# ETA COCYCLES, RELATIVE PAIRINGS AND THE GODBILLON-VEY INDEX THEOREM 

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#### Abstract

We prove a Godbillon-Vey index formula for longitudinal Dirac operators on a foliated bundle with boundary $(X, \mathcal{F})$; in particular, we define a Godbillon-Vey eta invariant on $\left(\partial X, \mathcal{F}_{\partial}\right)$, that is, a secondary invariant for longitudinal Dirac operators on type III foliations. Moreover, employing the Godb-illon-Vey index as a pivotal example, we explain a new approach to higher index theory on geometric structures with boundary. This is heavily based on the interplay between the absolute and relative pairings of $K$-theory and cyclic cohomology for an exact sequence of Banach algebras, which in the present context takes the form $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{A} \rightarrow \mathfrak{B} \rightarrow 0$ with $\mathfrak{J}$ dense and holomorphically closed in $C^{*}(X, \mathcal{F})$ and $\mathfrak{B}$ depending only on boundary data. Of particular importance is the definition of a relative cyclic cocycle $\left(\tau_{G V}^{r}, \sigma_{G V}\right)$ for the pair $\mathfrak{A} \rightarrow \mathfrak{B} ; \tau_{G V}^{r}$ is a cyclic cochain on $\mathfrak{A}$ defined through a regularization à la Melrose of the usual Godbillon-Vey cyclic cocycle $\tau_{G V} ; \sigma_{G V}$ is a cyclic cocycle on $\mathfrak{B}$, obtained through a suspension procedure involving $\tau_{G V}$ and a specific 1-cyclic cocycle (Roe's 1-cocycle). We call $\sigma_{G V}$ the eta cocycle associated to $\tau_{G V}$. The Atiyah-Patodi-Singer formula is obtained by defining a relative index class $\operatorname{Ind}\left(D, D^{\boldsymbol{\partial}}\right) \in K_{*}(\mathfrak{A}, \mathfrak{B})$ and establishing the equality $\left\langle\operatorname{Ind}(D),\left[\tau_{G V}\right]\right\rangle=\left\langle\operatorname{Ind}\left(D, D^{\partial}\right),\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right]\right\rangle$. The Godbillon-Vey eta invariant $\eta_{G V}$ is obtained through the eta cocycle $\sigma_{G V}$.


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## 1 Introduction

The Atiyah-Singer index theorem on closed compact manifolds is regarded nowadays as one of the milestones of modern Mathematics. The original result has branched into several directions, producing new ideas, new results as well as new connections between different fields of Mathematics and Theoretical Physics. One of these directions consists in considering elliptic differential operators on the following hierarchy of geometric structures:

- fibrations and operators that are elliptic in the fiber directions;
- Galois $\Gamma$-coverings and $\Gamma$-equivariant elliptic operators;
- measured foliations and operators that are elliptic along the leaves;
- general foliations and, again, operators that are elliptic along the leaves.

One pivotal example, going through all these situations, is the one of foliated bundles. Let $\Gamma \rightarrow \tilde{N} \rightarrow N$ be a Galois $\Gamma$-cover of a smooth compact manifold without boundary $N$, let $T$ be an oriented compact manifold on which $\Gamma$ acts by orientation preserving diffeomorphisms. We can consider the diagonal action of $\Gamma$ on $\tilde{N} \times T$ and the quotient space $Y:=\tilde{N} \times_{\Gamma} T$, which is a compact manifold, a bundle over $N$ and carries a foliation $\mathcal{F}$. This foliation is obtained by considering the images of the fibers of the trivial fibration $\tilde{N} \times T \rightarrow T$ under the quotient map $\tilde{N} \times T \rightarrow \tilde{N} \times{ }_{\Gamma} T$ and is known as a foliated bundle. We also consider $E \rightarrow Y$ a complex vector bundle on $Y$ and $\widehat{E} \rightarrow \tilde{N} \times T$ the $\Gamma$-equivariant vector bundle obtained by lifting $E$ to $\tilde{N} \times T$. We then consider a family of elliptic differential operators $\left(D_{\theta}\right)_{\theta \in T}$ on the product fibration $\tilde{N} \times T \rightarrow T$, acting on the sections of $\widehat{E}$, and we assume that it is $\Gamma$-equivariant; it therefore yields a leafwise differential operator $\left(D_{L}\right)_{L \in V / \mathcal{F}}$ on $Y$, which is elliptic along the leaves of $\mathcal{F}$. If $\operatorname{dim} T>0$ and $\Gamma=\{1\}$ then we are in the family situation. If $\operatorname{dim} T=0$ and $\Gamma \neq\{1\}$, then we are in the covering situation.

If $\operatorname{dim} T>0, \Gamma \neq\{1\}$ and $T$ admits a $\Gamma$-invariant Borel measure $\nu$ in the Lebesgue class, then we are in the measured foliation situation. Finally, if $\operatorname{dim} T>0, \Gamma \neq\{1\}$ then we are dealing with a general, typically type III, foliation. As an example of this latter type III situation we can consider $T=S^{1}, N$ a compact Riemann surface of genus $\geq 2, \tilde{N}=\mathbb{H}^{2}$ the hyperbolic plane, and $\Gamma=\pi_{1}(N)$ acting on $S^{1}$ by fractional linear transformations; we obtain a foliated bundle $(Y, \mathcal{F})$, where $Y$ is the unit tangent bundle of $N$ and $\mathcal{F}$ is the Anosov foliation of codimension one. It is known that the resulting foliation von Neumann algebra is the unique hyperfinite factor of type $\mathrm{III}_{1}$; in particular $(Y, \mathcal{F})$ is not measured.

In the first three cases, there is first of all a numeric index and the index theorems of Atiyah-Singer, Atiyah and Connes provide a geometric formula for it.

In this paper we are more generally interested in higher indices, numbers obtained by pairing the index class, an element in the K-theory of a suitable algebra, with cyclic cocycles of degree $>0$ defined on the same algebra. Notice that in the case of type III foliation, such as the example above, we must consider higher indices of degree $>0$ (indeed, there is no trace on the foliation von Neumann algebra and thus there is no numeric index).

The higher index problem can be stated as the problem of

- defining these higher indices;
- proving explicit geometric formulae for them, in the spirit of the original result of Atiyah and Singer;
- studying their stability properties.

It is important to observe that stability properties are obtained by considering the index class in the K-theory of a suitable $C^{*}$-algebra; in the case of foliated bundles, which is our concern here, one considers the foliation $C^{*}$-algebra $C^{*}(Y, \mathcal{F})$ and the K-theory groups $K_{*}\left(C^{*}(Y, \mathcal{F})\right)$. Equivalently, we can consider the index class in the K-theory of the Morita-equivalent algebra $C(T) \rtimes_{r} \Gamma$. The index class, however, is typically defined in a smaller algebra $C_{c}^{\infty}(Y, \mathcal{F}) \subset C^{*}(Y, \mathcal{F})$ and higher indices are easily obtained by pairing this class, call it $\operatorname{Ind}^{c}(D)$, with cyclic cocycles $\tau^{c}$ for $C_{c}^{\infty}(Y, \mathcal{F}) .{ }^{1}$ There is then a subtle point that can be stated in the following way: it might very well be possible to prove a formula for these numbers $\left\langle\operatorname{Ind}^{c}(D), \tau^{c}\right\rangle$ without connecting them with the $C^{*}$-algebraic index class $\operatorname{Ind}(D)$, which is the index class showing the most interesting geometric properties. In order to achieve a complete solution of the higher index problem for the cocycle $\tau^{c}$ one is usually confronted with the task of finding an intermediate subalgebra $\mathfrak{A}, C_{c}^{\infty}(Y, \mathcal{F}) \subset \mathfrak{A} \subset C^{*}(Y, \mathcal{F})$ which satisfies the following crucial properties: it is big enough to be holomorphically closed in $C^{*}(Y, \mathcal{F})$ and contain (specific) representatives of the $C^{*}$-index class $\operatorname{Ind}(D)$ but it is small enough that the cyclic cocycle $\tau^{c}$ extends from $C_{c}^{\infty}(Y, \mathcal{F})$ to $\mathfrak{A}$. Finding such an intermediate algebra can be a difficult task.

[^0]Connes' index theorem for $G$-proper manifolds [Con94], with $G$ an étale groupoid, gives a very satisfactory answer to the computation of the pairing between the index class $\operatorname{Ind}^{c}(D)$ for the small algebra and the cyclic cohomology classes $\left[\tau^{c}\right]$ of this same algebra. This higher index theorem applies in particular to a foliated bundle $\tilde{N} \times_{\Gamma} T$ (this is a $G$-proper manifold with $G$ equal to the groupoid $T \rtimes \Gamma$ ).

One fascinating higher index is the so-called Godbillon-Vey index on a codimension 1 foliation. In this case Connes proves the following [Con86]: there is an intermediate subalgebra $\mathcal{A}, C(T) \rtimes_{\mathrm{alg}} \Gamma \subset \mathcal{A} \subset C(T) \rtimes_{r} \Gamma$, which is holomorphically closed and contains the index class $\operatorname{Ind}(D)$; there is a cyclic 2-cocycle $\tau_{\mathrm{BT}}$ on $C(T) \rtimes_{\mathrm{alg}} \Gamma$ (the Bott-Thurston cocycle) which is extendable to $\mathcal{A}$; the general index formula for the pairing $\left\langle\operatorname{Ind}(D),\left[\tau_{\mathrm{BT}}\right]\right\rangle$ can be written down explicitly and it involves the Godbillon-Vey class of the foliation, $G V \in H^{3}(Y)$. This is a complete solution to the higher index problem. For the particular 3-dimensional example presented above this formula reads

$$
\begin{equation*}
\left\langle\operatorname{Ind}(D),\left[\tau_{\mathrm{BT}}\right]\right\rangle=\langle G V,[Y]\rangle=: \mathfrak{g v}(Y, \mathcal{F}) \tag{1.1}
\end{equation*}
$$

with $[Y]$ the fundamental homology class of $Y$ and $\mathfrak{g v}(Y, \mathcal{F})$ the Godbillon-Vey invariant of the foliation $(Y, \mathcal{F})$. Thus, a purely geometric invariant of the foliation $(Y, \mathcal{F}), \mathfrak{g v}(Y, \mathcal{F})$, is in fact a higher index. For geometric properties of the Godbil-lon-Vey invariant we refer the reader to the excellent survey of Ghys [Ghy89]. It is worth recalling here the remarkable result by Hurder and Katok [HK84] relating the Godbillon-Vey invariant to properties of the foliation von Neumann algebras; in our case, this result states that the von Neumann algebra of the foliation contains a nontrivial type III component if $\mathfrak{g v}(Y, \mathcal{F}) \neq 0$; thus the Godbillon-Vey invariant detects type III properties of the foliation von Neumann algebra.

An alternative treatment of the fascinating index formula (1.1) was given by Mor-iyoshi-Natsume in [MoN96]. In this work, a Morita-equivalent complete solution to the Godbillon-Vey index theorem is given. First of all, there is a cyclic 2-cocycle $\tau_{G V}$ on $C_{c}^{\infty}(Y, \mathcal{F})$ which can be paired with the index class $\operatorname{Ind}^{c}(D)$. Next, Moriyoshi and Natsume define a holomorphically closed subalgebra $\mathfrak{A}, C_{c}^{\infty}(Y, \mathcal{F}) \subset \mathfrak{A} \subset C^{*}(Y, \mathcal{F})$, containing the index class $\operatorname{Ind}(D)$ and such that $\tau_{G V}$ extends to $\mathfrak{A}$. The pairing $\left\langle\operatorname{Ind}(D),\left[\tau_{G V}\right]\right\rangle$, which is obtained in $[\mathrm{MoN96}]$ as a direct evaluation of the functional $\tau_{G V}$, is explicitly computed by expressing the index class through the graph projection $e_{D}$ associated to $D$, considering $s D, s>0$ and taking the limit as $s \downarrow 0$. Getzler's rescaling method is used crucially in establishing the analogue of (1.1):

$$
\begin{equation*}
\left\langle\operatorname{Ind}(D),\left[\tau_{\mathrm{GV}}\right]\right\rangle=\int_{Y} \omega_{G V} \tag{1.2}
\end{equation*}
$$

with $\omega_{G V}$ an explicit closed 3 -form on $Y$ such that $\left[\omega_{G V}\right]=G V \in H^{3}(Y)$. In particular, we find once again that $\left\langle\operatorname{Ind}(D),\left[\tau_{\mathrm{GV}}\right]\right\rangle=\mathfrak{g v}(Y, \mathcal{F})$.

Subsequently, Gorokhovsky and Lott [GL03] gave a superconnection proof of Connes' index theorem, including an explicit formula for the Godbillon-Vey higher
index. See also the Appendix of [GL06]. Yet another treatment was given by Gorokhovsky in his elegant paper [Gor06].

In the past 40 years this complex circle of ideas has been extended to some geometric structures with boundary. Let us give a short summary of these contributions, with an emphasis on the higher case. First, in the case of a single manifold and of a Dirac-type operator on it, such an index theorem is due to Atiyah-Patodi-Singer [APS75]. Assume that $D$ is an odd $\mathbb{Z}_{2}$-graded Dirac operator on a compact evendimensional manifold $M$ with boundary $\partial M=N$ acting on a $\mathbb{Z}_{2}$-graded bundle of Clifford module $E$. Assume all geometric structures to be of product type near the boundary. For simplicity, here and in what follows always assume the boundary operator $D^{\partial}$ to be invertible. Then the Dirac operator $D^{+}$with boundary conditions $\left\{u \in C^{\infty}\left(M, E^{+}\right)|u|_{\partial M} \in \operatorname{Ker} \Pi_{\geq}\right\}$with $\Pi_{\geq}=\chi_{[0, \infty)}\left(D^{\partial}\right)$, extends to a Fredholm operator; the index is given by the celebrated formula of Atiyah-Patodi-Singer

$$
\begin{equation*}
\operatorname{ind}_{\mathrm{APS}} D^{+}=\int_{M} \mathrm{AS}-\frac{1}{2} \eta\left(D^{\partial}\right) \tag{1.3}
\end{equation*}
$$

with AS the Atiyah-Singer form associated to $M$ and $E$ and $\eta\left(D^{\partial}\right)$ the eta invariant of the formally self-adjoint operator $D^{\partial}$, a spectral invariant measuring the asymmetry of the spectrum of $D^{\partial}$. The number $\eta\left(D^{\partial}\right)$ should be thought of as a secondary invariant of the boundary operator. The Atiyah-Patodi-Singer index is also equal to the $L^{2}$-index on the manifold with cylindrical end $V=((-\infty, 0] \times \partial M) \cup_{\partial M} M$. See Melrose' book [Mel93] for a thorough treatment of the APS index theorem from this point of view.

Let us move on in the hierarchy of geometric structures considered at the beginning of this Introduction. For families of Dirac operators on manifolds with boundary, the index theorem is due to Bismut and Cheeger [BC90a, BC90b] and, more generally, to Melrose and Piazza [MP97a, MP97b]. The numeric index theorem on Galois coverings of a compact manifold with boundary was established by Ramachandran [Ram93], whereas the corresponding higher index problem was solved by Leichtnam and Piazza [LP97,LP98], following a conjecture of Lott [Lot92]. See [LP04] for a survey. The numeric index theorem on measured foliations was established by Ramachandran in [Ram93]. See also [Ant09] for the cylindrical treatment. Finally, under a polynomial growth assumption on the group $\Gamma$, Leichtnam and Piazza [LP05] extended Connes' higher index theorem to foliated bundles with boundary, using an extension of Melrose' b-calculus and the Gorokhovsky-Lott superconnection approach. For general foliations, but always under a polynomial growth assumption, see also the recent contribution [Esf12]. Notice that, by a result of Plante, foliations with leaves of polynomial growth are measured.

The structure of the (higher) index formulae in all of these contributions is precisely the one displayed by the classic Atiyah-Patodi-Singer index formula recalled above, see (1.3). Thus there is a local contribution, which is the one appearing in the corresponding higher index formula in the closed case, and there is a boundarycorrection term, which is a higher eta invariant. This higher eta invariant should be
thought of as a secondary higher invariant of the operator on the boundary (indeed, the index class for the boundary operator is always zero). Geometric applications of the above results are given, for example, in [BG95, LP01, PS07a, PS07b, CW03].

Now, going back to the task of extending the Atiyah-Patodi-Singer index formula to more general geometric structures, we make the crucial observation that the polynomial growth assumption in [LP05] excludes many interesting (type III) examples and higher indices; in particular it excludes the possibility of proving a Atiyah-Patodi-Singer formula for the Godbillon-Vey higher index.

One primary objective of this article is to prove such a result, thus establishing the first instance of a higher APS index theorem on type III foliations. Consequently, we also define a Godbillon-Vey eta invariant on the boundary-foliation; this is a type III eta invariant, i.e. a type III secondary invariant for Dirac operators.

In tackling this specific index problem we also develop what we believe is a new approach to index theory on geometric structures with boundary, heavily based on the interplay between absolute and relative pairings. We think that this new method can be applied to a variety of situations.

Notice that relative pairings in K-theory and cyclic cohomology have already been successfully employed in the study of geometric and topological invariants of elliptic operators. We wish to mention here the paper by Lesch, Moscovici and Pflaum [LMP09b]; in this interesting article the absolute and relative pairings associated to a suitable short exact sequence of algebras (this is a short exact sequence of parameter dependent pseudodifferential operators) are used in order to define and study a generalization of the divisor flow of Melrose on a closed compact manifold, see [Mel95] and also [LPf00].

Let us give a very short account of our main results. First of all, it is clear from the structure of the Atiyah-Patodi-Singer index formula (1.3) that one of the basic tasks in the theory is to split precisely the interior contribution from the boundary contribution in the higher index formula. We look at operators on the boundary through the translation invariant operators on the associated infinite cylinder; by Fourier transform these two pictures are equivalent. We solve the Atiyah-Patodi-Singer higher index problem on a foliated bundle with boundary $\left(X_{0}, \mathcal{F}_{0}\right), X_{0}=\tilde{M} \times_{\Gamma} T$, by solving the associated $L^{2}$-problem on the associated foliation with cylindrical ends $(X, \mathcal{F})$. Thus, after explaining the geometric set-up in Section 2, we begin by defining a short exact sequence of $C^{*}$-algebras

$$
0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*}(X, \mathcal{F}) \rightarrow B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow 0
$$

This is an extension by the foliation $C^{*}$-algebra $C^{*}(X, \mathcal{F})$ of a suitable algebra of translation invariant operators on the cylinder; we call it the Wiener-Hopf extension. We briefly denote the Wiener-Hopf extension as $0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*} \rightarrow$ $B^{*} \rightarrow 0$. These $C^{*}$-algebras are the receptacle for the two $C^{*}$-index classes we will be working with. Thus, given a $\Gamma$-equivariant family of Dirac operators $\left(D_{\theta}\right)_{\theta \in T}$ with invertible boundary family $\left(D_{\theta}^{\partial}\right)_{\theta \in T}$ we prove that there exist an index class $\operatorname{Ind}(D) \in K_{*}\left(C^{*}(X, \mathcal{F})\right)$ and a relative index class $\operatorname{Ind}\left(D, D^{\partial}\right) \in K_{*}\left(A^{*}, B^{*}\right)$. The
higher Atiyah-Patodi-Singer index problem for the Godbillon-Vey cocycle consists in proving that there is a well defined paring $\left\langle\operatorname{Ind}(D),\left[\tau_{G V}\right]\right\rangle$ and giving a formula for it, with a structure similar to the one displayed by (1.3). Now, as in the case of Moriyoshi-Natsume, $\tau_{G V}$ is initially defined on the small algebra $J_{c}(X, \mathcal{F})$ of $\Gamma$-equivariant smoothing kernels of $\Gamma$-compact support; however, because of the structure of the parametrix on manifolds with cylindrical ends, there does not exist an index class in $K_{*}\left(J_{c}(X, \mathcal{F})\right)$. Hence, even defining the index pairing is not obvious. We shall solve this problem by showing that there exists a holomorphically closed intermediate subalgebra $\mathfrak{J}$ containing the index class $\operatorname{Ind}(D)$ but such that $\tau_{G V}$ extends. More on this in a moment. This point involves elliptic theory on manifolds with cylindrical ends in an essential way.

Once the higher Godbillon-Vey index is defined, we search for an index formula for it. Our main idea is to show that such a formula is a direct consequence of the equality

$$
\begin{equation*}
\left\langle\operatorname{Ind}(D),\left[\tau_{G V}\right]\right\rangle=\left\langle\operatorname{Ind}\left(D, D^{\partial}\right),\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right]\right\rangle \tag{1.4}
\end{equation*}
$$

where on the right hand side a new mathematical object, the relative Godbillon-Vey cocycle, appears. The relative Godbillon-Vey cocycle is built out of the usual Godb-illon-Vey cocycle by means of a very natural procedure. First, we proceed algebraically. Thus we first look at a subsequence of $0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*} \rightarrow B^{*} \rightarrow 0$ made of small algebras, call it $0 \rightarrow J_{c}(X, \mathcal{F}) \rightarrow A_{c} \rightarrow B_{c} \rightarrow 0 ; J_{c}(X, F)$ are, as above, the $\Gamma$ equivariant smoothing kernels of $\Gamma$-compact support; $B_{c}$ is made of $\Gamma \times \mathbb{R}$-equivariant smoothing kernels on the cylinder of $\Gamma \times \mathbb{R}$-compact support. The $A_{c}$ cyclic 2-cochain $\tau_{G V}^{r}$ is obtained from $\tau_{G V}$ through a regularization à la Melrose. The $B_{c}$ cyclic 3-cocycle $\sigma_{G V}$ is obtained by suspending $\tau_{G V}$ on the cylinder with Roe's 1-cocycle. We call this $\sigma_{G V}$ the eta cocycle associated to $\tau_{G V}$. One proves, but it is not quite obvious, that $\left(\tau_{G V}^{r}, \sigma_{G V}\right)$ is a relative cyclic 2 -cocycle for $A_{c} \rightarrow B_{c}$. We obtain in this way a relative cyclic cohomology class $\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right] \in H C^{2}\left(A_{c}, B_{c}\right)$. All of this is explained in Section 5; at the end of this section we also explain how this natural procedure can be extended to other higher indices, producing each time an associated eta cocycle. Once the algebraic theory is clarified, we need to pair the class $\left[\tau_{G V}\right] \in H^{2}\left(J_{c}\right)$ and the relative class $\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right] \in H C^{2}\left(A_{c}, B_{c}\right)$ with the corresponding index classes $\operatorname{Ind}(D) \in K_{*}\left(C^{*}(X, \mathcal{F})\right)$ and $\operatorname{Ind}\left(D, D^{\partial}\right) \in K_{*}\left(A^{*}, B^{*}\right)$ (in fact we shall have to consider the cocycles defined by suitable powers of the $S$-operation). To this end we construct an intermediate short exact subsequence $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{A} \rightarrow \mathfrak{B} \rightarrow 0$ of Banach algebras, sitting half-way between $0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*} \rightarrow B^{*} \rightarrow 0$ and $0 \rightarrow J_{c}(X, \mathcal{F}) \rightarrow A_{c} \rightarrow B_{c} \rightarrow 0$. Much work is needed in order to define such a subsequence, prove that

$$
\operatorname{Ind}(D) \in K_{*}(\mathfrak{J}) \cong K_{*}\left(C^{*}(X, \mathcal{F})\right), \quad \operatorname{Ind}\left(D, D^{\partial}\right) \in K_{*}(\mathfrak{A}, \mathfrak{B}) \cong K_{*}\left(A^{*}, B^{*}\right)
$$

and establishing, finally, that the Godbillon-Vey cyclic $\tau_{G V}$ and the relative cyclic $\left(\tau_{G V}^{r}, \sigma_{G V}\right)$ extend from $J_{c}$ and $A_{c} \rightarrow B_{c}$ to $\mathfrak{J}$ and $\mathfrak{A} \rightarrow \mathfrak{B}$. They therefore define cyclic cohomology classes $\left[\tau_{G V}\right]$ in $H C^{*}(\mathfrak{J})$ and $\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right]$ in $H C^{*}(\mathfrak{A}, \mathfrak{B})$.

We have now made sense of both sides of the equality (1.4), $\left\langle\operatorname{Ind}(D),\left[\tau_{G V}\right]\right\rangle=$ $\left\langle\operatorname{Ind}\left(D, D^{\partial}\right),\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right]\right\rangle$. The equality itself is proved by establishing an excision formula: if $\alpha_{\text {ex }}: K_{*}(\mathfrak{J}) \rightarrow K_{*}(\mathfrak{A}, \mathfrak{B})$ is the excision isomorphism, then $\alpha_{\text {ex }}(\operatorname{Ind}(D))=$ $\operatorname{Ind}\left(D, D^{\partial}\right)$ in $K_{*}(\mathfrak{A}, \mathfrak{B})$. The index formula is obtained by writing explicitly the relative pairing $\left\langle\operatorname{Ind}\left(D, D^{\partial}\right),\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right]\right\rangle$ in terms of the graph projection $e_{D}$, multiplying the operator $D$ by $s>0$ and taking the limit as $s \downarrow 0$. The final formula in the 3-dimensional case (always with an invertibility assumption on the boundary family) reads:

$$
\begin{equation*}
\left\langle\operatorname{Ind}(D),\left[\tau_{G V}\right]\right\rangle=\int_{X_{0}} \omega_{G V}-\eta_{G V} \tag{1.5}
\end{equation*}
$$

with $\omega_{G V}$ equal, as in the closed case, to (a representative of) the Godbillon-Vey class $G V$ and $\eta_{G V}$ expressed in terms of the eta cocycle and the graph projection associated to the cylindrical Dirac family $t D^{\mathrm{cyl}}$. Our main result is stated in Theorem 9.7.

