A note on the higher Atiyah–Patodi–Singer index theorem on Galois coverings

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Abstract. Let Γ be a finitely generated discrete group satisfying the rapid decay condition. We give a new proof of the higher Atiyah–Patodi–Singer theorem on a Galois Γ-coverings, thus providing an explicit formula for the higher index associated to a group cocycle $c \in Z^k(\Gamma; \mathbb{C})$ which is of polynomial growth with respect to a word-metric. Our new proof employs relative K-theory and relative cyclic cohomology in an essential way.

1. Introduction

Among the many results in index theory that have followed the original work of Atiyah and Singer, few have been as inspiring and central in the whole field as the higher index theorem on Galois Γ-coverings of Connes and Moscovici [7]. The theorem itself can be seen as a far reaching generalization of the family index theorem of Atiyah and Singer, in the sense that it can be reduced to it when $\Gamma = \mathbb{Z}^k$. This fundamental observation, due to Lustzig [24], and the heat-kernel proof of the family index theorem, due to Bismut [2], are at the basis of a different proof of the Connes–Moscovici index theorem, which was given by Lott in [20]. This new proof employs the superconnection formalism, suitably extended to the noncommutative framework, in an essential way. The work of Bismut and Lott opened the way to versions of these theorems on manifolds with boundary, in the spirit of the seminal work of Atiyah, Patodi and Singer [1]. Contributions were given by Bismut–Cheeger [3, 4] and Melrose–Piazza [27, 28] for families and by Leichtnam and Piazza [13], based on a conjecture of Lott [21], for Galois coverings. Geometric applications of these index theorems were given in [12, 14, 15, 31, 32].

In this article we give a new proof of the higher Atiyah–Patodi–Singer index theorem on Galois Γ-coverings. This new proof is based in a crucial way on the excision isomorphism in K-theory and on the pairing between relative K-theory and

relative cyclic cohomology; the b-calculus of Melrose and his b-trace formula also play an important role. The ideas we employ have been already exploited successfully in [29], where a Godbillon–Vey index theorem on foliated bundles with boundary was established. For more on the use of the pairing between relative K-theory and relative cyclic cohomology see also [17,18]. Our task here is to transfer and adapt the ideas used in [29] to the context of Galois Γ-coverings, with Γ a finitely generated discrete group satisfying the (PC) and (RD) conditions (Polynomial Cohomology and Rapid Decay). Our main result provides a formula of Atiyah–Patodi–Singer type for the higher index $\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D})$ associated to $c \in Z^k(\Gamma;\mathbb{C})$; here $\mathcal{D}$ is the Mishchenko–Fomenko operator associated to a Γ-equivariant Dirac-type operator $\widetilde{D}$ on the total space of a Γ-covering with boundary. We assume, as usual, that the associated Dirac operator on the boundary, $\widetilde{D}_\partial$, is L^2-invertible. The higher index $\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D})$ is obtained by pairing the index class $\mathrm{Ind}(\mathcal{D})$ with a suitably defined cyclic cocycle τ_c associated to c (we shall of course be more precise later, we only want to give the main ideas here); one of the main steps in our proof is the production of a relative index class $\mathrm{Ind}(\mathcal{D},\mathcal{D}_\partial)$ and of a relative cyclic cocycle (τ_c^r,σ_c) and the proof of the following equality: $\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D}) = \langle \mathrm{Ind}(\mathcal{D},\mathcal{D}_\partial), [(\tau_c^r,\sigma_c)] \rangle$; one crucial technical problem we have to face is the extendability property for the relative cocycle (τ_c^r,σ_c). It should be noticed that compared to the original result in [13] our theorem has the advantage of providing the boundary correction term, i.e. the higher eta invariant $\eta_{(c,\Gamma)}(\mathcal{D}_\partial)$, in a more explicit form; indeed, our higher eta invariant comes already paired, whereas in [13] the higher eta invariant is the result of a pairing between a rather abstract object, the higher eta invariant of Lott,

$$\eta_{\mathrm{Lott}}(\mathcal{D}_\partial) \in \widehat{\Omega}_*(\mathcal{B}^\infty)/\overline{[\widehat{\Omega}_*(\mathcal{B}^\infty),\widehat{\Omega}_*(\mathcal{B}^\infty)]}$$

and a cyclic cocyle t_c associated to c. In fact, an application of our index formula is a precise expression for the number $\langle \eta_{\mathrm{Lott}}(\mathcal{D}_\partial), t_c \rangle$ appearing in [13].

The paper is organised as follows. We start in Section 2 with a few geometric preliminaries, including a brief discussion on relative and absolute cyclic (co)homology. We then move on in Section 3 and define the index class $\mathrm{Ind}(\mathcal{D})$, see Subsection 3.1; we express this index class in terms of the Wassermann projector in Subsection 3.2; in Subsection 3.3 we define the relative index class $\mathrm{Ind}(\mathcal{D},\mathcal{D}_\partial)$ and prove that corresponds to $\mathrm{Ind}(\mathcal{D})$ via excision. In Section 4 we define the higher indeces and we compare them with the ones defined by Leichtnam and Piazza in [13], proving that they are in fact equal. In Section 5 we show how to define a *relative* cyclic cocycle starting from a $c \in Z^k(\Gamma;\mathbb{C})$. In the following section, Section 6, we prove that under the two assumptions (PC) and (RD), as in Connes–Moscovici [7], our relative cocycles are continuous on the relevant algebra. Finally in Section 7 we state and prove our main result, Theorem 7.2.

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2. Preliminaries

2.1. Manifolds with boundary and cylindrical ends. Let (M_0, g_0) be a compact even dimensional riemannian manifold with boundary; the metric is assumed to be of product type in a collar neighborhood $U \cong [0, 2] \times \partial M_0$ of the boundary: thus g_0 restricted to U can be written through the above isomorphism as $dt^2 + g_\partial$, with g_∂ a riemannian metric on ∂M_0. We consider the associated manifold with cylindrical ends $M := M_0 \cup_{\partial M_0} ((-\infty, 0] \times \partial M_0)$, endowed with the extended metric g. The coordinate along the cylinder will be denoted by t. We will also consider the b-version of (M, g), obtained by performing the change of variable $\log x = t$. This is a b-riemannian manifold with product b-metric

$$\frac{dx^2}{x^2} + g_\partial$$

near the boundary. We shall freely pass from the b-picture to the cylindrical-end picture, without employing two different notations. (Our arguments will actually apply to the more general case of *exact b-metrics*, or, equivalently, manifolds with asymptotic cylindrical ends; we shall not insist on this point.)[1]

Let $\widetilde{M}_0$ be a Galois Γ-covering of M_0; we let $\tilde{g}_0$ be the lifted metric. We also consider $\partial \widetilde{M}_0$, the boundary of $\widetilde{M}_0$. We consider $\widetilde{M} := \widetilde{M}_0 \cup_{\partial \widetilde{M}_0} ((-\infty, 0] \times \partial \widetilde{M}_0)$, endowed with the extended metric $\tilde{g}$ and the obviously extended Γ action along the cylindrical end. Notice that we obtain in this way a Γ-covering of manifolds with cylindrical ends

$$\Gamma \to \widetilde{M} \to M. \tag{2.1}$$

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[1]In this article we basically give the b-calculus of Melrose, and its generalisations, as known; the basic reference is of course Melrose' book [26]; short survey-articles that can be used as an introduction to the subject are, for example, [25, Sections 1.3 and 4.2], [23], [8].

With a small abuse we introduce the notation:

$$\mathrm{cyl}(\partial \widetilde{M}) := \mathbb{R} \times \partial \widetilde{M}_0 \,, \quad \mathrm{cyl}^-(\partial \widetilde{M}) := (-\infty, 0] \times \partial \widetilde{M}_0 \tag{2.2}$$

and

$$\mathrm{cyl}^+(\partial \widetilde{M}) := [0, +\infty) \times \partial \widetilde{M}_0 \,. \tag{2.3}$$

(The abuse of notation is in writing $\mathrm{cyl}(\partial \widetilde{M})$ for $\mathbb{R} \times \partial \widetilde{M}_0$ whereas we should really write $\mathrm{cyl}(\partial \widetilde{M}_0)$.)

We assume the existence of a bundle of Clifford modules E, endowed with a hermitian metric h for which the Clifford action is unitary, and equipped with a Clifford connection. We assume product structures near the boundary throughout.

2.2. Dirac operators.

Associated to the above structures there is a generalized Dirac operator D on M_0 with product structure near the boundary. We denote by D_∂ the operator induced on the boundary. We employ the same symbol for the associated b-Dirac operator on M. We denote by $\widetilde{D}$ and $\widetilde{D}_\partial$ the Γ-equivariant b-Dirac operators on $\widetilde{M}$ and $\partial \widetilde{M}$. We also have D_{cyl} on $\mathbb{R} \times \partial M_0 \equiv \mathrm{cyl}(\partial M)$ and $\widetilde{D}_{\mathrm{cyl}}$ on $\mathbb{R} \times \partial \widetilde{M}_0 \equiv \mathrm{cyl}(\partial \widetilde{M})$. Next we consider $\Lambda := C_r^* \Gamma$, the reduced group C^*-algebra and $\mathcal{B}^\infty \subset \Lambda$ the Connes–Moscovici algebra (we recall its definition in Section 6); we denote by $\mathcal{D}_\Lambda$ and $\mathcal{D}_\infty$ the Dirac operators obtained by twisting D by the Mishchenko bundle $\mathcal{V} := \tilde{M} \times_\Gamma \Lambda$ and the $\mathcal{B}^\infty$-Mishchenko bundle $\mathcal{V}^\infty := \tilde{M} \times_\Gamma \mathcal{B}^\infty$. Unless confusion should arise we denote the latter simply by $\mathcal{D}$. We refer for example to [11, Section 1] for more details on these geometric preliminaries on Dirac operators. We shall make the following fundamental assumption

Assumption 2.1. *There exists a $\delta > 0$ such that*

$$\mathrm{spec}_{L^2}(\widetilde{D}_\partial) \cap [-\delta, \delta] = \emptyset \tag{2.4}$$

It should be noticed that because of the self-adjointness of $\widetilde{D}_\partial$, assumption (2.4) implies the L^2-invertibility of $\widetilde{D}_{\mathrm{cyl}}$. A detailed proof of this implication is given, for example, in [16] (but in a more general situation) and we refer the reader to that paper for details (see page 188 there, between (11) and (12), and then page 189); the idea is to conjugate the operator $\widetilde{D}_{\mathrm{cyl}}$,

$$\widetilde{D}_{\mathrm{cyl}} = \begin{pmatrix} 0 & -\frac{\partial}{\partial t} + \widetilde{D}_\partial \\ \frac{\partial}{\partial t} + \widetilde{D}_\partial & 0 \end{pmatrix},$$

by Fourier transform in t, $\mathcal{F}_{t \to \lambda}$, obtaining

$$\begin{pmatrix} 0 & -i\lambda + \widetilde{D}_\partial \\ i\lambda + \widetilde{D}_\partial & 0 \end{pmatrix}. \tag{2.5}$$

In the b-calculus-picture (2.5) is the indicial family $I(\widetilde{D}_{\mathrm{cyl}}, \lambda)$ of $\widetilde{D}_{\mathrm{cyl}}$ and it is obtained through Mellin transform of the corresponding cylindrical b operator. The

self-adjointness of $\widetilde{D}_\partial$ implies that (2.5), i.e. $I(\widetilde{D}_{\mathrm{cyl}}, \lambda)$, is L^2-invertible for each $\lambda \in \mathbb{R} \setminus \{0\}$; the invertibility of $\widetilde{D}_\partial$ then implies that (2.5) is L^2-invertible for each $\lambda \in \mathbb{R}$. Conjugating back the inverse of (2.5) one obtains an operator which provides an L^2-inverse of $\widetilde{D}_{\mathrm{cyl}}$. Essentially the same argument shows the invertibility of $\mathcal{D}_{\mathrm{cyl}}$ in the $\mathcal{B}^\infty$-Mishchenko–Fomenko b-calculus with bounds (to be introduced in Section 3). The details are as follows: $\mathcal{D}_{\Lambda,\partial}$ is a self-adjoint regular unbounded operator on the Hilbert Λ-module of L^2-sections of $(E \otimes \mathcal{V})_\partial$: $L^2(\partial M_0, (E \otimes \mathcal{V})_\partial)$. Hence, because of Assumption (2.4), see for example [12, Lemmas 2.1 and 3.1] we know that for each $\lambda \in \mathbb{R}$ the operator

$$\begin{pmatrix} 0 & -i\lambda + \mathcal{D}_{\Lambda,\partial} \\ i\lambda + \mathcal{D}_{\Lambda,\partial} & 0 \end{pmatrix}. \tag{2.6}$$

is invertible with inverse a Λ-pseudodifferential operator of order -1. Using the arguments in [13] and in [14, Appendix], in turn based on the work of Lott [22], we conclude that the indicial family of $\mathcal{D}_{\mathrm{cyl}}$, the latter being a differential operator of order 1 in the $\mathcal{B}^\infty$-Mishchenko–Fomenko b-calculus, is invertible for each $\lambda \in \mathbb{R}$, with inverse $I(\mathcal{D}_{\mathrm{cyl}}, \lambda)^{-1} \in \Psi_{\mathcal{B}^\infty}^{-1} \, \forall \lambda \in \mathbb{R}$. Proceeding now as in [26], Sections 5.7 and 5.16, we conclude, using the inverse Mellin transform, that there exists an inverse of $\mathcal{D}_{\mathrm{cyl}}$ and that this inverse is an element in $\mathcal{B}^\infty$-Mishchenko–Fomenko b-calculus with ϵ-bounds, with $\epsilon < \delta$ and δ as in (2.4).

2.3. Cyclic homology. In this paper we use the periodic version of cyclic homology and cohomology. In this section we briefly recall definitions and notations we use. The general references for this material are [10, 17, 18, 19].

Let $\mathcal{A}$ be a complex unital algebra. Set $C_k(\mathcal{A}) = \mathcal{A} \otimes (\mathcal{A}/\mathbb{C}1)^{\otimes k}$ for $k \geq 0$, $C_k(\mathcal{A}) = 0$ for $k < 0$. Since the algebras we consider will be Fréchet algebras, the tensor product is understood to be completed so that $C_k(\mathcal{A})$ is a Fréchet space. The space of normalized periodic cyclic chains of degree $l \in \mathbb{Z}$ is defined by

$$CC_l(\mathcal{A}) = \prod_{n \in \mathbb{Z}} C_{l+2n}(\mathcal{A}).$$

The boundary is given by $b + B$ where b and B are the Hochschild and Connes boundaries of the cyclic complex. The homology of this complex is denoted $HC_\bullet(\mathcal{A})$.

If $\mathcal{A}$ is not necessarily unital denote by $\mathcal{A}^+$ its unitalisation and set $CC_l(\mathcal{A}) = CC_l(\mathcal{A}^+)/CC_l(\mathbb{C})$. For a unital $\mathcal{A}$ this complex is quasiisomorphic to the one previously described.

If $I \colon \mathcal{A} \to \mathcal{G}$ is a homomorphism of algebras, one can consider the relative cyclic complex $CC_\bullet(\mathcal{A}, \mathcal{G})$ which is the shifted cone of the morphism of cyclic complexes induced by I, see [17]. Explicitly,

$$CC_k(\mathcal{A}, \mathcal{G}) = CC_k(\mathcal{A}) \oplus CC_{k+1}(\mathcal{G}),$$

with the differential given by

$$(\alpha, \gamma) \mapsto ((b + B)\alpha, -I(\alpha) - (b + B)\gamma), \alpha \in CC_k(\mathcal{A}), \gamma \in CC_{k+1}(\mathcal{G}).$$

In a dual manner we also consider the cyclic cohomology associated to $\mathcal{A}$. For a unital $\mathcal{A}$ and $k \geq 0$ $C^k(\mathcal{A})$ denotes the space of continuous $k + 1$ linear forms ϕ on $\mathcal{A}$ with the property that $\phi(a_0, \ldots, a_{i-1}, 1, a_i, \ldots a_{k-1}) = 0, 1 \leq i \leq k$. We set $C^k(\mathcal{A}) = 0$ for $k < 0$.

$$CC^l(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} C^{l+2n}(\mathcal{A}).$$

and the differential is given by the (transposed of) $b + B$. for $k \geq 0$. There is a natural pairing $\langle \cdot, \cdot \rangle$ between $CC^l(\mathcal{A})$ and $CC_l(\mathcal{A})$ which induces a pairing

$$\langle \, , \, \rangle_{HC} : HC_\bullet(\mathcal{A}) \otimes HC^\bullet(\mathcal{A}) \to \mathbb{C} \tag{2.7}$$

If $I : \mathcal{A} \to \mathcal{G}$ is a homomorphism, the relative cohomological complex $CC^\bullet(\mathcal{A}, \mathcal{G})$ is given by

$$CC^k(\mathcal{A}, \mathcal{G}) = CC^k(\mathcal{A}) \oplus CC^{k+1}(\mathcal{G}),$$

with the differential given by

$$(\phi, \psi) \mapsto ((b + B)\phi - I^*\psi, -(b + B)\psi), \phi \in CC^k(\mathcal{A}), \psi \in CC^{k+1}(\mathcal{G}).$$

The pairing between $CC_\bullet(\mathcal{A}, \mathcal{G})$ and $CC^\bullet(\mathcal{A}, \mathcal{G})$ is given by

$$\langle (\alpha, \gamma), (\phi, \psi) \rangle = \langle \alpha, \phi \rangle + \langle \gamma, \psi \rangle ; \tag{2.8}$$

it induces a pairing

$$\langle \, , \, \rangle_{HC} : HC_\bullet(\mathcal{A}, \mathcal{G}) \otimes HC^\bullet(\mathcal{A}, \mathcal{G}) \to \mathbb{C} \tag{2.9}$$

Recall that for an algebra $\mathcal{A}$ we have a Chern character in cyclic homology ch: $K_0(\mathcal{A}) \to HC_0(\mathcal{A})$. It is defined by the following formula. Let $P, Q \in M_{n \times n}(\mathcal{A}^+)$ be two idempotents in $n \times n$ matrices of the algebra $\mathcal{A}^+$ such that $P - Q \in M_{n \times n}(\mathcal{A})$. Note that this pair of idempotents represents a class $[P] - [Q] \in K_0(\mathcal{A})$. Then

$$\mathrm{Ch}\,(P - Q) = \mathrm{tr}(P - Q)$$
$$+ \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{n!} \mathrm{tr}\left(\left(P - \frac{1}{2}\right) \otimes P^{\otimes(2n)} - \left(Q - \frac{1}{2}\right) \otimes Q^{\otimes(2n)}\right) \tag{2.10}$$

We will use the notation $\mathrm{Ch}\,(P - Q)$ for the cyclic cycle defined above and $\mathrm{ch}\,([P] - [Q])$ for its class in cyclic homology $HC_0(\mathcal{A})$.

