easy to see that the leading term in the expansion of $(f\omega)_{\delta, \tilde{\mathcal{L}}_k}$ at a stratum Y_{n-k-2} is the

leading term of $\omega_{\delta, \tilde{L}_k}$ multiplied by $f|_{Y_{n-k-2}} \in \mathcal{C}^\infty(Y_{n-k-2})$, so $\omega \in \mathcal{D}_{\tilde{\mathcal{L}}}(d)$ implies $f\omega \in \mathcal{D}_{\tilde{\mathcal{L}}}(d)$.

Definition 5.3. Let us define a pre-sheaf on \hat{X} by assigning to each open set $\mathcal{U} \subseteq \hat{X}$ the vector space

$$\mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{U}) := \{\omega \in \mathcal{D}_{\tilde{\mathcal{L}}}(d) : \text{supp } \omega \subseteq (\mathcal{U} \cap \hat{X}^{\text{reg}})\}$$

and assigning to each inclusion $j : \mathcal{V} \hookrightarrow \mathcal{U}$ of open sets the restriction map

$$j^* : \mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{U}) \rightarrow \mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{V}).$$

This presheaf is filtered by differential form degree and the exterior derivative makes it into a complex of presheaves. We sheafify and denote the corresponding sheaf complex by $\mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega^\bullet \in \text{Sh}^\bullet(\hat{X})$.

It is worth pointing out that sheafification is necessary here. The presheaf $\mathcal{U} \mapsto \mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{U})$ is not a sheaf as it does not satisfy the unique gluing condition. This is true even at the level of zero-forms. Indeed, if x is a function that vanishes linearly on a singular stratum and $\{\mathcal{V}_\alpha : \alpha \in \mathcal{A}\}$ is an open cover of the regular part of \hat{X} such that each $\overline{\mathcal{V}}_\alpha \subseteq \hat{X}^{\text{reg}}$, then the sections

$$\frac{1}{x^2} \in \mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{V}_\alpha)$$

coincide on overlaps. However, as $\frac{1}{x^2}$ is not an L^2 -function on \hat{X}^{reg} , these local sections do not glue together to an element of $\mathcal{D}_{\tilde{\mathcal{L}}}(\hat{X})$. Thus in general, for $\mathcal{U} \subseteq \hat{X}$,

$$\Gamma(\mathcal{U}, \mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega^\bullet) \neq \mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{U}).$$

This makes the following result remarkable.

Lemma 5.4. (1) *The space of global sections of the sheaf $\mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega^\bullet$ coincides with the space of global sections of the presheaf $\mathcal{U} \mapsto \mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{U})$,*

$$\Gamma(\hat{X}, \mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega^\bullet) = \mathcal{D}_{\tilde{\mathcal{L}}}(\hat{X}).$$

(2) *The sheaf $\mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega^\bullet$ satisfies*

$$\Gamma(\mathcal{U}, \mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega^\bullet) = \{\omega \in L_{\text{loc}}^2(\mathcal{U}) : \text{every } p \in \mathcal{U} \text{ has a neighborhood } \mathcal{V} \subseteq \mathcal{U} \text{ such that } \omega|_{\mathcal{V}} \in \mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{V})\}.$$

Proof. We follow [9, §4].

We prove (2) first. Note that the presheaf

$$\begin{aligned} \mathcal{U} \mapsto \mathcal{E}_{\tilde{\mathcal{L}}}(\mathcal{U}) &= \{\omega \in L_{\text{loc}}^2(\mathcal{U}) : \text{every } p \in \mathcal{U} \text{ has a neighborhood } \mathcal{V} \subseteq \mathcal{U} \\ &\quad \text{such that } \omega|_{\mathcal{V}} \in \mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{V})\} \end{aligned}$$

is in fact a sheaf, and the natural inclusions

$$\mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{U}) \rightarrow \mathcal{E}_{\tilde{\mathcal{L}}}(\mathcal{U})$$