Observe that by Fourier transform the Godbillon-Vey eta invariant $\eta_{G V}$ only depends on the boundary family $D^{\partial} \equiv\left(D_{\theta}^{\partial}\right)_{\theta \in T}$. Notice, finally, that this is a complete solution to the Godbillon-Vey higher index problem on foliated bundles with boundary, in the spirit of Connes and Moriyoshi-Natsume.

The paper is organized as follows. In Section 2 we explain our geometric data. Section 3 is devoted to a discussion of the operators involved in our analysis. In Section 4 we define the Wiener-Hopf extension $0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*} \rightarrow B^{*} \rightarrow 0$. In Section 5 we restrict our analysis to a subsequence $0 \rightarrow J_{c} \rightarrow A_{c} \rightarrow B_{c} \rightarrow 0$ of small dense subalgebras. We begin by defining the eta 1-cocycle associated to the usual trace-cocycle $\tau_{0}$. This is nothing but Roe's 1-cocycle $\sigma_{1}$; we define a relative 0 -cocycle for $A_{c} \rightarrow B_{c}$ by considering $\left(\tau_{0}^{r}, \sigma_{1}\right)$, with $\tau_{0}^{r}$ the regularized trace of Melrose ( $b$-trace). We also discuss the relation of all this with Melrose' formula for the $b$-trace of a commutator. Next we pass to the Godbillon-Vey cocycle $\tau_{G V}$, defining the associated eta 3-cocycle $\sigma_{G V}$ on $B_{c}$ and the associated relative Godbil-lon-Vey cocycle $\left(\tau_{G V}^{r}, \sigma_{G V}\right)$, with $\tau_{G V}^{r}$ defined via Melrose' regularization. In the last Subsection of this Section we also discuss briefly more general relative cocycles. In Section 6 we construct the intermediate short exact sequence $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{A} \rightarrow \mathfrak{B} \rightarrow 0$. In Section 7 we define the index class Ind $D \in K_{0}\left(C^{*}(X, \mathcal{F})\right)$ and the relative index class $\operatorname{Ind}\left(D, D^{\partial}\right) \in K_{0}\left(A^{*}, B^{*}\right)$ and we prove that they correspond under excision. In Section 8 we prove that the two Godbillon-Vey cocycles extend to the subalgebras in the exact sequence $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{A} \rightarrow \mathfrak{B} \rightarrow 0$. We also show how to smooth-out the two $C^{*}$-index classes and define them directly in $K_{*}(\mathfrak{J})$ and $K_{*}(\mathfrak{A}, \mathfrak{B})$. In Section 9 we then proceed to state and prove the main result of the paper. We also make some further remarks; in particular we explain how to get the classic Atiyah-Patodi-Singer index theorem from these relative-pairings arguments. The proof of the APS formula using relative-pairings techniques and the Roe's 1 -cocycle $\sigma_{1}$ was obtained by the
first author in 1988 and announced in [Mor98]. Long and technical proofs have been collected in a separate Section, Section 10.

## Table of symbols

| $N$ | A closed manifold | 2.1 |
| :---: | :---: | :---: |
| $\Gamma \rightarrow \tilde{N} \rightarrow N$ | A Galois covering on $N$ | 2.1 |
| $T$ | A smooth oriented compact manifold with a left $\Gamma$ action | 2.1 |
| $Y=\tilde{N} \times{ }_{\Gamma} T$ | A foliated bundle on $N$ | 2.1 |
| $(Y, \mathcal{F})$ | The induced foliation on $Y$ | 2.1 |
| M | A closed manifold with boundary | 2.2 |
| $\tilde{M}$ | A Galois $\Gamma$-over of $M$ | 2.2 |
| $X_{0}=\tilde{M} \times{ }_{\Gamma} T$ | A foliated bundle on $M$ | 2.2 |
| $\left(X_{0}, \mathcal{F}_{0}\right)$ | The induced foliation on $X_{0}$ | 2.2 |
| $\mathcal{F}_{\boldsymbol{\partial}}$ | The foliation induced on $\partial X_{0}$ | 2.2 |
| V | $M$ with a cylindrical end attached | 2.2 |
| $\tilde{V}$ | $\tilde{M}$ with a cylindrical end attached | 2.3 |
| X | $\tilde{V} \times{ }_{\Gamma} T$ | 2.3 |
| $\operatorname{cyl}(\partial X)$ | $\mathbb{R} \times \partial X_{0}$ | 2.3 |
| $G=(\tilde{V} \times \tilde{V} \times T) / \Gamma$ | Holonomy groupoid | 2.4 |
| $\operatorname{END}(E)$ | The endomorphism bundle | 2.4 |
| $\omega_{G V}$ | The Godbillon-Vey 3-form | 2.5 |
| $C_{c}(X, \mathcal{F})=C_{c}(G)$ | Compactly supported continuous functions on $G$ | 3.1 |
| $D=\left(D_{\theta}\right)_{\theta \in T}$ | A $\Gamma$-equivariant family of Dirac operators | 3.2 |
| $D^{\partial}=\left(D_{\theta}^{\partial}\right)_{\theta \in T}$ | The boundary family defined by $D^{+}$ | 3.2 |
| $D^{\text {cyl }}$ | The family of operators induced by $D^{\partial}$ on $\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ | 3.2 |
| $C^{*}(X, \mathcal{F})$ | The foliation $C^{*}$-algebra | 4.1 |
| $W^{*}(X, \mathcal{F})=\operatorname{End}_{\Gamma}(\mathcal{H})$ | The foliation von Neumann algebra | 4.2 |
| $G_{\text {cyl }}$ | Holonomy groupoid on $\operatorname{cyl}(\partial X)=\mathbb{R} \times \partial X_{0}$ | 4.3 |
| $B_{c}=B_{c}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ | A $*$-algebra of translation invariant smoothing operators | 4.3 |
| $B^{*}=B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ | The $C^{*}$-closure of $B_{c}$ | 4.3 |
| $\chi_{\mathbb{R}}^{0}, \chi_{\mathbb{R}}$ | The characteristic function of $(-\infty, 0]$ and its smooth approximation | 4.4 |
| $\chi$, $\chi_{\text {cyl }}$ | Induced smooth functions on $X$ and $\operatorname{cyl}(\partial X)$ | 4.4 |
| $\chi^{\lambda}, \chi_{\text {cyl }}^{\lambda}$ | $(-\infty,-\lambda]$-characteristic functions on $X$ and $\operatorname{cyl}(\partial X)$ | 4.4 |
| $A^{*}=A^{*}(X ; \mathcal{F})$ | The Wiener-Hopf extension | 4.4 |
| $J_{c}, A_{c}, B_{c}$ | Small subalgebras of $C^{*}(X, \mathcal{F}), A^{*}(X, \mathcal{F})$, $B^{*}(\operatorname{cyl}(\partial X))$ | 5.2 |
| $\omega_{\Gamma}$ | The weight associated to a transverse measure | 5.10 |
| $\tau_{G V}$ | The Godbillon-Vey cyclic cocycle | 5.10 |
| $\sigma_{G V}$ | The eta cocycle associated to $\tau_{G V}$ | 5.11 |


| $\omega_{\Gamma}^{r}$ | The regularized weight | 5.12 |
| :--- | :--- | :--- |
| $\tau_{G V}^{r}$ | The regularized Godbillon-Vey cyclic cochain | 5.12 |
| $\left(\tau_{G V}^{r}, \sigma_{G V}\right)$ | The relative Godbillon-Vey cocycle | 5.12 |
| $\mathcal{I}_{m}(X, \mathcal{F})$ | Schatten-type ideal | 6.2 |
| $\mathcal{J}_{m}(X, \mathcal{F})$ | A Banach subalgebra of $\mathcal{I}_{m}(X, \mathcal{F})$ | 6.2 |
| $\mathcal{B}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ | A Banach subalgebra of $B^{*}$ | 6.4 |
| $\mathcal{A}_{m}(X, \mathcal{F})$ | A Banach subalgebra of $A^{*}$ | 6.4 |
| $\mathfrak{J}_{\mathbf{m}}, \mathfrak{A}_{\mathbf{m}}, \mathfrak{B}_{\mathbf{m}}$ | Smooth subalgebra of $\mathcal{J}_{m}(X, \mathcal{F}), \mathcal{A}_{m}(X, \mathcal{F}), \mathcal{B}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right)$ | 6.5 .2 |
| $e_{D}$ | The graph projection associated to $D$ | 7.2 .2 |
| $\operatorname{Ind}(D)$ | The Connes-Skandalis index class | 7.3 |
| $\operatorname{Ind}\left(D, D^{\partial}\right)$ | The relative index class | 7.4 |
| $\operatorname{Ind}$ |  |  |

The results of this paper were announced in the July 2009 preprint [MoP09]. ${ }^{2}$

## 2 Geometry of Foliated Bundles

2.1 Closed manifolds. We shall denote by $N$ a closed orientable compact manifold. We consider a Galois $\Gamma$-cover $\Gamma \rightarrow \tilde{N} \rightarrow N$, with $\Gamma$ acting on the right, and $T$, a smooth oriented compact manifold with a left action of $\Gamma$ which is assumed to be by orientation preserving diffeomorphisms and locally faithful, as in [MoN96], that is: if $\gamma \in \Gamma$ acts as the identity map on an open set in $T$, then $\gamma$ is the identity element in $\Gamma$. See also [BP09]. We set $Y=\tilde{N} \times{ }_{\Gamma} T$; thus $Y$ is the quotient of $\tilde{N} \times T$ by the $\Gamma$-action

$$
(y, \theta) \gamma:=\left(y \gamma, \gamma^{-1} \theta\right) .
$$

$Y$ is foliated by the images under the quotient map of the fibers of the trivial fibration $\tilde{N} \times T \rightarrow T$ and is referred to as a foliated $T$-bundle. We use the notation $(Y, \mathcal{F})$ when we want to stress the foliated structure of $Y$. Finally, we consider $E \rightarrow Y$, a hermitian complex vector bundle on $Y$ as well as $\widehat{E} \rightarrow \tilde{N} \times T$, the latter being the $\Gamma$-equivariant vector bundle induced from $E$ lifted to $\tilde{N} \times T$.
2.2 Manifolds with boundary. Let now $(M, g)$ be a Riemannian manifold with boundary; the metric is assumed to be of product type in a collar neighborhood $U \cong[0,2] \times \partial M$ of the boundary. Let $\tilde{M}$ be a Galois $\Gamma$-cover of $M$; we let $\tilde{g}$ be the lifted metric. We also consider $\partial \tilde{M}$, the boundary of $\tilde{M}$. Let $X_{0}=\tilde{M} \times_{\Gamma} T$; this is a manifold with boundary and the boundary $\partial X_{0}$ is equal to $\partial \tilde{M} \times_{\Gamma} T$. $\left(X_{0}, \mathcal{F}_{0}\right)$ denotes the associated foliated bundle. The leaves of $\left(X_{0}, \mathcal{F}_{0}\right)$ are manifolds with boundary endowed with a product-type metric. The boundary $\partial X_{0}$ inherits a foliation $\mathcal{F}_{\partial}$. The cylinder $\mathbb{R} \times \partial X_{0}$ also inherits a foliation $\mathcal{F}_{\text {cyl }}$, obtained by crossing the leaves of $\mathcal{F}_{\partial}$ with $\mathbb{R}$. Similar considerations apply to the half cylinders $(-\infty, 0] \times \partial X_{0}$ and $[0,+\infty) \times \partial X_{0}$. We shall consider a complex hermitian vector bundle on $X_{0}$ and
${ }^{2}$ This announcement is now published, but with a different title. See [MoP11].
we shall assume the usual product structure near the boundary: we adopt without further comments the identification explained, for example, in [Mel93] and adopted also in [MP97a] and [LP05].
2.3 Manifolds with cylindrical ends: Notation. We consider $\tilde{V}:=\tilde{M} \cup_{\partial \tilde{M}}$ $((-\infty, 0] \times \partial \tilde{M})$, endowed with the extended metric and the obviously extended $\Gamma$ action along the cylindrical end. Notice incidentally that we obtain in this way a $\Gamma$-covering

$$
\begin{equation*}
\Gamma \rightarrow \tilde{V} \rightarrow V, \quad \text { with } \quad V:=M \cup_{\partial M}((-\infty, 0] \times \partial M) \tag{2.1}
\end{equation*}
$$

We consider $X:=\tilde{V} \times_{\Gamma} T$; this is a foliated bundle, with leaves manifolds with cylindrical ends. We denote by $(X, \mathcal{F})$ this foliation. Notice that $X=X_{0} \cup_{\partial X_{0}}$ $\left((-\infty, 0] \times \partial X_{0}\right) ;$ moreover the foliation $\mathcal{F}$ is obtained by extending $\mathcal{F}_{0}$ on $X_{0}$ to $X$ via the product cylindrical foliation $\mathcal{F}_{\text {cyl }}$ on $(-\infty, 0] \times \partial X_{0}$. We can write more suggestively $(X, \mathcal{F})=\left(X_{0}, \mathcal{F}_{0}\right) \cup_{\left(\partial X_{0}, \mathcal{F}_{\partial}\right)}\left((-\infty, 0] \times \partial X_{0}, \mathcal{F}_{\text {cyl }}\right)$. For $\lambda>0$ we shall also consider the finite cylinder $\tilde{V}_{\lambda}=\tilde{M} \cup_{\partial \tilde{M}}([-\lambda, 0] \times \partial \tilde{M})$ and the resulting foliated manifold $\left(X_{\lambda}, \mathcal{F}_{\lambda}\right)$. Finally, with a small abuse, ${ }^{3}$ we introduce the notation:

$$
\begin{gather*}
\operatorname{cyl}(\partial X):=\mathbb{R} \times \partial X_{0}, \quad \operatorname{cyl}^{-}(\partial X):=(-\infty, 0] \times \partial X_{0} \quad \text { and } \\
\operatorname{cyl}^{+}(\partial X):=[0,+\infty) \times \partial X_{0} . \tag{2.2}
\end{gather*}
$$

The foliations induced on $\operatorname{cyl}(\partial X), \operatorname{cyl}^{ \pm}(\partial X)$ by $\mathcal{F}_{\partial}$ will be denoted by $\mathcal{F}_{\text {cyl }}, \mathcal{F}_{\text {cyl }}^{ \pm}$; we obtain in this way foliated bundles $\operatorname{cyl}\left(\partial X, \mathcal{F}_{\text {cyl }}\right),\left(\operatorname{cyl}^{-}(\partial X), \mathcal{F}_{\text {cyl }}^{-}\right)$and $\left(\operatorname{cyl}^{+}(\partial X)\right.$, $\mathcal{F}_{\text {cyl }}^{+}$).
2.4 Holonomy groupoid. We consider the groupoid $G:=(\tilde{V} \times \tilde{V} \times T) / \Gamma$ with $\Gamma$ acting diagonally; $G^{(0)}:=X$ and the source map and the range map are defined by $s\left[x, x^{\prime}, \theta\right]=\left[x^{\prime}, \theta\right], r\left[x, x^{\prime}, \theta\right]=[x, \theta]$. Since the action on $T$ is assumed to be locally faithful, we know that $(G, r, s)$ is isomorphic to the holonomy groupoid of the foliation $(X, \mathcal{F})$. In the sequel, we shall call $(G, r, s)$ the holonomy groupoid. If $E \rightarrow X$ is a complex vector bundle on $X$, with product structure along the cylindrical end as above, then we can consider the bundle $\left(s^{*} E\right)^{*} \otimes r^{*} E$ over $G$; this bundle is sometime denoted $\operatorname{END}(E)$. If $F$ is a second complex vector bundle on $X$, we can likewise consider the bundle $\operatorname{HOM}(E, F):=\left(s^{*} E\right)^{*} \otimes r^{*} F$. Finally, we consider the maps $\hat{r}, \hat{s}: \tilde{V} \times \tilde{V} \times T \rightarrow \tilde{V} \times T, \hat{r}\left(x, x^{\prime}, \theta\right)=(x, \theta), \hat{s}\left(x, x^{\prime}, \theta\right)=\left(x^{\prime}, \theta\right)$ and, more importantly, the bundles $\operatorname{END}(\widehat{E}):=\left(\hat{s}^{*} \widehat{E}\right)^{*} \otimes\left(\hat{r}^{*} \widehat{E}\right)$ and $\operatorname{HOM}(\widehat{E}, \widehat{F}):=\left(\hat{s}^{*} \widehat{E}\right)^{*} \otimes\left(\hat{r}^{*} \widehat{F}\right)$.
2.5 The Godbillon-Vey differential form. Following [MoN96], we describe the explicit representative of the Godbillon-Vey class as a differential form in terms of the modular function of the holonomy groupoid. Let $X_{0}=\tilde{M} \times{ }_{\Gamma} T$. In this section $X_{0}$ can also be a closed manifold, namely, a compact manifold without boundary. Assume that $T$ is one-dimensional, and take an arbitrary 1-form $\omega$ on $X_{0}$ defining the
${ }^{3}$ The abuse of notation is in writing $\operatorname{cyl}(\partial X)$ for $\mathbb{R} \times \partial X_{0}$ whereas we should really write $\operatorname{cyl}\left(\partial X_{0}\right)$.
codimension-one foliation $\mathcal{F}_{0}$. Due to the integrability condition, there exists a 1 -form $\eta$ such that $d \omega=\eta \wedge \omega$. The Godbillon-Vey class for $\mathcal{F}_{0}$ is, by definition, the de Rham cohomology class given by $\eta \wedge d \eta$, denoted by $G V$; thus $G V:=[\eta \wedge d \eta] \in H_{\mathrm{dR}}^{3}\left(X_{0}\right)$. We shall explain another description of $G V$ in terms of the modular function of the holonomy groupoid.

Consider the product space $\tilde{M} \times T$, which is a covering of $X_{0}$. Choose a volume form $d \theta$ on $T$; it is in general impossible to choose $d \theta$ to be, in addition, $\Gamma$-invariant. Then $d \theta$ yields a defining 1 -form for the foliation (which is in fact a fibration) obtained by lifting the foliation $\mathcal{F}_{0}$. The de Rham complex on $\tilde{M} \times T$ is isomorphic to the graded tensor product $\Omega^{*}(\tilde{M}) \otimes \Omega^{*}(T)$ and accordingly the exterior differential on $\tilde{M} \times T$ splits as

$$
\begin{equation*}
d_{\tilde{M} \times T}=d+(-1)^{p} d_{T} \tag{2.3}
\end{equation*}
$$

on $\Omega^{p}(\tilde{M}) \otimes \Omega^{q}(T)$, with $d$ and $d_{T}$ denoting respectively the exterior differentials along $\tilde{M}$ and $T$. Let us take the volume forms $\omega$ and $\Omega$ respectively on $M$ and $X_{0}$ and take the pullbacks $\tilde{\omega}$ and $\tilde{\Omega}$ to $\tilde{M}$ and $\tilde{M} \times T$. These are $\Gamma$-invariant volume forms. The modular function of the holonomy groupoid is defined as the Radon-Nikodym derivative of the two volume forms on $\tilde{M} \times T$ :

$$
\begin{equation*}
\psi=\frac{\tilde{\omega} \times d \theta}{\tilde{\Omega}} \tag{2.4}
\end{equation*}
$$

Notice that $\psi$ has values in $\mathbb{R}^{+}$since $\Gamma$ acts by orientation preserving diffeomorphisms. Set

$$
\begin{equation*}
\phi=\log \psi . \tag{2.5}
\end{equation*}
$$

Proposition 2.6 ([MoN96], p. 504). The 3-form $\omega_{G V}=d \phi \wedge d d_{T} \phi=-d \phi \wedge d_{T} d \phi$ is $\Gamma$-invariant and closed on $\tilde{M} \times T$. The Godbillon-Vey class of $\mathcal{F}_{0}$ is represented by $\omega_{G V}$ in $H_{\mathrm{dR}}^{3}\left(X_{0}\right)$.

## 3 Operators

3.1 Equivariant families of integral operators. Let $C_{c}(X, \mathcal{F})$ denote the space of compactly supported continuous functions on the holonomy groupoid $G$; thus $C_{c}(X, \mathcal{F}):=C_{c}(G)$. More generally we set

$$
\begin{equation*}
C_{c}(X, \mathcal{F} ; E):=C_{c}\left(G,\left(s^{*} E\right)^{*} \otimes r^{*} E\right) \equiv C_{c}(G, \operatorname{END}(E)) \tag{3.1}
\end{equation*}
$$

equipped with the $*$-algebra structure induced from the convolution product. Given an additional vector bundle $F$, we also set

$$
\begin{equation*}
C_{c}(X, \mathcal{F} ; E, F):=C_{c}\left(G,\left(s^{*} E\right)^{*} \otimes r^{*} F\right) \equiv C_{c}(G, \operatorname{HOM}(E, F)) \tag{3.2}
\end{equation*}
$$

which is a left module over $C_{c}(X, \mathcal{F} ; E)$ and a right module over $C_{c}(X, \mathcal{F} ; F)$. Restricting ourselves to the space of compactly supported smooth sections on $G$, we similarly define $C_{c}^{\infty}(X, \mathcal{F}):=C_{c}^{\infty}(G), C_{c}^{\infty}(X, \mathcal{F} ; E)$ and $C_{c}^{\infty}(X, \mathcal{F} ; E, F)$.

The *-algebra $C_{c}(X, \mathcal{F})$ can be defined also as the space of continuous functions on $\tilde{V} \times \tilde{V} \times T$ that are $\Gamma$-invariant and admit $\Gamma$-compact support, i.e. admit support which is compact in $(\tilde{V} \times \tilde{V} \times T) / \Gamma$. A similar description holds for $C_{c}(X, \mathcal{F} ; E)$. Notice, in particular, that given an element $k$ in $C_{c}(X, \mathcal{F} ; E)$ there exists a constant $\lambda(k) \equiv \lambda>0$ such that $k$ is identically zero outside $\tilde{V}_{\lambda} \times \tilde{V}_{\lambda} \times T \subset \tilde{V} \times \tilde{V} \times T$. Thus an element $k \in C_{c}(X, \mathcal{F})$ give rise to an equivariant family of integral operators $(k(\theta))_{\theta \in T}$ in a natural way.
3.2 Dirac operators. We begin with a closed foliated bundle $(Y, \mathcal{F})$, with $Y=$ $\tilde{N} \times{ }_{\Gamma} T$. We are also given a $\Gamma$-equivariant complex vector bundle $\widehat{E}$ on $\tilde{N} \times T$, or, equivalently, a complex vector bundle on $Y$. We assume that $\widehat{E}$ has a $\Gamma$-equivariant Clifford structure along $\tilde{N}$. We obtain in this way a $\Gamma$-equivariant family of Dirac operators $\left(D_{\theta}\right)_{\theta \in T}$ that will be simply denoted by $D$. ${ }^{4}$

If $\left(X_{0}, \mathcal{F}_{0}\right), X_{0}=\tilde{M} \times_{\Gamma} T$, is a foliated bundle with boundary, as in the previous section, then we shall assume the relevant geometric structures to be of producttype near the boundary. If $(X, \mathcal{F})$ is the associated foliated bundle with cylindrical ends, then we shall extend all the structure in a constant way along the cylindrical ends. We shall eventually assume $\tilde{M}$ to be of even dimension, the bundle $\widehat{E}$ to be $\mathbb{Z}_{2}$-graded and the Dirac operator to be odd and formally self-adjoint. We denote by $D^{\partial} \equiv\left(D_{\theta}^{\partial}\right)_{\theta \in T}$ the boundary family defined by $D^{+}$. This is a $\Gamma$-equivariant family of formally self-adjoint first order elliptic differential operators on a complete manifold. We denote by $D^{\text {cyl }}$ the operator induced by $D^{\partial} \equiv\left(D_{\theta}^{\partial}\right)_{\theta \in T}$ on the cylindrical foliated manifold $\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) ; D^{\text {cyl }}$ is $\mathbb{R} \times \Gamma$-equivariant. We refer to [MoN96,LP05] and also [BP09] for precise definitions.
3.3 Pseudodifferential operators. Let $(Y, \mathcal{F}), Y=\tilde{N} \times_{\Gamma} T$, be a closed foliated bundle. Given vector bundles $E$ and $F$ on $Y$ with lifts $\widehat{E}, \widehat{F}$ on $\tilde{N} \times T$, we can define the space of $\Gamma$-compactly supported pseudodifferential operators of order $m$, denoted here, with a small abuse of notation, $\Psi_{c}^{m}(G ; E, F) .{ }^{5}$ An element $P \in$ $\Psi_{c}^{m}(G ; E, F)$ should be thought of as a smooth $\Gamma$-equivariant family of pseudodifferential operators, $(P(\theta))_{\theta \in T}$ with Schwartz kernel $K_{P}$, a distribution on $G$, of compact support. See [MoN96] and [BP09] for more details; we remark that in these two references the more general case in which $T$ is only a topological $\Gamma$-space, (so that $(Y, \mathcal{F})$ is a foliated space) is allowed.