Assume for a moment that $\mathcal{A}$ is a *unital* Fréchet algebra and p_t, $t \in [0, 1]$ is a smooth path of idempotents in $M_{n \times n}(\mathcal{A})$. Then

$$\mathrm{Ch}(p_1) - \mathrm{Ch}(p_0) = (b + B)\,\mathrm{Tch}(p_t).$$

Here the components of the chain $\mathrm{Tch} = \sum_{n=0}^{\infty} \mathrm{Tch}_{2n+1}$ are given by

$$\mathrm{Tch}_1(p_t) = -\int_0^1 \mathrm{tr}(p_t \otimes [\dot{p}_t, p_t])\,dt$$

$$\mathrm{Tch}_{2n+1}(p_t) = (-1)^n \frac{(2n)!}{n!} \int_0^1 \sum_{i=0}^{2n} (-1)^{i+1}\,\mathrm{tr}\left(p_t - \frac{1}{2}\right)$$
$$\otimes\, p_t^{\otimes i} \otimes [\dot{p}_t, p_t] \otimes p_t^{\otimes(2n-i)}\,dt,$$

where $\dot{p}_t = \frac{dp_t}{dt}$.

If $\mathcal{A}$ and $\mathcal{G}$ are Fréchet algebras, unital or not, and $I \colon \mathcal{A} \to \mathcal{G}$ is a continuous homomorphism, then an element in the relative group $K_0(\mathcal{A}, \mathcal{G}) = K_0(\mathcal{A}^+, \mathcal{G}^+)$ can be represented by a triple (e_1, e_0, p_t) with e_0 and e_1 projections in $M_{n \times n}(\mathcal{A}^+)$, and p_t a smooth family of projections in $M_{n \times n}(\mathcal{G}^+)$, $t \in [0, 1]$, satisfying $I(e_i) = p_i$ for $i = 0, 1$. Here I denotes also the induced homomorphism $\mathcal{A}^+ \to \mathcal{G}^+$.

There is a Chern character $\mathrm{ch} \colon K_0(\mathcal{A}, \mathcal{G}) \to HC_0(\mathcal{A}, \mathcal{G})$ given by

$$\mathrm{ch}([(e_1, e_0, p_t)]) = (\mathrm{Ch}(e_1 - e_0), -\mathrm{Tch}(p_t)).$$

We will also use pairings between K-theory and cyclic cohomology given by

$$\langle [e_1] - [e_0], [\tau] \rangle := \langle \mathrm{ch}([e_1] - [e_0]), [\tau] \rangle_{HC} \tag{2.11}$$

in the absolute case and by

$$\langle [(e_1, e_0, p_t)], [(\tau, \sigma)] \rangle := \langle \mathrm{ch}([(e_1, e_0, p_t)]), [(\tau, \sigma)] \rangle_{HC}. \tag{2.12}$$

in the relative case.

2.4. Noncommutative de Rham homology. For a unital algebra A let $\Omega^\bullet A$ be the free unital differential graded algebra generated by A. The differential in $\Omega^\bullet A$ is denoted by d. $\Omega^k A$ is the span of expressions of the form $a_0 da_1 \ldots da_k$. Set $\overline{\Omega}^\bullet A = \Omega^\bullet A / [\Omega^\bullet A, \Omega^\bullet A]$. d defines a map $\overline{\Omega}^\bullet A \to \overline{\Omega}^{\bullet+1} A$. Then Karoubi's homology $\overline{H}_\bullet(A)$ is the cohomology of the complex $\left(\overline{\Omega}^\bullet A, d\right)$. The Chern character

$K_0(A) \to \prod_i \overline{H}_{2i}(A)$ is defined as follows. Let $e \in M_{n \times n}(A)$ be an idempotent. Then the Chern character of $[e]$ is represented by the form

$$\mathrm{Ch}_K(e) = \mathrm{trace} \exp(-e\,de\,de).$$

Consider now the reduced cyclic complex $\overline{C}_\lambda^\bullet(A)$. In degree ℓ it consists of $(\ell + 1)$-linear functionals ϕ on A satisfying

$$\phi(a_\ell, a_0, a_1, \ldots, a_{\ell-1}) = (-1)^\ell \phi(a_0, a_1, \ldots, a_{\ell-1}, a_\ell), \quad \phi(1, a_1, \ldots a_{\ell-1}) = 0.$$

The differential is given by b. $\overline{C}_\lambda^\bullet(A)$ is naturally a subcomplex of $CC^\bullet(A)$, as B vanishes on $\overline{C}_\lambda^\bullet(A)$. The cohomology of $\overline{C}_\lambda^\bullet(A)$ is denoted $\overline{H}_\lambda^\bullet(A)$. By the above discussion there is a natural map $\iota \colon \overline{H}_\lambda^\bullet(A) \to HC^\bullet(A)$.

There is a natural pairing between $\overline{H}_\bullet(A)$ and $\overline{H}_\lambda^\bullet(A)$ given by

$$\left\langle \sum a_0 da_1 \ldots da_\ell, [\tau] \right\rangle_K := \ell! \sum \tau(a_0, a_1, \ldots, a_\ell).$$

Then for $[\tau] \in \overline{H}_\lambda^\bullet(A)$ we have

$$\langle \mathrm{Ch}_k(e), [\tau] \rangle_K = \langle [e], \iota[\tau] \rangle := \langle \mathrm{ch}[e], \iota[\tau] \rangle_{HC}. \tag{2.13}$$

2.5. Group cohomology. Let Γ be a discrete group. The homogeneous complex $C_{hom}^\bullet(\Gamma, \mathbb{C})$ computing the cohomology of Γ can be described as follows:

$$C_{hom}^k(\Gamma, \mathbb{C}) = \{\phi \colon \Gamma^{k+1} \to \mathbb{C} \mid \phi(gg_0, \ldots gg_k) = \phi(g_0, \ldots, g_k)\}.$$

The differential is given by

$$\partial\phi(g_0, \ldots, g_k) = \sum_i (-1)^i \phi(g_0, \ldots, g_{i-1}, g_{i+1} \ldots g_k)$$

This complex is isomorphic to the nonhomogeneous complex

$$C_{nhom}^k(\Gamma, \mathbb{C}) = \{c \colon \Gamma^k \to \mathbb{C}\}$$

with the differential

$$\delta c(g_1, \ldots, g_{k+1}) = c(g_2, \ldots, g_{k+1}) + \sum (-1)^i c(g_1, \ldots, g_i g_{i+1} \ldots, g_{k+1})$$
$$+ (-1)^{k+1} c(g_1, \ldots, g_k).$$

The isomorphism of complexes is given by $I \colon C_{nhom}^\bullet(\Gamma, \mathbb{C}) \to C_{hom}^\bullet(\Gamma, \mathbb{C})$:

$$I(c)(g_0, \ldots, g_k) = c(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{k-1}^{-1} g_k)$$

with the inverse map

$$I^{-1}(\phi)(g_1, \ldots, g_k) = \phi(1, g_1, g_1 g_2, \ldots, g_1 \ldots g_k).$$

One can consider the subcomplex $C^\bullet_{hom,\Lambda}(\Gamma,\mathbb{C}) \subset C^\bullet_{hom}(\Gamma,\mathbb{C})$ defined by

$$C^\bullet_{hom,\Lambda}(\Gamma,\mathbb{C}) = \{\phi \in\subset C^\bullet_{hom}(\Gamma,\mathbb{C}) \mid \phi(g_{\sigma(0)}, g_{\sigma(1)}, \ldots, g_{\sigma(k)})$$
$$= \operatorname{sgn}\sigma\,\phi(g_0,\ldots,g_k) \text{ for every } \sigma \in S_{k+1}\}.$$

The inclusion $C^\bullet_{hom,\Lambda}(\Gamma,\mathbb{C}) \subset C^\bullet_{hom}(\Gamma,\mathbb{C})$ is a quasiisomorphism.

In this paper we will be working with the complex

$$C^\bullet(\Gamma,\mathbb{C}) \subset C^\bullet_{nhom}(\Gamma,\mathbb{C}) \tag{2.14}$$

which is the image of $C^\bullet_{hom,\Lambda}(\Gamma,\mathbb{C})$ under the map I^{-1}.

$Z^\bullet(\Gamma,\mathbb{C}) = \operatorname{Ker}\big(\delta\colon C^\bullet(\Gamma,\mathbb{C}) \to C^{\bullet+1}(\Gamma,\mathbb{C})\big)$ denotes the subspace of group cocycles. We note several immediate properties of the cochains in $C^\bullet(\Gamma,\mathbb{C})$:

Lemma 2.2. *Let $c \in C^\bullet(\Gamma,\mathbb{C})$.*

(1) *c is normalised, i.e. $c(g_1,\ldots,g_k) = 0$ if $g_i = 1$ for some i or $g_1 g_2 \cdots g_k = 1$.*

(2) *Let $g_{ij} \in \Gamma$, i, $j = 0, 1, \ldots, m$ be such that $g_{ij} g_{jk} = g_{ik}$ for every i, j, k. Then the expression $c(g_{i_0 i_1}, g_{i_1 i_2}, \ldots g_{i_{k-1} i_k})$ is antisymmetric in $i_0, i_1, \ldots, i_k$.*

(3) *If $g_1 g_2 \ldots g_{k+1} = 1$, then*

$$c(g_2,\ldots,g_{k+1}) = (-1)^k c(g_1,\ldots,g_k).$$

3. Index classes

3.1. The index class $\operatorname{Ind}_\infty(\mathcal{D})$. Let $\epsilon > 0$ be strictly smaller than δ, the width of the spectral gap for the boundary operator appearing in 2.1. We introduce

- $A := \Psi_b^{-\infty,\epsilon}(M, E) + \Psi^{-\infty,\epsilon}(M, E)$, the sum of the smoothing operators in the b-calculus with ϵ-bounds and the residual operators in the b-calculus with ϵ-bounds.

- $J := \Psi^{-\infty,\epsilon}(M, E)$

We know that A is an algebra and that J is an ideal in A (see [26] and, for this particular result, [27, Theorem 4]).

- $G := \Psi_{b,\mathbb{R}+}^{-\infty,\epsilon}(\overline{N^+(\partial M)}, E) + \Psi_{\mathbb{R}+}^{-\infty,\epsilon}(\overline{N^+(\partial M)}, E)$, the $\mathbb{R}^+$-invariant smoothing operators in the b-calculus with ϵ-bounds on the compactified positive normal bundle to the boundary. Here we have abused notation and used E to denote the extension to the normal bundle of the restriction of E to the boundary.

We know, see [26], that there is a short exact sequence of algebras

$$0 \to J \to A \xrightarrow{I} G \to 0$$

with I denoting the map equal to the *indicial operator* on $\Psi_b^{-\infty,\epsilon}(M, E)$ and equal to zero on $\Psi^{-\infty,\epsilon}(M, E)$.

We then consider (we write MF for Mishchenko–Fomenko):

- the algebra $\mathfrak{A} := \Psi_b^{-\infty,\epsilon}(M, E \otimes \mathcal{V}^\infty) + \Psi^{-\infty,\epsilon}(M, E \otimes \mathcal{V}^\infty)$, the sum of the smoothing operators in the $\mathcal{B}^\infty$-MF b-calculus with ϵ-bounds and the residual operators in the $\mathcal{B}^\infty$-MF b-calculus with ϵ-bounds;
- the ideal $\mathfrak{J}$ in $\mathfrak{A}$ equal to the residual operators $\Psi^{-\infty,\epsilon}(M, E \otimes \mathcal{V}^\infty)$;
- the algebra $\mathfrak{G}$ of $\mathbb{R}^+$-invariant smoothing operators in the $\mathcal{B}^\infty$-MF b-calculus with ϵ-bounds on the compactified positive normal bundle to the boundary

Considering the map $I : \mathfrak{A} \to \mathfrak{G}$ equal to zero on the residual operators and equal to the indicial operator on the smoothing operators in the $\mathcal{B}^\infty$-MF b-calculus with bounds, we get a short exact sequence

$$0 \to \mathfrak{J} \to \mathfrak{A} \xrightarrow{I} \mathfrak{G} \to 0$$

One can prove, see [13] and [14, Appendix], that $\mathcal{D}^+$ is invertible modulo elements in $\mathfrak{J}$; if $\mathcal{Q}$ is a parametrix with remainders $\mathcal{S}_\pm$ then we can consider the Connes–Skandalis projector

$$P_\mathcal{Q} := \begin{pmatrix} \mathcal{S}_+^2 & \mathcal{S}_+(I + \mathcal{S}_+)\mathcal{Q} \\ \mathcal{S}_-\mathcal{D}^+ & I - \mathcal{S}_-^2 \end{pmatrix}. \tag{3.1}$$

We obtain in this way an index class

$$\mathrm{CS}_\infty(\mathcal{D}) := [P_\mathcal{Q}] - [e_1] \in K_0(\mathfrak{J}) \ \text{ with } \ e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2}$$

see, for example, [6] (II.9.α) and [7] (p. 353).

We denote by $\mathrm{CS}_\Lambda(\mathcal{D})$ (recall that $\Lambda = C_r^*\Gamma$) the image of this class in $K_0(C_r^*\Gamma)$ through the homomorphism ι_* associated to the natural inclusion $\iota : \mathfrak{J} \to \mathbb{K}(\mathcal{E}_{\mathrm{MF}})$; here $\mathcal{E}_{\mathrm{MF}}$ is the Λ-Hilbert module given by $L^2(M, E \otimes \mathcal{V})$. It is clear that this is the Connes–Skandalis class associated to a parametrix for $\mathcal{D}_\Lambda$. We shall often denote this class simply by $\mathrm{CS}(\mathcal{D})$; thus

$$\mathrm{CS}(\mathcal{D}) := \iota_*(\mathrm{CS}_\infty(\mathcal{D})) \tag{3.3}$$

As in Connes–Moscovici [7, Section 5], we can use a trivializing open cover of M_0, with k trivializing open sets, a partition of unity associated to it and a collection of local sections in order to define an isometric embedding

$$C^\infty(M, E \otimes \mathcal{V}^\infty) \xrightarrow{U} C^\infty(M, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k)) \tag{3.4}$$

with the trivializing open cover extended to M in the obvious way.

Then $\theta(A) := UAU^*$ defines an algebra homomorphism between the algebra $\mathfrak{A}$, i.e. $\Psi_b^{-\infty,\epsilon}(M, E \otimes \mathcal{V}^\infty) + \Psi^{-\infty,\epsilon}(M, E \otimes \mathcal{V}^\infty)$, and the algebra $\mathcal{A}$ defined by

$$\mathcal{A} := \Psi_b^{-\infty,\epsilon}(M, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k)) + \Psi^{-\infty,\epsilon}(M, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k)) \qquad (3.5)$$

and obtained by considering the relevant MF-calculi with values in the trivial bundle $\mathcal{B}^\infty \otimes \mathbb{C}^k$. We obtain also

$$\mathcal{J} := \Psi^{-\infty,\epsilon}(M, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k))$$

and

$$\mathcal{G} := \Psi_{b,\mathbb{R}^+}^{-\infty,\epsilon}(\overline{N^+(\partial M)}, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k)) + \Psi_{\mathbb{R}^+}^{-\infty,\epsilon}(\overline{N^+(\partial M)}, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k))$$

and we know that there is a short exact sequence of algebras

$$0 \to \mathcal{J} \to \mathcal{A} \xrightarrow{I} \mathcal{G} \to 0$$

We can similarly define a homomorphism $\theta_{\mathrm{cyl}} : \mathfrak{G} \to \mathcal{G}$ and a simple argument with coverings shows that there exists a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{J} & \longrightarrow & \mathfrak{A} & \xrightarrow{I} & \mathfrak{G} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta_{\mathrm{cyl}}} \\
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{A} & \xrightarrow{I} & \mathcal{G} & \longrightarrow & 0
\end{array}
\qquad (3.6)
$$

Let $\Theta : K_0(\mathfrak{J}) \to K_0(\mathcal{J})$ be the homomorphism defined by θ; as explained in [7] this homomorphism is well-defined, independent of the choices we have made in its definition.