form a map of presheaves and so induce a map of sheaves, $F : \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet \rightarrow \mathcal{E}_{\widehat{\mathcal{L}}}$. It suffices to see that, for every $x \in \widehat{X}$, the induced map on stalks $F_x : \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega_x^\bullet \rightarrow (\mathcal{E}_{\widehat{\mathcal{L}}})_x$ is an isomorphism which we show as follows: Assume $F_x(s)$ is zero. Then it has a representative $\omega \in \mathcal{E}_{\widehat{\mathcal{L}}}(\mathcal{U})$ with $\omega = 0$ in a neighborhood \mathcal{V} of x . Thus $0 \in \mathcal{D}_{\widehat{\mathcal{L}}}(\mathcal{V})$ represents s , i.e. $s = 0$, so F_x is injective. Next, if $t \in (\mathcal{E}_{\widehat{\mathcal{L}}})_x$ is represented by $\eta \in \mathcal{E}_{\widehat{\mathcal{L}}}(\mathcal{U})$ then there is a neighborhood \mathcal{V} of x in \mathcal{U} such that $\eta|_{\mathcal{V}} \in \mathcal{D}_{\widehat{\mathcal{L}}}(\mathcal{V})$. So the germ of $\eta|_{\mathcal{V}}$ at x , $s \in \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega_x^\bullet$, satisfies $F_x(s) = t$.

Now to prove (1), we start with the canonical map

$$\phi : \mathcal{D}_{\widehat{\mathcal{L}}}(\widehat{X}) \rightarrow \Gamma(\widehat{X}, \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet(\widehat{X}))$$

sending an L^2 -differential form ω to the section

$$\widehat{X} \ni x \mapsto \omega_x \in \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega_x^\bullet,$$

where ω_x is the germ of ω at x . In particular, $\phi(\omega)(x) = \phi(\eta)(x)$ means that there is an open set in \widehat{X} containing x on which $\omega = \eta$. Thus if $\phi(\omega) = 0$ then there are open sets covering \widehat{X} on which ω vanishes and hence $\omega = 0$, so ϕ is injective.

Having established (2), we know that

$$\Gamma(\widehat{X}, \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet(\widehat{X})) = \{\omega \in L_{\text{loc}}^2(\widehat{X}) : \text{every } p \in \widehat{X} \text{ has a neighborhood } \mathcal{V} \subseteq \widehat{X} \text{ such that } \omega|_{\mathcal{V}} \in \mathcal{D}_{\widehat{\mathcal{L}}}(\mathcal{V})\}.$$

Let $\omega \in \Gamma(\widehat{X}, \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet(\widehat{X}))$ and for each $x \in \widehat{X}$ let \mathcal{V}_x be an open neighborhood such that $\omega \in \mathcal{D}_{\widehat{\mathcal{L}}}(\mathcal{V}_x)$. These open sets cover \widehat{X} , so by compactness there is a finite sub-cover

$$\widehat{X} = \mathcal{V}_{x_1} \cup \dots \cup \mathcal{V}_{x_N}.$$

By Lemma 5.2 there is a partition of unity $\{\psi_1, \dots, \psi_N\} \subseteq \mathcal{C}_{\Phi}^\infty(\widetilde{X})$ subordinate to the cover $\{\beta^{-1}\mathcal{V}_{x_1}, \dots, \beta^{-1}\mathcal{V}_{x_N}\}$ of \widetilde{X} . We have

$$\omega = \sum \psi_j \omega|_{\mathcal{V}_{x_j}} \text{ over } \widehat{X}^{\text{reg}}$$

and since each summand is in $\mathcal{D}_{\widehat{\mathcal{L}}}(\widehat{X})$, so is ω . This shows that ϕ is surjective. \square

From Lemma 5.4(2), the fact that $\mathcal{D}_{\widehat{\mathcal{L}}}$ is closed under multiplication by elements of $\mathcal{C}^\infty(\widehat{X})$ implies that sections of $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ are closed under multiplication by elements of $\mathcal{C}^\infty(\widehat{X})$. A useful consequence is that the sheaf $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ is soft — that is, every section of $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ on a closed set extends to a section of $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ over all of \widehat{X} . Indeed, if $K \subseteq \widehat{X}$ is a closed set and $s \in \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet(K)$ then s can be represented by a section $\tilde{s} \in \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet(\mathcal{U})$ for some neighborhood \mathcal{U} of K and by Lemma 5.2 we can find $f \in \mathcal{C}^\infty(\widehat{X})$ equal to one on K and vanishing outside of \mathcal{U} . Thus fs is an extension of s from K to a section of $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet(\widehat{X})$.