The space $\Psi_{c}^{\infty}(G ; E, E):=\bigcup_{m \in \mathbb{Z}} \Psi_{c}^{m}(G ; E, E)$ is a filtered algebra and will be simply denoted by $\Psi_{c}^{\infty}(G ; E)$. Moreover, assuming $E$ and $F$ to be hermitian and

[^1]assigning to $P$ its formal adjoint $P^{*}=\left(P_{\theta}^{*}\right)_{\theta \in T}$ gives $\Psi_{c}^{\infty}(G ; E)$ the structure of an involutive algebra; the formal adjoint of an element $P \in \Psi_{c}^{m}(G ; E, F)$ is in general an element in $\Psi_{c}^{m}(G ; F, E)$. We call $\Psi_{c}^{-\infty}(G ; E):=\bigcap_{m \in \mathbb{Z}} \Psi_{c}^{m}(G ; E)$ the algebra of $\Gamma$-equivariant smoothing operators with $\Gamma$-compact support.

## 4 Wiener-Hopf extensions

4.1 Foliation $C^{*}$-algebras. The foliation $C^{*}$-algebra $C^{*}(X, \mathcal{F})$ is defined as the completion of $C_{c}(X, \mathcal{F})$ with respect to $\|k\|_{C^{*}}:=\sup _{\theta \in T}\|k(\theta)\|$, the norm on the right hand side being equal to the $L^{2}$-operator norm on $L^{2}(\tilde{V} \times\{\theta\})$. A similar definition holds for $C_{c}(X, \mathcal{F} ; E)$. For more on this foundational material see, for example, $[\mathrm{MoN} 96]$ and [BP09]. It is proved in [MoN96] that $C^{*}(X, \mathcal{F} ; E)$ is isomorphic to the $C^{*}$-algebra of compact operators of a $C(T) \rtimes \Gamma$-Hilbert module $\mathcal{E}$. The Hilbert module $\mathcal{E}$ is obtained by completing $C_{c}^{\infty}(\tilde{V} \times T, \widehat{E})$, endowed with its $C(T) \rtimes_{\mathrm{alg}} \Gamma$-module structure and $C(T) \rtimes_{\mathrm{alg}} \Gamma$-valued inner product, with respect to the $C(T) \rtimes \Gamma$-norm. Once again, see [MoN96] and [BP09] for details: summarizing

$$
\begin{equation*}
C^{*}(X, \mathcal{F} ; E) \cong \mathbb{K}(\mathcal{E}) \subset \mathcal{L}(\mathcal{E}) \tag{4.1}
\end{equation*}
$$

4.2 Foliation von Neumann algebras. Consider the family of Hilbert spaces $\mathcal{H}:=\left(\mathcal{H}_{\theta}\right)_{\theta \in T}$, with $\mathcal{H}_{\theta}:=L^{2}\left(\tilde{V} \times\{\theta\}, E_{\theta}\right)$. Then $C_{c}(\tilde{V} \times T)$ is a continuous field inside $\mathcal{H}$, that is, a linear subspace in the space of measurable sections of $\mathcal{H}$ satisfying a certain number of properties (see [Con82], p. 576 for the details). Let $\operatorname{End}(\mathcal{H})$ the space of measurable families of bounded operators $T=\left(T_{\theta}\right)_{\theta \in T}$, where the norms $T_{\theta}$ are measurable function and essentially bounded. Then $\operatorname{End}(\mathcal{H})$ is a $C^{*}$-algebra, in fact a von Neumann algebra, equipped with the norm

$$
\|T\|:=\operatorname{ess} \cdot \sup \left\{\left\|T_{\theta}\right\| ; \theta \in T\right\}
$$

with $\left\|T_{\theta}\right\|$ the operator norm. We also denote by $\operatorname{End}_{\Gamma}(\mathcal{H})$ the subalgebra of $\operatorname{End}(\mathcal{H})$ consisting of $\Gamma$-equivariant measurable families of operators. This is a von Neumann algebra which is, by definition, the foliation von Neumann algebra associated to $(X, \mathcal{F})$; it is often denoted $W^{*}(X, \mathcal{F})$. We set $C_{\Gamma}^{*}(\mathcal{H})$ the closure of $\Gamma$-equivariant families $T=\left(T_{\theta}\right)_{\theta \in T} \in \operatorname{End}_{\Gamma}(\mathcal{H})$ preserving the continuous field $C_{c}(\tilde{V} \times T)$. In [MoN96], Section 2, it is proved that the foliation $C^{*}$-algebra $C^{*}(X, \mathcal{F})$ is isomorphic to a $C^{*}$-subalgebra of $C_{\Gamma}^{*}(\mathcal{H}) \subset \operatorname{End}_{\Gamma}(\mathcal{H}) .{ }^{6}$ Notice, in particular, that an element in $C^{*}(X, \mathcal{F})$ can be considered as a $\Gamma$-equivariant family of operators.
4.3 Translation invariant operators. Recall $\operatorname{cyl}(\partial X):=\mathbb{R} \times \partial X_{0} \equiv(\mathbb{R} \times$ $\partial \tilde{M}) \times_{\Gamma} T$ with $\Gamma$ acting trivially in the $\mathbb{R}$-direction of $\mathbb{R} \times \partial \tilde{M}$. We consider the foliated cylinder $\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ and its holonomy groupoid $G_{\text {cyl }}:=((\mathbb{R} \times \partial \tilde{M}) \times$ $(\mathbb{R} \times \partial \tilde{M}) \times T) / \Gamma$ (source and range maps are clear). Let $\mathbb{R}$ act trivially on $T$; then
${ }^{6}$ The $C^{*}$-algebra $C_{\Gamma}^{*}(\mathcal{H})$ was denoted $\mathfrak{B}$ in [MoN96].
$(\mathbb{R} \times \partial \tilde{M}) \times(\mathbb{R} \times \partial \tilde{M}) \times T$ has a $\mathbb{R} \times \Gamma$-action, with $\mathbb{R}$ acting by translation on itself. We define a ${ }^{*}$-algebra $B_{c}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \equiv B_{c}$ to be

$$
\begin{align*}
B_{c}:= & \{k \in C((\mathbb{R} \times \partial \tilde{M}) \times(\mathbb{R} \times \partial \tilde{M}) \times T) ; k \text { is } \mathbb{R} \times \Gamma \text {-invariant, } k \text { has } \mathbb{R} \\
& \times \Gamma \text {-compact support }\} \tag{4.2}
\end{align*}
$$

The product is by convolution. An element $\ell$ in $B_{c}$ defines a $\Gamma$-equivariant family $(\ell(\theta))_{\theta \in T}$ of translation-invariant operators. The completion of $B_{c}$ with respect to the $C^{*}$-norm (the sup over $\theta$ of the operator norm of $\ell(\theta)$ ) gives us a $C^{*}$-algebra that will be denoted by $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ or more briefly by $B^{*}$. Notice that we can in fact define $B^{*}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ for any foliated flat bundle $(Y, \mathcal{F})$, with $Y=\tilde{N} \times{ }_{\Gamma} T$.

Proposition 4.3. Let $(Y, \mathcal{F})$, with $Y=\tilde{N} \times_{\Gamma} T$, a foliated flat bundle without boundary. Let us denote by $\mathbb{R}_{\Delta}$ the group $\mathbb{R}$ acting diagonally by translation on $\mathbb{R} \times \mathbb{R}$. Consider the quotient group $(\mathbb{R} \times \mathbb{R}) / \mathbb{R}_{\Delta}$ which is isomorphic to $\mathbb{R}$. Consider the quotient groupoid $G_{\text {cyl }} / \mathbb{R}_{\Delta}$. Then $B^{*}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right)=C^{*}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$ and we have the following $C^{*}$-isomorphisms:

$$
\begin{equation*}
C^{*}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right) \cong C^{*}\left((\mathbb{R} \times \mathbb{R}) / \mathbb{R}_{\Delta}\right) \otimes C^{*}(Y, \mathcal{F}) \cong C^{*}(\mathbb{R}) \otimes C^{*}(Y, \mathcal{F}) \tag{4.4}
\end{equation*}
$$

Proof. The holonomy groupoid for $\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ is $G_{\text {cyl }}=(\mathbb{R} \times \tilde{N} \times \mathbb{R} \times \tilde{N} \times T) / \Gamma$; directly from the definition we see that $B^{*}$ is the $C^{*}$-algebra for the quotient groupoid $G_{\text {cyl }} / \mathbb{R}_{\Delta}$ which is clearly isomorphic to $(\mathbb{R} \times \mathbb{R}) / \mathbb{R}_{\Delta} \times(\tilde{N} \times \tilde{N} \times T) / \Gamma \equiv$ $(\mathbb{R} \times \mathbb{R}) / \mathbb{R}_{\Delta} \times G(Y, \mathcal{F})$. From these isomorphisms we can immediately end the proof.

REMARK 4.5. We can interpret $B^{*}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ as the compact operators of a suitable Hilbert $C^{*}$-module. Consider $\mathbb{R} \times N \times T$ with its natural $\Gamma \times \mathbb{R}$-action; consider $C_{c}^{\infty}(\mathbb{R} \times \tilde{N} \times T)$; we can complete it to a Hilbert $C^{*}$-module $\mathcal{E}_{\text {cyl }}$ over $(C(T) \rtimes \Gamma) \otimes C^{*} \mathbb{R}$. Proceeding as in [MoN96] one can prove that there is a $C^{*}$-algebra isomorphism $B^{*}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right) \simeq \mathbb{K}\left(\mathcal{E}_{\text {cyl }}\right)$. In particular, we see that $B^{*}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ can be seen as an ideal in the $C^{*}$-algebra $\mathcal{L}\left(\mathcal{E}_{\text {cyl }}\right)$.
4.4 Wiener-Hopf extensions. Recall the Hilbert $C(T) \rtimes \Gamma$-module $\mathcal{E}$ and the $C^{*}$-algebras $\mathbb{K}(\mathcal{E})$ and $\mathcal{L}(\mathcal{E})$; see [Bla 98$\left.]\right)$. Since the $C(T) \rtimes \Gamma$-compact operators $\mathbb{K}(\mathcal{E})$ are an ideal in $\mathcal{L}(\mathcal{E})$ we have the classical short exact sequence of $C^{*}$-algebras

$$
0 \rightarrow \mathbb{K}(\mathcal{E}) \hookrightarrow \mathcal{L}(\mathcal{E}) \xrightarrow{\pi} \mathcal{Q}(\mathcal{E}) \rightarrow 0
$$

with $\mathcal{Q}(\mathcal{E})=\mathcal{L}(\mathcal{E}) / \mathbb{K}(\mathcal{E})$ the Calkin algebra. Let $\chi_{\mathbb{R}}^{0}: \mathbb{R} \rightarrow \mathbb{R}$ be the characteristic function of $(-\infty, 0]$; let $\chi_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function with values in $[0,1]$ such that:

$$
\chi_{\mathbb{R}}(t)= \begin{cases}1 & \text { for } \quad t \leq-\epsilon  \tag{4.6}\\ 0 & \text { for } t \geq 0\end{cases}
$$

for given $\epsilon>0$. Let $\chi$ be the smooth function induced by $\chi_{\mathbb{R}}$ on $X$; when we want to exhibit the dependence on $\epsilon$ clearly, we shall denote it by $\chi_{\epsilon}$. Similarly, we consider $\chi_{\text {cyl }}$, the smooth function induced by $\chi_{\mathbb{R}}$ on $\operatorname{cyl}(\partial X)$. Finally, let $\chi^{0}$ and $\chi_{\text {cyl }}^{0}$ be the functions induced by the characteristic function $\chi_{\mathbb{R}}^{0}$ on $X$ and $\operatorname{cyl}(\partial X)$ respectively. For $\lambda>0$, we shall also make use of the real functions $\chi^{\lambda}$ and $\chi_{\text {cyl }}^{\lambda}$, induced on $X$ and $\operatorname{cyl}(\partial X)$ by $\chi_{(-\infty,-\lambda]}^{\mathbb{R}}$, the characteristic function of $(-\infty,-\lambda]$ in $\mathbb{R}$; thus $\chi^{\lambda}$ is equal to 0 on the interior of $X_{\lambda}$ and equal to 1 on its complement in $X$. Similarly: $\chi_{\text {cyl }}^{\lambda}$ is equal to zero on $(\lambda,+\infty) \times \partial X_{0}$ and equal to one on $(-\infty, \lambda] \times \partial X_{0}$.

Lemma 4.7. There exists a bounded linear map

$$
\begin{equation*}
s: B^{*}=B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}\right) \rightarrow \mathcal{L}(\mathcal{E}) \tag{4.8}
\end{equation*}
$$

extending $s_{c}: B_{c} \rightarrow \mathcal{L}(\mathcal{E}), s_{c}(\ell):=\chi^{0} \ell \chi^{0}$ Here $\chi^{0}$ stands for the multiplication operator and $s_{c}(\ell)$ is defined as the composite of those operators. ${ }^{7}$ Moreover, the composition $\rho=\pi s$ induces an injective $C^{*}$-homomorphism

$$
\begin{equation*}
\rho: B^{*} \rightarrow \mathcal{Q}(\mathcal{E}) . \tag{4.9}
\end{equation*}
$$

See Section 10.1 for a detailed proof of Lemma 4.7; there we also explain why $s_{c}$ is well defined. A key tool in the proof of the Lemma is the following Sublemma, stated here for later use but proved in Section 10.1:

Sublemma 4.10. Let $\ell \in B_{c}$. Then $\chi^{\lambda} \ell\left(1-\chi^{\lambda}\right),\left(1-\chi^{\lambda}\right) \ell \chi^{\lambda}$ and $\left[\chi^{\lambda}, \ell\right]$ are all of $\Gamma$-compact support on $\operatorname{cyl}(\partial X)$.

In the sequel we shall use repeatedly this simple but crucial result.
We now consider $\operatorname{Im} \rho$ as a $C^{*}$-subalgebra in $\mathcal{Q}(\mathcal{E})$ and identify it with $B^{*} \equiv$ $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ via $\rho$. Set

$$
A^{*}(X ; \mathcal{F}):=\pi^{-1}(\operatorname{Im} \rho) \quad \text { with } \pi \text { the projection } \quad \mathcal{L}(\mathcal{E}) \rightarrow \mathcal{Q}(\mathcal{E})
$$

Recalling the identification $C^{*}(X, \mathcal{F})=\mathbb{K}(\mathcal{E})$, we thus obtain a short exact sequence of $C^{*}$-algebras: $0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*}(X ; \mathcal{F}) \xrightarrow{\pi} B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow 0$ where the quotient map is still denoted by $\pi$.

Definition 4.11. The short exact sequence of $C^{*}$-algebras

$$
\begin{equation*}
0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*}(X ; \mathcal{F}) \xrightarrow{\pi} B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow 0 \tag{4.12}
\end{equation*}
$$

is by definition the Wiener-Hopf extension of $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$.
Notice that (4.12) splits as a short exact sequence of Banach spaces, since we can choose $s: B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow A^{*}(X ; \mathcal{F})$, the map in the statement of Lemma 4.7, as a section. So $A^{*}(X ; \mathcal{F}) \cong C^{*}(X, \mathcal{F}) \oplus s\left(B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)\right)$ as Banach spaces.

There is also a linear map $t: A^{*}(X, \mathcal{F}) \rightarrow C^{*}(X, \mathcal{F})$ which is obtained as follows: if $k \in A^{*}(X ; \mathcal{F})$, then $k$ is uniquely expressed as $k=a+s(\ell)$ with $a \in C^{*}(X, \mathcal{F})$
${ }^{7}$ For the precise meaning of this composition, see Section 10.1.
and $\pi(k)=\ell \in B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. Thus, $\pi(k)=\left[\chi^{0} \ell \chi^{0}\right] \in \mathcal{Q}(\mathcal{E})$ for one (and only one) $\ell \in B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ since $\rho$ is injective. We set

$$
\begin{equation*}
t(k):=k-s \pi(k)=k-\chi^{0} \ell \chi^{0} \tag{4.13}
\end{equation*}
$$

Then $t(k) \in C^{*}(X, \mathcal{F})$.
Remark 4.14. A standard Wiener-Hopf extension of $C^{*} \mathbb{R}$ is obtained as follows. Let $C^{*} \mathbb{R}$ act on the Hilbert space $\mathcal{H}=L^{2}(\mathbb{R})$ by convolutions. Recall that $\chi_{\mathbb{R}}^{0}$ is the characteristic function of $(-\infty, 0]$ and denote by $\mathcal{H}_{o}$ the subspace $L^{2}(-\infty, 0] \subset \mathcal{H}$. Then the same proof of Lemma 4.7 can be applied to prove that there exists an injective homomorphism $\rho_{\mathbb{R}}: C^{*} \mathbb{R} \rightarrow \mathcal{Q}\left(\mathcal{H}_{o}\right)$ with $\rho_{\mathbb{R}}(\ell)=\pi_{o}\left(\chi_{\mathbb{R}}^{0} \ell \chi_{\mathbb{R}}^{0}\right)$, where $\mathcal{Q}\left(\mathcal{H}_{o}\right)$ denotes the Calkin algebra and $\pi_{o}$ the projection from the bounded operators on $\mathcal{H}_{o}$ onto $\mathcal{Q}\left(\mathcal{H}_{o}\right)$. Set $\mathcal{E}_{o}=\pi_{o}^{-1}\left(\operatorname{Im} \rho_{\mathbb{R}}\right)$. Exactly in the same manner as before, one has a short exact sequence $0 \rightarrow \mathbb{K}_{o} \rightarrow \mathcal{E}_{o} \xrightarrow{\pi_{o}} C^{*} \mathbb{R} \rightarrow 0$, where $\mathbb{K}_{o}$ denotes the compact operators on $\mathcal{H}_{o}$. It is called a Wiener-Hopf extension of $C^{*} \mathbb{R}$. What we are going to construct is a slightly larger algebra than this. Observe that $\mathcal{Q}\left(\mathcal{H}_{o}\right)$ is naturally embedded in the Calkin algebra $\mathcal{Q}(\mathcal{H})$. Thus one has another injective homomorphism $\hat{\rho}_{\mathbb{R}}: C^{*} \mathbb{R} \rightarrow \mathcal{Q}(\mathcal{H})$. Set $\mathcal{E}=\pi^{-1}\left(\operatorname{Im} \hat{\rho}_{\mathbb{R}}\right)$ with $\pi$ the projection onto $\mathcal{Q}(\mathcal{H})$. It then induces an extension of $C^{*}$-algebras: $0 \rightarrow \mathbb{K} \rightarrow \mathcal{E} \xrightarrow{\pi} C^{*} \mathbb{R} \rightarrow 0$ where $\mathbb{K}$ is the compact operators on $\mathcal{H}$. Obviously it contains the above extension. Now recall the definition of the Extension group $\operatorname{Ext}\left(C^{*} \mathbb{R}\right)$ and its additive structure; see Douglas [Dou80] for instance. It is easily verified that the second extension is exactly the one corresponding to the sum of $\mathcal{E}_{o}$ and the trivial extension; hence the resulting extension class is the same as that of $\mathcal{E}_{o}$. Therefore, the second extension also deserves to be called a Wiener-Hopf extension.

Let us consider the simplest case, namely a foliation consists of a single leaf $X$, which is a complete manifold with cylindrical end. It turns out that our extension (4.12) is isomorphic to the second extension tensored with the algebra of compact operators. This can be proved by observing the property $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \cong$ $C^{*} \mathbb{R} \otimes \mathbb{K}$ in (4.4). Thus we call the short exact sequence (4.12) the Wiener-Hopf extension of $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$.

## 5 Relative Pairings and Eta Cocycles: The Algebraic Theory

5.1 Introductory remarks. On a closed foliated bundle $(Y, \mathcal{F})$ with holonomy groupoid $G$, the Godbillon-Vey cyclic cocycle is initially defined on the "small" algebra $\Psi_{c}^{-\infty}(G, E) \subset C^{*}(Y, \mathcal{F} ; E)$ of $\Gamma$-equivariant smoothing operators of $\Gamma$-compact support, which can be described as $\Psi_{c}^{-\infty}(G, E)=C_{c}^{\infty}\left(G,\left(s^{*} E\right)^{*} \otimes r^{*} E\right)$.

Since the index class defined using a pseudodifferential parametrix is already well defined in the $K$-group $K_{*}\left(\Psi_{c}^{-\infty}(G, E)\right)$, the pairing between the the Godbillon-Vey cyclic cocycle and the index class is well-defined.

In a second stage, the cocycle is continuously extended to a dense holomorphically closed subalgebra $\mathfrak{A} \subset C^{*}(Y, \mathcal{F})$; there are at least two reasons for doing this.

First, as already remarked in the Introduction, it is only by going to the $C^{*}$-algebraic index that the well known properties for the signature and the spin Dirac operator of a metric of positive scalar curvature hold. The second reason for this extension rests on the structure of the index class which is employed in the proof of the higher index formula, i.e. either the graph projection or the Wassermann projection; in both cases $\Psi_{c}^{-\infty}(G, E)$ is too small to contain these particular representatives of the index class and one is therefore forced to find an intermediate subalgebra $\mathfrak{A}$,

$$
\begin{equation*}
\Psi_{c}^{-\infty}(G, E) \subset \mathfrak{A} \subset C^{*}(Y, \mathcal{F} ; E) \tag{5.1}
\end{equation*}
$$

$\mathfrak{A}$ is big enough for the two particular representatives of the index class to belong to it but small enough for the Godbillon-Vey cyclic cocycle to extend; moreover, being dense and holomorphically closed it has the crucial property of having the same $K$-theory as $C^{*}(Y, \mathcal{F} ; E)$.

Let now $(X, \mathcal{F})$ be a foliated bundle with cylindrical ends. For notational simplicity, unless confusion should arise, let us not write the bundle $E$ in our algebras. In this section we shall select "small" subalgebras $J_{c} \subset C^{*}(X, \mathcal{F}), A_{c} \subset A^{*}(X, \mathcal{F}), B_{c} \subset$ $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$, with $J_{c}$ an ideal in $A_{c}$, so that there is a short exact sequence $0 \rightarrow J_{c} \hookrightarrow A_{c} \xrightarrow{\pi_{c}} B_{c} \rightarrow 0$ which is a subsequence of $0 \rightarrow C^{*}(X, \mathcal{F}) \hookrightarrow A^{*}(X ; \mathcal{F}) \xrightarrow{\pi}$ $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow 0$.