Definition 3.1. Inspired directly by [7] we define

$$\mathrm{Ind}_\infty(\mathcal{D}) := \Theta(\mathrm{CS}_\infty(\mathcal{D})) \in K_0(\mathcal{J}) \qquad (3.7)$$

where we recall that $\mathcal{J} = \Psi^{-\infty,\epsilon}(M, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k))$. We can also define $\mathrm{Ind}(\mathcal{D}) := \iota_*(\mathrm{Ind}_\infty(\mathcal{D})) \in K_0(C_r^*\Gamma)$ with ι equal to the composition of inclusions

$$\Psi^{-\infty,\epsilon}(M, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k)) \to \Psi^{-\infty,\epsilon}(M, E \otimes (C_r^*\Gamma \otimes \mathbb{C}^k)) \to \mathbb{K}(\mathcal{E}_{\mathrm{MF}}^\otimes)$$

with $\mathcal{E}_{\mathrm{MF}}^\otimes$ equal to the $C_r^*\Gamma$-Hilbert module $L^2(M, E \otimes (C_r^*\Gamma \otimes \mathbb{C}^k))$.

3.2. The Connes–Moscovici idempotent.
There are descriptions of the class $\mathrm{CS}_\infty(\mathcal{D}) \in K_0(\mathfrak{J})$, and thus of the index class $\mathrm{Ind}(\mathcal{D})_\infty \in K_0(\mathcal{J})$, that are particularly useful in computations.

First, for motivation, consider a closed compact manifold N and a Galois Γ-covering $\widetilde{N}$. Consider $\mathrm{CS}(\mathcal{D}) \in K_0(C_r^*\Gamma)$. We can make two specific choices of parametrices for $\mathcal{D}^+$ in (3.1):

$$\mathcal{Q}_e := (I + \mathcal{D}^-\mathcal{D}^+)^{-1}\mathcal{D}^- \quad \text{and} \quad \mathcal{Q}_V =:= \frac{I - \exp(-\frac{1}{2}\mathcal{D}^-\mathcal{D}^+)}{\mathcal{D}^-\mathcal{D}^+}\mathcal{D}^+ \qquad (3.8)$$

with $I - \mathcal{Q}_e\mathcal{D}^+ = (I + \mathcal{D}^-\mathcal{D}^+)^{-1}$, $I - \mathcal{D}^+\mathcal{Q}_e = (I + \mathcal{D}^+\mathcal{D}^-)^{-1}$ and $I - \mathcal{Q}_V\mathcal{D}^+ = \exp(-\frac{1}{2}\mathcal{D}^-\mathcal{D}^+)$, $I - \mathcal{D}^+\mathcal{Q}_V = \exp(-\frac{1}{2}\mathcal{D}^+\mathcal{D}^-)$.

The first choice of parametrix produces the graph projection

$$e_\mathcal{D} = \begin{pmatrix} (I + \mathcal{D}^-\mathcal{D}^+)^{-1} & (I + \mathcal{D}^-\mathcal{D}^+)^{-1}\mathcal{D}^- \\ \mathcal{D}^+(I + \mathcal{D}^-\mathcal{D}^+)^{-1} & \mathcal{D}^+(I + \mathcal{D}^-\mathcal{D}^+)^{-1}\mathcal{D}^- \end{pmatrix}. \qquad (3.9)$$

The choice of $\mathcal{Q}_V$ produces the idempotent

$$V_\mathcal{D} = \begin{pmatrix} e^{-\mathcal{D}^-\mathcal{D}^+} & e^{-\frac{1}{2}\mathcal{D}^-\mathcal{D}^+}\left(\frac{I - e^{-\mathcal{D}^-\mathcal{D}^+}}{\mathcal{D}^-\mathcal{D}^+}\right)\mathcal{D}^- \\ e^{-\frac{1}{2}\mathcal{D}^+\mathcal{D}^-}\mathcal{D}^+ & I - e^{-\mathcal{D}^+\mathcal{D}^-} \end{pmatrix}.$$

We call this the Connes–Moscovici idempotent (see [7]).

One can also consider the Wassermann projection $W_\mathcal{D}$,

$$W_\mathcal{D} := \begin{pmatrix} e^{-\mathcal{D}^-\mathcal{D}^+} & e^{-\frac{1}{2}\mathcal{D}^-\mathcal{D}^+}\left(\frac{I - e^{-\mathcal{D}^-\mathcal{D}^+}}{\mathcal{D}^-\mathcal{D}^+}\right)^{\frac{1}{2}}\mathcal{D}^- \\ e^{-\frac{1}{2}\mathcal{D}^+\mathcal{D}^-}\left(\frac{I - e^{-\mathcal{D}^+\mathcal{D}^-}}{\mathcal{D}^+\mathcal{D}^-}\right)^{\frac{1}{2}}\mathcal{D}^+ & I - e^{-\mathcal{D}^+\mathcal{D}^-} \end{pmatrix},$$

$$\hspace{12cm} (3.10)$$

homotopic to $V_\mathcal{D}$ via

$$P_\mathcal{D}(s) := \begin{pmatrix} e^{-\mathcal{D}^-\mathcal{D}^+} & e^{-\frac{1}{2}\mathcal{D}^-\mathcal{D}^+}\left(\frac{I - e^{-\mathcal{D}^-\mathcal{D}^+}}{\mathcal{D}^-\mathcal{D}^+}\right)^{\frac{1}{2}+s}\mathcal{D}^- \\ e^{-\frac{1}{2}\mathcal{D}^+\mathcal{D}^-}\left(\frac{I - e^{-\mathcal{D}^+\mathcal{D}^-}}{\mathcal{D}^+\mathcal{D}^-}\right)^{\frac{1}{2}-s}\mathcal{D}^+ & I - e^{-\mathcal{D}^+\mathcal{D}^-} \end{pmatrix},$$

$$\hspace{12cm} (3.11)$$

with $s \in [0, 1/2]$. See [7], before Lemma (2.5).

The same formula (3.11) with $s \in [-1/2, 1/2]$ defines a homotopy between $V_\mathcal{D}$ and $V_\mathcal{D}^*$. From the discussion above we obtain the following equality of elements in $K_0(C_r^*\Gamma)$:

$$[P_\mathcal{Q}] - [e_1] = [e_\mathcal{D}] - [e_1] = [V_\mathcal{D}] - [e_1] = [V_\mathcal{D}^*] - [e_1] \qquad (3.12)$$

We will also need to consider a symmetrized idempotent $\widetilde{V}_\mathcal{D} := V_\mathcal{D} \oplus V_\mathcal{D}^*$. If we set $\widetilde{e}_1 := e_1 \oplus e_1$, we have

$$[\widetilde{V}_\mathcal{D}] - [\widetilde{e}_1] = 2([V_\mathcal{D}] - [e_1]).$$

Well known properties of the heat operator and of the pseudodifferential calculus imply that actually $[V_\mathcal{D}] - [e_1]$ and $[\widetilde{V}_\mathcal{D}] - [\widetilde{e}_1]$ belong to $K_0(\Psi^{-\infty}(N, E \otimes \mathcal{V}^\infty))$.

Summarizing, we can set

$$\mathrm{CS}_\infty(\mathcal{D}) := [V_\mathcal{D}] - [e_1] = [V_\mathcal{D}^*] - [e_1] \text{ in } K_0(\Psi^{-\infty}(N, E \otimes \mathcal{V}^\infty)). \quad (3.13)$$

The last equality follows from the fact that $\mathcal{B}^\infty$ is closed under holomorphic functional calculus in $C_r^*\Gamma$. Indeed, the images of the classes $[V_\mathcal{D}] - [e_1]$ and $[V_\mathcal{D}^*] - [e_1]$ are equal in $K_0(C_r^*\Gamma)$, see (3.12), and the map

$$K_0(\Psi^{-\infty}(N, E \otimes \mathcal{V}^\infty)) \to K_0(C_r^*\Gamma)$$

is an isomorphism, given that $\Psi^{-\infty}(N, E)$ is holomorphically closed in the compacts of $L^2(N, E)$ and $\mathcal{V}^\infty$ is holomorphically closed in $C_r^*\Gamma$.

Consider now the isometric embedding

$$C^\infty(M, E \otimes \mathcal{V}^\infty) \xrightarrow{U} C^\infty(M, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k))$$

recalled in (3.4); one can check that $U\mathcal{D}U^* = \mathcal{D}^\otimes$, with $\mathcal{D}^\otimes$ equal to the operator D twisted by the trivial bundle $\mathcal{B}^\infty \otimes \mathbb{C}^r$. This implies that $\theta(V_\mathcal{D}) = V_{\mathcal{D}\otimes}$ and $\theta(V_\mathcal{D}^*) = V_{\mathcal{D}\otimes}^*$. We obtain immediately:

$$\begin{aligned}
\mathrm{Ind}_\infty(\mathcal{D}) := \Theta(\mathrm{CS}_\infty(\mathcal{D})) &= [V_{\mathcal{D}\otimes}] - [e_1] \\
&= [V_{\mathcal{D}\otimes}^*] - [e_1] \text{ in } K_0(\Psi^{-\infty}(N, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^r)))
\end{aligned}$$
$$(3.14)$$

Let us now pass to a b-manifold M and to an operator $\mathcal{D}$ satisfying the invertibility assumption on the boundary. First of all, recall how the (true) parametrix of $\mathcal{D}^+$ is constructed. We shall be somewhat brief on this point since this procedure is explained in detail in many places; in particular we shall not be particularly precise about the gradings and the identifications on the boundary. One begins by finding a symbolic parametrix $\mathcal{Q}_\sigma$ to $\mathcal{D}^+$, with remainders $\mathcal{R}_\sigma^\pm$. Next, by fixing a cut-off function χ on the collar neighborhood of the boundary, equal to 1 on the boundary, we define a section $s : \mathfrak{G} \to \mathfrak{A}$ to the indicial homomorphism $I : \mathfrak{A} \to \mathfrak{G}$. s associates to a translation invariant operator G on the cylinder an operator on the manifold with cylindrical end; the latter is obtained by pre-multiplying and post-multiplying G by the cut-off function χ. The (true) parametrix of $\mathcal{D}^+$ is defined as $\mathcal{Q}^b = \mathcal{Q}_\sigma - \mathcal{Q}'$ with $\mathcal{Q}'$ equal to $s((I(D^+)^{-1}I(\mathcal{R}_\sigma^-))$. Then, with this definition, one can check, using the b-calculus, that $\mathcal{D}^+\mathcal{Q}^b = I - \mathcal{S}^-$ and $\mathcal{Q}^b\mathcal{D}^+ = I - \mathcal{S}^+$ with $\mathcal{S}^\pm$ residual operators. Now, going back to the classes $\mathrm{CS}(\mathcal{D})$ and $\mathrm{CS}_\infty(\mathcal{D})$ it is clear that we can define the Connes–Skandalis projection using the (true) parametrix obtained through the above procedure but starting with the symbolic parametrices $\mathcal{Q}_e$ and $\mathcal{Q}_V$

appearing in (3.8). Recall that the remainders $\mathcal{R}_\sigma^\pm$ of these two symbolic parametrices are given by the following two equations

$$I - \mathcal{Q}_e \mathcal{D}^+ = (I + \mathcal{D}^- \mathcal{D}^+)^{-1}, \qquad I - \mathcal{D}^+ \mathcal{Q}_e = (I + \mathcal{D}^+ \mathcal{D}^-)^{-1}$$

and $$\quad I - \mathcal{Q}_V \mathcal{D}^+ = \exp(-\frac{1}{2}\mathcal{D}^- \mathcal{D}^+), \qquad I - \mathcal{D}^+ \mathcal{Q}_V = \exp(-\frac{1}{2}\mathcal{D}^+ \mathcal{D}^-).$$

The construction just explained produces then two different (true) parametrices $\mathcal{Q}_e^b$ and $\mathcal{Q}_V^b$ and two different projectors that we denote respectively $e_{\mathcal{D}}^b$ and $V_{\mathcal{D}}^b$. Let us see the specific structure of these two projectors, starting with $e_{\mathcal{D}}^b$. Recall that $I(\mathcal{D}^\pm) = \mathcal{D}_{\text{cyl}}^\pm = \pm x \partial_x + \mathcal{D}_\partial$. By definition

$$\mathcal{Q}_e' := -\chi((\mathcal{D}_{\text{cyl}}^+)^{-1}(I + \mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-)^{-1})\chi. \tag{3.15}$$

Then, a simple computation gives

$$\mathcal{Q}_e' \mathcal{D}^+ = -\chi(I + \mathcal{D}_{\text{cyl}}^- \mathcal{D}_{\text{cyl}}^+)^{-1}\chi + \chi(\mathcal{D}_{\text{cyl}}^+)^{-1}(I + \mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-)^{-1} \, cl(d\chi) \tag{3.16}$$

$$\mathcal{D}^+ \mathcal{Q}_e' = -\chi(I + \mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-)^{-1}\chi - cl(d\chi)(\mathcal{D}_{\text{cyl}}^+)^{-1}(I + \mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-)^{-1}\chi. \tag{3.17}$$

This means that $\mathcal{S}_e^+ := I - \mathcal{Q}_e^b \mathcal{D}^+ = I - (\mathcal{Q}_e - \mathcal{Q}_e') \, \mathcal{D}^+ = (I + \mathcal{D}^- \mathcal{D}^+)^{-1} + \mathcal{Q}_e' \mathcal{D}^+$, which we know from the b-calculus to be residual, is given by

$$\mathcal{S}_e^+ = (I + \mathcal{D}^- \mathcal{D}^+)^{-1} - \chi(I + \mathcal{D}_{\text{cyl}}^- \mathcal{D}_{\text{cyl}}^+)^{-1}\chi + \chi(\mathcal{D}_{\text{cyl}}^+)^{-1}(I + \mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-)^{-1} \, cl(d\chi). \tag{3.18}$$

Similarly

$$\mathcal{S}_e^- = (I + \mathcal{D}^+ \mathcal{D}^-)^{-1} - \chi(I + \mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-)^{-1}\chi - cl(d\chi)(\mathcal{D}_{\text{cyl}}^+)^{-1}(I + \mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-)^{-1}\chi \tag{3.19}$$

Substituting $\mathcal{S}_e^\pm$ and $\mathcal{Q}_e^b$ at the place of $\mathcal{S}^\pm$ and $\mathcal{Q}$ into the expression of the Connes–Skandalis projection

$$\begin{pmatrix} \mathcal{S}_+^2 & \mathcal{S}_+(I + \mathcal{S}_+)\mathcal{Q} \\ \mathcal{S}_- \mathcal{D}^+ & I - \mathcal{S}_-^2 \end{pmatrix}.$$

we obtain $e_{\mathcal{D}}^b$. The precise form of $e_{\mathcal{D}}^b$ plays a role in the proof of the excision correspondence (3.34) explained in Theorem 3.3 below.

Similarly, by definition,

$$\mathcal{Q}_V' := -\chi((\mathcal{D}_{\text{cyl}}^+)^{-1} \exp(-\frac{1}{2}\mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-)\chi. \tag{3.20}$$

and this gives us

$$\mathcal{Q}_V' \mathcal{D}^+ = -\chi \exp(-\frac{1}{2}\mathcal{D}_{\text{cyl}}^- \mathcal{D}_{\text{cyl}}^+)\chi + \chi(\mathcal{D}_{\text{cyl}}^+)^{-1} \exp(-\frac{1}{2}\mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-) \, cl(d\chi) \tag{3.21}$$

$$\mathcal{D}^+ \mathcal{Q}_W' = -\chi \exp(-\frac{1}{2}\mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-)^{-1})\chi - cl(d\chi)(\mathcal{D}_{\text{cyl}}^+)^{-1} \exp(-\frac{1}{2}\mathcal{D}_{\text{cyl}}^+ \mathcal{D}_{\text{cyl}}^-)\chi. \tag{3.22}$$

This means that

$$\mathcal{S}_V^+ := I - \mathcal{Q}_V^b \mathcal{D}^+ = I - (\mathcal{Q}_V - \mathcal{Q}_V') \, D^+ = \exp(-\tfrac{1}{2}\mathcal{D}^+\mathcal{D}^-) + \mathcal{Q}_V' \, D^+,$$

a residual operator, is given by

$$\mathcal{S}_V^+ = \exp(-\tfrac{1}{2}\mathcal{D}^+\mathcal{D}^-) - \chi \exp(-\tfrac{1}{2}\mathcal{D}_{\mathrm{cyl}}^+\mathcal{D}_{\mathrm{cyl}}^-)\chi + \chi(D_{\mathrm{cyl}}^+)^{-1}\exp(-\tfrac{1}{2}\mathcal{D}_{\mathrm{cyl}}^+\mathcal{D}_{\mathrm{cyl}}^-)\, cl(d\chi). \tag{3.23}$$

It is important to remark that this is precisely the expression in (3.18) once we substitute $\exp(-\tfrac{1}{2}\mathcal{D}_{\mathrm{cyl}}^+\mathcal{D}_{\mathrm{cyl}}^-)$ for $(I + \mathcal{D}^+\mathcal{D}^-)^{-1}$; this will play a role in the excision argument to be given below. A similar expression can be found for $\mathcal{S}_V^-$. Substituting $\mathcal{S}_V^\pm$ and $\mathcal{Q}_V^b$ at the place of $\mathcal{S}^\pm$ and $\mathcal{Q}$ into the expression of the Connes–Skandalis projection we obtain $V_{\mathcal{D}}^b$.