Lemma 5.5. *The complex of sheaves $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ is constructible with respect to the stratification of \widehat{X} .*

Proof. Recall [17, §1.4] that a complex of sheaves is constructible with respect to a given stratification if its restriction to each stratum has locally constant cohomology sheaves, with finitely generated stalks. Let \mathcal{U} be a distinguished neighborhood of a point $p \in \widehat{X}$. If $p \in \widehat{X}^{\text{reg}}$, then \mathcal{U} is a ball and by the Poincaré lemma its reduced de Rham cohomology vanishes; otherwise if p is on a singular stratum Y , $\mathcal{U} \cong \mathbb{B}^h \times C(Z)$, and the cohomology was computed in [2, §7]: for a fixed flat trivialization $W(Y)|_{\mathcal{U}} \cong \mathcal{U} \times W(Y)_p$, we have the generalized Poincaré lemma

$$H^k(\Gamma(\mathcal{U}, \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet)) = \begin{cases} \mathcal{H}_{\widehat{\mathcal{L}}(Z)}^k(Z) & \text{if } k < \frac{1}{2} \dim Z, \\ W(Y)_p & \text{if } k = \frac{1}{2} \dim Z, \\ 0 & \text{if } k > \frac{1}{2} \dim Z. \end{cases} \quad (5.5)$$

In either case we note that all distinguished neighborhoods of a given point have the same cohomology which hence coincides with the stalk cohomology. Since \mathcal{U} is a distinguished neighborhood of all points in $\mathcal{U} \cap Y$, the local cohomology of $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ is locally constant. Since the groups $\mathcal{H}_{\widehat{\mathcal{L}}(Z)}^k(Z)$, $W(Y)_p$ are finitely generated, the sheaf complex $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ is constructible. \square

Before stating Theorem 5.6 we make a comment. Since soft resolutions can be used to compute Ri_{k*} , we shall see in the proof of the next theorem that $Ri_{k*}i_k^* \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet = i_{k*}i_k^* \mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ for all k . To check the stalk vanishing condition of perversity \overline{p} for \mathbf{A}^\bullet , the sheafification of a presheaf A^\bullet at the stratum Y_{n-k} , it is enough to show that

$$H^j(A^\bullet(\mathcal{U})) = 0 \text{ if } j > \overline{p}(k)$$

for all distinguished neighborhoods \mathcal{U} of points in Y_{n-k} . Indeed, we have

$$\mathbb{H}^j(\mathbf{A}^\bullet)_x = H^j(\mathbf{A}^\bullet_x) = H^j(\lim_{\mathcal{U} \ni x} A^\bullet(\mathcal{U})) = \lim_{\mathcal{U} \ni x} H^j(A^\bullet(\mathcal{U})). \quad (5.6)$$

Theorem 5.6. *Let (\widehat{X}, g) be a smoothly stratified pseudomanifold with a suitably scaled iie-metric and let $\widetilde{\mathcal{L}}^\bullet$ be an analytic mezzoperversity. The sheaf $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ is in $RP(\widehat{X})$.*

Proof. The constructibility of the sheaf complex $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ is established in Lemma 5.5. We check that $\mathbf{L}_{\widehat{\mathcal{L}}}^2 \Omega^\bullet$ satisfies the axioms (RP1)–(RP4). The normalization and lower bound axioms are automatic and, by the comments above, (RP3) follows from the local Poincaré lemma (5.5), so we only need to check (RP4).

To check (RP4) first let us note that each \mathcal{U}_k is locally compact and countable at infinity. (The latter means that it can be written as a countable union of compact sets, (K_n) , with $K_\ell \subseteq K_{\ell+1}^\circ$. One can take the sub-level sets $K_\ell = \{x \geq \frac{1}{\ell}\}$ of a boundary defining function x for Y_{n-k} as the family of compact sets.) We know

that $\mathbf{S}^\bullet = \mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega^\bullet$ is soft and hence c -soft. So $i_k^* \mathbf{S}^\bullet$ is a c -soft sheaf on \mathcal{U}_k , and by [18, §IV.2, Corollary 2.3], a c -soft sheaf on a locally compact set countable at infinity is soft. Now since soft resolutions can be used to compute Ri_{k*} , we have $Ri_{k*} i_k^* \mathbf{S}^\bullet = i_{k*} i_k^* \mathbf{S}^\bullet$. Then using (5.6) we have at a point $x \in Y_{n-k}$:

$$H^j(i_{k*} i_k^* \mathbf{S}^\bullet)_x = \lim_{\mathcal{V} \ni x} H^j \Gamma(i_k^{-1}(\mathcal{V}), i_k^* \mathbf{S}^\bullet) = \lim_{\mathcal{V} \ni x} H^j \Gamma(\mathcal{V} \cap \mathcal{U}_k, i_k^* \mathbf{S}^\bullet).$$