We shall then proceed to define the two relevant Godbillon-Vey cyclic cocycles and study, algebraically, their main properties. As in the closed case, we shall eventually need to find an intermediate short exact sequence, sitting between the two, call it $0 \rightarrow \mathfrak{J} \hookrightarrow \mathfrak{A} \rightarrow \mathfrak{B} \rightarrow 0$, with constituents big enough for the index classes to belong to them but small enough for the two cyclic cocycles to extend; this is quite a delicate point and it will be explained in Section 6. We anticipate that, in contrast with the closed case, the ideal $J_{c}$ in the small subsequence will be too small even for the index class defined by a pseudodifferential parametrix. This has to do with the non-locality of the parametrix on manifolds with boundary; it is a phenomenon that was explained in detail in [LP05]; we shall come back to it in Section 7.
5.2 Small dense subalgebras. Set $J_{c}:=C_{c}^{\infty}(X, \mathcal{F})$; see Section 3.1. Redefine

$$
\begin{aligned}
B_{c}:= & \left\{k \in C^{\infty}((\mathbb{R} \times \partial \tilde{M}) \times(\mathbb{R} \times \partial \tilde{M}) \times T) ; k \text { is } \mathbb{R} \times \Gamma \text {-invariant, } k \text { has } \mathbb{R}\right. \\
& \times \Gamma \text {-compact support }\}
\end{aligned}
$$

see Section 4.3 (we pass from continuous to smooth functions). We now define $A_{c}$; consider the functions $\chi^{\lambda}, \chi_{\text {cyl }}^{\lambda}$ induced on $X$ and $\operatorname{cyl}(\partial X)$ by the real function $\chi_{(-\infty,-\lambda]}$ (the characteristic function of the interval $\left.(-\infty,-\lambda]\right)$. We shall say that $k$ is in $A_{c}$ if it is a smooth function on $\tilde{V} \times \tilde{V} \times T$ which is $\Gamma$-invariant and for which there exists $\lambda \equiv \lambda(k)>0$, such that

- $k-\chi^{\lambda} k \chi^{\lambda}$ is of $\Gamma$-compact support
- there exists $\ell \in B_{c}$ such that $\chi^{\lambda} k \chi^{\lambda}=\chi_{\mathrm{cyl}}^{\lambda} \ell \chi_{\mathrm{cyl}}^{\lambda}$ on $((-\infty,-\lambda] \times \partial \tilde{M}) \times$ $((-\infty,-\lambda] \times \partial \tilde{M}) \times T$

Lemma 5.2. $A_{c}$ is a $*$-subalgebra of $A^{*}(X, \mathcal{F})$.
Proof. Let $k, k^{\prime} \in A_{c}$. Write, with a small abuse of notation, $k=a+\chi^{\lambda} \ell \chi^{\lambda}$ with $a$ of $\Gamma$-compact support and $\ell \in B_{c}$ and similarly for $k^{\prime}$. Observe first of all that if $\mu>\lambda$, so that $-\mu<-\lambda$, then $\left(\chi^{\lambda} \ell \chi^{\lambda}-\chi^{\mu} \ell \chi^{\mu}\right)$ is also of $\Gamma$-compact support (since $\ell$ is of $\mathbb{R} \times \Gamma$-compact support). Thus we can assume that $k=a+\chi^{\mu} \ell \chi^{\mu}, k^{\prime}=a^{\prime}+\chi^{\mu} \ell^{\prime} \chi^{\mu}$. We compute: $k k^{\prime}=a a^{\prime}+a \chi^{\mu} \ell^{\prime} \chi^{\mu}+\chi^{\mu} \ell \chi^{\mu} a^{\prime}+\chi^{\mu} \ell \chi^{\mu} \chi^{\mu} \ell^{\prime} \chi^{\mu}$. The first summand on the right hand side is again of $\Gamma$-compact support; the second and the third summand are also of $\Gamma$-compact support since $\ell$ and $\ell^{\prime}$ are of $\mathbb{R} \times \Gamma$-compact support; the last term can be written as $\chi^{\mu} \ell \ell^{\prime} \chi^{\mu}+\left(\chi^{\mu} \ell\left(\chi^{\mu}-1\right)\right)\left(\left(\chi^{\mu}-1\right) \ell^{\prime} \chi^{\mu}\right)$. Thus $k k^{\prime}-\chi^{\mu} \ell \ell^{\prime} \chi^{\mu}=$ $a a^{\prime}+\left(\chi^{\mu} \ell\left(\chi^{\mu}-1\right)\right)\left(\left(\chi^{\mu}-1\right) \ell^{\prime} \chi^{\mu}\right)$; now, by Sublemma 4.10 both $\left(\chi^{\mu} \ell\left(\chi^{\mu}-1\right)\right)$ and $\left(\left(\chi^{\mu}-1\right) \ell^{\prime} \chi^{\mu}\right)$ are of $\Gamma$-compact support. Thus $k k^{\prime}-\chi^{\mu} \ell \ell^{\prime} \chi^{\mu}$ is of $\Gamma$-compact support as required. Finally, consider $\nu \in \mathbb{R}, \nu>\mu$ and let $F\left(p, p^{\prime}, \theta\right):=\chi^{\nu}(p)\left(1-\chi^{\mu}\right)\left(p^{\prime}\right)$, a function on $W \times W \times T$ which is $\theta$-independent. Since $\ell$ and $\ell^{\prime}$ are of $\mathbb{R} \times \Gamma$ compact support, we can choose $\nu>\mu$ so that that $\operatorname{supp} \ell \cap \operatorname{supp} F=\emptyset$. Thus $\chi^{\nu}\left(\chi^{\mu} \ell\left(\chi^{\mu}-1\right)\right)=\chi^{\nu} \ell\left(\chi^{\mu}-1\right)$ is equal to zero. We conclude that for such a $\nu$ we do get $\chi^{\nu} k k^{\prime} \chi^{\nu}=\chi^{\mu} \ell \ell^{\prime} \chi^{\mu}$ and the proof is complete.

We thus have:
Proposition 5.3. Let $\pi_{c}:=\left.\pi\right|_{A_{c}}$. Then there is a short exact sequence of ${ }^{*}$-algebras

$$
\begin{equation*}
0 \rightarrow J_{c} \hookrightarrow A_{c} \xrightarrow{\pi_{c}} B_{c} \rightarrow 0 . \tag{5.4}
\end{equation*}
$$

REmARK 5.5. Notice that the image of $A_{c}$ through $\left.t\right|_{A_{c}}$ is not contained in $J_{c}$ since $\chi^{0}$ is not even continuous. Similarly, the image of $B_{c}$ through $\left.s\right|_{B_{c}}$ is not contained in $A_{c}$.

REmark 5.6. Using the foliated $b$-calculus developed in [LP05] and Melrose' indicial operator in the foliated context, it is possible to define a slightly bigger dense subsequence. We shall briefly comment on this in Section 5.7.
5.3 Relative cyclic cocycles. Let $A$ be a $k$-algebra over $k=\mathbb{C}$. The cyclic cohomology group $H C^{*}(A)$ [Con85] (see also [Tsy83]) is the cohomology groups of cochain complex $\left(C_{\lambda}^{n}, b\right)$, where $C_{\lambda}^{n}$ is the space of $(n+1)$-linear functionals $\varphi$ on $A$ satisfying the condition $\varphi\left(a_{1}, a_{2}, \ldots, a_{n}, a_{0}\right)=(-1)^{n} \varphi\left(a_{0}, \ldots, a_{n+1}\right), \forall a_{i} \in A$ and with $b$ the Hochschild coboundary map defined by

$$
(b \varphi)\left(a_{0}, \ldots, a_{n+1}\right)=\sum_{j=0}^{n}(-1)^{j} \varphi\left(a_{0}, \ldots, a_{j} a_{j+1}, \ldots, a_{n+1}\right)+(-1)^{n+1} \varphi\left(a_{n+1} a_{0}, \ldots, a_{n}\right) .
$$

Given a second algebra $B$ together with a surjective homomorphism $\pi: A \rightarrow B$, one can define a relative cochain complex

$$
C_{\lambda}^{n}(A, B):=\left\{(\tau, \sigma): \tau \in C_{\lambda}^{n}(A), \sigma \in C_{\lambda}^{n+1}(B)\right\}
$$

with coboundary map given by

$$
(\tau, \sigma) \longrightarrow\left(b \tau-\pi^{*} \sigma, b \sigma\right) .
$$

A relative cochain $(\tau, \sigma)$ is thus a cocycle if $b \tau=\pi^{*} \sigma$ and $b \sigma=0$. One obtains in this way a relative cyclic cohomology group $H C^{*}(A, B)$. If $A$ and $B$ are Fréchet algebra, then we can also define a topological (relative) cyclic cohomology group. More detailed information are given, for example, in [LMP09b].
5.4 Roe's 1-cocycle. In this subsection, and in the next two, we study a particular but important example. We assume that $T$ is a point and that $\Gamma=\{1\}$, so that we are really considering a compact manifold $X_{0}$ with boundary $\partial X_{0}$ and associated manifold with cylindrical end $X$; we keep denoting the cylinder $\mathbb{R} \times \partial X_{0}$ by $\operatorname{cyl}(\partial X)$ (thus, as before, we omit the subscript 0 ). The algebras appearing in the short exact sequence (5.4) are now given by $J_{c}=C_{c}^{\infty}(X \times X)$ and
$B_{c}=\left\{k \in C^{\infty}\left(\left(\mathbb{R} \times \partial X_{0}\right) \times\left(\mathbb{R} \times \partial X_{0}\right)\right) ; k\right.$ is $\mathbb{R}$-invariant, $k$ has compact $\mathbb{R}$-support $\}$.
Finally, a smooth function $k$ on $X \times X$ is in $A_{c}$ if there exists a $\lambda \equiv \lambda(k)>0$ such that
(i) $k-\chi^{\lambda} k \chi^{\lambda}$ is of compact support on $X \times X$;
(ii) $\exists \ell \in B_{c}$ such that $\chi^{\lambda} k \chi^{\lambda}=\chi_{\text {cyl }}^{\lambda} \chi_{\text {cyl }}^{\lambda}$ on $\left((-\infty,-\lambda] \times \partial X_{0}\right) \times\left((-\infty,-\lambda] \times \partial X_{0}\right)$.

For such a $k \in A_{c}$ we set $\pi_{c}:=\left.\pi\right|_{A_{c}}$. Since $k-\chi^{0} \ell \chi^{0}$ admits compact support, it belongs to $C^{*}(G)$ (in this case this is just equal to the compact operators on $\left.L^{2}(X)\right)$. Hence it follows that $\pi(k)=\ell$ and thus $\pi_{c}(k)=\ell$; so we have the short exact sequence of $*$-algebras $0 \rightarrow J_{c} \hookrightarrow A_{c} \xrightarrow{\pi_{c}} B_{c} \rightarrow 0$ (The Wiener-Hopf short exact sequence (4.12) now reads as $\left.0 \rightarrow \mathcal{K}\left(L^{2}(X)\right) \rightarrow A^{*}(X) \xrightarrow{\pi} B^{*}(\operatorname{cyl}(\partial X)) \rightarrow 0\right)$. All of this has an obvious generalization if instead of functions we consider sections of the bundle END $E:=E \boxtimes E^{*} \rightarrow X \times X$, with $E$ a complex vector bundle on $X$.

We shall define below a relative cyclic 0-cocycle associated to the homomorphism $\pi_{c}: A_{c} \rightarrow B_{c}$. To this end we start by defining a cyclic 1-cocycle $\sigma_{1}$ for the algebra $B_{c}$; this is directly inspired from work of John Roe (indeed, a similarly defined 1-cocycle plays a fundamental role in his index theorem on partitioned manifolds [Roe88]). It should be noticed that $\sigma_{1}$ is in fact defined on $B_{c}(\operatorname{cyl}(Y))$, with $Y$ any closed compact manifold.

Consider the characteristic function $\chi_{\text {cyl }}^{\lambda}, \lambda>0$, induced on the cylinder $\operatorname{cyl}(Y)$ by the real function $\chi_{(-\infty,-\lambda]}^{\mathbb{R}}$. For notational convenience, unless absolutely necessary, we shall use the simpler notation $\chi^{\lambda}$.

We define $\sigma_{1}^{\lambda}: B_{c} \times B_{c} \rightarrow \mathbb{C}$ as

$$
\begin{equation*}
\sigma_{1}^{\lambda}\left(\ell_{0}, \ell_{1}\right):=\operatorname{Tr}\left(\ell_{0}\left[\chi^{\lambda}, \ell_{1}\right]\right) \tag{5.7}
\end{equation*}
$$

We need to show that the definition is well posed.

Proposition 5.8. The operators $\left[\chi^{\lambda}, \ell_{0}\right]$ and $\ell_{0}\left[\chi^{\lambda}, \ell_{1}\right]$ are trace class $\forall \ell_{0}, \ell_{1} \in B_{c}$ (and $\operatorname{Tr}\left[\chi^{\lambda}, \ell_{0}\right]=0$ ). In particular $\sigma_{1}^{\lambda}\left(\ell_{0}, \ell_{1}\right)$ is well defined.

Proof. We already know, see Sublemma 4.10, that the operator $\left[\chi^{\lambda}, \ell_{1}\right]$ is expressed by a kernel on the cylinder which is of compact support. Indeed, in the proof of Sublemma 4.10, which is given in Section 10, we have explicitly written down the kernel $k$ corresponding to $\left[\chi^{\lambda}, \ell\right]$ as

$$
k\left(y, s, y^{\prime}, s^{\prime}\right)= \begin{cases}\ell\left(y, y^{\prime}, s-s^{\prime}\right) & \text { if } s \leq-\lambda, \quad s^{\prime} \geq-\lambda  \tag{5.9}\\ -\ell\left(y, y^{\prime}, s-s^{\prime}\right) & \text { if } s^{\prime} \leq-\lambda, \quad s \geq-\lambda \\ 0 & \text { otherwise }\end{cases}
$$

where $y, y^{\prime} \in Y, s, s^{\prime} \in \mathbb{R}$ and where we have used the $\mathbb{R}$-invariance of $\ell$ in order to write $\ell\left(s, y, s^{\prime}, y^{\prime}\right) \equiv \ell\left(y, y^{\prime}, s-s^{\prime}\right)$. Choose now a smooth compactly supported function $\varphi$ on $\operatorname{cyl}(Y) \times \operatorname{cyl}(Y)$, equal to 1 on the support of $k$. Let $k_{0}$ be the smooth compactly supported kernel obtained by multiplying $k$ by $\varphi ; k_{0}$ is clearly trace class. Now, multiplication by $\chi^{\lambda}$ is a bounded operator so the operators given by $\chi^{\lambda} k_{0}$ and $k_{0} \chi^{\lambda}$ are also trace class. Since $\left[\chi^{\lambda}, k_{0}\right]=\left[\chi^{\lambda}, \ell\right]$, we conclude that $\left[\chi^{\lambda}, \ell\right]$ is trace class; since $\ell_{0}$ defines a bounded operator, we also see immediately that the trace of $\ell_{0}\left[\chi^{\lambda}, \ell_{1}\right]$ is well defined. Finally, it remains to justify that $\operatorname{Tr}\left[\chi^{\lambda}, \ell\right]=0$; this is now clear, since $\operatorname{Tr}\left[\chi^{\lambda}, \ell\right]=\operatorname{Tr}\left[\chi^{\lambda}, k_{0}\right]=0$. The proposition is proved.

Proposition 5.10. The value $\operatorname{Tr}\left(\ell_{0}\left[\chi^{\lambda}, \ell_{1}\right]\right)$ is independent of $\lambda$ and will simply be denoted by $\sigma_{1}\left(\ell_{0}, \ell_{1}\right)$. The functional $\sigma_{1}: B_{c} \times B_{c} \rightarrow \mathbb{C}$ is a 1-cyclic cocycle.

Proof. In order to prove the independence on $\lambda$ we make crucial use of the $\mathbb{R}$-invariance of $\ell_{j}$. We write $\ell_{j}\left(y, y^{\prime}, s, s^{\prime}\right) \equiv \ell_{j}\left(y, y^{\prime}, s-s^{\prime}\right)$. We compute:

$$
\begin{aligned}
\sigma_{1}^{\lambda}\left(\ell_{0}, \ell_{1}\right)= & \operatorname{Tr}\left(\ell_{0} \chi^{\lambda} \ell_{1}-\ell_{0} \ell_{1} \chi^{\lambda}\right) \\
= & \int_{Y \times Y} d y d y^{\prime} \int_{\mathbb{R} \times \mathbb{R}} d s d s^{\prime}\left[\ell_{0}\left(y, y^{\prime}, s-s^{\prime}\right) \chi^{\lambda}\left(s^{\prime}\right) \ell_{1}\left(y^{\prime}, y, s^{\prime}-s\right)\right. \\
& \left.-\ell_{0}\left(y, y^{\prime}, s-s^{\prime}\right) \ell_{1}\left(y^{\prime}, y, s^{\prime}-s\right) \chi^{\lambda}(s)\right] \\
= & \int_{Y \times Y} d y d y^{\prime}\left(\int_{\mathbb{R}} d s \int_{-\infty}^{-\lambda} d s^{\prime}-\int_{-\infty}^{-\lambda} d s \int_{\mathbb{R}} d s^{\prime}\right) \ell_{0}\left(y, y^{\prime}, s-s^{\prime}\right) \ell_{1}\left(y^{\prime}, y, s^{\prime}-s\right) \\
= & \int_{Y \times Y} d y d y^{\prime}\left(\int_{-\lambda}^{+\infty} d s \int_{-\infty}^{-\lambda} d s^{\prime}-\int_{-\infty}^{-\lambda} d s \int_{-\lambda}^{+\infty} d s^{\prime}\right) \ell_{0}\left(y, y^{\prime}, s-s^{\prime}\right) \ell_{1}\left(y^{\prime}, y, s^{\prime}-s\right) \\
= & \int_{Y \times Y} d y d y^{\prime}\left(\int_{0}^{+\infty} d t \int_{-\infty}^{0} d t^{\prime}-\int_{-\infty}^{0} d t \int_{0}^{+\infty} d t^{\prime}\right) \ell_{0}\left(y, y^{\prime}, t-t^{\prime}\right) \ell_{1}\left(y^{\prime}, y, t^{\prime}-t\right) .
\end{aligned}
$$

Thus $\operatorname{Tr}\left(\ell_{0}\left[\chi^{\lambda}, \ell_{1}\right]\right)$ is independent of $\lambda$ since we have proved that $\forall \lambda$ it is equal to $\operatorname{Tr}\left(\ell_{0}\left[\chi^{0}, \ell_{1}\right]\right)$. In particular we record that

$$
\begin{equation*}
\sigma_{1}^{\lambda}\left(\ell_{0}, \ell_{1}\right)=\operatorname{Tr}\left(\ell_{0}\left[\chi^{0}, \ell_{1}\right]\right) \tag{5.11}
\end{equation*}
$$

We shall denote $\sigma_{1}^{\lambda}$ as $\sigma_{1}$. In order to show that $\sigma_{1}$ is a cyclic cocycle we begin by recalling that $\operatorname{Tr}\left[\chi^{\lambda}, \ell\right]=0 \forall \ell \in B_{c}$. Thus we have $\sigma_{1}\left(\ell_{0}, \ell_{1}\right)+\sigma_{1}\left(\ell_{1}, \ell_{0}\right)=$ $\operatorname{Tr}\left(\ell_{0}\left[\chi^{0}, \ell_{1}\right]\right)+\operatorname{Tr}\left(\left[\chi^{0}, \ell_{0}\right] \ell_{1}\right)=\operatorname{Tr}\left(\left[\chi^{0}, \ell_{0} \ell_{1}\right]\right)=0$ proving that $\sigma_{1}$ is a cyclic cochain. Next we compute

$$
\begin{aligned}
b \sigma_{1}\left(\ell_{0}, \ell_{1}, \ell_{2}\right) & \left.=\operatorname{Tr}\left(\ell_{0} \ell_{1}\left[\chi^{0}, \ell_{2}\right]\right)+\ell_{0}\left[\chi^{0}, \ell_{1} \ell_{2}\right]+\ell_{2} \ell_{0}\left[\chi^{0}, \ell_{1}\right]\right) \\
& =\operatorname{Tr}\left(-\ell_{0}\left[\chi^{0}, \ell_{1}\right] \ell_{2}+\ell_{2} \ell_{0}\left[\chi^{0}, \ell_{1}\right]\right)=\operatorname{Tr}\left(\left[\ell_{2}, \ell_{0}\left[\chi^{0}, \ell_{1}\right]\right]\right)=0 .
\end{aligned}
$$

Remark 5.12. We point out that following expression for $\sigma_{1}$ :

$$
\begin{equation*}
\sigma_{1}\left(\ell_{0}, \ell_{1}\right)=\frac{1}{2} \operatorname{Tr}\left(\chi^{0}\left[\chi^{0}, \ell_{0}\right]\left[\chi^{0}, \ell_{1}\right]\right) \tag{5.13}
\end{equation*}
$$

The proof of (5.13) is elementary (just apply repeatedly the fact that $1=\chi^{0}+(1-$ $\left.\chi^{0}\right)$ ) and for the sake of brevity we omit it. The advantage of this new expression for $\sigma_{1}$ is that it makes the extension to certain dense subalgebras easier to deal with. (Notice, for example, that $\sigma_{1}$ is now defined under the weaker assumption that $\left[\chi^{0}, \ell_{j}\right]$ is Hilbert-Schmidt.) The right hand side of (5.13) is in fact the original definition by Roe.
5.5 Melrose' regularized integral. Recall that our immediate goal is to define a 0 -relative cyclic cocycle for the homomorphism $\pi_{c}: A_{c} \rightarrow B_{c}$ appearing in the short exact sequence of the previous section. Having defined a 1-cocycle $\sigma_{1}$ on $B_{c}$ we now need to define a 0 -cochain on $A_{c}$. Our definition will be a simple adaptation of the definition of the $b$-trace in Melrose' $b$-calculus (but since our algebra $A_{c}$ is very small, we can give a somewhat simplified treatment). Recall that for $\lambda>0$ we are denoting by $X_{\lambda}$ the compact manifold obtained attaching $[-\lambda, 0] \times \partial X_{0}$ to our manifold with boundary $X_{0}$.

So, let $k \in A_{c}$ with $\pi_{c}(k)=\ell \in B_{c}$. Since $\ell$ is $\mathbb{R}$-invariant on the cylinder $\mathbb{R} \times \partial X_{0}$ we can write $\ell\left(y, y^{\prime}, s\right)$ with $y, y^{\prime} \in \partial X_{0}, s \in \mathbb{R}$. Set

$$
\begin{equation*}
\tau_{0}^{r}(k):=\lim _{\lambda \rightarrow+\infty}\left(\int_{X_{\lambda}} k(x, x) \mathrm{dvol}_{g}-\lambda \int_{\partial X_{0}} \ell(y, y, 0) \mathrm{dvol}_{g_{\partial}}\right) \tag{5.14}
\end{equation*}
$$

where the superscript $r$ stands for regularized. (The $b$-superscript would be of course more appropriate; unfortunately it gets confused with the $b$ operator in cyclic cohomology.) It is elementary to see that the limit exists; in fact, because of the very particular definition of $A_{c}$ the function

$$
\varphi(\lambda):=\int_{X_{\lambda}} k(x, x) \mathrm{dvol}_{g}-\lambda \int_{\partial X_{0}} \ell(y, y, 0) \mathrm{dvol}_{g_{\partial}}
$$

becomes constant for large values of $\lambda$. The proof is elementary and thus omitted. $\tau_{0}^{r}$ defines a 0 -cochain on $A_{c}$.

Remark 5.15. Notice that (5.14) is nothing but Melrose's regularized integral, in the cylindrical language, for the restriction of $k$ to the diagonal of $X \times X$.

We shall also need the following
Lemma 5.16. If $k \in A_{c}$ then $t(k)$, which is a priori a compact operator, is in fact trace class. Moreover

$$
\begin{equation*}
\tau_{0}^{r}(k)=\operatorname{Tr}(t(k)) . \tag{5.17}
\end{equation*}
$$

We remark once again that $t(k)$ is not an element in $J_{c}$.
Proof. We first need the following:
Sublemma 5.18. Let $\chi$ is the characteristic function of a measurable set $K$ in $X$. If $a \in J_{c}$, then $k=\chi a \chi$ is of trace class and the trace is obtained as $\operatorname{Tr}(k)=$ $\int_{K} a(x, x) d x$.

Proof. Since $a$ gives rise to a smoothing operator with compact support, it is of trace class. Recall that the algebra of trace class operators forms an ideal in the algebra of bounded operators. Thus $k$ is of trace class and we can assume that $k=b c$ with $b$ and $c$ operators of Hilbert-Schmidt class. Then

$$
\operatorname{Tr}(k)=\left\langle b, c^{*}\right\rangle_{2}=\int_{X \times X} b(x, y) c(y, x) d x d y=\int_{K} a(x, x) d x
$$

with $\langle,\rangle_{2}$ denoting the inner product for operators of Hilbert-Schmidt class.
Write $k=a+\chi^{\lambda} \ell \chi^{\lambda}$ with $a \in J_{c}$ and $\ell \in B_{c}$ as in Section 5.2. There exists a compactly supported smooth function $\sigma$ on $X$, depending on $\ell$, such that $\chi^{\lambda} \ell \chi^{\lambda}-\chi^{0} \ell \chi^{0}=$ $\chi^{\lambda} \sigma \ell \sigma \chi^{\lambda}-\chi^{0} \sigma \ell \sigma \chi^{0}$ since the support of $\chi^{\lambda}-\chi^{0}$ is compact. Note that we can choose the same $\ell$ in 4.13 and Section 5.2. Thus $t(k)=k-\chi^{0} \ell \chi^{0}=a+\chi^{\lambda} \sigma \ell \sigma \chi^{\lambda}-\chi^{0} \sigma \ell \sigma \chi^{0}$ is of trace class due to the sublemma above. Therefore, we have

$$
\operatorname{Tr}(t(k))=\int_{X} a(x, x) d x-\int_{X_{\lambda} \backslash X_{0}} \ell(y, y, 0) d y d t=\tau_{0}^{r}(k)
$$

for a sufficiently large $\lambda$. This completes the proof.
5.6 Melrose' regularized integral and Roe's 1-cocycle define a relative 0-cocycle. We finally consider the relative 0-cochain $\left(\tau_{0}^{r}, \sigma_{1}\right)$ for the pair $A_{c} \xrightarrow{\pi_{c}}$ $B_{c}$.

Proposition 5.19. The relative 0-cochain $\left(\tau_{0}^{r}, \sigma_{1}\right)$ is a relative 0 -cocycle. It thus defines an element $\left[\left(\tau_{0}^{r}, \sigma_{1}\right)\right]$ in the relative group $H C^{0}\left(A_{c}, B_{c}\right)$.