One gets as before, $[e_{\mathcal{D}}^b] - [e_1] = [V_{\mathcal{D}}^b] - [e_1] = [(V_{\mathcal{D}}^b)^*] - [e_1] \in K_0(C_r^*\Gamma)$. Thus we can set: $\mathrm{CS}(\mathcal{D}) := [V_{\mathcal{D}}^b] - [e_1] \equiv [e_{\mathcal{D}}^b] - [e_1]$ in $K_0(C_r^*\Gamma)$. One can check, using the MF-b-calculus with bounds, that the b-Connes–Moscovici projection $V_{\mathcal{D}}^b$ belongs to $\mathfrak{J}^+$; crucial, here, is the information that $\mathcal{D}_\partial$ is invertible in the $\mathcal{B}^\infty$-MF-calculus. (The graph projector, on the other hand, belongs to $\Psi^{-1,\epsilon}(M, E \otimes \mathcal{V}^\infty)^+$.) We shall choose the incarnation of the class $\mathrm{CS}_\infty(\mathcal{D})$ given by $[V_{\mathcal{D}}^b] - [e_1]$; put it differently

$$\mathrm{CS}_\infty(\mathcal{D}) := [V_{\mathcal{D}}^b] - [e_1] \text{ in } K_0(\mathfrak{J}). \tag{3.24}$$

Notice, finally, that from the invertibility of $\mathcal{D}_\partial$ follows the invertibility of the boundary operator of $\mathcal{D}^\otimes$ (this is a simple consequence of $U^*U = \mathrm{Id}$); proceeding as in (3.14) we obtain

$$\mathrm{Ind}_\infty(\mathcal{D}) := \Theta\left([V_{\mathcal{D}}^b] - [e_1]\right) = [V_{\mathcal{D}^\otimes}^*] - [e_1] \text{ in } K_0(\mathcal{J}). \tag{3.25}$$

3.3. The relative index class $\mathrm{Ind}_\infty(\mathcal{D}, \mathcal{D}_\partial)$. Excision.

Let $0 \to J \to A \xrightarrow{\pi} B \to 0$ be a short exact sequence of Fréchet algebras. Recall that $K_0(J) := K_0(J^+, J) \cong \mathrm{Ker}(K_0(J^+) \to \mathbb{Z})$ and that $K_0(A^+, B^+) = K_0(A, B)$. For the definition of relative K-groups we refer, for example, to [5, 9, 17]. Recall that a relative K_0-element for $A \xrightarrow{\pi} B$ with unital algebras A, B is represented by a triple (P, Q, p_t) with P and Q idempotents in $M_{n \times n}(A)$ and $p_t \in M_{n \times n}(B)$ a path of idempotents connecting $\pi(P)$ to $\pi(Q)$. The excision isomorphism

$$\alpha_{\mathrm{ex}} : K_0(J) \longrightarrow K_0(A, B) \tag{3.26}$$

is given by $\alpha_{\mathrm{ex}}([(P, Q)]) = [(P, Q, \mathbf{c})]$ with $\mathbf{c}$ denoting the constant path (this is not necessarily the 0-path, given that we are taking J^+).

In this paper we are interested in the relative groups $K_0(\mathfrak{A}, \mathfrak{G})$ and $K_0(\mathcal{A}, \mathcal{G})$ associated respectively to $0 \to \mathfrak{J} \to \mathfrak{A} \xrightarrow{I} \mathfrak{G} \to 0$ and $0 \to \mathcal{J} \to \mathcal{A} \xrightarrow{I} \mathcal{G} \to 0$.

Consider the Connes–Moscovici projections $V_{\mathcal{D}}$ and $V_{\mathcal{D}_{\mathrm{cyl}}}$ associated to $\mathcal{D}$ and $\mathcal{D}_{\mathrm{cyl}}$. With $e_1 := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ consider the triple

$$(V_{\mathcal{D}}, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}})}) , \quad t \in [1, +\infty], \quad \text{with} \quad q_t := \begin{cases} V_{(t\mathcal{D}_{\mathrm{cyl}})} & \text{if } t \in [1, +\infty) \\ e_1 & \text{if } t = \infty \end{cases} \quad (3.27)$$

Similarly, we can consider the triple

$$(V_{\mathcal{D}}^*, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}})}^*) , \quad t \in [1, +\infty]. \tag{3.28}$$

Proposition 3.2. *Under assumption* (2.4) *the Connes–Moscovici idempotents* $V_{\mathcal{D}}$ *and* $V_{\mathcal{D}_{\mathrm{cyl}}}$ *define by the formula* (3.27)*, a relative class in* $K_0(\mathfrak{A}, \mathfrak{G})$ *associated with the short exact sequence* $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{A} \xrightarrow{I} \mathfrak{G} \rightarrow 0$. *We denote this class by* $[V_{\mathcal{D}}, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}})}]$. *Similarly the adjoint idempotents define a relative class* $[V_{\mathcal{D}}^*, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}})}^*] \in K_0(\mathfrak{A}, \mathfrak{G})$. *These two classes are equal.*

Proof. The existence of the two classes follows from the invertibility assumption and well known properties of the pseudodifferential calculus. Their equality follows from the homotopy (3.11) and the fact that $\mathcal{B}^\infty$ is closed under the holomorphic functional calculus. $\qquad\square$

We set
$$\mathrm{CS}_\infty(\mathcal{D}, \mathcal{D}_\partial) := [V_{\mathcal{D}}, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}})}] \in K_0(\mathfrak{A}, \mathfrak{G}) \tag{3.29}$$

The homomorphisms θ and θ_{cyl} define through (3.6) a homomorphism

$$\Theta_{\mathrm{rel}} : K_0(\mathfrak{A}, \mathfrak{G}) \rightarrow K_0(\mathcal{A}, \mathcal{G})$$

which is well defined, independent of choices; we set

$$\mathrm{Ind}_\infty(\mathcal{D}, \mathcal{D}_\partial) := \Theta_{\mathrm{rel}}(\mathrm{CS}_\infty(\mathcal{D}, \mathcal{D}_\partial)) \in K_0(\mathcal{A}, \mathcal{G}). \tag{3.30}$$

Notice that, as in (3.25),

$$\mathrm{Ind}_\infty(\mathcal{D}, \mathcal{D}_\partial) = [V_{\mathcal{D}^\otimes}, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}}^\otimes)}] = [V_{\mathcal{D}^\otimes}^*, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}}^\otimes)}^*]. \tag{3.31}$$

Theorem 3.3. *Let* $\alpha_{\mathrm{ex}} : K_0(\mathfrak{J}) \rightarrow K_0(\mathfrak{A}, \mathfrak{G})$ *be the excision isomorphism for the short exact sequence* $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{A} \xrightarrow{I} \mathfrak{G}$. *Then*

$$\alpha_{\mathrm{ex}}(\mathrm{CS}_\infty(\mathcal{D})) = \mathrm{CS}_\infty(\mathcal{D}, \mathcal{D}_\partial) \in K_0(\mathfrak{A}, \mathfrak{G}). \tag{3.32}$$

Consequently, if $\beta_{\mathrm{ex}} : K_0(\mathcal{J}) \rightarrow K_0(\mathcal{A}, \mathcal{G})$ *is the excision isomorphism for the short exact sequence* $0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \xrightarrow{I} \mathcal{G}$. *Then*

$$\beta_{\mathrm{ex}}(\mathrm{Ind}_\infty(\mathcal{D})) = \mathrm{Ind}_\infty(\mathcal{D}, \mathcal{D}_\partial) \in K_0(\mathcal{A}, \mathcal{G}). \tag{3.33}$$

Proof. It is sufficient to prove (3.32) which can be rewritten as

$$\alpha_{\mathrm{ex}}([V_{\mathcal{D}}^b] - [e_1]) = [V_{\mathcal{D}}, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}})}].$$

Indeed, if we can give an argument justifying this equality, then we can also prove that

$$\beta_{\mathrm{ex}}([V_{\mathcal{D}\otimes}^b] - [e_1]) = [V_{\mathcal{D}\otimes}, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}}^{\otimes})}];$$

on the left we have $\beta_{\mathrm{ex}}(\mathrm{Ind}_\infty(\mathcal{D}))$ whereas on the right we have $\Theta_{\mathrm{rel}}([V_{\mathcal{D}}, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}})}])$ which is precisely $\Theta_{\mathrm{rel}}(\mathrm{CS}_\infty(\mathcal{D}, \mathcal{D}_\partial))$; thus

$$\beta_{\mathrm{ex}}(\mathrm{Ind}_\infty(\mathcal{D})) = \mathrm{Ind}(\mathcal{D}, \mathcal{D}_\infty).$$

as required.

In order to show the equality $\alpha_{\mathrm{ex}}([V_{\mathcal{D}}^b] - [e_1]) = [V_{\mathcal{D}}, e_1, V_{(t\mathcal{D}_{\mathrm{cyl}})}]$ we can adapt the proof of the equality

$$\alpha_{\mathrm{ex}}([e_{\mathcal{D}}^b] - [e_1]) = [e_{\mathcal{D}}, e_1, e_{(t\mathcal{D}_{\mathrm{cyl}})}], \tag{3.34}$$

given in [29], keeping in mind the remark given right after (3.23). It is here that the specific structures of $e_{\mathcal{D}}^b$ and $V_{\mathcal{D}}^b$ are used. Since the details are elementary but somewhat lengthy we omit them.   □

4. Cyclic cocycles and higher indices

Given a group cohomology class ξ of $H^k(\Gamma; \mathbb{C})$, we choose a representative cocycle c in $C^k(\Gamma; \mathbb{C})$. Thus, see Lemma (2.2) and the discussion preceding it, c is normalized, namely: $c(g_1, g_2, \ldots, g_k) = 0$ if any $g_i = 1$ or $g_1 g_2 \cdots g_k = 1$. Consider the algebra $\mathcal{J}$ with $r = 1$; an element $S \in \mathcal{J}$ is, in particular, a continuous section of the bundle $\mathrm{END}(E) \otimes \mathcal{B}_\infty$ on $M \times M$. Equivalently, from the inclusion $\Psi^{-\infty, \epsilon}(M, E \otimes \mathcal{B}_\infty) \subset \Psi^{-\infty, \epsilon}(M, E \otimes C_r^*\Gamma)$ and the fact that $C_r^*\Gamma$ is contained in $\ell^2(\Gamma)$, we can see that S is a function on Γ with values in $\Psi^{-\infty, \epsilon}(M, E)$, denoted $\Gamma \ni g \to S(g) \in \Psi^{-\infty, \epsilon}(M, E)$. We shall have to be precise about the continuity properties of this function, but for the time being we work on the dense subalgebra $\mathcal{J}_f$ of $\mathcal{J}$ given by the elements of compact support in Γ; put it differently we work with the algebraic tensor product

$$\mathcal{J}_f := \Psi^{-\infty, \epsilon}(M, E) \otimes \mathbb{C}\Gamma \subset \mathcal{J}.$$

Before passing to the next definition, recall that elements in $\Psi^{-\infty, \epsilon}(M, E)$ are trace class on L^2. Hence it makes sense to give the following

Definition 4.1. For $S_i \in \mathcal{J}_f$ we set

$$\tau_c(S_0 + \omega \cdot 1, S_1, \ldots S_k) = \sum_{g_0 g_1 \cdots g_k = 1} \mathrm{Tr}(S_0(g_0) S_1(g_1) \cdots S_k(g_k)) c(g_1, g_2, \ldots, g_k).$$

We know, see [6], that τ_c defines a cyclic cocycle for $\mathcal{J}_f$.

Assumption 4.2 (Extendability). *τ_c extends from $\mathcal{J}_f$ to $\mathcal{J}$.*

Proposition 4.3. *If Γ is Gromov hyperbolic then we can choose a representative c of ξ so that τ_c extends.*

Proof. This will be proved later, see Proposition 6.1 and its proof. □

Recall the pairing between K-groups and cyclic cohomology groups and more particularly between the K_0-group and the cyclic cohomology group of even degree. See the definition in (2.11).

Definition 4.4. If τ_c satisfies the extendability assumption then we define the higher index associated to c as

$$\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D}) := \langle \mathrm{Ind}_\infty(\mathcal{D}), \tau_c \rangle \tag{4.1}$$

We can now state the following:

The main goal of this paper is to prove a Atiyah–Patodi–Singer formula for $\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D})$.

To this end we recall one of the main steps in the proof of the higher index theorem of Connes–Moscovici. Let N be a closed compact manifold and $\widetilde{N} \xrightarrow{\pi} N$ a Galois Γ-covering. Consider the idempotent $V_{\mathcal{D}\otimes}$ and the index class

$$\mathrm{Ind}_\infty(\mathcal{D}) = [V_{\mathcal{D}\otimes}] - [e_1] \in K_0(\Psi^{-\infty}(N, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^r)).$$

For τ_c extendable and of degree k, $k = 2p$, we have:

$$\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D}) = \mathrm{const}_k \ \tau_c(V_{u\mathcal{D}\otimes}, \dots, V_{u\mathcal{D}\otimes}),$$

where

$$\mathrm{const}_k = (-1)^p \frac{(2p)!}{p!}, \quad k = 2p \tag{4.2}$$

and $u > 0$. The following Proposition is crucial and employs Getzler-rescaling in an essential way. Recall the data needed in order to construct the map (3.4):

- A good open cover $\mathcal{U} = \{U_1, \dots, U_r\}$.
- Continuous sections $s_i \colon U_i \to \widetilde{N}$ of the projection $\widetilde{N} \to N$.
- A partition of unity χ_i, supp $\chi_i \subset U_i$, $\sqrt{\chi_i}$ smooth.

Given a Γ-cocycle c of degree k we can use this data in order to construct a closed differential form ω_c as follows. For every i, j let $g_{ij} \in \Gamma$ be the unique element such that $g_{ij} s_j(x) = s_i(x)$ for every $x \in U_i \cap U_j$. Then set

$$\omega_c = \sum_{i_0, i_1, \ldots, i_k} c(g_{i_1 i_2}, \ldots, g_{i_k i_0}) \chi_{i_0} d\chi_{i_1} \cdots d\chi_{i_k}$$
$$= \sum_{i_0, i_1, \ldots, i_k} c(g_{i_0 i_1}, \ldots, g_{i_{k-1} i_k}) \chi_{i_0} d\chi_{i_1} \cdots d\chi_{i_k}$$

The form ω_c defined by the above equation is closed and $[\omega_c] = \nu^*[c]$ where $\nu \colon N \to B\Gamma$ is the classifying map. Here we use the isomorphism $H^\bullet(B\Gamma, \mathbb{C}) \simeq H^\bullet(\Gamma, \mathbb{C})$. We can give another description of the form ω_c as in [20]. The sections s_i induce diffeomorphisms $s_i \colon U_i \to \widetilde{U}_i = s_i(U_i) \subset \widetilde{N}$. One then constructs the functions $\tilde{\chi}_i \in C_0^\infty(\widetilde{U}_i)$ by $\tilde{\chi}_i = (s_i^{-1})^* \chi_i$. Set $h = \sum_i \tilde{\chi}_i \in C_0^\infty(\widetilde{N})$. Then the function h has the property that

$$\sum_{g \in \Gamma} g \cdot h = 1,$$

where $g \cdot f(x) = f(g^{-1} x)$. Let $\widetilde{\omega}_c \in \Omega^*(\widetilde{N})$ be the differential form given by

$$\widetilde{\omega}_c = \sum_{g_i \in \Gamma} d(g_1 \cdot h) \cdots d(g_k \cdot h) c(g_1, g_1^{-1} g_2, \ldots, g_{k-1}^{-1} g_k) \,.$$

This form is Γ-equivariant and moreover $\widetilde{\omega}_c = \pi^*(\omega_c)$.

Proposition 4.5. *For any $u > 0$ we have*

$$\mathrm{const}_k \cdot \tau_c(V_{u\mathcal{D}\otimes}, \ldots, V_{u\mathcal{D}\otimes}) = \int_N \mathrm{AS} \wedge \omega_c \,. \tag{4.3}$$

with AS *equal to the Atiyah-Singer integrand.*

This theorem has been proved by Connes–Moscovici in [7]. In that paper they used the Getzler's calculus to compute short-time asymptotics of Wassermann projection. In [30] Moscovici and Wu show that the same method applies to a wider class of idempotents, in particular to the symmetrized idempotent

$$\widetilde{V}_{u\mathcal{D}\otimes} := V_{u\mathcal{D}\otimes} \oplus V_{u\mathcal{D}\otimes}^*.$$

This is the result that we will need for the calculation of the short-time limit. We note that a different method, also based on Getzler's rescaling, but using instead the superconnection techniques, has been used by J. Lott in [20].