Next notice that, for any distinguished neighborhood \mathcal{V} of x we have

$$\Gamma(\mathcal{V} \cap \mathcal{U}_k, i_k^* \mathbf{S}^\bullet) = \{\omega \in L_{\text{loc}}^2(\mathcal{V} \cap \mathcal{U}_k) : \text{every } p \in \mathcal{V} \cap \mathcal{U}_k \text{ has a neighborhood } \mathcal{W} \subseteq \mathcal{V} \cap \mathcal{U}_k \text{ such that } \omega|_{\mathcal{W}} \in \mathcal{D}_{\tilde{\mathcal{L}}}(\mathcal{W})\}.$$

Here the domain $\mathcal{V} \cap \mathcal{U}_k$ can be identified with $\mathbb{B}^h \times (C(Z_{n-k}) \setminus x) \cong \mathbb{B}^h \times (0, 1) \times Z_{n-k}$ and hence this is the L^2 -cohomology of $\mathbb{B}^h \times (0, 1) \times Z_{n-k}$ with boundary conditions $\tilde{\mathcal{L}}_{k-1}(Z_{n-k})$. Thus by the local computation above we have

$$H^j \Gamma(\mathcal{V} \cap \mathcal{U}_k, i_k^* \mathbf{S}^\bullet) = \mathcal{H}_{\tilde{\mathcal{L}}_{k-1}(Z_{n-k})}^j(Z_{n-k}).$$

The adjunction map from \mathbf{S}_x^\bullet to $Ri_{k*} i_k^* \mathbf{S}_x^\bullet$ corresponds to the restriction map r from \mathcal{V} to $\mathcal{V} \cap \mathcal{U}_k$, and the computations above show that

$$r : H^j \Gamma(\mathcal{V}, \mathbf{S}^\bullet) \rightarrow H^j \Gamma(\mathcal{V} \cap \mathcal{U}_k, i_k^* \mathbf{S}^\bullet)$$

is an isomorphism for $j \leq \overline{m}(k)$, so we have verified axiom (RP4). \square

Thus for any analytic mezzoperversity $\tilde{\mathcal{L}}$ we now know from Theorem 2.2 that there exists a topological mezzoperversity \mathcal{L} such that

$$\mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega^\bullet \text{ is quasi-isomorphic to } \mathcal{P}(\mathcal{L}). \quad (5.7)$$

By the proof of Theorem 2.2, the topological mezzoperversity \mathcal{L} consists of the degree $\overline{n}(k)$ -cohomology sheaves of $\mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega^\bullet$ over the non-Witt singular strata, and by the local computation (5.5), this is the sheaf of flat sections of the flat bundle $W(Y_{n-k})$. Thus we can identify an analytic mezzoperversity with a topological one by replacing each flat bundle with its sheaf of flat sections.

Conversely, given a topological mezzoperversity we show that it can be realized as an analytic mezzoperversity. It suffices to work with the Deligne sheaf of a topological mezzoperversity \mathcal{L} and show that we can identify it with $\mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega^\bullet$ for some analytic mezzoperversity $\tilde{\mathcal{L}}$. We construct $\mathbf{L}_{\tilde{\mathcal{L}}}^2 \Omega_k^\bullet \in D(\mathcal{U}_k)$ inductively over k . Over \mathcal{U}_3 we have an isomorphism

$$\mathcal{P}(\mathcal{L})|_{\mathcal{U}_3} = \mathcal{P}_3(\mathcal{L}) \cong \mathbf{L}^2 \Omega_3^\bullet$$

where $\mathbf{L}^2 \Omega_3^\bullet$ is the L^2 sheaf without any boundary conditions. Recall that $\mathcal{P}_4(\mathcal{L})$ is constructed over \mathcal{U}_4 by

$$\mathcal{P}_4(\mathcal{L}) = \tau_{\leq \overline{n}(3)}(Ri_{3*} \mathcal{P}_3, \mathbf{W}(Y_{n-3})),$$