Proof. We need to show that $b \sigma_{1}=0$ and that $b \tau_{0}^{r}=\left(\pi_{c}\right)^{*} \sigma_{1}$. The first has already been proved, so we concentrate on the second. We compute: $b \tau\left(k, k^{\prime}\right)=\tau_{0}^{r}\left(k k^{\prime}-k^{\prime} k\right)$.

Write $k=a+\chi^{\mu} \ell \chi^{\mu}, k^{\prime}=a^{\prime}+\chi^{\mu} \ell^{\prime} \chi^{\mu}$ as we did in the proof of Lemma 5.2. Then we need to show that

$$
\begin{equation*}
\tau_{0}^{r}\left(k k^{\prime}-k^{\prime} k\right)=\sigma_{1}\left(\pi_{c} k, \pi_{c} k^{\prime}\right)=\sigma_{1}\left(\ell, \ell^{\prime}\right) \tag{5.20}
\end{equation*}
$$

There are several proofs of this fundamental relation. One proof of (5.20) employs Melrose' formula for the $b$-trace of a commutator; we shall give the details in the next Subsection. Here we propose a different proof that has the advantage of extending to more general situations. Following the proof of Lemma 5.2, we can write

$$
k k^{\prime}=\left(a a^{\prime}+a \chi^{\mu} \ell^{\prime} \chi^{\mu}+\chi^{\mu} \ell \chi^{\mu} a^{\prime}-\chi^{\mu} \ell\left(1-\chi^{\mu}\right) \ell^{\prime} \chi^{\mu}\right)+\chi^{\mu} \ell \ell^{\prime} \chi^{\mu}
$$

Notice that the first summand is trace class; this is obvious for the first term $a a^{\prime}$ and clear for the next two terms; the fourth term, viz. $-\chi^{\mu} \ell\left(1-\chi^{\mu}\right) \ell^{\prime} \chi^{\mu}$ is trace class because $\chi^{\mu} \ell\left(1-\chi^{\mu}\right)$ is trace class and $\ell^{\prime} \chi^{\mu}$ is bounded (see the proof of Proposition 5.8). A similar expression can be written for $k^{\prime} k$. Using first Lemma 5.16 and then the definition of $t$, we obtain easily

$$
\begin{aligned}
\tau_{0}^{r}\left(k k^{\prime}-k^{\prime} k\right)= & \operatorname{Tr}\left(t\left(k k^{\prime}-k^{\prime} k\right)\right) \\
= & \operatorname{Tr}\left(\left[a, a^{\prime}\right]+\left[\chi^{\mu} \ell \chi^{\mu}, a^{\prime}\right]+\left[a, \chi^{\mu} \ell^{\prime} \chi^{\mu}\right]-\chi^{\mu} \ell\left(1-\chi^{\mu}\right) \ell^{\prime} \chi^{\mu}\right. \\
& \left.+\chi^{\mu} \ell^{\prime}\left(1-\chi^{\mu}\right) \ell \chi^{\mu}\right) \\
= & \operatorname{Tr}\left(-\chi^{\mu} \ell\left(1-\chi^{\mu}\right) \ell^{\prime} \chi^{\mu}+\chi^{\mu} \ell^{\prime}\left(1-\chi^{\mu}\right) \ell \chi^{\mu}\right)=\sigma_{1}^{\mu}\left(\ell, \ell^{\prime}\right)=\sigma_{1}\left(\ell, \ell^{\prime}\right)
\end{aligned}
$$

The Proposition is proved.
5.7 Melrose' 1-cocycle and the relative cocycle condition via the $b$-trace formula. The results in this Subsection will not be used in the sequel.

As we have anticipated in the previous subsection, the equation $b \tau_{0}^{r}=\pi_{c}^{*} \sigma_{1}$ is nothing but a compact way of rewriting Melrose' formula for the $b$-trace of a commutator. We wish to explain this point here.

First of all, since it will cost us nothing, we consider a slightly larger subsequence of dense subalgebras. We hinted to this subsequence in Remark 5.6; we explain it here for $T=$ point and $\Gamma=\{1\}$ even though it exists in the general foliated case. Thus, following the notations of the $b$-calculus, we set

$$
A_{c}^{b}:=\Psi_{b, c}^{-\infty}(X, E), \quad B_{c}^{b}:=\Psi_{b, I, c}^{-\infty}\left(\overline{N_{+} \partial X},\left.E\right|_{\partial}\right), \quad J_{c}^{b}:=\rho_{\mathrm{ff}} \Psi_{b, c}^{-\infty}(X, E)
$$

and consider

$$
\begin{equation*}
0 \longrightarrow J_{c}^{b} \longrightarrow A_{c}^{b} \xrightarrow{\pi_{c}^{b}} B_{c}^{b} \longrightarrow 0, \tag{5.21}
\end{equation*}
$$

with $\pi_{c}^{b}$ equal to Melrose' indicial operator $I(\cdot)$. This sequence is certainly larger than the one we have defined, viz. $0 \rightarrow J_{c} \hookrightarrow A_{c} \xrightarrow{\pi_{c}} B_{c} \rightarrow 0$ (indeed the latter corresponds to the subsequence of (5.21) obtained by restricting (5.21) to the subalgebra $\left\{k \in A_{c}^{b}: k-\left.k\right|_{\text {ff }}\right.$ has support in the interior of $\left.\widehat{X} \times_{b} \widehat{X}\right\}$, with ff denoting the front face of the $b$-stretched product).

Let $\tau_{0}^{r}$ be equal to the $b$ - $\operatorname{Trace}: \tau_{0}^{r}:={ }^{b} \mathrm{Tr}$. Observe that $\sigma_{1}$ also defines a 1-cocycle on $B_{c}^{b}$. We can thus consider the relative 0 -cochain $\left(\tau_{0}^{r}, \sigma_{1}\right)$ for the homomorphism $A_{c}^{b} \xrightarrow{I(\cdot)} B_{c}^{b}$; in order to prove that this is a relative 0-cocycle it remains to to show that $b \tau_{0}^{r}\left(k, k^{\prime}\right)=\sigma_{1}\left(I(k), I\left(k^{\prime}\right)\right)$, i.e.

$$
\begin{equation*}
{ }^{b} \operatorname{Tr}\left[k, k^{\prime}\right]=\operatorname{Tr}\left(I(k)\left[\chi^{0}, I\left(k^{\prime}\right)\right]\right) . \tag{5.22}
\end{equation*}
$$

Recall here that Melrose' formula for the $b$-trace of a commutator is

$$
\begin{equation*}
{ }^{b} \operatorname{Tr}\left[k, k^{\prime}\right]=\frac{i}{2 \pi} \int_{\mathbb{R}} \operatorname{Tr}_{\partial X}\left(\partial_{\mu} I(k, \mu) \circ I\left(k^{\prime}, \mu\right)\right) d \mu \tag{5.23}
\end{equation*}
$$

with $\mathbb{C} \ni z \rightarrow I(k, z)$ denoting the indicial family of the operator $k$, i.e. the Fourier transform of the indicial operator $I(k)$.

Inspired by the right hand side of (5.23) we consider an arbitrary compact manifold $Y$, the algebra $B_{c}^{b}(\operatorname{cyl}(Y))$ and the functional

$$
\begin{equation*}
\mathfrak{s}_{1}\left(\ell, \ell^{\prime}\right):=\frac{i}{2 \pi} \int_{\mathbb{R}} \operatorname{Tr}_{Y}\left(\partial_{\mu} \hat{\ell}(\mu) \circ \hat{\ell}^{\prime}(\mu)\right) d \mu \tag{5.24}
\end{equation*}
$$

That this is a cyclic 1-cocycle follows by elementary arguments (it also follows from the Proposition below). Formula (5.24) defines what should be called Melrose' 1-cocycle

Proposition 5.25. Roe's 1-cocycle $\sigma_{1}$ and Melrose 1-cocycle $\mathfrak{s}_{1}$ coincide:

$$
\begin{equation*}
\sigma_{1}\left(\ell, \ell^{\prime}\right):=\operatorname{Tr}\left(\ell\left[\chi^{0}, \ell^{\prime}\right]\right)=\frac{i}{2 \pi} \int_{\mathbb{R}} \operatorname{Tr}_{Y}\left(\partial_{\mu} \hat{\ell}(\mu) \circ \hat{\ell}^{\prime}(\mu)\right) d \mu=: \mathfrak{s}_{1}\left(\ell, \ell^{\prime}\right) . \tag{5.26}
\end{equation*}
$$

Proof. In order to prove (5.26) we shall employ the Hilbert transformation $\mathcal{H}$ : $L^{2}(\mathbb{R}) \rightarrow L^{2}(\mathbb{R}):$

$$
\mathcal{H}(f):=\lim _{\epsilon \downarrow 0} \frac{i}{\pi} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y} d y
$$

The crucial observation is that if we denote by $F: L^{2}(\mathbb{R}) \rightarrow L^{2}(\mathbb{R})$ the Fourier transformation, then

$$
\begin{equation*}
F \circ \mathcal{H} \circ F^{-1}=-F^{-1} \circ \mathcal{H} \circ F=1-2 \chi_{\mathbb{R}}^{0} \tag{5.27}
\end{equation*}
$$

where the right hand side denotes, as usual, the multiplication operator. Using this, we see that $\operatorname{Tr}\left(\ell\left[\chi^{0}, \ell^{\prime}\right]\right)=\frac{1}{2} \int_{\mathbb{R}} \operatorname{Tr}_{Y}\left(\hat{\ell}(\mu)\left[\mathcal{H}, \hat{\ell}^{\prime}\right](\mu)\right) d \mu$. Using the definition of the Hilbert transform $\mathcal{H}$ one checks that $[\mathcal{H}, \hat{\ell}]$ is the integral operator with kernel function equal to $-i / \pi \omega(u, v)$, with $\omega(u, v)=(\hat{\ell}(u)-\hat{\ell}(v)) /(u-v)$. This implies that $\frac{1}{2} \int_{\mathbb{R}} \operatorname{Tr}_{Y}\left(\hat{\ell}(\mu)\left[\mathcal{H}, \hat{\ell}^{\prime}\right](\mu)\right) d \mu$ is equal to $-\frac{i}{2 \pi} \int_{\mathbb{R}} \operatorname{Tr}_{Y}\left(\hat{\ell}(\mu) \circ \partial_{\mu} \hat{\ell}^{\prime}(\mu)\right) d \mu$ which is equal to the right hand side of (5.26) once we integrate by parts.

Proposition 5.25 and Melrose' formula imply at once the relative 0 -cocycle condition for $\left(\tau_{0}^{r}, \sigma_{1}\right)$ : indeed using first Proposition 5.25 and then Melrose' formula we get:

$$
\begin{aligned}
\sigma_{1}\left(I(k), I\left(k^{\prime}\right)\right) & :=\operatorname{Tr}\left(I(k)\left[\chi^{0}, I\left(k^{\prime}\right)\right]\right)=\frac{i}{2 \pi} \int_{\mathbb{R}} \operatorname{Tr}_{\partial X}\left(\partial_{\mu} I(k, \mu) \circ I\left(k^{\prime}, \mu\right)\right) d \mu \\
& ={ }^{b} \operatorname{Tr}\left[k, k^{\prime}\right]=b \tau_{0}^{r}\left(k, k^{\prime}\right) .
\end{aligned}
$$

Thus $I^{*}\left(\sigma_{1}\right)=b \tau_{0}^{r}$ as required.
Conclusions. We have established the following:

- the right hand side of Melrose' formula defines a 1-cocycle $\mathfrak{s}_{1}$ on $B_{c}(\operatorname{cyl}(Y)), Y$ any closed compact manifold;
- Melrose 1-cocycle $\mathfrak{s}_{1}$ equals Roe's 1-cocycle $\sigma_{1}$;
- Melrose' formula itself can be interpreted as a relative 0-cocycle condition for the 0 -cochain $\left(\tau_{0}^{r}, \mathfrak{s}_{1}\right) \equiv\left(\tau_{0}^{r}, \sigma_{1}\right)$.
5.8 Philosophical remarks. We wish to recollect the information obtained in the Sections 5.4, 5.5, 5.6 and start to explain our approach to Atiyah-Patodi-Singer higher index theory.

On a closed compact orientable Riemannian smooth manifold $Y$ let us consider the algebra of smoothing operators $J_{c}(Y):=C^{\infty}(Y \times Y)$. Then the functional $\left.J_{c}(Y) \ni k \rightarrow \int_{Y} k\right|_{\Delta}$ dvol defines a 0-cocycle $\tau_{0}$ on $J_{c}(Y)$; indeed by Lidski's theorem the functional is nothing but the functional analytic trace of the integral operator corresponding to $k$ and we know that the trace vanishes on commutators; in formulae, $b \tau_{0}=0$. The 0 -cocycle $\tau_{0}$ plays a fundamental role in the proof of the Atiyah-Singer index theorem, but we leave this aside for the time being.

Let now $X$ be a smooth orientable manifold with cylindrical ends, obtained from a manifold with boundary $X_{0}$; let $\operatorname{cyl}(\partial X)=\mathbb{R} \times \partial X_{0}$. We have then defined algebras $A_{c}(X), B_{c}(\operatorname{cyl}(\partial X))$ and $J_{c}(X)$ fitting into a short exact sequence $0 \rightarrow J_{c}(X) \rightarrow$ $A_{c}(X) \xrightarrow{\pi_{c}} B_{c}(\operatorname{cyl}(\partial X)) \rightarrow 0$.

Corresponding to the 0 -cocycle $\tau_{0}$ in the closed case we can define two important 0 -cocycles on a manifold with cylindrical ends $X$ :

- We can consider $\tau_{0}$ on $J_{c}(X)=C_{c}^{\infty}(X \times X)$; this is well defined and does define a 0-cocycle.
- Starting with the 0 -cocycle $\tau_{0}$ on $J_{c}(X)$ we define a relative 0 -cocycle $\left(\tau_{0}^{r}, \sigma_{1}\right)$ for $A_{c}(X) \xrightarrow{\pi_{c}} B_{c}(\operatorname{cyl}(\partial X))$. The relative 0-cocycle $\left(\tau_{0}^{r}, \sigma_{1}\right)$ is obtained through the following two fundamental steps.
(1) We define a 0 - cochain $\tau_{0}^{r}$ on $A_{c}(X)$ by replacing the integral with Melrose' regularized integral.
(2) We define a 1 -cocycle $\sigma_{1}$ on $B_{c}(\operatorname{cyl}(\partial X))$ by taking a suspension of $\tau_{0}$ through the linear map $\delta(\ell):=\left[\chi^{0}, \ell\right]$. In other words, $\sigma_{1}\left(\ell_{0}, \ell_{1}\right)$ is obtained from $\tau_{0} \equiv \mathrm{Tr}$ by considering $\left(\ell_{0}, \ell_{1}\right) \rightarrow \tau_{0}\left(\ell_{0}\left[\chi^{0}, \ell_{1}\right]\right) \equiv \tau_{0}\left(\ell_{0} \delta\left(\ell_{1}\right)\right)$.

Definition 5.28. We shall also call Roe's 1-cocycle $\sigma_{1}$ the eta 1-cocycle corresponding to the 0 -cocycle $\tau_{0}$.

In order to justify the wording of this definition we need to show that all this has something to do with the eta invariant and its role in the Atiyah-Patodi-Singer index formula. This will be explained in Section 9.1.
5.9 Cyclic cocycles on graded algebras endowed with commuting derivations. In this subsection we collect some general facts about cyclic cocycles on algebras endowed with commuting derivations. These results will be repeatedly used in the sequel.

The following Lemma is obvious.
Lemma 5.29. Let $A^{0}$ be an algebra and $A^{1}$ a bimodule over $A^{0}$. Consider $\Omega:=$ $A^{0} \oplus A^{1}$ as a linear space; define a multiplication on $\Omega$ by

$$
\alpha \beta=\left(a_{0} b_{0}, a_{0} b_{1}+a_{1} b_{0}\right) \quad \text { for } \quad \alpha=\left(a_{0}, a_{1}\right), \beta=\left(b_{0}, b_{1}\right) \quad \text { in } \quad A^{0} \oplus A^{1} .
$$

Then $\Omega$ is a graded algebra, with the grading on $\Omega$ defined by $\operatorname{deg} a_{i}=i$ for $a_{i} \in$ $A^{i}, i=0,1$. Observe that $\Omega$ is not a ${ }^{*}$-algebra in general.

Definition 5.30. A linear map $\delta: \Omega \rightarrow \Omega$ is called a derivation of degree $k$ if it satisfies:
(1) $\delta(\alpha \beta)=(\delta \alpha) \beta+\alpha(\delta \beta)$ for $\alpha, \beta \in \Omega$;
(2) $\operatorname{deg}(\delta \alpha)=\operatorname{deg} \alpha+k$.

Let $\delta_{\Omega}: \Omega \rightarrow \Omega$ be a derivation on $\Omega$. Suppose that $\delta_{\Omega}$ is of degree 0 and denote by $\delta: A^{0} \rightarrow A^{0}$ and $\delta^{\prime}: A^{1} \rightarrow A^{1}$ the restrictions $\left.\delta_{\Omega}\right|_{A^{0}},\left.\delta_{\Omega}\right|_{A^{1}}$ respectively. Then the derivation property of $\delta_{\Omega}$ is equivalent to the following three properties: (i) $\delta\left(a_{0} b_{0}\right)=$ $\left(\delta a_{0}\right) b_{0}+a_{0}\left(\delta b_{0}\right)$; (ii) $\delta^{\prime}\left(a_{0} b_{1}\right)=\left(\delta a_{0}\right) b_{1}+a_{0}\left(\delta^{\prime} b_{1}\right)$; (iii) $\delta^{\prime}\left(a_{1} b_{0}\right)=\left(\delta^{\prime} a_{1}\right) b_{0}+a_{1}\left(\delta b_{0}\right)$, for $a_{i}, b_{i} \in A^{i}$. We also observe that giving a derivation $\delta_{\Omega}$ of degree 1 is equivalent to giving a linear map $\delta: A^{0} \rightarrow A^{1}$ with $\delta\left(a_{0} b_{0}\right)=\left(\delta a_{0}\right) b_{0}+a_{0}\left(\delta b_{0}\right)$ in such a way that $\delta_{\Omega} a_{0}=\delta a_{0}$ and $\delta_{\Omega} a_{1}=0$ for $a_{i} \in A^{i}$.

Finally, let $\omega: A^{1} \rightarrow \mathbb{C}$ be a linear map such that

$$
\begin{equation*}
\omega\left(a_{0} a_{1}\right)=\omega\left(a_{1} a_{0}\right) \quad \text { for } \quad a_{i} \in A^{i} . \tag{5.31}
\end{equation*}
$$

We shall call such a linear map a bimodule trace map on $A^{1}$. The following Lemma is obvious

Lemma 5.32. A bimodule trace map $\omega$ on $A^{1}$ extends to a trace map $\varpi: \Omega \rightarrow \mathbb{C}$ such as $\left.\varpi\right|_{A^{0}}=0$ and $\left.\varpi\right|_{A^{1}}=\omega$.

Let $\delta_{i}, i=1, \ldots, k$, be derivations on $\Omega:=A^{0} \oplus A^{1}$ and let $\varpi$ be a trace map on $\Omega$. We consider the following $(k+1)$-multilinear map on $\Omega$ :

$$
\begin{equation*}
\tau\left(a_{0}, \ldots, a_{k}\right):=\frac{1}{k!} \sum_{\alpha \in \mathfrak{S}_{k}} \operatorname{sign}(\alpha) \varpi\left(a_{0} \delta_{\alpha(1)} a_{1} \ldots \delta_{\alpha(k)} a_{k}\right) \quad \text { for } \quad a_{i} \in \Omega \tag{5.33}
\end{equation*}
$$

In order to get a nontrivial map, we may assume that one of $\delta_{i}$ is of degree 1 and the others are of degree 0 . Here $\mathfrak{S}_{k}$ denotes the symmetric group of order $k$ and $\operatorname{sign}(\alpha)$ is the signature of permutation $\alpha \in \mathfrak{S}_{k}$.

Proposition 5.34. Assume that
(1) the derivations $\delta_{i}$ are pairwise commuting, i.e. $\delta_{i} \delta_{j}=\delta_{j} \delta_{i}$ for $1 \leq i, j \leq k$;
(2) $\varpi\left(\delta_{i} a\right)=0$ for $a \in \Omega$ and $1 \leq i \leq k$.

Then $\tau$ defined as in (5.33) gives rise to a cyclic $k$-cocycle on $\Omega$.
Proof. First we verify the cyclic condition. The second assumption and the derivation property imply that

$$
\begin{aligned}
\tau\left(a_{k}, a_{0}, \ldots, a_{k-1}\right)= & \frac{1}{k!} \sum_{\alpha \in \mathfrak{S}_{k}} \operatorname{sign}(\alpha) \varpi\left(a_{k} \delta_{\alpha(1)} a_{0} \ldots \delta_{\alpha(k)} a_{k-1}\right) \\
= & \frac{1}{k!} \sum_{\alpha \in \mathfrak{S}_{k}} \operatorname{sign}(\alpha)\left\{\varpi\left(a_{k} \delta_{\alpha(1)} a_{0} \ldots \delta_{\alpha(k)} a_{k-1}\right)\right. \\
& \left.-\varpi\left(\delta_{\alpha(1)}\left(a_{k} a_{0} \delta_{\alpha(2)} a_{1} \ldots \delta_{\alpha(k)} a_{k-1}\right)\right)\right\} \\
= & -\frac{1}{k!} \sum_{\alpha \in \mathfrak{S}_{k}} \operatorname{sign}(\alpha) \varpi\left(\left(\delta_{\alpha(1)} a_{k}\right) a_{0} \delta_{\alpha(2)} a_{1} \ldots \delta_{\alpha(k)} a_{k-1}\right) \\
& -\frac{1}{k!} \sum_{\alpha \in \mathfrak{S}_{k}} \operatorname{sign}(\alpha) \sum_{i=1}^{k-1} \varpi\left(a_{k} a_{0} \delta_{\alpha(2)} a_{1} \ldots \delta_{\alpha(1)} \delta_{\alpha(i+1)} a_{i} \ldots \delta_{\alpha(k)} a_{k-1}\right) .
\end{aligned}
$$

The second summand in the last term vanishes. In fact the signatures are opposite to each other for $\alpha$ and $\alpha \circ(1, i+1)$; thus the values cancel out due to assumption (1). Observing that the signature of the cyclic permutation $(1,2, \ldots, k)$ is equal to $(-1)^{k-1}$, the trace property implies that
$\tau\left(a_{k}, a_{0}, \ldots, a_{k-1}\right)=\frac{(-1)^{k}}{k!} \sum_{\alpha \in \mathfrak{S}_{k}} \varpi\left(a_{0} \delta_{\alpha(1)} a_{1} \ldots \delta_{\alpha(k)} a_{k}\right)=(-1)^{k} \tau\left(a_{0}, a_{1}, \ldots, a_{k}\right)$.
Second we prove the cocycle condition. Due to the derivation and trace properties again we obtain

$$
\begin{aligned}
b \tau\left(a_{0}, \ldots, a_{k+1}\right)= & \sum_{i=0}^{k}(-1)^{i} \tau\left(a_{0}, \ldots, a_{i} a_{i+1}, \ldots, a_{k+1}\right)+(-1)^{k+1} \tau\left(a_{k+1} a_{0}, a_{1}, \ldots, a_{k}\right) \\
= & \frac{(-1)^{k}}{k!} \sum_{\alpha \in \mathfrak{S}_{k}} \operatorname{sign}(\alpha)\left\{\varpi\left(a_{0} \delta_{\alpha(1)} a_{1} \ldots\left(\delta_{\alpha(k)} a_{k}\right) a_{k+1}\right)\right. \\
& \left.-\varpi\left(a_{k+1} a_{0} \delta_{\alpha(1)} a_{1} \ldots \delta_{\alpha(k)} a_{k}\right)\right\}
\end{aligned}
$$

and the last term is zero. This completes the proof.
5.10 The Godbillon-Vey cyclic 2-cocycle $\boldsymbol{\tau}_{G V} . \quad$ Let $(Y, \mathcal{F}), Y=\tilde{N} \times{ }_{\Gamma} T$, be a compact foliated bundle without boundary. We take directly $T=S^{1}$. Let $E \rightarrow Y$ a hermitian complex vector bundle on $Y$. Let $G$ be the holonomy groupoid associated to $Y$, namely $G=(\tilde{N} \times \tilde{N} \times T) / \Gamma$. Consider again the convolution algebra $\Psi_{c}^{-\infty}(G, E):=C_{c}^{\infty}\left(G,\left(s^{*} E\right)^{*} \otimes r^{*} E\right)$ of equivariant smoothing families with $\Gamma$-compact support. On $\Psi_{c}^{-\infty}(G, E)$ there exists a remarkable 2-cocycle, denoted by $\tau_{G V}$, and known as the Godbillon-Vey cyclic cocycle. It was defined by Moriyoshi and Natsume in [MoN96], following seminal work of Connes. Here we shall simply recall the very basic facts leading to the definition of $\tau_{G V}$.