We recall now some of the steps in Connes–Moscovici's proof this theorem, using slight modification from [30], referring the reader to [7, 20, 30] for details. We start by noticing that as τ_c extends to pair with the K-theory of $C_r^*\Gamma$, we have

$$\tau_c(V_{u\mathcal{D}\otimes}, \ldots, V_{u\mathcal{D}\otimes}) = \frac{1}{2} \tau_c(\widetilde{V}_{u\mathcal{D}\otimes}, \ldots, \widetilde{V}_{u\mathcal{D}\otimes}).$$

Consider the cochain $\tilde{\tau}_c$ on the smoothing operators $\Psi^{-\infty}(N)$ given by

$$\tilde{\tau}_c(A_0, A_1, \ldots, A_k) = \int\limits_{N^{k+1}} \operatorname{tr} A_0(x_0, x_1) \cdots A_k(x_k, x_0)\phi_c(x_0, \ldots, x_k)dx_0 \cdots dx_k$$

where

$$\phi_c(x_0, \ldots, x_k) = \sum_{i_0, i_1, \ldots, i_k} c(g_{i_1 i_2}, \ldots, g_{i_k i_0})\chi_{i_0}(x_0)\chi_{i_1}(x_1) \cdots \chi_{i_k}(x_k).$$

For $k > 0$ $\tilde{\tau}_c$ is extended to the unitalization of $\Psi^{-\infty}(N)$ by $\tilde{\tau}_c(A_0, A_1, \ldots, A_k) = 0$ if one of $A_i = 1$. To prove the proposition one first establishes equality

$$\tau_c(\widetilde{V}_{u\mathcal{D}\otimes}, \ldots, \widetilde{V}_{u\mathcal{D}\otimes}) = \tilde{\tau}_c(\widetilde{V}_{uD}, \widetilde{V}_{uD}, \ldots, \widetilde{V}_{uD}) + O(u^\infty) \text{ as } u \to 0$$

where D is the Dirac operator on N. (Notice that an inspection of the arguments in [7] shows that the trace identity is not used in this proof; this will be important when we shall want to extend this result to b-manifolds.) In the next step one uses Getzler's calculus to show that

$$\lim_{u \to 0} \operatorname{const}_k \cdot \tilde{\tau}_c(\widetilde{V}_{uD}, \widetilde{V}_{uD}, \ldots, \widetilde{V}_{uD}) = 2\int_N \operatorname{AS} \wedge \omega_c.$$

In fact, Connes and Moscovici (for the case of Wassermann projection) and Moscovici and Wu obtain a local result, computing the limit of the corresponding trace density. Later in the paper we shall deal with the case of manifolds with cylindrical ends.

We end this section by discussing the compatibility of our definition with the one appearing in the work of the third author and Leichtnam. For the latter we consider the Mishchenko–Fomenko index class $\operatorname{Ind}_{\mathrm{MF},\infty}(\mathcal{D}) \in K_*(\mathcal{B}^\infty)$. Recall that this is obtained through a $\mathcal{B}^\infty$-MF decomposition theorem; thus there exist finitely generated projective $\mathcal{B}^\infty$-submodules $\mathcal{L}_\infty \subset H_b^\infty(M, E^+ \otimes \mathcal{V}^\infty)$ and $\mathcal{N}_\infty \subset H_b^\infty(M, E^- \otimes \mathcal{V}^\infty)$, with $H_b^\infty := \cap_{k \in \mathbb{N}} H_b^k$, and decompositions

$$\mathcal{L}_\infty \oplus \mathcal{L}_\infty^\perp = H_b^\infty(M, E^+ \otimes \mathcal{V}^\infty), \qquad \mathcal{N}_\infty \oplus \mathcal{D}^+(\mathcal{L}_\infty^\perp) = H_b^\infty(M, E^- \otimes \mathcal{V}^\infty)$$

so that $\mathcal{D}^+$ is block diagonal and invertible when restricted to $\mathcal{L}_\infty^\perp$. We refer the reader to [13, Theorem 12.7] and [14, Appendix] for the precise statement. The Mishchenko–Fomenko $\mathcal{B}^\infty$-index is, by definition,

$$\operatorname{Ind}_{\mathrm{MF},\infty}(\mathcal{D}) := [\mathcal{L}_\infty] - [\mathcal{N}_\infty] \in K_*(\mathcal{B}^\infty).$$

We can consider the Karoubi-Chern character of this class, with values in the noncommutative de Rham homology of $\mathcal{B}^\infty$:

$$\operatorname{Ch}_K(\operatorname{Ind}_{\mathrm{MF},\infty}(\mathcal{D})) \in \overline{H}_\bullet(\mathcal{B}^\infty).$$

Fix $c \in Z^{2p}(\Gamma; \mathbb{C})$, a normalized group cocycle with associated reduced cyclic cocycle $t_c \in \overline{C}_\lambda^{2p}(\mathbb{C}\Gamma) \subset CC^{2p}(\mathbb{C}\Gamma)$. Assume now that Γ satisfies the (RD) condition and that c is of polynomial growth; then t_c extends from $\mathbb{C}\Gamma$ to $\mathcal{B}^\infty$ and since $\overline{H}_\bullet(\mathcal{B}^\infty)$ can be paired with (reduced) cyclic cohomology, see Subsection 2.4, we obtain a number $\langle \mathrm{Ch}_K(\mathrm{Ind}_{\mathrm{MF},\infty}(\mathcal{D})), t_c \rangle_K$.

Proposition 4.6. *Under the above assumptions on Γ and c, and with the notation introduced so far, the following equality holds:*

$$\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D}) = \langle \mathrm{Ch}_K(\mathrm{Ind}_{\mathrm{MF},\infty}(\mathcal{D})), t_c \rangle_K. \qquad (4.4)$$

Proof. Let Π_+ be the orthogonal projection onto $\mathcal{L}_\infty$ and let Π_- be the projection onto $\mathcal{N}_\infty$ along $\mathcal{D}(\mathcal{L}_\infty^\perp)$. It is proved in [13, Theorem 12.7] that these elements are residual. Thus

$$P := \begin{pmatrix} \Pi_+ & 0 \\ 0 & \mathrm{Id} - \Pi_- \end{pmatrix} \in \mathfrak{J}^+,$$

with $\mathfrak{J}^+$ denoting the unitalization of $\mathfrak{J}$ and $[P] - [e_1] \in K_0(\mathfrak{J})$. By choosing as a parametrix of $\mathcal{D}^+$ the Green operator defined by the Mishchenko–Fomenko decomposition, i.e. the operator equal to 0 on $\mathcal{N}_\infty$ and equal to $(\mathcal{D}^+|_{\mathcal{L}_\infty^\perp})^{-1}$ on $\mathcal{D}^+(\mathcal{L}_\infty^\perp)$, we easily see that

$$\mathrm{CS}_\infty(\mathcal{D}) = [P] - [e_1] \in K_0(\mathfrak{J}).$$

Thus

$$\mathrm{Ind}_\infty(\mathcal{D}) = \left[\begin{pmatrix} \theta(\Pi_+) & 0 \\ 0 & \mathrm{Id} - \theta(\Pi_-) \end{pmatrix} \right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \right]$$

which implies

$$\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D}) = \langle \left[\begin{pmatrix} \theta(\Pi_+) & 0 \\ 0 & \mathrm{Id} - \theta(\Pi_-) \end{pmatrix} \right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \right], \tau_c \rangle.$$

Recall the isometric embedding U, see (3.4), that we rewrite in the b-context as $H_b^\infty(M, E \otimes \mathcal{V}^\infty) \xrightarrow{U} H_b^\infty(M, E \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k))$; this identifies $\mathcal{L}_\infty$ and $\mathcal{N}_\infty$ with two finitely generated projective $\mathcal{B}^\infty$-modules $\mathcal{L}_\infty^\otimes$ and $\mathcal{N}_\infty^\otimes$ in $H^\infty(M, E^\pm \otimes (\mathcal{B}^\infty \otimes \mathbb{C}^k))$ and $\theta(\Pi_\pm)$ are projections onto $\mathcal{L}_\infty^\otimes$ and $\mathcal{N}_\infty^\otimes$. There are natural connections on these finitely generated projective modules, obtained by compressing with $\theta(\Pi_\pm)$ the trivial connection d_Γ induced by the differential in the Γ-direction, $d : \mathcal{B}^\infty \to \Omega_1(\mathcal{B}^\infty)$. Thus we can compute the right hand side of (4.4) by using $\mathcal{L}_\infty^\otimes$ and $\mathcal{N}_\infty^\otimes$ endowed with the connections $\theta(\Pi_\pm) d_\Gamma \theta(\Pi_\pm)$. Recall now that the definition of the pairing between $K_0(\mathcal{J})$ and $HC^{2\star}(\mathcal{J})$ is through the Connes–Chern

character from K-theory to cyclic homology, see (2.7)

$$\left\langle \left[\begin{pmatrix} \theta(\Pi_+) & 0 \\ 0 & \mathrm{Id} - \theta(\Pi_-) \end{pmatrix} \right] - [e_1], \tau_c \right\rangle$$

$$:= \left\langle \mathrm{Ch} \left[\begin{pmatrix} \theta(\Pi_+) & 0 \\ 0 & \mathrm{Id} - \theta(\Pi_-) \end{pmatrix} \right] - [e_1], \tau_c \right\rangle_{HC} \quad (4.5)$$

where $e_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$ and where we recall that given an idempotent p in $\mathcal{J}$ one defines

$$\mathrm{Ch}(p) = p + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{k!} (p - \frac{1}{2}) \otimes p^{\otimes 2k}$$

and similarly for an idempotent p in $M_{r \times r}(\mathcal{J})$. What appears above, in (4.5), is the left hand side of (4.4); unwinding this expression one can show easily that the number we get is precisely equal to $\langle \mathrm{Ch}(\mathcal{L}_\infty^\otimes), t_c \rangle_K - \langle \mathrm{Ch}(\mathcal{N}_\infty^\otimes), t_c \rangle_K$ with the first Chern character computed with the connection $\theta(\Pi_+) \, d_\Gamma \, \theta(\Pi_+))$ and the second one with $\theta(\Pi_-) \, d_\Gamma \, \theta(\Pi_-)$. Since, as just explained, this is in turn equal to the right hand side of (4.4), we conclude that the proof of the proposition is complete. $\square$

5. The relative cyclic cocycle (τ_c^r, σ_c) associated to a group cocyle

Consider now the algebra $\mathcal{A}$ with $r = 1$; an element $A \in \mathcal{A}$ is a function on Γ with values in $\Psi_b^{-\infty,\epsilon}(M, E) + \Psi^{-\infty,\epsilon}(M, E)$, denoted $\Gamma \ni g \to A(g) \in \Psi_b^{-\infty,\epsilon}(M, E) + \Psi^{-\infty,\epsilon}(M, E)$. We first work on the dense subalgebra $\mathcal{A}_f$ of $\mathcal{A}$ given by the elements of compact support in Γ, i.e.

$$\mathcal{A}_f := (\Psi_b^{-\infty,\epsilon}(M, E) + \Psi^{-\infty,\epsilon}(M, E)) \otimes \mathbb{C}\Gamma.$$

Definition 5.1. For $A_i \in \mathcal{A}_f$ we set

$$\tau_c^r(A_0 + \omega \cdot 1, A_1, \ldots A_k) = \sum_{g_0 g_1 \cdots g_k = 1} {}^b \mathrm{Tr}(A_0(g_0) A_1(g_1) \cdots A_k(g_k)) c(g_1, g_2, \ldots, g_k).$$

$$(5.1)$$

with ${}^b \mathrm{Tr}$ equal to Melrose b-trace, see [26].

Recall the definition of a double complex $(C^*(\mathcal{A}), B + b)$ for an arbitrary algebra $\mathcal{A}$ over $\mathbb{C}$; as already explained, the cochain complex $(CC^n(\mathcal{A}), B + b)$, consists of multilinear mappings $\tau : \mathcal{A}^+ \otimes \mathcal{A}^{\otimes n} \to \mathbb{C}$ with the Hochschild coboundary map $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ and $B : C^{n+1}(\mathcal{A}) \to C^n(\mathcal{A})$

Lemma 5.2. *In the double complex $(CC^*(\mathcal{A}_f), b + B)$ one has $B\tau_c^r = 0$.*

Proof. This is obvious. $\square$

Consider now $\mathcal{G}_f$ which is nothing but the algebraic tensor product $G \otimes \mathbb{C}\Gamma$, with $G = \Psi_{b,\mathbb{R}^+}^{-\infty,\epsilon}(\overline{N^+(\partial M)}, E) + \Psi_{\mathbb{R}^+}^{-\infty,\epsilon}(\overline{N^+(\partial M)}, E)$. Recall that there exists a (surjective) homomorphism $I : \mathcal{A}_f \to \mathcal{G}_f$, the indicial operator.

Lemma 5.3. *Let* σ_c *be the cochain on* $\mathcal{G}_f$ *defined by*

$$\sigma_c(B_0 + \omega \cdot 1, B_1, \ldots, B_{k+1})$$

$$:= (-1)^{k+1} \sum_{g_0 \cdots g_{k+1}=1} \frac{i}{2\pi} \int d\lambda \, \mathrm{Tr}\left(\widehat{B}_0(\lambda)(g_0) \cdots \widehat{B}_k(\lambda)(g_k) \frac{d\widehat{B}_{k+1}(\lambda)(g_{k+1})}{d\lambda}\right)$$

$$\cdot c(g_1, g_2, \ldots, g_k)$$

Then $b\tau_c^r = I^*\sigma_c$. *We call* σ_c *the* **eta cocycle** *associated to* c.

Proof. We observe first of all that

$$\tau_c^r(A_0 A_1, A_2, \ldots, A_{k+1})$$
$$= \sum_{\gamma g_2 \cdots g_{k+1}=1} {}^b \mathrm{Tr}((A_0 A_1)(\gamma) A_2(g_2) \cdots A_{k+1}(g_{k+1})) c(g_2, g_3, \ldots, g_{k+1})$$
$$= \sum_{\substack{\gamma g_2 \cdots g_{k+1}=1; \\ \gamma = g_0 g_1}} {}^b \mathrm{Tr}(A_0(g_0) A_1(g_1) A_2(g_2) \cdots A_{k+1}(g_{k+1})) c(g_2, g_3, \ldots, g_{k+1})$$
$$= \sum_{g_0 g_1 g_2 \cdots g_{k+1}=1} {}^b \mathrm{Tr}(A_0(g_0) A_1(g_1) A_2(g_2) \cdots A_{k+1}(g_{k+1})) c(g_2, g_3, \ldots, g_{k+1})$$

We also observe that

$$\tau_c^r(A_0, \ldots, A_i A_{i+1}, \ldots, A_{k+1})$$
$$= \sum_{g_0 \cdots g_{k+1}=1} {}^b \mathrm{Tr}(A_0(g_0) \cdots A_{k+1}(g_{k+1})) c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1})$$

and that

$$\tau_c^r(A_{k+1} A_0, A_1, \ldots, A_k)$$
$$= \sum_{g_0 \cdots g_{k+1}=1} {}^b \mathrm{Tr}(A_{k+1}(g_{k+1}) A_0(g_0) A_1(g_1) \cdots A_k(g_k)) c(g_1, g_2, \ldots, g_k)$$
$$= \sum_{g_0 \cdots g_{k+1}=1} {}^b \mathrm{Tr}(A_0(g_0) A_1(g_1) \cdots A_{k+1}(g_{k+1})) c(g_1, g_2, \ldots, g_k)$$
$$+ \sum_{g_0 \cdots g_{k+1}=1} {}^b \mathrm{Tr}[A_{k+1}(g_{k+1}), A_0(g_0) A_1(g_1) \cdots A_k(g_k)] c(g_1, g_2, \ldots, g_k).$$

Adding up and using the fact that c is a cocycle we see that

$$b\tau_c^r(A_0, \ldots, A_{k+1}) = (-1)^{k+1} \sum_{g_0 \cdots g_{k+1}=1} {}^b \text{Tr}[A_{k+1}(g_{k+1}), A_0(g_0)A_1(g_1)\cdots A_k(g_k)]$$
$$\cdot c(g_1, g_2, \ldots, g_k)$$

Thus, using the b-trace identity of Melrose we find:

$$b\tau_c^r(A_0, \ldots, A_{k+1}) = (-1)^{k+1} \sum_{g_0 \cdots g_{k+1}=1} \frac{i}{2\pi} \int d\lambda \, \text{Tr}\left(I(A_0, \lambda)(g_0) \cdots \right.$$
$$\left. \cdots I(A_k, \lambda)(g_k)\frac{dI(A_{k+1}, \lambda)(g_{k+1})}{d\lambda}\right)c(g_1, g_2, \ldots, g_k)$$

We end the proof by computing $b\tau_c^r(1, A_1, \ldots, A_{k+1})$. We have:

$$b\tau_c^r(1, A_1, \ldots, A_{k+1})$$
$$= \tau_c^r(A_1, \ldots, A_{k+1}) + (-1)^{k+1}\tau_c^r(A_{k+1}, A_1, \ldots, A_k)$$
$$= \sum_{g_1 \cdots g_{k+1}=1} {}^b \text{Tr}(A_1(g_1)\cdots A_{k+1}(g_{k+1}))c(g_2, \ldots, g_{k+1})$$
$$+ (-1)^{k+1} \sum_{g_1 \cdots g_{k+1}=1} {}^b \text{Tr}(A_{k+1}(g_{k+1})A_1(g_1)\cdots A_k(g_k))c(g_1, \ldots, g_k)$$
$$= (-1)^{k+1} \sum_{g_1 \cdots g_{k+1}=1} {}^b \text{Tr}[A_{k+1}(g_{k+1}), A_1(g_1)\cdots A_k(g_k)]c(g_1, \ldots, g_k)$$
$$= (-1)^{k+1} \sum_{g_1 \cdots g_{k+1}=1} \frac{i}{2\pi} \int d\lambda \, \text{Tr}\left(I(A_1, \lambda)(g_1) \cdots \right.$$
$$\left. \cdots I(A_k, \lambda)(g_k)\frac{dI(A_{k+1}, \lambda)(g_{k+1})}{d\lambda}\right)c(g_1, g_2, \ldots, g_k)$$

where we have used Lemma 2.2, part 3 in the penultimate step. The Lemma is proved. $\square$

Lemma 5.4. *For the cochain σ_c we have*

$$b\sigma_c = 0, \ B\sigma_c = 0$$

Proof. This is an immediate consequence of the definitions and of the previous Lemma, given that I^* is injective. $\square$

Summarizing, we have proved the following: $(\tau_c^r, \sigma_c) \in C^k(\mathcal{A}_f) \oplus C^{k+1}(\mathcal{G}_f)$ and

$$\begin{pmatrix} b + B & -I^* \\ 0 & -(b + B) \end{pmatrix} \begin{pmatrix} \tau_c^r \\ \sigma_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We restate all this in the following important theorem.