where $\mathbf{W}(Y_{n-3})$ is a subsheaf of $\mathbf{H}^{\overline{n}(3)}(Ri_{3*}\mathcal{P}_3)$ over Y_{n-3} . But \mathcal{P}_3 is quasi-isomorphic to $\mathbf{L}^2\Omega_3^\bullet$, and we can identify the sheaf

$$\mathbf{H}^{\overline{n}(3)}(Ri_{3*}\mathbf{L}^2\Omega_3^\bullet)|_{Y_{n-3}}$$

with the sheaf of flat sections of the flat bundle $\mathcal{H}^{\text{mid}}(H_{n-3}/Y_{n-3})$. Indeed, the sections of the former sheaf over any distinguished neighborhood are the flat sections of $\mathcal{H}^{\text{mid}}(H_{n-3}/Y_{n-3})$ (by the local Poincaré lemma). Thus $\mathbf{W}(Y_{n-3})$ is a subsheaf of the locally constant sheaf of flat sections of $\mathcal{H}^{\text{mid}}(H_{n-3}/Y_{n-3}) \rightarrow Y_{n-3}$. This allows us to identify, for some flat sub-bundle $W(Y_{n-3}) \subseteq \mathcal{H}^{\text{mid}}(H_{n-3}/Y_{n-3})$,

$$\mathbf{W}(Y_{n-3}) = \text{flat sections of } W(Y_{n-3}).$$

Now over \mathcal{U}_4 we have both $\mathcal{P}_4(\mathcal{L}_4)$ and $\mathbf{L}_{\tilde{\mathcal{L}}_4}^2\Omega^\bullet$, where $\tilde{\mathcal{L}}_4 = \{W(Y_{n-3})\}$. As mentioned in (5.7), since the topological perversity \mathcal{L}_4 corresponds to the analytic perversity $\tilde{\mathcal{L}}_4$, we have

$$\mathcal{P}_4(\mathcal{L}_4) \cong \mathbf{L}_{\tilde{\mathcal{L}}_4}^2\Omega^\bullet \text{ over } \mathcal{U}_4.$$

It is now clear that one can proceed inductively and show that each $\mathbf{W}(Y_{n-k}) \in \mathcal{L}$ can be identified with the sheaf of flat sections of a flat sub-bundle $W(Y_{n-k})$ of

$$\mathcal{H}_{\tilde{\mathcal{L}}_k(Z_{n-k})}^{\text{mid}}(H_{n-k}/Y_{n-k}) \rightarrow Y_{n-k}.$$

We have thus shown:

Theorem 5.7. *On a compact smoothly stratified pseudomanifold \hat{X} , every topological perversity \mathcal{L} corresponds to an analytic perversity $\tilde{\mathcal{L}}$. The hypercohomology of the Deligne sheaf of \mathcal{L} can be computed as the L^2 -de Rham cohomology with Cheeger ideal boundary conditions induced by $\tilde{\mathcal{L}}$,*

$$\mathbb{H}^j(\hat{X}; \mathcal{P}(\mathcal{L})) = \mathbb{H}^j(\hat{X}; \mathbf{L}_{\tilde{\mathcal{L}}}^2\Omega^\bullet) = H^j(\mathcal{D}_{\tilde{\mathcal{L}}}(\hat{X})) = \mathbb{H}_{\tilde{\mathcal{L}}}^j(\hat{X}).$$

Moreover, these cohomology groups are stratified homotopy invariant.

Here the second equality comes from Lemma 5.4 and the final one is a definition. The stratified homotopy invariance is a consequence of Sec. 3.5 of [3].

Finally note that the Verdier dual of the sheaf $\mathbf{L}_{\tilde{\mathcal{L}}}^2\Omega^\bullet$ is, up to a shift, quasi-isomorphic to the sheaf corresponding to the dual mezzoperversity. Indeed, if \mathcal{L} is the topological mezzoperversity corresponding to $\tilde{\mathcal{L}}$, $\mathcal{D}\mathcal{L}$ is the dual topological mezzoperversity and $\widetilde{\mathcal{D}\mathcal{L}}$ is the corresponding analytic mezzoperversity, then in the derived category we have

$$\mathcal{D}(\mathbf{L}_{\tilde{\mathcal{L}}}^2\Omega^\bullet)[-n] = \mathcal{D}\mathcal{P}(\mathcal{L})[-n] = \mathcal{P}(\mathcal{D}\mathcal{L}) = \mathbf{L}_{\mathcal{D}\mathcal{L}}^2\Omega^\bullet.$$

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