Recall from Section 2.5, the modular function $\psi$ on $\tilde{N} \times T$ defined by $\tilde{\omega} \wedge d \theta=\psi \tilde{\Omega}$, where $\tilde{\omega}$ and $\tilde{\Omega}$ denote $\Gamma$-invariant volume forms on $\tilde{N}$ and $\tilde{N} \times T$ respectively.

There is a well defined derivation $\delta_{2}$ on the algebra $\Psi_{c}^{-\infty}(G, E)$ :

$$
\begin{equation*}
\delta_{2}(P)=[\phi, P] \quad \text { with } \quad \phi=\log \psi . \tag{5.35}
\end{equation*}
$$

We observe here that $\phi$ is neither $\Gamma$-invariant nor compactly supported in general. It is even possible that $\phi$ is unbounded. Recall next the bundle $\widehat{E}^{\prime}$ on $\tilde{N} \times T$ introduced in [MoN96]: this is the same vector bundle as $\widehat{E}$ but equipped with a new $\Gamma$-equivariant structure. See [MoN96]. There is a natural bijective correspondence between $\Psi_{c}^{-\infty}(G, E)$ and $\Psi_{c}^{-\infty}\left(G, E^{\prime}\right)$. Take $\Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right)$ as in Section 3.3. Using the above identification we consider $\Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right)$ as a bimodule over $\Psi_{c}^{-\infty}(G, E)$. Let $\dot{\phi}$ denote the partial derivative of $\phi$ in the direction of $S^{1}$. There is a well-defined bimodule derivation $\delta_{1}: \Psi_{c}^{-\infty}(G, E) \rightarrow \Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right)$ :

$$
\begin{equation*}
\delta_{1}(P)=[\dot{\phi}, P] \quad \text { with } \quad \phi=\log \psi . \tag{5.36}
\end{equation*}
$$

There is also a linear map $\delta_{2}^{\prime}: \Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right) \rightarrow \Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right)$ defined in a similar way to (5.35). Then one can verify that

$$
\begin{equation*}
\delta_{1}\left(\delta_{2}(P)\right)=\delta_{2}^{\prime}\left(\delta_{1}(P)\right) . \tag{5.37}
\end{equation*}
$$

Recall, finally, that there is a weight $\omega_{\Gamma}$ defined on the algebra $\Psi_{c}^{-\infty}(G ; E)$ :

$$
\begin{equation*}
\omega_{\Gamma}(k)=\int_{Y(\Gamma)} \operatorname{Tr}_{(y, \theta)} k(y, y, \theta) d y d \theta \tag{5.38}
\end{equation*}
$$

In this formula, $Y(\Gamma)$ is a fundamental domain in $\tilde{N} \times T$ for the diagonal free action of $\Gamma$ on $\tilde{N} \times T$; the kernel $k$ is restricted to $\Delta_{\tilde{N}} \times T \subset \tilde{N} \times \tilde{N} \times T$ where $\Delta_{\tilde{N}}$ is the diagonal set in $\tilde{N} \times \tilde{N}$ and $\Delta_{\tilde{N}} \times T \cong \tilde{N} \times T$; and $\operatorname{Tr}_{(y, \theta)}$ denotes the trace map on $\operatorname{End}\left(\widehat{E}_{(y, \theta)}\right)$ (If the measure on $T$ is $\Gamma$-invariant, then this weight is a trace; however, we do not make this assumption here).

We shall be interested in the linear functional ${ }^{8}$ defined on the bimodule $\Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right)$ by the analogue of (5.38). Following what has been explained in

[^2]the previous subsection, we call this linear functional on $\Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right)$ a bimodule trace; this is justified by the next fundamental equation, which is proved in [MoN96]:
\[

$$
\begin{equation*}
k \in \Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right), k^{\prime} \in \Psi_{c}^{-\infty}(G ; E) \equiv \Psi_{c}^{-\infty}\left(G ; E^{\prime}\right) \Rightarrow \omega_{\Gamma}\left(k k^{\prime}\right)=\omega_{\Gamma}\left(k^{\prime} k\right) . \tag{5.39}
\end{equation*}
$$

\]

It is also important to recall that the bimodule trace $\omega_{\Gamma}$ on $\Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right)$ satisfies the following Stokes formula:

$$
\begin{equation*}
k \in \Psi_{c}^{-\infty}(G ; E) \Rightarrow \omega_{\Gamma}\left(\delta_{1}(k)\right)=0 \quad \text { and } \quad k \in \Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right) \Rightarrow \omega_{\Gamma}\left(\delta_{2}^{\prime}(k)\right)=0 . \tag{5.40}
\end{equation*}
$$

Definition 5.41. When $\operatorname{dim} T=1$, the Godbillon-Vey cyclic 2-cocycle on $\Psi_{c}^{-\infty}$ $(G ; E) \equiv C_{c}^{\infty}\left(G,\left(s^{*} E\right)^{*} \otimes r^{*} E\right)$ is defined to be

$$
\begin{align*}
\tau_{G V}\left(a_{0}, a_{1}, a_{2}\right) & =\frac{1}{2!} \sum_{\alpha \in \mathfrak{S}_{2}} \operatorname{sign}(\alpha) \omega_{\Gamma}\left(a_{0} \delta_{\alpha(1)} a_{1} \delta_{\alpha(2)} a_{2}\right) \\
& =\frac{1}{2}\left\{\omega_{\Gamma}\left(a_{0} \delta_{1} a_{1} \delta_{2} a_{2}\right)-\omega_{\Gamma}\left(a_{0} \delta_{2} a_{1} \delta_{1} a_{2}\right)\right\} . \tag{5.42}
\end{align*}
$$

We remark that the Godbillon-Vey cocycle in [MoN96] is equal to twice the above cocycle.
Proposition 5.43. The 3-functional $\tau_{G V}$ does satisfy

$$
\begin{equation*}
b \tau_{G V}=0, \quad \tau\left(a_{0}, a_{1}, a_{2}\right)=\tau\left(a_{1}, a_{2}, a_{0}\right), \forall a_{j} \in \Psi_{c}^{-\infty}(G ; E) . \tag{5.44}
\end{equation*}
$$

Proof. This is certainly proved in [MoN96]. We give a proof here by using general results of the previous subsection; this will serve as a guide for the more complicated situation we will consider later. Recall the definition of $\tau_{G V}: \tau_{G V}\left(a_{0}, a_{1}, a_{2}\right):=$ $\frac{1}{2}\left\{\omega_{\Gamma}\left(a_{0} \delta_{1} a_{1} \delta_{2} a_{2}\right)-\omega_{\Gamma}\left(a_{0} \delta_{2} a_{1} \delta_{1} a_{2}\right)\right\}$ where $\delta_{1} a=[\dot{\phi}, a]$ and $\delta_{2} a=[\phi, a]$. Let $A^{0}$ be the algebra $\Psi_{c}^{-\infty}(G ; E)$ and $A^{1}$ the $A^{0}$-bimodule $\Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right)$ introduced above. Proceeding as in Section 5.9, we construct a graded algebra $\Omega$ out of $A^{0}$ and $A^{1}$ as in Lemma 5.29. We denote this algebra by $\Omega(G) ;$ thus $\Omega(G):=\Psi_{c}^{-\infty}(G ; E) \oplus$ $\Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right)$. Then, according to the explanations given in Section 5.9, there exist extensions of our derivations to

$$
\begin{equation*}
\delta_{j}: \Omega(G) \rightarrow \Omega(G), j=1,2 \quad \text { with } \quad \delta_{1} a=[\dot{\phi}, a], \quad \delta_{2} a=[\phi, a] \tag{5.45}
\end{equation*}
$$

with $\delta_{1}$ a derivation of degree 1 and $\delta_{2}$ of degree 0 . Here we employed the same notation for these extensions. On the other hand, we know that the functional $\omega_{\Gamma}$ : $\Psi_{c}^{-\infty}\left(G ; E, E^{\prime}\right) \rightarrow \mathbb{C}$ defined by (5.38) induces a bimodule trace map on $A^{1}$ due to (5.39). Thus Lemma 5.32 implies that there exists a trace map $\tau_{\Gamma}: \Omega(G) \rightarrow \mathbb{C}$ with $\tau_{\Gamma}(a)=\omega_{\Gamma}(a)$ for $a \in A^{1}$ and $\tau_{\Gamma}(a)=0$ for $a \in A^{0}$. Now the relation (5.37) shows that the derivations $\delta_{1}, \delta_{2}$ in (5.45) commute with each other, whereas (5.40) implies that $\tau_{\Gamma}\left(\delta_{j} a\right)=0$ for $a \in \Omega(G)$ and $j=1,2$. Thus, directly from Proposition 5.34, we obtain a cyclic 2-cocycle

$$
\tau_{2}\left(a_{0}, a_{1}, a_{2}\right)=\frac{1}{2}\left\{\tau_{\Gamma}\left(a_{0} \delta_{1} a_{1} \delta_{2} a_{2}\right)-\tau_{\Gamma}\left(a_{0} \delta_{2} a_{1} \delta_{1} a_{2}\right)\right\} .
$$

Thus $\tau_{G V}$ is also a cyclic cocycle on $A^{0} \equiv \Psi_{c}^{-\infty}(G ; E)$ since it is nothing but the restriction of $\tau_{2}$ to the subalgebra $A^{0} \subset \Omega(G)$. Thus the proof is completed.

We now go back to a foliated bundle $(X, \mathcal{F})$ with cylindrical ends, where $X=$ $\tilde{V} \times_{\Gamma} T$ as in Section 2. We consider the small subalgebras introduced in Section 5.2. Note that the weight $\omega_{\Gamma}$ is well defined on $J_{c}(X, \mathcal{F})$; thus the 2 -cocycle $\tau_{G V}$ can be defined on $J_{c}(X, \mathcal{F})$, which we call the Godbillon-Vey cyclic cocycle for $(X, \mathcal{F})$.
5.11 The eta 3 -cocycle $\sigma_{G V}$ corresponding to $\tau_{G V}$. Now we apply the general philosophy explained at the end of the previous Section. Let $\chi^{0}$ be the usual characteristic function of $(-\infty, 0] \times \partial X_{0}$ in $\operatorname{cyl}(\partial X)=\mathbb{R} \times \partial X_{0}$. Write $\operatorname{cyl}(\partial X)=$ $(\mathbb{R} \times \partial \tilde{M}) \times_{\Gamma} T$ with $\Gamma$ acting trivially on the $\mathbb{R}$ factor. Let cyl $(\Gamma)$ be a fundamental domain for the action of $\Gamma$ on $(\mathbb{R} \times \partial \tilde{M}) \times T$; finally, let $\omega_{\Gamma}^{\text {cyl }}$ be the corresponding trace map on the bimodule defined similarly to (5.38). Recall $\delta(\ell):=\left[\chi^{0}, \ell\right]$; recall that we passed from the 0-cocycle $\tau_{0} \equiv \operatorname{Tr}$ to the 1-eta cocycle on the cylindrical algebra $B_{c}$ by considering $\left(\ell_{0}, \ell_{1}\right) \rightarrow \tau_{0}\left(\ell_{0} \delta\left(\ell_{1}\right)\right)$. We referred to this operation as a suspension.

We are thus led to suspend Definition 5.41, thus defining the following 4-linear functional on the algebra $B_{c}$.

Definition 5.46. The eta functional $\sigma_{G V}$ associated to the Godbillon-Vey 2-cocycle $\tau_{G V}$ is given by the 4 -linear functional on $B_{c}$

$$
\begin{equation*}
\sigma_{G V}\left(\ell_{0}, \ell_{1}, \ell_{2}, \ell_{3}\right):=\frac{1}{3!} \sum_{\alpha \in \mathfrak{S}_{3}} \operatorname{sign}(\alpha) \omega_{\Gamma}^{\mathrm{cyl}}\left(\ell_{0} \delta_{\alpha(1)} \ell_{1} \delta_{\alpha(2)} \ell_{2} \delta_{\alpha(3)} \ell_{3}\right) \tag{5.47}
\end{equation*}
$$

with

$$
\begin{equation*}
\delta_{3} \ell:=\left[\chi^{0}, \ell\right], \quad \delta_{2} \ell:=\left[\phi_{\partial}, \ell\right] \quad \text { and } \quad \delta_{1} \ell:=\left[\dot{\phi}_{\partial}, \ell\right] \tag{5.48}
\end{equation*}
$$

and $\phi_{\partial}$ equal to the restriction of the modular function to the boundary, extended in a constant way along the cylinder. We shall prove that this is a cyclic 3 -cocycle for the algebra $B_{c}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. More generally, formula (5.47) defines the GodbillonVey eta 3-cocycle on $B_{c}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ with $Y=\tilde{N} \times_{\Gamma} T$ any closed foliated $T$-bundle, not necessarily arising as a boundary. Here, as usual, we don't write the bundle $E$ in the notation. In this case $\delta_{2} \ell:=\left[\phi_{Y}, \ell\right]$ and $\delta_{1} \ell:=\left[\dot{\phi}_{Y}, \ell\right]$ with $\phi_{Y}$ the logarithm of a modular function on $Y$ extended in a constant way along the cylinder.

We must justify the well-posedness of this definition. To this end, remark that each sum will contain an element of type $\delta_{3}\left(\ell_{j}\right):=\left[\chi^{0}, \ell_{j}\right]$. This is a kernel of $\Gamma$ compact support (we have already justified this claim in Sublemma 4.10) which is, of course, not translation invariant. Since the other three operators appearing in the composition $\left(\ell_{0} \delta_{\alpha(1)}\left(\ell_{1}\right) \delta_{\alpha(2)}\left(\ell_{2}\right) \delta_{\alpha(3)}\left(\ell_{3}\right)\right)$ are $(\mathbb{R} \times \Gamma)$-equivariant and of $(\mathbb{R} \times \Gamma)$-compact support, we can conclude easily that each term appearing in the definition of $\sigma_{G V},\left(\ell_{0} \delta_{\alpha(1)}\left(\ell_{1}\right) \delta_{\alpha(2)}\left(\ell_{2}\right) \delta_{\alpha(3)}\left(\ell_{3}\right)\right)$, is in fact of $\Gamma$-compact support. Indeed, recall that a kernel that is $\Gamma$-equivariant and of $\Gamma$-compact support, such
as $\delta_{3}\left(\ell_{j}\right)=\left[\chi^{0}, \ell_{j}\right]$ above, can be considered as a compactly supported function on the holonomy groupoid $G_{\text {cyl }}$ for $\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. On the other hand, a kernel that is $(\mathbb{R} \times \Gamma)$-equivariant and of $(\mathbb{R} \times \Gamma)$-compact support corresponds to a compactly supported function on $G_{\text {cyl }} / \mathbb{R}_{\Delta}$, which admits a $\mathbb{R}$-compact support once lifted to $G_{\mathrm{cyl}}$; see Proposition 4.3. We then take the convolution product of these kernels. A simple argument on support implies that the resulting kernel corresponds to a compactly supported function on $G_{\mathrm{cyl}}$ and hence the kernel itself is of $\Gamma$-compact support on $(\mathbb{R} \times \partial \tilde{M}) \times(\mathbb{R} \times \partial \tilde{M}) \times T$.

Summarizing, $\omega_{\Gamma}^{\text {cyl }}\left(\ell_{0} \delta_{\alpha(1)}\left(\ell_{1}\right) \delta_{\alpha(2)}\left(\ell_{2}\right) \delta_{\alpha(3)}\left(\ell_{3}\right)\right)$ is finite and the definition of $\sigma_{G V}$ is well posed. In fact, we can define, as we did for $\sigma_{1}$, the 3-cochain $\sigma_{G V}^{\lambda}$ by employing the characteristic function $\chi^{\lambda}$. However, one checks as in Proposition 5.10 that the value of $\sigma_{G V}^{\lambda}$ does not depend on $\lambda$.

Proposition 5.49. Let $Y=\tilde{N} \times_{\Gamma} T$ be an arbitrary foliated $T$-bundle without boundary. The eta functional $\sigma_{G V}$ on $B_{c}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ is cyclic and is a cocycle: $b \sigma_{G V}=0$; it thus defines a cyclic 3-cocycle on the algebra $B_{c}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$.

Proof. We wish to apply Proposition 5.34 as we did in the proof of the 2-cyclic-cocycle property for $\tau_{G V}$, see Proposition 5.43. However, we need to deal with a small complication, having to do with the fact that $\chi^{0}$ is not smooth and that $\left[\chi^{0}, \ell\right]$ is no longer translation invariant. Recall the groupoid $G_{\text {cyl }}:=\operatorname{cyl}(\tilde{N}) \times \operatorname{cyl}(\tilde{N}) \times T / \Gamma$ which is nothing but $G_{Y} \times \mathbb{R} \times \mathbb{R}$ with $G_{Y}$ the holonomy groupoid for $Y=\tilde{N} \times{ }_{\Gamma} T$. Define
$L_{c}^{\infty}\left(G_{\mathrm{cyl}}\right)=\left\{k: G_{\mathrm{cyl}} \rightarrow \mathbb{C} \mid k\right.$ is measurable, essentially bounded and of $\Gamma$-compact support $\}$.
More generally, let $E$ be a vector bundle on $Y$ with lift $\widehat{E}$ on $\tilde{N} \times T$; we pull back $E$ to $\operatorname{cyl}(Y)$ through the obvious projection obtaining a vector bundle $E_{\text {cyl }}$. We can then consider in a natural way $L_{c}^{\infty}\left(G_{\mathrm{cyl}} ; E_{\mathrm{cyl}}\right)$ and $\left.L_{c}^{\infty}\left(G_{\mathrm{cyl}}\right) ; E_{\mathrm{cyl}}, E_{\mathrm{cyl}}^{\prime}\right)$. We omit the obvious details. Recall also

$$
B_{c}\left(G_{\mathrm{cyl}}\right) \equiv B_{c}:=\left\{\ell: G_{\mathrm{cyl}} \rightarrow \mathbb{C} \mid \ell \text { is smooth, } \mathbb{R} \times \Gamma \text { - invariant and of } \mathbb{R} \times \Gamma \text { - support }\right\} .
$$

We also have $B_{c}\left(G_{\mathrm{cyl}} ; E\right)$ (this is the algebra on which $\sigma_{G V}$ is defined) and $B_{c}\left(G_{\mathrm{cyl}} ; E, E^{\prime}\right)$. We set now

$$
\begin{aligned}
& A^{0}:=B_{c}\left(G_{\mathrm{cyl}} ; E\right) \oplus L_{c}^{\infty}\left(G_{\mathrm{cyl}} ; E_{\mathrm{cyl}}\right) \\
& A^{1}:=B_{c}\left(G_{\mathrm{cyl}} ; E, E^{\prime}\right) \oplus L_{c}^{\infty}\left(G_{\mathrm{cyl}} ; E_{\mathrm{cyl}}, E_{\mathrm{cyl}}^{\prime}\right)
\end{aligned}
$$

First, observe here that $A^{0}$ and $A^{1}$ are naturally considered as subspaces in $\operatorname{End}(\mathcal{H})$ and $\operatorname{Hom}\left(\mathcal{H}, \mathcal{H}^{\prime}\right)$ respectively, where we recall that $\mathcal{H}=\left(\mathcal{H}_{\theta}\right)_{\theta \in T}, \mathcal{H}_{\theta}=L^{2}(\operatorname{cyl}(\tilde{N}) \times$ $\left.\{\theta\}, E_{\mathrm{cyl}, \theta}\right)$ and similarly for $\mathcal{H}^{\prime}$; indeed, each summand of $A^{0}$, for example, is in $\operatorname{End}(\mathcal{H})$ and the direct sum holds because of the support conditions. Next we observe that $A^{0}$ is in fact as a subalgebra of $\operatorname{End}(\mathcal{H})$, since the product of $k \in B_{c}\left(G_{\mathrm{cyl}} ; E\right)$ and $k^{\prime} \in L_{c}^{\infty}\left(G_{\mathrm{cyl}} ; E_{\mathrm{cyl}}\right)$ is an element in $L_{c}^{\infty}\left(G_{\mathrm{cyl}} ; E_{\mathrm{cyl}}\right)$. Moreover, for the same
reason, $A^{1}$ has a bimodule structure over $A^{0}$, inherited from the one of $\operatorname{Hom}\left(\mathcal{H}, \mathcal{H}^{\prime}\right)$ over $\operatorname{End}(\mathcal{H})$. The direct sum $\Omega:=A^{0} \oplus A^{1}$, with the product defined in Lemma 5.29, is the graded algebra to which we want to apply Proposition 5.34.

We can define three derivations $\delta_{1}, \delta_{2}$ and $\delta_{3}$ as in (5.48). We consider $\delta_{1}$ as a derivation of degree 1 , mapping $A^{0}$ to $A^{1}$ and vanishing on $A^{1}$; we consider $\delta_{2}$ and $\delta_{3}$ as derivations of degree 0 , preserving $A^{0}$ and $A^{1}$ respectively. Notice that since $\phi_{\partial}$ and $\dot{\phi}_{\partial}$ are translation invariant on the cylinder, $\delta_{1}$ and $\delta_{2}$ are diagonal with respect to the direct sum decomposition of $A^{0}$ and $A^{1}$. As far as $\delta_{3}$ is concerned, we remark that using (5.9) we see that $\delta_{3}$ maps $B_{c} \oplus L_{c}^{\infty}$ into $L_{c}^{\infty}$ both on $A^{0}$ and $A^{1}$. It is clear that these three derivations are pairwise commuting. Finally, we define a bimodule trace map on $A^{1}$ by employing the bimodule trace $\omega_{\Gamma}^{\text {cyl }}$ appearing in Definition 5.46; this is well defined on $L_{c}^{\infty}\left(G_{\mathrm{cyl}} ; E, E^{\prime}\right)$ since elements in this space have $\Gamma$-compact support. We can then define $\omega: A^{1} \rightarrow \mathbb{C}$ by $\omega(\alpha)=\omega_{\Gamma}^{\text {cyl }}(k)$ if $\alpha=(\ell, k) \in A^{1} \equiv B_{c}\left(G_{\mathrm{cyl}} ; E, E^{\prime}\right) \oplus L_{c}^{\infty}\left(G_{\mathrm{cyl}} ; E_{\text {cyl }}, E_{\text {cyl }}^{\prime}\right)$. We know that $\omega\left(\delta_{j} \alpha\right)=0$ if $j=1,2$. On the other hand, always for $\alpha=(\ell, k)=\ell+k \in A^{1}$, we have

$$
\begin{aligned}
\omega\left(\delta_{3} \alpha\right)= & \omega\left(\left[\chi^{0}, \ell+k\right]\right)=\omega_{\Gamma}^{\mathrm{cyl}}\left(\left[\chi^{0}, \ell+k\right]\right) \\
= & \int_{\operatorname{cyl}(\Gamma)} \operatorname{Tr}_{(y, s, \theta)}\left(\chi^{0}(s)(\ell(y, y, 0, \theta)+k(y, y, s, s, \theta))\right. \\
& \left.\quad-(\ell(y, y, 0, \theta)+k(y, y, s, s, \theta)) \chi^{0}(s)\right) d y d \theta
\end{aligned}
$$

and the last term is zero. Thus, we also have Stokes formula for the derivation $\delta_{3}$. Now we define $\tau_{0}$ from $\omega$ as in Lemma (5.32) so that all the conditions in the hypothesis of Proposition 5.34 are satisfied. Finally, we point out that $B_{c}$ is a subalgebra of $A^{0}$ : proceeding exactly as in the proof of Proposition 5.34 we can now check that $\sigma_{G V}$ is indeed a cyclic 3-cocycle on $B_{c}$.
5.12 The relative Godbillon-Vey cyclic cocycle ( $\tau_{G V}^{r}, \sigma_{G V}$ ). We now apply our strategy as in Section 5.8. Thus starting with the cyclic cocycle $\tau_{G V}$ on $J_{c}(X, \mathcal{F})$ we first consider the 3 -linear functional on $A_{c}(X, \mathcal{F})$ given by

$$
\psi_{G V}^{r}\left(k_{0}, k_{1}, k_{2}\right):=\frac{1}{2!} \sum_{\alpha \in \mathfrak{S}_{2}} \operatorname{sign}(\alpha) \omega_{\Gamma}^{r}\left(a_{0} \delta_{\alpha(1)} a_{1} \delta_{\alpha(2)} a_{2}\right)
$$

with $\omega_{\Gamma}^{r}$ the regularized weight corresponding to $\omega_{\Gamma}$. The regularized weight $\omega_{\Gamma}^{r}$ is defined as follows. Let us consider $X=\tilde{V} \times_{\Gamma} T$, the quotient of $\tilde{V} \times T$ with free $\Gamma$-action, and denote $X(\Gamma)$ a fundamental domain for this $\Gamma$-covering. We can take $X(\Gamma)$ as $F \times T$, with $F$ a fundamental domain for the Galois covering $\Gamma \rightarrow \tilde{V} \rightarrow$ $V, \tilde{V}=\tilde{M} \cup_{\partial \tilde{M}}((-\infty, 0] \times \partial \tilde{M})$ and $V:=M \cup_{\partial M}((-\infty, 0] \times \partial M)$. See Section 2.3. Thus $F$ has a cylindrical end, with cross section $F_{\partial}$ Then, using the usual notations, we set

$$
\begin{equation*}
\omega_{\Gamma}^{r}(k):=\lim _{\lambda \rightarrow+\infty}\left(\int_{F_{\lambda} \times T} k(x, x, \theta) d x d \theta-\lambda \int_{F_{\partial} \times T} \ell(y, y, 0, \theta) d y d \theta\right) \tag{5.50}
\end{equation*}
$$

with $\pi_{c}(k)=\ell$. Here we have used the translation invariance of $\ell$ in order to write $\ell$ as a function of $\left(y, y^{\prime}, s, \theta\right)$ with $s \in \mathbb{R}$.