Theorem 5.5. *Let c be a normalized group cocycle for Γ. The cochain τ_c^r defined in (5.1) and the cochain σ_c defined in Lemma 5.3 define together a relative cyclic cocycle (τ_c^r, σ_c) for $\mathcal{A}_f \xrightarrow{I} \mathcal{G}_f$.*

6. Continuity properties of the cocycles (τ_c^r, σ_c) and τ_c

Let us recall the definition of the Connes–Moscovici algebra $\mathcal{B}^\infty \subset C_r^*\Gamma$. See [7] and [33] for more details. Fix a word metric $|\cdot|$ on Γ. Define an unbounded operator D on $\ell^2(\Gamma)$ by setting $D(e_\gamma) = |\gamma|e_\gamma$ where $(e_\gamma)_{\gamma \in \Gamma}$ denotes the standard orthonormal basis of $\ell^2(\Gamma)$. Then consider the unbounded derivation $\delta(T) = [D, T]$ on $B(\ell^2(\Gamma))$ and set

$$\mathcal{B}^\infty = \{T \in C_r^*(\Gamma) \mid \forall k \in \mathbb{N}, \ \delta^k(T) \in B(\ell^2(\Gamma))\}.$$

It is not difficult to prove that $\mathbb{C}\Gamma \subset \mathcal{B}^\infty$ and that $\mathcal{B}^\infty$ is dense in $C_r^*\Gamma$. We endow $\mathcal{B}^\infty$ with the topology defined by the restriction of the $C_r^*\Gamma$-norm and the sequence of seminorms

$$\|T\|_j := \|\delta^j(T)\| \tag{6.1}$$

where on the right hand side we have the operator norm in $B(\ell^2(\Gamma))$. $\mathcal{B}^\infty$ is a Fréchet (locally m-convex) algebra and it is closed under holomorphic functional calculus in $C_r^*\Gamma$.

For the continuity properties of the relative cocycle (τ_c^r, σ_c) and of the cocycle τ_c we shall not work directly with the seminorms defining $\mathcal{B}^\infty$ but will employ instead norms $\nu_k(\cdot)$ on $C_r^*\Gamma$ which, as proved in [7, Lemma (6.4) (i)], is continuous on $\mathcal{B}^\infty$. Let us recall the definition: if $a \in C_r^*\Gamma$ and $k \in \mathbb{N}$ we define

$$\nu_k(a) = \left(\sum_{g \in \Gamma} (1 + |g|)^{2k} |a(g)|^2 \right)^{1/2} \tag{6.2}$$

Recall also that a finitely generated discrete group Γ satisfies the rapid decay condition (RD) if there exists $k \in \mathbb{N}$ and $C > 0$ such that

$$\|a\|_{C_r^*(\Gamma)}^2 \leq C \sum_{g \in \Gamma} (1 + |g|)^{2k} |a(g)|^2, \quad \forall a \in \mathbb{C}\Gamma.$$

It is a non-trivial result that Gromov hyperbolic groups satisfy the (RD) condition; moreover, for each $\xi \in H^\bullet(\Gamma; \mathbb{C})$ there exists a polynomially bounded cocycle $c \in Z^\bullet(\Gamma; \mathbb{C})$ such that $\xi = [c]$.

The main goal of this section is to establish the following proposition.

Proposition 6.1. *If Γ is a finitely generated discrete group satisfying the rapid decay condition (RD) and $c \in Z^k(\Gamma; \mathbb{C})$ has polynomial growth with respect to a word metric $|\cdot|$ then*

$$\tau_c^r \text{ extends continuously from } \mathcal{A}_f \text{ to } \mathcal{A}; \tag{6.3}$$

$$\sigma_c \text{ extends continuously from } \mathcal{G}_f \text{ to } \mathcal{G}; \tag{6.4}$$

$$\tau_c \text{ extends continuously from } \mathcal{J}_f \text{ to } \mathcal{J}. \tag{6.5}$$

Moreover, the extended pair (τ_c^r, σ_c) is a relative cyclic cocycle for $\mathcal{A} \xrightarrow{I} \mathcal{G}$.

Proof. The last statement follows by continuity from the corresponding statement for $\mathcal{A}_f \xrightarrow{I} \mathcal{G}_f$. We thus turn to (6.3), (6.4), (6.5). The main difficulty in establishing (6.3) comes from the use of the b-Trace in the definition of τ_c^r. Crucial in our argument will be the following Proposition, due to Lesch, Moscovici and Pflaum, see [18, Proposition 2.6].

Before stating it, we introduce some notation. Let $\phi \in C^\infty(M)$ be a function equal to t on the cylindrical end $(-\infty, 0] \times \partial M_0 \subset M$. Let $\mathcal{V}$ be a vector field equal to $\partial/\partial t$ on the cylindrical end. In particular $\mathcal{V}(\phi) = 1$ on the cylindrical end. Let $\chi := 1 - \mathcal{V}(\phi) \in C_c^\infty(M_0 \setminus \partial M_0)$.

Proposition 6.2 (Lesch–Moscovici–Pflaum). *If $P \in A := \Psi_b^{-\infty,\epsilon}(M, E) + \Psi^{-\infty,\epsilon}(M, E)$ then*

$$^b\text{Tr}(P) = -\text{Tr}(\phi[\mathcal{V}, P]) + \text{Tr}(\chi P). \tag{6.6}$$

Consequently, the b-Trace of P is the difference of the traces of two trace-class operators naturally associated to P. On the basis of this Proposition and of the next Lemma (in particular its proof), we give the following

Definition 6.3. If $P \in A$, with $A := \Psi_b^{-\infty,\epsilon}(M, E) + \Psi^{-\infty,\epsilon}(M, E)$ then

$$|||P|||^2 := \|\chi P\|_1^2 + \|\phi[\mathcal{V}, P]\|_1^2 + \|[\mathcal{V}, P]\|_1^2 + \|[\phi, P]\|^2 + \|P\|^2 \tag{6.7}$$

with the last two norms denoting the L^2-operator norm.

Lemma 6.4. *If $P_j \in \Psi_b^{-\infty,\epsilon}(M, E) + \Psi^{-\infty,\epsilon}(M, E)$, $j \in \{0, 1, \ldots, k\}$, then there exists $C > 0$ such that*

$$|^b\text{Tr}(P_0 P_1 \cdots P_k)| \le C |||P_0||| \cdots |||P_k|||. \tag{6.8}$$

Proof. Using formula (6.6) we see that

$$
\begin{aligned}
|^b\,&\mathrm{Tr}(P_0 P_1 \cdots P_k)| \\
&\leq |\mathrm{Tr}(\phi[\mathcal{V}, P_0 P_1 \cdots P_k])| + |\mathrm{Tr}(\chi P_0 P_1 \cdots P_k)| \\
&\leq \sum_i |\mathrm{Tr}(\phi P_0 \cdots [\mathcal{V}, P_i] \cdots P_k)| + |\mathrm{Tr}(\chi P_0 P_1 \cdots P_k)| \\
&\leq \sum_{j<i} |\mathrm{Tr}(\phi P_0 \cdots [\phi, P_j] \cdots [\mathcal{V}, P_i] \cdots P_k)| \\
&\qquad + \sum_i |\mathrm{Tr}(P_0 \cdots \phi[\mathcal{V}, P_i] \cdots P_k)| + |\mathrm{Tr}(\chi P_0 P_1 \cdots P_k)| \\
&\leq \sum_{j<i} \|P_0\| \cdots \widehat{j} \cdots \widehat{i} \cdots \|P_k\| \|[\phi, P_j]\| \|[\mathcal{V}, P_i]\|_1 \\
&\qquad + \sum_i \|P_0\| \cdots \|P_{i-1}\| \|\phi[\mathcal{V}, P_i]\|_1 \cdots \|P_k\| + \|\chi P_0\|_1 \|P_1\| \cdots \|P_k\| \\
&\leq C \||P_0|\| \cdots \||P_k|\| \,.
\end{aligned}
$$

$\square$

We now introduce norms on $\mathcal{A}_f := \Psi_b^{-\infty,\epsilon}(M, E \otimes \mathbb{C}\Gamma) + \Psi^{-\infty,\epsilon}(M, E \otimes \mathbb{C}\Gamma)$. If $\mathcal{P} \in \mathcal{A}_f$ then, as already remarked,

$$
\mathcal{P} = \sum_{g \in \Gamma} P(g)g
$$

with $P(g) \in A$ and where the sum is finite. We set

$$
\||\mathcal{P}|\|_k^2 := \sum_{g \in \Gamma} \||P(g)|\|^2 (1 + |g|)^{2k} \tag{6.9}
$$

Lemma 6.5. *Let $k \in \mathbb{N}$. If $\mathcal{P} \in \mathcal{A}$ then $\||\mathcal{P}|\|_k < \infty$. Consequently, $\mathcal{A}$ is contained is the closure of $\mathcal{A}_f$ with respect to the norm $\|| \cdot |\|_k$.*

We give the proof of this Lemma below; see "End of the proof of Proposition 6.1."

We now recall the following fundamental result, due to Connes–Moscovici and Jolissant.

Lemma 6.6. *Let Γ be a discrete finitely generated group satisfying the rapid decay condition (RD). Let $c \in Z^k(\Gamma; \mathbb{C})$ be polynomially bounded. Let $f_j \in \mathbb{C}\Gamma$, $j \in \{0, 1, \ldots, k\}$. Then there exists $m \in \mathbb{N}$ and $C > 0$ such that*

$$
\sum_{g_0 \cdots g_k = 1} |f_0(g_0) f_1(g_1) \cdots f_k(g_k) c(g_1, \ldots, g_k)| \leq C \nu_m(f_0) \cdots \nu_m(f_k) \,. \tag{6.10}
$$

Granted Lemma 6.5 we can now conclude the proof of (6.3). Indeed, let $\mathcal{P}_0, \ldots, \mathcal{P}_k$ be elements in $\mathcal{A}_f$ and consider $f_j \in \mathbb{C}\Gamma$ defined by $f_j(g) := |||P_j(g)|||$; then

$$
\begin{aligned}
|\tau_c^r(\mathcal{P}_0, \ldots, \mathcal{P}_k)| &\leq \sum_{g_0 g_1 \cdots g_k = 1} |^b\operatorname{Tr}(\mathcal{P}_0(g_0) \cdots \mathcal{P}_k(g_k) c(g_1, \ldots, g_k)| \\
&\leq C \sum_{g_0 g_1 \cdots g_k = 1} |||\mathcal{P}_0(g_0)||| \cdots |||\mathcal{P}_k(g_k)||| \, |c(g_1, \ldots, g_k)| \\
&\leq C' \nu_m(f_0) \cdots \nu_m(f_k) = C' |||\mathcal{P}_0|||_m \cdots |||\mathcal{P}_k|||_m
\end{aligned}
$$

where we have used Lemma 6.4 in the second inequality and Lemma 6.6 in the third inequality. This shows that there exists $m \in \mathbb{N}$ such that τ_c^r extends continuously to the closure of $\mathcal{A}_f$ with respect to the $||| \cdot |||_m$-norm. By Lemma 6.5 we conclude that τ_c^r extends continuously to $\mathcal{A}$, which is the content of (6.3).

A similar, easier, argument proves (6.5), the extension of τ_c from $\mathcal{J}_f$ to $\mathcal{J}$ for groups satisfying (RD) and group cocycles that are polynomially bounded. Indeed, recall that elements in the residual calculus are trace class; moreover the following Lemma holds:

Lemma 6.7. *Let $k \in \mathbb{N}$. If $\mathcal{R} = \sum_{g \in \Gamma} R(g) g \in \mathcal{J}$, then*

$$
\sum_{g \in \Gamma} ||R(g)||_1^2 (1 + |g|)^{2k} < \infty. \tag{6.11}
$$

Notice that this implies that

$$
\sum_{g \in \Gamma} ||R(g)||^2 (1 + |g|)^{2k} < \infty. \tag{6.12}
$$

Consequently $\mathcal{J}$ is contained in the closure of $\mathcal{J}_f$ with respect to the norms defined by the left hand side of (6.11) and (6.12). Adapting (in an easier situation) the arguments given above for (6.3) we conclude that (6.5) holds.

End of the proof of Proposition 6.1. We need to establish Lemma 6.5, Lemma 6.7 as well as (6.4).

We shall first establish results for an element S in G (where we recall that G is the space of $\mathbb{R}^+$-invariant operators in the b-calculus with ϵ-bounds in the compactified positive normal bundle to the boundary) and then, in the Mishchenko–Fomenko context, for an element $\mathcal{S}$ in $\mathcal{G}$. By making the substitution $t = \log x$, we will be equivalently looking at translation invariant operators on the infinite cylinder; the estimates appearing in the definition of calculus with bounds translate then into weighted exponential bounds, i.e. with respect to $e^{|t|\epsilon}$, at $t = \pm\infty$. More generally, we can consider any smooth closed compact manifold N, not necessarily a boundary,

and the infinite cylinder $N \times \mathbb{R}$; all the arguments that will be given below apply to this general setting. An operator S in G can then be seen as a Schwartz kernel $K_S(x, y, t)$ on $N \times N \times \mathbb{R}$, acting as a convolution operator in the t-variable. In order to simplify the notation we shall often write K, and not K_S, for the Schwartz kernel of S. We denote by $\widehat{K}(\lambda) := \mathcal{F}_{t \to \lambda}(K)$ the Fourier transform, in t, of the kernel K; this is a smooth family of smoothing kernels on $N \times N$ which is rapidly decreasing, with all its derivatives, in λ as $\lambda \to \pm\infty$. We make a small abuse of notation and keep the same symbol for the smoothing kernel $\widehat{K}_S(\lambda)$ and the smoothing operator it defines on N.

We begin by establishing a number of elementary results about S, K_S and $\widehat{K}_S(\lambda)$. First notice that $\widehat{K}_S : \mathbb{R} \to B(\mathcal{H})$, with $\mathcal{H} = L^2(N)$; the family $\widehat{K}_S$ acts in a natural way on $L^2(\mathbb{R}, \mathcal{H})$ (by multiplication in the $\mathbb{R}$ variable and by its natural action on $\mathcal{H}$) and as such has a norm $\|\widehat{K}_S\|_{B(L^2(\mathbb{R}, \mathcal{H}))}$. Since Fourier transform interchanges convolution and multiplication we clearly have

$$\|S\|_{L^2(\mathbb{R}_t \times N)} = \|\widehat{K}_S\|_{B(L^2(\mathbb{R}_\lambda, \mathcal{H}))}. \tag{6.13}$$

On the other hand we observe that

$$\|\widehat{K}_S\|_{B(L^2(\mathbb{R}_\lambda, \mathcal{H}))} \le \sup_{\lambda \in \mathbb{R}} \|\widehat{K}_S(\lambda)\|_{B(\mathcal{H})}. \tag{6.14}$$

Indeed, using K instead of K_S, we have for any $f \in L^2(\mathbb{R}, \mathcal{H})$:

$$
\begin{aligned}
\langle \widehat{K} f, \widehat{K} f \rangle_{L^2(\mathbb{R}, \mathcal{H})} &= \int_{\mathbb{R}} \langle \widehat{K}(\lambda) f(\lambda), \widehat{K}(\lambda) f(\lambda) \rangle_{\mathcal{H}} \, d\lambda \\
&\le \int_{\mathbb{R}} \|\widehat{K}(\lambda)\|_{B(\mathcal{H})}^2 \|f(\lambda)\|_{\mathcal{H}}^2 \, d\lambda \\
&\le \sup_{\lambda \in \mathbb{R}} \|\widehat{K}(\lambda)\|_{B(\mathcal{H})}^2 \int_{\mathbb{R}} \|f(\lambda)\|_{\mathcal{H}}^2 \, d\lambda \\
&= \left(\sup_{\lambda \in \mathbb{R}} \|\widehat{K}(\lambda)\|_{B(\mathcal{H})}^2 \right) \|f\|_{L^2(\mathbb{R}, \mathcal{H})}^2
\end{aligned}
$$

Putting (6.13) and (6.14) together and using a well known inequality we obtain

$$\|S\|_{L^2(\mathbb{R}_t \times N)} \le \sup_{\lambda \in \mathbb{R}} \|\widehat{K}_S(\lambda)\|_{B(\mathcal{H})} \le \sup_{\lambda \in \mathbb{R}} \|\widehat{K}_S(\lambda)\|_{\mathrm{HS}} \tag{6.15}$$

with $\|\cdot\|_{\mathrm{HS}}$ denoting the Hilbert–Schmidt norm.

Now, let H be any Hilbert space, for example the Hilbert space of Hilbert–Schmidt operators on $L^2(N)$. For any smooth (non-vanishing) rapidly decreasing function $\kappa : \mathbb{R} \to H$ we have,

$$
\begin{aligned}
\|\kappa(\lambda)\| = \left\| \int_{\mathbb{R}} \widehat{\kappa}(\xi) e^{i\xi\lambda} d\xi \right\| &\leq \int_{\mathbb{R}} \|\widehat{\kappa}(\xi)\| d\xi \\
&= \int_{\mathbb{R}} \left\langle \widehat{\kappa}(\xi) \sqrt{1+\xi^2}, \frac{\widehat{\kappa}(\xi)}{\|\widehat{\kappa}(\xi)\|} \frac{1}{\sqrt{1+\xi^2}} \right\rangle d\xi \\
&\leq C \left(\int_{\mathbb{R}} \|\widehat{\kappa}(\xi)\|^2 (1+\xi^2) d\xi \right)^{1/2} \\
&= C \left(\|\kappa\|_{L^2(\mathbb{R}, H)} + \|\kappa'\|_{L^2(\mathbb{R}, H)} \right)^{1/2}
\end{aligned}
$$

where $C = \sqrt{\int_{\mathbb{R}} \frac{1}{1+\xi^2} d\xi} = \sqrt{\pi}$. In particular, we can apply this to $\widehat{K_S} : \mathbb{R} \to H$, with $H = S_2(L^2(N))$, the Hilbert space of Hilbert–Schmidt operators on $L^2(N)$, obtaining the existence of $C > 0$

$$
\sup_{\lambda \in \mathbb{R}} \|\widehat{K_S}(\lambda)\|_{\mathrm{HS}}^2 \leq C \left(\int_{\mathbb{R}} \|\widehat{K_S}(\lambda)\|_{\mathrm{HS}}^2 d\lambda + \int_{\mathbb{R}} \left\| \frac{d}{d\lambda} \widehat{K_S}(\lambda) \right\|_{\mathrm{HS}}^2 d\lambda \right) \quad (6.16)
$$

Thus, there exists $C > 0$ such that

$$
\|S\|_{L^2(\mathbb{R}_t \times N)}^2 \leq C \left(\int_{\mathbb{R}} \|\widehat{K_S}(\lambda)\|_{\mathrm{HS}}^2 d\lambda + \int_{\mathbb{R}} \left\| \frac{d}{d\lambda} \widehat{K_S}(\lambda) \right\|_{\mathrm{HS}}^2 d\lambda \right). \quad (6.17)
$$

Notice that the right hand side is nothing but

$$
C \left(\int_{N \times N \times \mathbb{R}} |\widehat{K_S}(y, y', \lambda)|^2 \, dy \, dy' \, d\lambda + \int_{N \times N \times \mathbb{R}} \left| \frac{d}{d\lambda} \widehat{K_S}(y, y', \lambda) \right|^2 \, dy \, dy' \, d\lambda \right).
$$

Using elementary properties of the Fourier transform we conclude that the following Lemma holds true:

Lemma 6.8. *For a translation invariant smoothing operator on $\mathbb{R} \times N$ with weighted exponential bounds at infinity we have*

$$
\|S\|_{L^2(\mathbb{R} \times N)}^2 \leq C \left(\int_{N \times N \times \mathbb{R}} |K_S(y, y', t)|^2 (1+t^2) \, dy \, dy' \, dt \right) \quad (6.18)
$$

for some universal constant C.

We can now end the proof of Lemma 6.5. Our goal is to show that if $\mathcal{P} \in \mathcal{A}$ then $\sum_{g \in \Gamma} |||\mathcal{P}(g)|||(1 + |g|)^{2k}$ is finite. Here $||| \quad |||$ is the norm introduced in

Definition 6.3. With respect to the notation introduced in that definition we observe that:

$$P \in A \Rightarrow \chi P \in J \,, \quad [\mathcal{V}, P] \in J \,, \quad \phi[\mathcal{V}, P] \in J \qquad (6.19)$$

We also remark that

$$P \in A \Rightarrow [\phi, P] \in A \,. \qquad (6.20)$$

On the basis of (6.19) (6.20) we conclude that it suffices to show that

$$\mathcal{P} \in \mathcal{A} \Rightarrow \sum_{g \in \Gamma} \|\mathcal{P}(g)\|(1 + |g|)^{2k} < \infty \qquad (6.21)$$

and

$$\mathcal{R} \in \mathcal{J} \Rightarrow \sum_{g \in \Gamma} \|\mathcal{R}(g)\|_1(1 + |g|)^{2k} < \infty \qquad (6.22)$$

the latter being in fact the content of Lemma 6.7.

We now prove (6.18). To this end we fix a cut-off function near the boundary, equal to 1 on the boundary and equal to 0 outside a collar neighborhood of the boundary. Using this cut-off function we can define a section $s : \mathcal{G} \to \mathcal{A}$ to the indicial homomorphism $I : \mathcal{A} \to \mathcal{G}$. If $\mathcal{P} \in \mathcal{A}$ then we know that we can write $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1$ with $\mathcal{P}_0 = s(I(\mathcal{P}))$ and $\mathcal{P}_1 \in \mathcal{J}$. Put it differently, we write $\mathcal{P}$ in terms of its Taylor series at the front face. We then have

$$\sum_{g \in \Gamma} \|\mathcal{P}(g)\|(1 + |g|)^{2k} \leq C \sum_{g \in \Gamma} (\|I(\mathcal{P})(g)\|(1 + |g|)^{2k} + \|\mathcal{P}_1(g)\|(1 + |g|)^{2k})$$

We consider the two distinct series

$$\sum_{g \in \Gamma} \|I(\mathcal{P})(g)\|(1 + |g|)^{2k} \quad \text{and} \quad \sum_{g \in \Gamma} \|\mathcal{P}_1(g)\|(1 + |g|)^{2k}$$

and we show that they are both convergent; this will suffice.

Using Lemma 6.8 the term on the left can be bounded by

$$\sum_{g \in \Gamma} \int_{N \times N \times \mathbb{R}_t} |K_{I(\mathcal{P})(g)}(y, y', t)|^2 (1 + t^2)(1 + |g|)^{2k} \, dy \, dy' \, dt$$

$$\leq C \sum_{g \in \Gamma} \int_{N \times N \times \mathbb{R}_t} |K_{I(\mathcal{P})(g)}(y, y', t)|^2 \exp\left(\frac{\epsilon}{2}|t|\right) (1 + |g|)^{2k} \, dy \, dy' \, dt$$

with $N = \partial M_0$. Now, by assumption, $I(\mathcal{P})$ is a translation invariant smoothing operator in the $\mathcal{B}^\infty$-Mishchenko–Fomenko calculus with ϵ-bounds, thus, in particular

$$\sum_{g \in \Gamma} |K_{I(\mathcal{P})(g)}(y, y', t)|^2 \exp(\epsilon|t|)(1 + |g|)^{2k}$$

is convergent and uniformly bounded in $N \times N \times \mathbb{R}$.

We can integrate this series with respect to the finite measure $dy\, dy'\, \exp(-\frac{\epsilon}{2}|t|)dt$ and obtain a finite number; since we can interchange the summation and the integration we conclude that

$$\sum_{g\in\Gamma} \int_{N\times N\times\mathbb{R}_t} |K_{I(\mathcal{P})(g)}(y, y', t)|^2 \exp\left(\frac{\epsilon}{2}|t|\right)(1+|g|)^{2k}\, dy\, dy'\, dt < \infty$$

and this implies that

$$\sum_{g\in\Gamma} \|I(\mathcal{P})(g)\|(1+|g|)^{2k} < \infty \tag{6.23}$$

as required.

Next we tackle the sum $\sum_{g\in\Gamma} \|\mathcal{P}_1(g)\|(1+|g|)^{2k}$ or, more generally, the sum $\sum_{g\in\Gamma} \|\mathcal{R}(g)\|(1+|g|)^{2k}$ for any element $\mathcal{R}$ in $\mathcal{J}$. This is very similar to the closed case analyzed in [7], given that the elements $\mathcal{R}(g)$ are residual. Indeed, if R is residual, $R \in J := \Psi^{-\infty,\epsilon}(M)$, then, in particular, R is a Hilbert–Schmidt operator, in fact, even trace class. This means that

$$\|R\|^2 \le \|R\|_{\mathrm{HS}}^2 = \int_{M\times M} |K_R|^2 d\,\mathrm{vol}_{M\times M} \tag{6.24}$$

Let now $e(\epsilon) \in C^\infty(M)$ be a non vanishing function equal to 1 on M_0 and equal to $\exp(\epsilon|t|)$ along the cylindrical end $(-\infty, 1] \times \partial M_0$. Consider $\mathcal{R} \in \Psi^{-\infty,\epsilon}(M, \mathcal{B}^\infty)$; then, in particular,

$$\sum_{g\in\Gamma} (|K_{\mathcal{R}(g)}|^2 (e(\epsilon) \boxtimes e(\epsilon)))(p, p'))(1+|g|)^{2k}, \quad p, p' \in M$$

is convergent and uniformly bounded in $M \times M$. We can integrate this series with respect to the finite measure $(e(\epsilon) \boxtimes e(\epsilon))^{-1} d\,\mathrm{vol}_{M\times M}$; interchanging summation and integration we conclude that

$$\sum_{g\in\Gamma} \int_{M\times M} |K_{\mathcal{R}(g)}|^2 d\,\mathrm{vol}_{M\times M} (1+|g|)^{2k} < \infty$$

Thus

$$\sum_{g\in\Gamma} \|\mathcal{R}(g)\|(1+|g|)^{2k} \le \sum_{g\in\Gamma} \|\mathcal{R}(g)\|_{\mathrm{HS}}(1+|g|)^{2k}$$

$$= \sum_{g\in\Gamma} \int_{M\times M} |K_{\mathcal{R}(g)}|^2 d\,\mathrm{vol}_{M\times M} (1+|g|)^{2k} < \infty$$

which is what we wanted to show. Summarizing, we have established (6.18).

Regarding (6.22): we know that if R is residual then R is trace class. We want to estimate $\|R\|_1$. Write $R = ((1 + \Delta)^{-\ell}\rho)(\rho^{-1}(1 + \Delta)^{\ell}R)$ with $\ell > \dim M$ and $\rho := e(\epsilon/2)$ (thus $\rho \in C^{\infty}(M)$ is a non vanishing function equal to 1 on M_0 and equal to $\exp((\epsilon/2)|t|)$ along the cylindrical end $(-\infty, 1] \times \partial M_0$). Then $(1 + \Delta)^{-\ell}\rho$ is trace class and we have

$$\|R\|_1 \leq \|(1 + \Delta)^{-\ell}\rho)\|_1 \|\rho^{-1}(1 + \Delta)^{\ell}R\| \leq C \|\rho^{-1}(1 + \Delta)^{\ell}R\|$$

The term $\|\rho^{-1}(1 + \Delta)^{\ell}R\|$ can be treated exactly as above, given that $(1 + \Delta)^{\ell}R$ is still residual and the term ρ^{-1} can be absorbed easily in the estimates. Proceeding as above, using the hypothesis that $\mathcal{R} \in \Psi^{-\infty,\epsilon}(M, \mathcal{B}^{\infty})$, we conclude that (6.22) holds true.

We are left with the task of proving (6.4), i.e. that σ_c extends continuously from $\mathcal{G}_f$ to $\mathcal{G}$. Recall the definition of σ_c on $\mathcal{G}_f$. If $B_j \in \mathcal{G}_f$ and $\widehat{B}_j(\lambda) = \sum_{g \in \Gamma} \widehat{B}_j(\lambda)(g)g$, $j = 0, \ldots k + 1$, then

$$\sigma_c(B_0 + \omega \cdot 1, B_1, \ldots, B_{k+1})$$
$$:= (-1)^{k+1} \sum_{g_0 \cdots g_{k+1}=1} \frac{i}{2\pi} \int d\lambda \, \mathrm{Tr}\left(\left(\widehat{B}_0(\lambda)(g_0)\cdots\right.\right.$$
$$\left.\left.\cdots \widehat{B}_k(\lambda)(g_k)\frac{d\widehat{B}_{k+1}(\lambda)(g_{k+1})}{d\lambda}\right)c(g_1, g_2, \ldots, g_k)\right)$$

Let

$$f_j(\lambda, g) := \|\widehat{B}_k(\lambda)(g)\|, \ j \in \{0, 1, \ldots, k\} \ \text{ and } \ f_{k+1}(\lambda, g) := \left\|\frac{d\widehat{B}_{k+1}(\lambda)(g)}{d\lambda}\right\|_1.$$

We obtain corresponding elements $f_\ell(\lambda) \in \mathbb{C}\Gamma$, $\ell \in \{0, 1, \ldots, k + 1\}$. Well known estimates for the trace-class norm, together with Lemma 6.6, give the existence of $m \in \mathbb{N}$ such that

$$\sum_{g_0 \cdots g_{k+1}} \mathrm{Tr}\left(\left(\widehat{B}_0(\lambda)(g_0)\cdots\widehat{B}_k(\lambda)(g_k)\frac{d\widehat{B}_{k+1}(\lambda)(g_{k+1})}{d\lambda}\right)c(g_1, g_2, \ldots, g_k)\right)$$
$$\leq v_m(f_0(\lambda))\cdots v_m(f_{k+1}(\lambda))$$

Easy arguments show that in order to complete the proof of (6.4) it suffices to show the following:

Claim. *If $B_j \in \mathcal{G}$ then $v_m^2(f_j(\lambda))$, $j \in \{0, \ldots, k + 1\}$, is finite and bounded by $1/(1 + \lambda^2)$.*

Recall that if $S \in G$ then we have proved the following estimate (see (6.16) through (6.18)):

$$
\begin{aligned}
\sup_{\lambda \in \mathbb{R}} \|\widehat{K_S}(\lambda)\|_{\mathrm{HS}} &\leq C \left(\int_{\mathbb{R}} \|\widehat{K_S}(\lambda)\|_{\mathrm{HS}}^2 d\lambda + \int_{\mathbb{R}} \left\| \frac{d}{d\lambda} \widehat{K_S}(\lambda) \right\|_{\mathrm{HS}}^2 d\lambda \right) \\
&= C \left(\int_{N \times N \times \mathbb{R}} |\widehat{K_S}(y, y', \lambda)|^2 \, dy \, dy' \, d\lambda \right. \\
&\qquad\qquad \left. + \int_{N \times N \times \mathbb{R}} \left| \frac{d}{d\lambda} \widehat{K_S}(y, y', \lambda) \right|^2 \, dy \, dy' \, d\lambda \right) \\
&= C \left(\int_{N \times N \times \mathbb{R}} |K_S(y, y', t)|^2 (1 + t^2) \, dy \, dy' \, dt \right)
\end{aligned}
$$

where, as before, we make a small abuse of notation and keep the same symbol for the smoothing kernel $\widehat{K_S}(\lambda)$ and the smoothing operator it defines on N. A similar argument shows that, more generally,

$$
\sup_{\lambda \in \mathbb{R}} \lambda^{2\ell} \|\widehat{K_S}(\lambda)\|_{\mathrm{HS}} \leq C \int_{N \times N \times \mathbb{R}} |\partial_t^\ell K_S(y, y', t)|^2 (1 + t^2) dy \, dy' \, dt \, . \tag{6.25}
$$

In particular, taking $\ell = 0$ and $\ell = 1$ and adding we obtain the estimate

$$
\begin{aligned}
\|\widehat{K_S}(\lambda)\|_{\mathrm{HS}} \leq \frac{C}{1 + \lambda^2} \left(\int_{N \times N \times \mathbb{R}} |K_S(y, y', t)|^2 (1 + t^2) dy \, dy' \, dt \right. \\
\left. + \int_{N \times N \times \mathbb{R}} |\partial_t K_S(y, y', t)|^2 (1 + t^2) dy \, dy' \, dt \, . \right)
\end{aligned} \tag{6.26}
$$

and, hence,

$$
\begin{aligned}
\|\widehat{K_S}(\lambda)\|_{\mathrm{HS}} \leq \frac{C}{1 + \lambda^2} \left(\int_{N \times N \times \mathbb{R}} |K_S(y, y', t)|^2 \exp\left(\frac{\epsilon}{2}|t|\right) dy \, dy' \, dt \right. \\
\left. + \int_{N \times N \times \mathbb{R}} |\partial_t K_S(y, y', t)|^2 \exp\left(\frac{\epsilon}{2}|t|\right) dy \, dy' \, dt \, . \right)
\end{aligned} \tag{6.27}
$$