Notice that, as in Section 5.5 the function $\varphi(\lambda):=\int_{F_{\lambda} \times T} k(x, x, \theta) d x d \theta-$ $\lambda \int_{F_{\partial} \times T} \ell(y, y, 0, \theta) d y d \theta$ becomes constant for $\lambda \gg 0$.

Next we consider the cyclic cochain associated to $\psi_{G V}^{r}$ :

$$
\begin{equation*}
\tau_{G V}^{r}\left(k_{0}, k_{1}, k_{2}\right):=\frac{1}{3}\left(\psi_{G V}^{r}\left(k_{0}, k_{1}, k_{2}\right)+\psi_{G V}^{r}\left(k_{1}, k_{2}, k_{0}\right)+\psi_{G V}^{r}\left(k_{2}, k_{0}, k_{1}\right)\right) . \tag{5.51}
\end{equation*}
$$

The next Proposition is crucial:
Proposition 5.52. The pair of cyclic cochains $\left(\tau_{G V}^{r}, \sigma_{G V}\right) \in C_{\lambda}^{2}\left(A_{c}, B_{c}\right)$ is a relative cocycle: thus

$$
\begin{equation*}
b \sigma_{G V}=0 \quad \text { and } \quad b \tau_{G V}^{r}=\left(\pi_{c}\right)^{*} \sigma_{G V} \tag{5.53}
\end{equation*}
$$

We have proved the first equation in the general (non-bounding) case in Proposition 5.49. Notice that in the present context, with $Y=\partial X$, we can prove the first equation using the second; indeed $\pi_{c}$ is surjective and thus the induced cochain map $\pi_{c}^{*}$ is injective; thus $b \tau_{G V}^{r}=\left(\pi_{c}\right)^{*} \sigma_{G V}$ implies that $b\left(\pi_{c}\right)^{*} \sigma_{G V}=0$ so that $b \sigma_{G V}=0$.

We shall present a proof of the second equation of Proposition 5.52 in Section 10.
For later use we also state and prove the analogue of formula (5.17):
Proposition 5.54. Let $t: A^{*}(X, \mathcal{F}) \rightarrow C^{*}(X, \mathcal{F})$ be the section introduced in Section 4.4. If $k \in A_{c} \subset A^{*}(X, \mathcal{F})$ then $t(k)$ has finite weight. Moreover, for the regularized weight $\omega_{\Gamma}^{r}: A_{c} \rightarrow \mathbb{C}$ we have

$$
\begin{equation*}
\omega_{\Gamma}^{r}=\omega_{\Gamma} \circ t \tag{5.55}
\end{equation*}
$$

Proof. The proof is virtually identical to the one establishing (5.17). Write $k=$ $a+\chi^{\lambda} \ell \chi^{\lambda}$ with $a \in J_{c}$ and $\ell \in B_{c}$. Remark that the support of $\chi^{\lambda}-\chi^{0}$ is compact. Thus $t(k)=k-\chi^{0} \ell \chi^{0}=a+\chi^{\lambda} \ell \chi^{\lambda}-\chi^{0} \ell \chi^{0}$ has certainly finite weight, given that it is of $\Gamma$-compact support. Thus,

$$
\omega_{\Gamma}(t(k))=\int_{F \times T} a(x, x, \theta) d x d \theta-\int_{F_{\lambda} \times T \backslash F_{0} \times T} \ell(y, y, 0, \theta) d y d t d \theta=\omega_{\Gamma}^{r}(k)
$$

for a sufficiently large $\lambda$. This completes the proof.
5.13 Eta $(\boldsymbol{n}+\mathbf{1})$-cocycles. The goal of this subsection is to generalize the results stated above for the Godbillon-Vey cyclic 2-cocycle $\tau_{G V}$ to the more general cyclic $n$-cocycles $\tau\left(a_{0}, \ldots, a_{n}\right):=\frac{1}{n!} \sum_{\alpha \in \mathfrak{S}_{n}} \operatorname{sign}(\alpha) \varpi\left(a_{0} \delta_{\alpha(1)} a_{1} \ldots \delta_{\alpha(n)} a_{n}\right)$ considered in Proposition 5.34. We hope to use these results in a future project, in collaboration with Sacha Gorokhovsky, where our aim will be to extend the results of this paper to all Gelfand-Fuchs classes, employing the very interesting results in [CM90]. Here we shall explain how to construct, starting with such a $\tau$, an eta $(n+1)$-cyclic cocycle $\sigma$ on $B_{c}$ and a relative cocycle $\left(\tau^{r}, \sigma\right)$ on $\left(A_{c}, B_{c}\right)$; proofs will
be given in Section 10.2. For the sake of clarity we shall make our notation slightly more precise.

Let $J_{c}, B_{c}$ and $A_{c}$ be the algebras defined in Section 5.2, fitting into the exact sequence $0 \rightarrow J_{c} \rightarrow A_{c} \xrightarrow{\pi_{c}} B_{c} \rightarrow 0$. Starting with a foliated bundle $(X, \mathcal{F})$ with cylindrical ends, we consider the following situation:

- There exist (graded) algebras $\Omega_{J}, \Omega_{A}$ and $\Omega_{B}$, which contain $J_{c}, A_{c}$ and $B_{c}$ as subalgebras, respectively and which fit into a short exact sequence:

$$
0 \rightarrow \Omega_{J} \rightarrow \Omega_{A} \xrightarrow{\pi_{\Omega}} \Omega_{B} \rightarrow 0 .
$$

Here $\Omega_{J}$ is the analogue of the algebra $\Omega(G)$ constructed in (5.45) for closed foliated bundles.

- One has a trace map $\tau_{\Gamma}^{\text {cyl }}: \Omega_{B} \rightarrow \mathbb{C}$ and a linear functional $\tau_{\Gamma}^{r}: \Omega_{A} \rightarrow \mathbb{C}$. The second map is not a trace map; however, it satisfies

$$
\tau_{\Gamma}^{r}\left(\kappa \kappa^{\prime}-\kappa^{\prime} \kappa\right)=\tau_{\Gamma}^{\mathrm{cyl}}\left(\lambda\left[\chi^{0}, \lambda^{\prime}\right]\right)
$$

for $\kappa, \kappa^{\prime} \in \Omega_{A}$ with $\pi_{\Omega}(\kappa)=\lambda$ and $\pi_{\Omega}\left(\kappa^{\prime}\right)=\lambda^{\prime}$ in $\Omega_{B}$. Observe that the above identity implies that $\tau_{\Gamma}^{r}$ is a trace map once restricted to $\Omega_{J}$ since $\pi_{\Omega}(\kappa)=0$ for $\kappa \in \Omega_{J}$. We then require that the restriction coincides with the trace map $\tau_{\Gamma}$ on $\Omega_{J}$, which is defined in the proof of Proposition 5.44. We call $\tau_{\Gamma}^{r}$ a regularized trace associated to $\tau_{\Gamma}$ on $\Omega_{J}$ and to $\tau_{\Gamma}^{\mathrm{cyl}}$ on $\Omega_{B}$.

- There exist derivations $\delta_{i}^{A}$ on $\Omega_{A}$ with $1 \leq i \leq n$, which preserve $\Omega_{J}$ and satisfy $\delta_{i}^{A} \delta_{j}^{A}=\delta_{j}^{A} \delta_{i}^{A}$ for $1 \leq i, j \leq n$. There also exist derivations $\delta_{i}^{B}$ on $\Omega_{B}$ with $1 \leq i \leq n$, which are compatible with $\delta_{i}^{A}$ in the sense that $\pi_{\Omega} \delta_{i}^{A}=\delta_{i}^{B} \pi_{\Omega}$ (thus $\delta_{i}^{B}$ are pairwise commuting since $\delta_{i}^{A}$ are). In the sequel we often suppress the suffix and simply denote them by the same letter $\delta_{i}$.
- Stokes' formulas hold: one has $\tau_{\Gamma}^{r}\left(\delta_{i}^{A} \kappa\right)=0$ and $\tau_{\Gamma}^{\mathrm{cyl}}\left(\delta_{i}^{B} \lambda\right)=0$ for $\kappa \in \Omega_{A}$ and $\lambda \in \Omega_{B}$ with $1 \leq i \leq n$.

We shall now produce eta cocycles and relative cocycles in this setting. Note that for eta cocycles we shall proceed more generally, precisely as we did for Godb-illon-Vey, thus considering $B_{c}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ for $Y=\widetilde{N} \times_{\Gamma} T$ a foliated $T$-bundle without boundary; in this more general situation we shall assume that the derivations $\delta_{j}^{B}, j=1, \ldots, n$, are pairwise commuting.

First we take a cyclic $n$-cochain on $J_{c}$ of the following form:

$$
\begin{equation*}
\tau_{n}\left(a_{0}, \ldots, a_{n}\right)=\frac{1}{n!} \sum_{\alpha \in \mathfrak{S}_{n}} \operatorname{sign}(\alpha) \tau_{\Gamma}\left(a_{0} \delta_{\alpha(1)} a_{1} \ldots \delta_{\alpha(n)} a_{n}\right) \tag{5.56}
\end{equation*}
$$

for $a_{i} \in J_{c}$. Due to Proposition 5.34 it is a cyclic cocycle. Now simply replacing $\tau_{\Gamma}$ by the regularized trace $\tau_{\Gamma}^{r}$, we extend $\tau_{n}$ to $A_{c}$ by the same formula:

$$
\psi\left(k_{0}, \ldots, k_{n}\right)=\frac{1}{n!} \sum_{\alpha \in \mathfrak{S}_{n}} \operatorname{sign}(\alpha) \tau_{\Gamma}^{r}\left(k_{0} \delta_{\alpha(1)} k_{1} \ldots \delta_{\alpha(n)} k_{n}\right)
$$

for $k_{i} \in A_{c}$. However, the resulting multilinear map $\psi$ is not a cyclic cochain anymore. Thus, in order to get a cyclic one we set $\psi^{(i)}\left(k_{0}, \ldots, k_{n}\right)=\psi\left(k_{i}, \ldots, k_{n}, k_{0}, \ldots, k_{i-1}\right)$ with $i=0,1 \ldots, n$ and define a cyclic $n$-cochain on $A_{c}$ to be

$$
\begin{equation*}
\tau_{n}^{r}\left(k_{0}, \ldots, k_{n}\right)=\frac{1}{n+1} \sum_{i=0}^{n}(-1)^{n(i+1)} \psi^{(i)}\left(k_{0}, \ldots, k_{n}\right) \tag{5.57}
\end{equation*}
$$

Now we have the following two Propositions:
Proposition 5.58. Let $Y=\widetilde{N} \times_{\Gamma} T$ be a foliated $T$-bundle without boundary. Introduce a new derivation $\delta_{n+1}(\ell):=\left[\chi^{0}, \ell\right]$ for $\ell \in B_{c}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ and define a $(n+1)$-linear functional $\sigma_{n+1}$ by

$$
\sigma_{n+1}\left(\ell_{0}, \ldots, \ell_{n+1}\right)=\frac{1}{(n+1)!} \sum_{\beta \in \mathfrak{S}_{n+1}} \operatorname{sign}(\beta) \tau_{\Gamma}^{\mathrm{cyl}}\left(\ell_{0} \delta_{\beta(1)} \ell_{1} \ldots \delta_{\beta(n+1)} \ell_{n+1}\right)
$$

Then $\sigma_{n+1}$ is a cyclic $(n+1)$-cocycle on $B_{c}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$.
Proof. The proof follows directly from Proposition 5.34, once we take into account the modifications that were made in order to prove that $\sigma_{G V}$ is a cyclic 3 -cocycle.

Proposition 5.59. Let $(X, \mathcal{F})$ be a foliated $T$-bundle with cylindrical ends. Let $\tau_{n}^{r}$ be the cyclic n-cochain on $A_{c}$ defined in (5.57). Consider the eta $(n+1)$-cocycle $\sigma_{n+1}$ defined in the above Proposition. Then the relative cocycle condition is satisfied: one has $b \sigma_{n+1}=0$, which we already know, and $b \tau_{n}^{r}=\left(\pi_{c}\right)^{*} \sigma_{n+1}$.

Notice once again that in this bounding situation the equation $b \sigma_{n+1}=0$ on $B_{c}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ is in fact a consequence of $b \tau_{n}^{r}=\left(\pi_{c}\right)^{*} \sigma_{n+1}$ (using that $\pi_{c}^{*}$ is injective).

Summarizing: starting with $\tau_{n}$ of the form (5.56) we have obtained an eta cocycle $\sigma_{n+1}$ and a relative cocycle $\left(\tau_{n}^{r}, \sigma_{n+1}\right)$. For example, given a codimension $q$ foliated bundle $(Y, \mathcal{F})$, there exist derivations $\delta_{i}$ with $1 \leq i \leq q+1$ obtained from [ $\left.d_{T} \phi, a\right]$ and $[\phi, a]$ where $d_{T}$ is the exterior differentiation along transversal and $\phi$ the logarithm of the modular function. Then a Godbillon-Vey cyclic $(q+1)$-cocycle can be constructed on $J_{c}=C_{c}^{\infty}(Y, \mathcal{F})$ by a similar formula to (5.41). Thanks to the above results it is then possible to obtain an eta $(q+2)$-cocycle and a relative cyclic $(q+1)$-cocycle associated to such a generalized Godbillon-Vey cyclic $(q+1)$-cocycle.

## 6 Smooth Subalgebras

6.1 Summary of this section. The goal of this whole Section is to define the subsequence

$$
0 \rightarrow \mathfrak{J} \hookrightarrow \mathfrak{A} \rightarrow \mathfrak{B} \rightarrow 0
$$

of $0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*}(X, \mathcal{F}) \rightarrow B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow 0$ we have alluded to in the Introduction and in Section 5.1. Since the definitions are somewhat involved, we have decided to give here a brief account of the main definitions and of the main results of the whole Section; this summary will be enough for understanding the main ideas in the proof of our main theorem.

Step 1. We begin by defining Schatten-type ideals $\mathcal{I}_{m}(X, \mathcal{F}) \subset C^{*}(X, \mathcal{F})$; these are for each $m \geq 1$ dense and holomorphically closed subalgebras of $C^{*}(X, \mathcal{F})$. (We shall eventually fix $m$ greater than dimension of the leaves.) By imposing that the kernels in $\mathcal{I}_{m}(X, \mathcal{F})$ define bounded operators when multiplied by a function that goes like $\left(1+s^{2}\right)$ on the cylindrical end, we obtain the Banach algebras $\mathcal{J}_{m}(X, \mathcal{F}) \subset C^{*}(X, \mathcal{F})$; these are still dense and holomorphically closed.

Step 2. Next we define dense holomorphically closed subalgebras $\mathcal{B}_{m}(\operatorname{cyl}(\partial X)$, $\left.\mathcal{F}_{\text {cyl }}\right) \subset B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)\left(\right.$ often simply denoted $\left.\mathcal{B}_{m}\right)$.

To this end we first define $\mathrm{OP}^{-1}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$, the closure of $\Psi_{\mathbb{R} . c}^{-1}\left(G_{\mathrm{cyl}}\right) \subset$ $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ with respect to the norm $\|\mid P\| \|:=\max \left(\|P\|_{-n,-n-1},\|P\|_{n+1, n}\right)$, where on the right hand side we have the norm for operators between Sobolev spaces and where $n$ is a fixed integer greater or equal to the dimension of the leaves. Next we define $\mathcal{D}_{m}$ as those elements in $\mathrm{OP}^{-1}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ for which (a certain closure of) the derivation $\left[\chi^{0}, \cdot\right]$ has values in $\mathcal{J}_{m}$. We then define $\mathcal{D}_{m, \alpha}$ as $\mathcal{D}_{m} \cap \operatorname{Dom}\left(\partial_{\alpha}\right)$ with $\partial_{\alpha}$ the closed derivation associated to the $\mathbb{R}$-action $\alpha_{t}$ defined by $\alpha_{t}(\ell):=e^{i t s} \ell e^{-i t s} . \mathcal{B}_{m}$ is obtained as a subalgebra of $\mathcal{D}_{m, \alpha}: \mathcal{B}_{m}=\{\ell \in$ $\mathcal{D}_{m, \alpha} \mid[f, \ell]$ and $[f,[f, \ell]]$ are bounded, with $\left.f(y, s)=\sqrt{1+s^{2}}\right\}$. We endow $\mathcal{B}_{m}$ with a Banach norm and we prove that it is a dense holomorphically closed subalgebra of $\mathcal{B}^{*}$ for each $m \geq 1$.

Step 3. We define $\mathcal{A}_{m}(X, \mathcal{F}):=\left\{k \in A^{*}(X, \mathcal{F}) ; \pi(k) \in \mathcal{B}_{m}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right), t(k) \in\right.$ $\left.\mathcal{J}_{m}(X, \mathcal{F})\right\}$ with $t: A^{*}(X, \mathcal{F}) \rightarrow C^{*}(X, \mathcal{F})$ defined in (4.13). We endow $\mathcal{A}_{m}$ with a norm that makes it a Banach subalgebra of $A^{*}$

Step 4. We prove that $\mathcal{J}_{m}$ is an ideal in $\mathcal{A}_{m}$ and that there is for each $m \geq 1$ a short exact sequence of Banach algebras $0 \rightarrow \mathcal{J}_{m}(X, \mathcal{F}) \rightarrow \mathcal{A}_{m}(X, \mathcal{F}) \rightarrow$ $\mathcal{B}_{m}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow 0$.

Step 5. Recall the function $\phi$, equal to the logarithm of the modular function. Recall the (algebraic) derivations $\delta_{1}:=[\dot{\phi}$,$] and \delta_{2}:=[\phi$,$] . We define suitable clo-$ sures $\bar{\delta}_{1}, \bar{\delta}_{2}$ of these two derivations and we define $\mathfrak{J}_{\mathrm{m}}$ as $\mathcal{J}_{m} \cap \operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right)$. We endow $\mathfrak{J}_{\mathbf{m}}$ with a Banach norm and we remark that it is a dense holomorphically closed subalgebra of $C^{*}(X, \mathcal{F})$. Similarly, we define suitable closures of the derivations $\delta_{1}:=\left[\dot{\phi}_{\partial},\right]$ and $\delta_{2}:=\left[\phi_{\partial},\right]$ on the cylinder and we define $\mathfrak{B}_{\mathrm{m}}$ as $\mathcal{B}_{m} \cap \operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right)$. We endow $\mathfrak{B}_{\mathrm{m}}$ with a Banach norm and we show that it is a dense holomorphically closed subalgebra of $B^{*}$. Finally, we define in a similar way the Banach algebra $\mathfrak{A}_{\mathrm{m}}$; this is a subalgebra of $A^{*}$.

Step 6. We prove that $\mathfrak{J}_{\mathrm{m}}$ is an ideal in $\mathfrak{A}_{\mathrm{m}}$ and that there is a short exact sequence of Banach algebras $0 \rightarrow \mathfrak{J}_{\mathrm{m}} \hookrightarrow \mathfrak{A}_{\mathrm{m}} \rightarrow \mathfrak{B}_{\mathrm{m}} \rightarrow 0$.

The subsequence we are interested in is obtained by taking $m=2 n+1$ in the above sequence, with $2 n$ equal to the dimension of the leaves in $(X, \mathcal{F})$.
6.2 Schatten ideals. Let $\chi_{\Gamma}$ be a characteristic function for a fundamental domain of $\Gamma \rightarrow \tilde{V} \rightarrow V$. Consider $C_{c}^{\infty}(G)=: J_{c}(X, \mathcal{F}) \equiv J_{c}$.

Definition 6.1. Let $k \in J_{c}$ be positive and self-adjoint. The Schatten norm $\|k\|_{m}$ of $k$ is defined as

$$
\begin{equation*}
\left(\|k\|_{m}\right)^{m}:=\sup _{\theta \in T}\left\|\chi_{\Gamma}(k(\theta))^{m} \chi_{\Gamma}\right\|_{1} \tag{6.2}
\end{equation*}
$$

with the $\left\|\|_{1}\right.$ denoting the usual trace-norm on the Hilbert space $\mathcal{H}_{\theta}$. Equivalently

$$
\begin{equation*}
\left(\|k\|_{m}\right)^{m}=\sup _{\theta \in T}\left\|\chi_{\Gamma}(k(\theta))^{m / 2}\right\|_{H S}^{2} \tag{6.3}
\end{equation*}
$$

with $\left\|\|_{H S}\right.$ denoting the usual Hilbert-Schmidt norm. In general, we set $\| k \|_{m}:=$ $\left\|\left(k k^{*}\right)^{1 / 2}\right\|_{m}$. The Schatten norm of $k \in J_{c}$ is easily seen to be finite for any $m \geq 1$.

Proposition 6.4. The following properties hold:
(1) if $1 / r=1 / p+1 / q$ then $\left\|k k^{\prime}\right\|_{r} \leq\|k\|_{p}\left\|k^{\prime}\right\|_{q}$;
(2) if $r \geq 1$ then $\left\|k k^{\prime}\right\|_{r} \leq\|k\|_{C^{*}}\left\|k^{\prime}\right\|_{r}$;
(3) if $p<q$ then $\|k\|_{p} \geq\|k\|_{q}$;
(4) if $p \geq 1$ then $\|k\|_{p} \geq\|k\|_{C^{*}}$.

The proof of the Proposition is easily given using standard properties of the Schatten norms on a Hilbert space.

Consider now $\chi_{\Gamma}$, the characteristic function of a fundamental domain for $\tilde{V}$. Define a map

$$
\begin{equation*}
\phi_{m}: C^{*}(X, \mathcal{F}) \rightarrow \operatorname{End}(\mathcal{H}) \tag{6.5}
\end{equation*}
$$

to be given by $\phi_{m}(k):=\left(\chi_{\Gamma}\left|T_{\theta}\right|^{m} \chi_{\Gamma}\right)_{\theta \in T}$ with $m \in \mathbb{N}$. It is a continuous map (although, obviously, not a linear operator), given as the composition of $\left(T_{\theta}\right)_{\theta \in T} \rightarrow$ $\left(\left|T_{\theta}\right|^{m}\right)_{\theta \in T}$ and left and right multiplication by $\chi_{\Gamma}$. Let $\mathcal{L}^{1}(\mathcal{H})$ be the subalgebra of $\operatorname{End}(\mathcal{H})$ (see Section 4.2) consisting of measurable families $T=\left(T_{\theta}\right)_{\theta \in T}$ such that $T_{\theta}$ is an operator of trace class for almost every $\theta$. It is a Banach subalgebra of $\operatorname{End}(\mathcal{H})$ with the norm

$$
\begin{equation*}
\|T\|_{1}:=\operatorname{ess} \cdot \sup \left\{\left\|T_{\theta}\right\|_{1} ; \theta \in T\right\} \tag{6.6}
\end{equation*}
$$

where $\left\|T_{\theta}\right\|_{1}$ denotes the trace norm. For $m \in \mathbb{N}, m \geq 1$ we set

$$
\begin{equation*}
\mathcal{I}_{m}(X, \mathcal{F}):=\left\{T \in C^{*}(X, \mathcal{F}) \mid \phi_{m}(T) \in \mathcal{L}^{1}(\mathcal{H})\right\} \tag{6.7}
\end{equation*}
$$

and denote by $\psi_{m}$ the restriction of $\phi_{m}$ to $\mathcal{I}_{m}(X, \mathcal{F})$, so that $\psi_{m}: \mathcal{I}_{m}(X, \mathcal{F}) \rightarrow$ $\mathcal{L}^{1}(\mathcal{H})$. We anticipate that we shall need to take a slightly smaller algebra; this smaller algebra will be denoted $\mathcal{J}_{m}(X, \mathcal{F})$.

It is clear that $\mathcal{I}_{m}(X, \mathcal{F})$ is closed under composition. We can prove that the graph of $\psi_{m}$ is a closed subset of $C^{*}(X, \mathcal{F}) \times \mathcal{L}^{1}(\mathcal{H})$ : indeed the graph of $\phi_{m}$ is a closed subset of $C^{*}(X, \mathcal{F}) \times \operatorname{End}(\mathcal{H})$ due to continuity, the inclusion of $C^{*}(X, \mathcal{F}) \times \mathcal{L}^{1}(\mathcal{H})$ into $C^{*}(X, \mathcal{F}) \times \operatorname{End}(\mathcal{H})$ is continuous and the graph of $\psi_{m}$ is the intersection of the graph of $\phi_{m}$ with $C^{*}(X, \mathcal{F}) \times \mathcal{L}^{1}(\mathcal{H})$.