Let now $\mathcal{S} \in \mathcal{G}$, $\mathcal{S} = \sum \mathcal{S}(g)g$, and let $f(\lambda)$ be the function on Γ defined by $f(\lambda)(g) := \|\widehat{\mathcal{S}}(\lambda)(g)\|$. Since $\mathcal{S}$ is a translation-invariant $\mathcal{B}^\infty$-smoothing operator

with ϵ-bounds we do know that for any $m \in \mathbb{N}$

$$\sum_{g \in \Gamma} |K_{\mathcal{S}(g)}(y, y', t)|^2 \exp(\epsilon|t|)(1+|g|)^{2m} + |\partial_t K_{\mathcal{S}(g)}(y, y', t)|^2 \exp(\epsilon|t|)(1+|g|)^{2m}$$

is convergent and uniformly bounded on $N \times N \times \mathbb{R}$. Proceeding precisely as in the steps leading to the proof of (6.23) and using (6.27) we conclude that the following fundamental estimate holds true:

$$v_m^2(f(\lambda)) \equiv \sum_{g \in \Gamma} \|\widehat{\mathcal{S}}(\lambda)(g)\|(1 + |g|)^{2m} \leq \frac{C}{1 + \lambda^2} \tag{6.28}$$

If now $\mathcal{S} \in \mathcal{G}$ and $h(\lambda)$ is the function on Γ defined by $h(\lambda)(g) := \|\|\widehat{\mathcal{S}}(\lambda)(g)\|_1$ then, similarly,

$$v_m^2(h(\lambda)) \equiv \sum_{g \in \Gamma} \|\widehat{\mathcal{S}}(\lambda)(g)\|_1(1 + |g|)^{2m} \leq \frac{C}{1 + \lambda^2} \tag{6.29}$$

Indeed, it suffices to observe as before that if $S \in G$ then for $k > \dim N$

$$\|\widehat{S}(\lambda)\|_1 \leq \|(1 + \Delta_N)^{-k}\|_1 \|(1 + \Delta_N)^k \widehat{S}(\lambda)\| \leq C \|(1 + \Delta_N)^k \widehat{S}(\lambda)\|$$

and the term on the right hand side can be analyzed as before. The proof of the claim, and thus of Proposition 6.1 is now complete. $\qquad \square$

7. The higher Atiyah–Patodi–Singer index formula

We are now ready to state and prove the main result of this paper. Let $c \in Z^k(\Gamma; \mathbb{C})$, $k = 2p$, be a normalized group cocycle. We assume that Γ satisfies the (RD)-condition and that c has polynomial growth. We know that under these assumptions the cyclic cocycle τ_c extends from $\mathcal{J}_f$ to $\mathcal{J}$ and our goal is to give a formula for the higher APS index

$$\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D}) := \langle \mathrm{Ind}_\infty(\mathcal{D}), [\tau_c] \rangle$$

with $\mathrm{Ind}_\infty(\mathcal{D}) \in K_0(\mathcal{J})$ the index class associated to $\mathcal{D}$.

Recall, see Subsection 3.3, that if $\mathcal{A}$ and $\mathcal{G}$ are Fréchet algebras and $I : \mathcal{A} \to \mathcal{G}$ denotes a bounded homomorphism, then the relative group $K_0(\mathcal{A}, \mathcal{G})$ is by definition $K_0(\mathcal{A}^+, \mathcal{G}^+)$; the latter is the abelian group obtained from equivalence classes of triplets (e_1, e_0, p_t) with e_0 and e_1 projections in $M_{n \times n}(\mathcal{A}^+)$, and p_t a continuous family of projections in $M_{n \times n}(\mathcal{G}^+)$, $t \in [0, 1]$, satisfying $I(e_i) = p_i$ for $i = 0, 1$. As

already explained, there is a paring $K_0(\mathcal{A}, \mathcal{G}) \times HC^{2p}(\mathcal{A}, \mathcal{G}) \to \mathbb{C}$, which in this case takes the form

$$\langle [(e_1, e_0, p_t)], [(\tau, \sigma)] \rangle = \text{const}_{2p} \left[\left(\tau(e_1, \ldots, e_1) - \tau(e_0, \ldots, e_0) \right. \right.$$

$$\left. \left. - \sum_{i=0}^{2p} \int_0^1 \sigma(p_t, \ldots, [\dot{p}_t, p_t], \ldots, p_t) dt \right) \right]$$

Here $\text{const}_{2p} := (-1)^p \frac{(2p)!}{p!}$ and the commutator appears at the i-th position in the i-th summand. We denote $\tau(e_i, \ldots, e_i)$ simply as $\tau(e_i)$.

We know that associated to a normalized group cocycle $c \in Z^k(\Gamma; \mathbb{C})$, $k = 2p$, there is a relative cyclic cocycle $[(\tau_c^r, \sigma_c)] \in HC^{2p}(\mathcal{A}, \mathcal{G})$ and a relative index class $\text{Ind}_\infty(\mathcal{D}, \mathcal{D}_\partial) \in K_0(\mathcal{A}, \mathcal{G})$. We can thus consider, in particular, the pairing $\langle \text{Ind}_\infty(\mathcal{D}, \mathcal{D}_\partial), [(\tau_c^r, \sigma_c)] \rangle$.

Our immediate goal is to show the following crucial identity:

$$\langle \text{Ind}_\infty(\mathcal{D}), [\tau_c] \rangle = \langle \text{Ind}_\infty(\mathcal{D}, \mathcal{D}_\partial), [(\tau_c^r, \sigma_c)] \rangle \tag{7.1}$$

The left hand side of formula (7.1) can be written in terms of the b-Connes–Moscovici projector $V_{\mathcal{D}\otimes}^b$ as

$$\langle [V_{\mathcal{D}\otimes}^b] - [e_1], \tau_c \rangle .$$

Recall that if $\beta_{\text{ex}} : K_0(\mathcal{J}) \to K_0(\mathcal{A}, \mathcal{G})$ is the excision isomorphism then

$$\beta_{\text{ex}}([V_{\mathcal{D}\otimes}^b] - [e_1]) = [V_{\mathcal{D}\otimes}^b, e_1, \mathbf{c}] ,$$

with $\mathbf{c}$ the constant path with value e_1. Since the derivative of the constant path is equal to zero and since, by its very definition, $\tau_c^r|_{\mathcal{J}} = \tau_c$, we obtain at once the crucial relation

$$\langle \beta_{\text{ex}}([V_{\mathcal{D}\otimes}^b] - [e_1]), [(\tau_c^r, \sigma_c)] \rangle = \langle [V_{\mathcal{D}\otimes}^b] - [e_1], [\tau_c] \rangle . \tag{7.2}$$

Now we use the excision formula, asserting that $\beta_{\text{ex}}([V_{\mathcal{D}\otimes}^b] - [e_1])$ is equal, *as a relative class*, to $[V_{\mathcal{D}\otimes}, e_1, V_{tD_{\text{cyl}}^\otimes}]$, $t \in [1, +\infty]$. Thus

$$\langle [V_{\mathcal{D}\otimes}, e_1, V_{tD_{\text{cyl}}^\otimes}], [(\tau_c^r, \sigma_c)] \rangle = \langle [V_{\mathcal{D}\otimes}^b] - [e_1], [\tau_c] \rangle$$

which is precisely $\langle \text{Ind}_\infty(\mathcal{D}, \mathcal{D}_\partial), [(\tau_c^r, \sigma_c)] \rangle = \langle \text{Ind}_\infty(\mathcal{D}), [\tau_c] \rangle$.

We therefore obtain

$$\text{Ind}_{(c, \Gamma)}(\mathcal{D}) := \langle \text{Ind}_\infty(\mathcal{D}), [\tau_c] \rangle$$

$$\equiv \langle [V_{\mathcal{D}\otimes}^b] - [e_1], [\tau_c] \rangle$$

$$= \langle \beta_{\text{ex}}([V_{\mathcal{D}\otimes}^b] - [e_1]), [(\tau_c^r, \sigma_c)] \rangle$$

$$= \langle [V_{\mathcal{D}\otimes}, e_1, V_{tD_{\text{cyl}}^\otimes}], [(\tau_c^r, \sigma_c)] \rangle .$$

In order to compute this expression we recall the equality of relative classes

$$[V_{\mathcal{D}\otimes}, e_1, V_{tD_{\text{cyl}}^\otimes}] = [V_{\mathcal{D}\otimes}^*, e_1, V_{tD_{\text{cyl}}^\otimes}^*], \ t \in [1, +\infty].$$

Therefore if we set $\widetilde{V}_{\mathcal{D}\otimes} := V_{\mathcal{D}\otimes} \oplus V_{\mathcal{D}\otimes}^*, \widetilde{e}_1 = e_1 \oplus e_1, \ p_t = V_{tD_{\text{cyl}}^\otimes} \oplus V_{tD_{\text{cyl}}^\otimes}^*$, we obtain a relative class $[\widetilde{V}_{\mathcal{D}\otimes}, \widetilde{e}_1, p_t]$ satisfying

$$\langle [V_{\mathcal{D}\otimes}, e_1, V_{tD_{\text{cyl}}^\otimes}], [(\tau_c^r, \sigma_c)] \rangle = \frac{1}{2} \langle [\widetilde{V}_{\mathcal{D}\otimes}, \widetilde{e}_1, p_t], [(\tau_c^r, \sigma_c)] \rangle$$

Using the definition of the relative pairing we can thus write

$$\text{Ind}_{(c,\Gamma)}(\mathcal{D}) = \frac{1}{2} \langle [\widetilde{V}_{\mathcal{D}\otimes}, \widetilde{e}_1, p_t], [(\tau_c^r, \sigma_c)] \rangle$$

$$:= \frac{\text{const}_{2p}}{2} \tau_c^r(\widetilde{V}_{\mathcal{D}\otimes} - \widetilde{e}_1) - \frac{\text{const}_{2p}}{2} \left[\sum_{i=0}^{2p} \int_1^\infty \sigma_c(p_t, \dots, [\dot{p}_t, p_t], \dots, p_t) dt \right],$$

$$= \frac{\text{const}_{2p}}{2} \tau_c^r(\widetilde{V}_{\mathcal{D}\otimes}) - \frac{\text{const}_{2p}}{2} \left[\sum_{i=0}^{2p} \int_1^\infty \sigma_c(p_t, \dots, [\dot{p}_t, p_t], \dots, p_t) dt \right]$$

The convergence at infinity of $\left[\sum_{i=0}^{2p} \int_1^\infty \sigma_c(p_t, \dots, [\dot{p}_t, p_t], \dots, p_t) dt \right]$ follows from the fact that the pairing is well defined but can also be proved directly, using the properties of the heat kernel and the invertibility of $\mathcal{D}_{\text{cyl}}$.

Replace now $\mathcal{D}$ by $u\mathcal{D}$, $u > 0$. We obtain, after a simple change of variable in the integral,

$$\text{const}_{2p} \left[\sum_{i=0}^{2p} \int_u^\infty \sigma_c(p_t, \dots, [\dot{p}_t, p_t], \dots, p_t) dt \right]$$

$$= -2 \langle \text{Ind}_\infty(u\mathcal{D}), [\tau_c] \rangle + \text{const}_{2p} \tau_c^r(u\widetilde{V}_{\mathcal{D}\otimes})$$

But the absolute pairing $\langle \text{Ind}_\infty(u\mathcal{D}), [\tau_c] \rangle$ in independent of u and equal to $\text{Ind}_{(c,\Gamma)}(\mathcal{D})$; thus

$$\text{const}_{2p} \left[\sum_{i=0}^{2p} \int_u^\infty \sigma_c(p_t, \dots, [\dot{p}_t, p_t], \dots, p_t) dt \right]$$

$$= -2 \langle \text{Ind}_\infty(\mathcal{D}), [\tau_c] \rangle + \text{const}_{2p} \tau_c^r(u\widetilde{V}_{\mathcal{D}\otimes}).$$

Now, by a well-known principle, see [26, Chapter 8] we know that the short-time behaviour of the b-trace of the heat-kernel is computable as in the closed case, using Getzler-rescaling. Thus, keeping in mind Proposition 4.5, we can prove that the

second summand of the right hand side converges as $u \downarrow 0$ to $2 \int_{M_0} \mathrm{AS} \wedge \omega_c$. Thus the limit

$$\frac{\mathrm{const}_{2p}}{2} \lim_{u \downarrow 0} \left[\sum_{i=0}^{2p} \int_u^\infty \sigma_c(p_t, \ldots, [\dot{p}_t, p_t], \ldots, p_t) dt \right]$$

exists and is equal to $\int_{M_0} \mathrm{AS} \wedge \omega_c - \mathrm{Ind}_{(c,\Gamma)}(\mathcal{D})$.

Definition 7.1. If Γ satisfies the (RD) condition and $c \in Z^k(\Gamma; \mathbb{C})$, $k = 2p$ is a normalized group cocycle (as in (2.14)) of polynomial growth, then we define the higher eta invariant associated to c and the boundary operator $\mathcal{D}_\partial$ as

$$\eta_{(c,\Gamma)}(\mathcal{D}_\partial) := \mathrm{const}_{2p} \left[\sum_{i=0}^{2p} \int_0^\infty \sigma_c(p_t, \ldots, [\dot{p}_t, p_t], \ldots, p_t) dt \right] \qquad (7.3)$$

with $p_t = V_{tD_{\mathrm{cyl}}^\otimes} \oplus V^*_{tD_{\mathrm{cyl}}^\otimes}$.

If N is any closed compact manifold, not necessarily a boundary, and $\Gamma \to \widetilde{N} \to N$ is a Galois Γ-covering, then it should be possible to prove, using Getzler rescaling, that the limit

$$\lim_{u \downarrow 0} \left[\sum_{i=0}^{2p} \int_u^\infty \sigma_c(p_t, \ldots, [\dot{p}_t, p_t], \ldots, p_t) dt \right]$$

exists. This would allow to define the higher eta invariant $\eta_{(c,\Gamma)}(\mathcal{D}_N)$ in general, even for non-bounding coverings.

The arguments given before Definition 7.1 prove the main result of this paper:

Theorem 7.2. *Let Γ be a finitely generated discrete group satisfying the (RD) condition and let $c \in Z^k(\Gamma; \mathbb{C})$, $k = 2p$, be a normalized group cocycle of polynomial growth. Let $\Gamma \to \widetilde{M}_0 \to M_0$ be a Galois Γ-covering of a compact even dimensional manifold with boundary M_0, endowed with a Riemannian metric g_0 and a bundle of unitary Clifford modules E_0 with Clifford connection ∇_0. We assume that all these structures are of product-type near the boundary. Let $\Gamma \to \widetilde{M} \to M$ be the associated Galois covering with cylindrical ends and let g, E and ∇ be the extended structures. Let D and $\widetilde{D}$ be the associated Dirac operators and let $\mathcal{D}$ be the operator D twisted by the $\mathcal{B}^\infty$-Mishchenko bundle. Let us make the assumption that $\widetilde{D}_\partial$ is L^2-invertible. Then there is a well defined higher index $\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D})$ and the following higher Atiyah–Patodi–Singer formula holds:*

$$\mathrm{Ind}_{(c,\Gamma)}(\mathcal{D}) = \int_{M_0} \mathrm{AS} \wedge \omega_c - \frac{1}{2} \eta_{(c,\Gamma)}(\mathcal{D}_\partial). \qquad (7.4)$$

Recall now that $\text{Ind}_{(c,\Gamma)}(\mathcal{D}) = \langle \text{Ch}(\text{Ind}_{\text{MF},\infty}(\mathcal{D})), t_c \rangle_K$, see (4.4); the Atiyah–Patodi–Singer formula for the right hand side, proved in [13, Theorem 12.7] and [14, Appendix], reads

$$\langle \text{Ch}(\text{Ind}_{\text{MF},\infty}(\mathcal{D})), t_c \rangle_K = \int_{M_0} \text{AS} \wedge \omega_c - \frac{1}{2} \langle \eta_{\text{Lott}}(\mathcal{D}_\partial), t_c \rangle \qquad (7.5)$$

with

$$\eta_{\text{Lott}}(\mathcal{D}_\partial) \in \widehat{\Omega}_*(\mathcal{B}^\infty)/\overline{[\widehat{\Omega}_*(\mathcal{B}^\infty), \widehat{\Omega}_*(\mathcal{B}^\infty)]}$$

the higher eta invariant of Lott [21]. By using the identity $\text{Ind}_{(c,\Gamma)}(\mathcal{D}) = \langle \text{Ch}(\text{Ind}_{\text{MF},\infty}(\mathcal{D})), t_c \rangle_K$ and by comparing the two APS index formulae, we obtain, as a corollary, the following interesting equality:

$$\langle \eta_{\text{Lott}}(\mathcal{D}_\partial), t_c \rangle = \eta_{(c,\Gamma)}(\mathcal{D}_\partial). \qquad (7.6)$$

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