Proposition 6.8. $\mathcal{I}_{m}(X, \mathcal{F})$ is a Banach algebra, an ideal inside $C^{*}(X, \mathcal{F})$ and is isomorphic to the completion of $J_{c}(X, \mathcal{F})$ with respect to the $m$-Schatten norm. In particular $\mathcal{I}_{m}(X, \mathcal{F})$ is a holomorphically closed dense subalgebra of $C^{*}(X, \mathcal{F})$.

Proof. We define a norm on $\mathcal{I}_{m}(X, \mathcal{F})$ by considering the graph norm associated to $\psi_{m}$, viz:

$$
\|T\|_{m}:=\|T\|_{C^{*}}+\left\|\psi_{m}(T)\right\|_{1} .
$$

Since the graph of $\psi_{m}$ is closed this is a complete Banach space. Moreover, by the analogue of Proposition 6.4 (stated for elements in $\operatorname{End}_{\Gamma}(\mathcal{H})$ ) we see that this graph norm satisfies $\|S T\|_{m} \leq\|S\|_{m}\|T\|_{m}$ so that $\mathcal{I}_{m}(X, \mathcal{F})$ is a Banach algebra. Next observe that, obviously, $J_{c}(X, \mathcal{F}) \subset \mathcal{I}_{m}(X, \mathcal{F})$; moreover, from the fourth inequality in Proposition 6.4 we see that on $J_{c}(X, \mathcal{F})$ the graph-norm and the Schatten norm introduced in Definition (6.1) are equivalent (thus the small abuse of notation); since $\mathcal{I}_{m}(X, \mathcal{F})$ contains $J_{c}(X, \mathcal{F})$ as a dense set and it is complete by the norm $\left\|\|_{m}\right.$, we conclude that the completion of $J_{c}(X, \mathcal{F})$ by the norm of Definition (6.1) is naturally isomorphic, as a Banach algebra, to $\mathcal{I}_{m}(X, \mathcal{F})$. The fact that $\mathcal{I}_{m}$ is an ideal in $C^{*}(X, \mathcal{F})$ follows easily from the inequality $\left\|k k^{\prime}\right\|_{m} \leq\|k\|\left\|k^{\prime}\right\|_{m}$. From the ideal property one can easily prove that $\mathcal{I}_{m}$ is closed under holomorphic functional calculus; indeed if $a \in \mathcal{I}_{m}$ and $f$ is a holomorphic function in a neighbourhood of $\operatorname{spec}(a)$ such that $f(0)=0$ then we can write $f(z)=z g(z)$ for some holomorphic function $g$ and thus $f(a)=a g(a)$ which therefore belongs to $\mathcal{I}_{m}$, given that $\mathcal{I}_{m}$ is an ideal.

Remark 6.9. For the elements in the ideals $\mathcal{I}_{p}(X, \mathcal{F})$ the inequalities of Proposition 6.4 continue to hold. In particular, if we have $T_{j} \in \mathcal{I}_{p}(X, \mathcal{F})$ for $j=1, \ldots, p$, then their composition $T_{1} \cdots T_{p} \in \mathcal{I}_{1}(X, \mathcal{F})$ and the product map $\mathcal{I}_{p}(X, \mathcal{F}) \times \cdots \times$ $\mathcal{I}_{p}(X, \mathcal{F}) \rightarrow \mathcal{I}_{1}(X, \mathcal{F})$ is continuous.

Recall now the weight $\omega_{\Gamma}$ defined on $J_{c}:=C_{c}^{\infty}\left(G,\left(s^{*} E\right)^{*} \otimes r^{*} E\right)$ :

$$
\begin{equation*}
\omega_{\Gamma}(k):=\int_{X(\Gamma)} \operatorname{Tr}_{(x, \theta)} k(x, x, \theta) d x d \theta \tag{6.10}
\end{equation*}
$$

where $\operatorname{Tr}_{(x, \theta)}$ denotes the trace map on $\operatorname{End}\left(E_{(x, \theta)}\right)$ identifying $\operatorname{End}\left(\widehat{E}_{(x, \theta)}\right)$ with $\operatorname{End}\left(E_{(x, \theta)}\right)$. Recall also that

$$
\begin{equation*}
\omega_{\Gamma}(k)=\int_{S^{1}} \operatorname{Tr}\left(\sigma_{\theta} k(\theta) \sigma_{\theta}\right) d \theta \tag{6.11}
\end{equation*}
$$

with $\sigma$ a compactly supported smooth function on $\tilde{N} \times S^{1}$ such that $\sum_{\gamma \in \Gamma} \gamma(\sigma)^{2}=$ $1, \sigma_{\theta}:=\left.\sigma\right|_{\tilde{V} \times\{\theta\}}$ and $\operatorname{Tr}$ denoting the usual trace functional on the Hilbert space $\mathcal{H}_{\theta}$.
Proposition 6.12. The weight $\omega_{\Gamma}$ in (6.10) extends continuously from $J_{c}$ to $\mathcal{I}_{1}$. In particular, if $k_{0}, k_{1}, \ldots, k_{p} \in \mathcal{I}_{p+1}$ then $\omega_{\Gamma}\left(k_{0} k_{1} \cdots k_{p}\right)$ is finite.

Proof. We need to prove that for an element $k \in J_{c}(X, \mathcal{F})$ we have $\left|\omega_{\Gamma}(k)\right| \leq C\|k\|_{1}$. However, this follows at once from the following two inequalities

$$
\left|\int_{\mathrm{FD}} \operatorname{Tr}_{x} k(x, x, \theta) d x\right| \leq\left\|\chi_{\Gamma} k_{\theta} \chi_{\Gamma}\right\|_{1}, \quad \int_{T}|f(\theta)| d \theta \leq \operatorname{vol}(T) \sup _{\theta}|f(\theta)| .
$$

Thus $\left|\omega_{\Gamma}(k)\right| \leq \operatorname{vol}(T)\|k\|_{1}$ as stated.
We shall now introduce the subalgebra of $C^{*}(X, \mathcal{F})$ that will be used in the proof of our index theorem. Consider on the cylinder $\mathbb{R} \times Y$ (with cylindrical variable $s$ ) the functions

$$
\begin{equation*}
f_{\mathrm{cyl}}(s, y):=\sqrt{1+s^{2}} \quad g_{\mathrm{cyl}}(s, y)=1+s^{2} . \tag{6.13}
\end{equation*}
$$

We denote by $f$ and $g$ smooth functions on $X$ equal to $f_{\text {cyl }}$ and $g_{\text {cyl }}$ on the open subset $(-\infty, 0) \times Y ; f$ and $g$ are well defined up to a compactly supported function. We set

$$
\begin{equation*}
\mathcal{J}_{m}(X, \mathcal{F}):=\left\{k \in \mathcal{I}_{m} \mid g k \text { and } k g \text { are bounded }\right\} \tag{6.14}
\end{equation*}
$$

We shall often simply write $\mathcal{J}_{m}$.
Proposition 6.15. $\mathcal{J}_{m}$ is a subalgebra of $\mathcal{I}_{m}$ and a Banach algebra with the norm

$$
\begin{equation*}
\|k\|_{\mathcal{J}_{m}}:=\|k\|_{m}+\|g k\|_{C^{*}}+\|k g\|_{C^{*}} \tag{6.16}
\end{equation*}
$$

Moreover $\mathcal{J}_{m}$ is holomorphically closed in $\mathcal{I}_{m}$ (and, therefore, in $C^{*}(X, \mathcal{F})$ ).
Proof. The subalgebra property is obvious, so we pass directly to the fact that $\mathcal{J}_{m}$ is a Banach algebra. It suffices to show that multiplication by $g$ on the left and on the right induces closed operators; namely if $k_{j} \rightarrow k, k_{j} g \rightarrow \ell_{1}, g k_{j} \rightarrow \ell_{2}$ for $k_{j} \in \Psi_{c}^{-1}(G)$, then $\ell_{1}=k g$ and $\ell_{2}=g k$. In fact, given $\xi \in C_{c}^{\infty}(\tilde{V} \times\{\theta\})$, one has

$$
\ell_{1}(\xi)=\left(\lim k_{j} g\right)(\xi)=\left(\lim k_{j}\right)(g \xi)=k g(\xi)
$$

noting that $g \xi \in C_{c}^{\infty}(\tilde{V} \times\{\theta\})$, which proves that $\ell_{1}=k g$. Similarly one has $\ell_{2}=g k$. This proves that $\left(\mathcal{J}_{m},\| \|_{\mathcal{J}_{m}}\right)$ is a Banach space. The Banach-algebra property of this norm follows easily from the Banach-algebra property of $\left\|\|_{m}\right.$ on $\mathcal{I}_{m}$ and $\left\|\|_{C^{*}}\right.$ on $C^{*}(X, \mathcal{F})$. Finally we show that $\mathcal{J}_{m}$ is holomorphically closed in $\mathcal{I}_{m}$. To this end we need to show that if $1+k \in \mathcal{J}_{m}^{+}:=\mathcal{J}_{m}+\mathbb{C} \cdot 1$ is invertible in $\mathcal{I}_{m}^{+}$, with $(1+k)^{-1}=1+k^{\prime}$ and $k^{\prime} \in \mathcal{I}_{m}$ then one has $k^{\prime} \in \mathcal{J}_{m}$. First we observe that $1=(1+k)\left(1+k^{\prime}\right)=1+k+k^{\prime}+k k^{\prime}$. Thus $k^{\prime}=-k-k k^{\prime}$. Similarly one has $k^{\prime}=-k-k^{\prime} k$ (using $1=\left(1+k^{\prime}\right)(1+k)$ ). Thus $g k^{\prime}=-g k-(g k) k^{\prime}$ and $k^{\prime} g=-k g-k^{\prime}(k g)$. Since the right hand sides are bounded so are $g k^{\prime}$ and $k^{\prime} g$. The Proposition is proved.

Remark 6.17. As usual, we have not included the vector bundle $E$ into the notation; however, strictly speaking, the notation for the Schatten ideals we have defined above should be $\mathcal{I}_{m}(X, \mathcal{F} ; E)$. With obvious changes we can also define $\mathcal{I}_{m}(X, \mathcal{F} ; E, F)$, with $F$ a hermitian vector bundle on $X$; in particular, given $E$ on $X=\tilde{V} \times_{\Gamma} T$, and thus $\widehat{E}$ on $\tilde{V} \times T$, we can define $\widehat{E}^{\prime}$, which is $\widehat{E}$ but with a new $\Gamma$-equivariant structure. We then have $\mathcal{I}_{m}\left(X, \mathcal{F} ; E, E^{\prime}\right)$. Notice that, by continuity, we have an isomorphism of Banach algebras $\mathcal{I}_{m}(X, \mathcal{F} ; E) \cong \mathcal{I}_{m}\left(X, \mathcal{F} ; E^{\prime}\right)$ as well as continuous maps $\mathcal{I}_{p}\left(X, \mathcal{F} ; E, E^{\prime}\right) \times \mathcal{I}_{q}(X, \mathcal{F} ; E) \rightarrow \mathcal{I}_{r}\left(X, \mathcal{F} ; E, E^{\prime}\right)$ and $\mathcal{I}_{p}(X, \mathcal{F} ; E) \times \mathcal{I}_{q}\left(X, \mathcal{F} ; E, E^{\prime}\right) \rightarrow \mathcal{I}_{r}\left(X, \mathcal{F} ; E, E^{\prime}\right)$ if $1 / r=1 / p+1 / q$. Moreover, the analogue of Proposition 6.12 holds for the bimodule trace $\omega_{\Gamma}: J_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right) \rightarrow \mathbb{C}$.
6.3 Closed derivations. In this Subsection we give some general results on derivations; this material plays an important role in the sequel. Let in general $T$ : $\mathcal{B}_{0} \rightarrow \mathcal{B}_{1}$ be a linear operator between Banach spaces with a domain $\operatorname{Dom}(T)$ which is assumed to be dense. Denote by $G_{T}$ the graph of $T$, namely the subspace $G_{T}:=$ $\left\{(u, T u) \in \mathcal{B}_{0} \oplus \mathcal{B}_{1} \mid u \in \operatorname{Dom}(T)\right\}$ and consider the closure $\overline{G_{T}}$. Also, denote by $p$ the projection $p: \mathcal{B}_{0} \oplus \mathcal{B}_{1} \rightarrow \mathcal{B}_{0}$ onto the first component. The following Lemma and Definition are well known:

Lemma 6.18. The following are equivalent:
(1) $\overline{G_{T}}$ is the graph of a linear operator $\bar{T}$, with $p\left(\overline{G_{T}}\right)$ equal to the domain of $\bar{T}$, which is an extension of $T$;
(2) set $p_{T}:=\left.p\right|_{\bar{G}_{T}}$; then $\operatorname{Ker} p_{T}=0$;
(3) for $u_{i} \in \operatorname{Dom}(T)$ with $u_{i} \rightarrow 0$ and $T u_{i} \rightarrow v$ one has $v=0$.

Definition 6.19. A linear operator $T: \mathcal{B}_{0} \rightarrow \mathcal{B}_{1}$ with dense domain $\operatorname{Dom}(T)$ is a closable operator if one of the properties of the Lemma above is satisfied. Then $\bar{T}$ is called the closure of $T$.

It is obvious that $\operatorname{Dom}(\bar{T})\left(=\operatorname{Im} p_{T}\right)$ becomes a Banach space if we equip it with the graph norm

$$
\begin{equation*}
\|u\|_{T}:=\|u\|_{0}+\|T u\|_{1}, \tag{6.20}
\end{equation*}
$$

with $\left\|\|_{i}\right.$ denoting the Banach norms on $\mathcal{B}_{i}$. It is also obvious that the closure $\bar{T}$ induces a bounded operator $\bar{T}:\left(\operatorname{Dom}(\bar{T}),\| \|_{T}\right) \rightarrow\left(\mathcal{B}_{1},\| \|_{1}\right)$.

Let now $A_{0}$ be a Banach algebra with norm $\left\|\|_{0}\right.$ and $A_{1}$ a $A_{0}$-bimodule with norm $\left\|\|_{1}\right.$. Let $\delta: A_{0} \rightarrow A_{1}$ be a closable derivation into the bimodule $A_{1}$, that is: $\delta$ is a closable operator that has the derivation property

$$
\begin{equation*}
\delta(a b)=(\delta a) b+a(\delta b), \quad \text { for } \quad a, b \in \operatorname{Dom}(\delta) . \tag{6.21}
\end{equation*}
$$

Denote by $\bar{\delta}: \operatorname{Dom}(\bar{\delta}) \rightarrow A_{1}$ the closure of $\delta$.
Proposition 6.22. Set $\mathfrak{A}:=\operatorname{Dom}(\bar{\delta})$.
(1) $\mathfrak{A}$ is a Banach algebra with respect to the graph norm;
(2) $\bar{\delta}$ induces a derivation $\mathfrak{A} \rightarrow A_{1}, \bar{\delta}(a b)=(\bar{\delta} a) b+a(\bar{\delta} b), a, b \in \mathfrak{A}$.

Proof. Let $a, b \in \operatorname{Dom}(\bar{\delta})$. Then there exist sequences $\left\{a_{i}\right\},\left\{b_{i}\right\}$ in $\operatorname{Dom}(\delta)$ such that $a_{i} \rightarrow a, \delta a_{i} \rightarrow \bar{\delta} a, b_{i} \rightarrow b$ and $\delta b_{i} \rightarrow \bar{\delta} b$ in $A_{0}$ and $A_{1}$ respectively. Since $A_{0}$ is a Banach algebra and $A_{1}$ is a bimodule over $A_{0}$, we have $a_{i} b_{i} \rightarrow a b$ and $\delta\left(a_{i} b_{i}\right)=\left(\delta a_{i}\right) b_{i}+a_{i}\left(\delta b_{i}\right) \rightarrow(\bar{\delta} a) b+a(\bar{\delta} b)$, which implies $(a b,(\bar{\delta} a) b+a(\bar{\delta} b)) \in \overline{G_{\delta}}$ and $\bar{\delta}(a b)=(\bar{\delta} a) b+a(\bar{\delta} b)$ since $\overline{G_{\delta}}$ is the graph of $\bar{\delta}$ by the previous Lemma. This proves that $a b \in \mathfrak{A}$ and hence $\mathfrak{A}$ is an algebra. Moreover $\bar{\delta}$ satisfies the derivation property. Finally, we note that

$$
\begin{aligned}
\|a b\|_{\bar{\delta}} & =\|a b\|_{0}+\|\bar{\delta}(a b)\|_{1} \leq\|a\|_{0}\|b\|_{0}+\|\bar{\delta} a\|_{1}\|b\|_{0}+\|a\|_{0}\|\bar{\delta} b\|_{1} \\
& \leq\left(\|a\|_{0}+\|\bar{\delta} a\|_{1}\right)\left(\|b\|_{0}+\|\bar{\delta} b\|_{1}\right)=\|a\|_{\bar{\delta}}\|b\|_{\bar{\delta}} .
\end{aligned}
$$

which proves that $\mathfrak{A}$ is a Banach algebra with respect to the graph norm of $\bar{\delta}$.
We shall also need the following simple but important Lemma. First we introduce the relevant objects. Let $B_{0}$ be a subalgebra of $A_{0}$ endowed with a Banach algebra norm, $\left\|\left\|\|_{B_{0}} \text {, satisfying }\right\| b_{0}\right\|_{B_{0}} \geq\left\|b_{0}\right\|_{A_{0}}$. Let $B_{1} \subset A_{1}$ be a $B_{0}$-bimodule with $\left\|b_{1}\right\|_{B_{1}} \geq\left\|b_{1}\right\|_{A_{1}}$. Observe that $A_{1}$ is then also a $B_{0}$-bimodule since

$$
\left\|b_{0} a_{1}\right\|_{A_{1}} \leq\left\|b_{0}\right\|_{A_{0}}\left\|a_{1}\right\|_{A_{1}} \leq\left\|b_{0}\right\|_{B_{0}}\left\|a_{1}\right\|_{A_{1}}
$$

and similarly $\left\|a_{1} b_{0}\right\|_{A_{1}} \leq\left\|a_{1}\right\|_{A_{1}}\left\|b_{0}\right\|_{B_{0}}$ for $b_{0} \in B_{0}, a_{1} \in A_{1}$. Then $B_{1}$ is a $B_{0^{-}}$ submodule of $A_{1}$ endowed with the above $B_{0}$-bimodule structure and moreover the inclusion is clearly bounded.

Lemma 6.23. Let $\bar{\delta}$ be a closed derivation from $\operatorname{Dom}(\bar{\delta}) \subset A_{0}$ to $A_{1}$. Set

$$
\operatorname{Dom}_{B}:=\bar{\delta}^{-1}\left(B_{1}\right) \cap B_{0} \equiv\left\{a \in \operatorname{Dom}(\bar{\delta}) \cap B_{0} \mid \bar{\delta} a \in B_{1}\right\} .
$$

Define $\delta_{B}: \operatorname{Dom}_{B} \rightarrow B_{1}$ as $\delta_{B}(b):=\bar{\delta}(b)$. Then $\delta_{B}$ is a closed derivation.
Proof. By hypothesis we know that the graph of $\bar{\delta}$ is a closed subspace of $A_{0} \oplus A_{1}$. Then, because of our assumptions, its intersection with $B_{0} \oplus B_{1}$ is a closed subset of $B_{0} \oplus B_{1}$ (indeed, it is the inverse image of the graph for the inclusion map, which is continuous). On the other hand, this intersection is easily seen to be the graph of $\delta_{B}$. The Lemma is proved.
6.4 Schatten extensions. Let $(Y, \mathcal{F}), Y:=\tilde{N} \times_{\Gamma} T$, be a foliated $T$-bundle without boundary; for example $Y=\partial X_{0}$. Consider $\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ the associated foliated cylinder. Recall the function $\chi_{\mathrm{cyl}}^{0}$ (often just $\chi^{0}$ ), the function on the cylinder induced by the characteristic function of $(-\infty, 0]$ in $\mathbb{R}$. Notice that the definition of Schatten norm also apply to $\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$, viewed as a foliated $T$-bundle with cylindrical ends. Let $\Psi_{\mathbb{R}, c}^{-p}\left(G_{\mathrm{cyl}}\right) \equiv \Psi_{c}^{-p}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$, see Proposition 4.3 , be the space of $\mathbb{R} \times \Gamma$-equivariant families of pseudodifferential operators of order $-p$ on the fibration $(\mathbb{R} \times \tilde{N}) \times T \rightarrow T$ with $\mathbb{R} \times \Gamma$-compact support. Consider an element
$\ell \in \Psi_{c}^{-p}\left(G_{c y l} / \mathbb{R}_{\Delta}\right)$; then we know that $\ell$ defines a bounded operator from the Sobolev field $\mathcal{E}^{k}$ to the Sobolev field $\mathcal{E}^{k+p}$. See [MoN96], Section 3. Let us denote, as in [MoN96], the operator norm of a bounded operator $L$ from $\mathcal{E}^{k}$ to $\mathcal{E}^{j}$ as $\|L\|_{j, k}$; notice the reverse order. For a $\mathbb{R} \times \Gamma$-invariant, $\mathbb{R} \times \Gamma$-compactly supported pseudodifferential operator of order $(-p), P$, we consider the norm

$$
\begin{equation*}
\|P\|_{p}:=\max \left(\|P\|_{-n,-n-p},\|P\|_{n+p, n}\right) \tag{6.24}
\end{equation*}
$$

with $n$ a fixed integer strictly greater than $\operatorname{dim} N$. We denote the closure of $\mid\|\cdot\| \|_{p}$ by $\mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$. We shall often write $\mathrm{OP}^{-p}$.

Proposition 6.25. $\mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ is a Banach algebra and a subalgebra of $B^{*}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$

Proof. It is proved in [MoN96], section 3, that the norm \|\| • \| $\|_{p}$ satisfies the Banach algebra inequality $\||P Q|\|_{p} \leq\left\|\left||P|\| \|_{p}\||Q|\|\right|_{p}\right.$. Thus $\mathrm{OP}^{-p}$ is indeed a Banach algebra. In order to prove that $\mathrm{OP}^{-p}$ is a subalgebra of $B^{*}$ we need the following

Lemma 6.26. $B^{*}$ coincides with the $C^{*}$-closure of $\Psi_{c}^{-p}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$.
Proof. Let $D$ be the Dirac operator on $\left(\operatorname{cyl} Y, \mathcal{F}_{\text {cyl }}\right)$. Applying the same arguments as in [MoN96] we can prove that $(D+\mathfrak{s})^{-1}$ belongs to $B^{*}$ (see the proof of Proposition 7.18 in Section 10.5 for the details). Given $\ell \in \Psi_{c}^{-p}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$, $p \geq 1$, we can write $\ell=\ell(D+\mathfrak{s})^{p}(D+\mathfrak{s})^{-p}$ where we know that $\ell(D+\mathfrak{s})^{p} \in \Psi_{c}^{0}\left(G_{\text {cyl }} / \mathbb{R}_{\Delta}\right)$ and $(D+\mathfrak{s})^{-p} \in B^{*}$. Now recall from Remark 4.5 that $B^{*}$ is an ideal in $\mathcal{L}\left(\mathcal{E}_{\text {cyl }}\right)$; thus the above equality proves that $\ell \in B^{*}$. On the other hand, obviously, $\Psi_{c}^{-p}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$ contains $B_{c} \equiv C_{c}^{\infty}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$. Thus $B^{*} \equiv C^{*}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$, which is by definition the $C^{*}$-closure of $B_{c}$, is contained in the $C^{*}$-closure of $\Psi_{c}^{-p}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$. Thus one has:

$$
B^{*} \equiv C^{*}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right) \subset C^{*} \text {-closure of } \Psi_{c}^{-p}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right) \subset B^{*}
$$

proving the Proposition.
Since the $C^{*}$-norm is dominated by the $\left|\left||~ \cdot \||_{p}\right.\right.$-norm, we can immediately conclude the proof of the Proposition.

Notation. From now until the end of this subsection we fix $p=1$ and, following [MoN96], we denote the corresponding norm simply as ||| • |||.

Consider now the bounded linear map $\partial_{3}^{\max }: B^{*} \rightarrow \operatorname{End}_{\Gamma} \mathcal{H}$ given by $\partial_{3}^{\max } \ell:=$ $\left[\chi^{0}, \ell\right]$. Consider in $B^{*}$ the Banach subalgebra $\mathrm{OP}^{-1}$ endowed with the Banach norm $\left\|\|\cdot\|\left|\mid\right.\right.$ and consider in $\operatorname{End}_{\Gamma} \mathcal{H}$ the subalgebra $\mathcal{J}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$. Let $\partial_{3}$ be the restriction of $\partial_{3}^{\max }$ to $\mathrm{OP}^{-1}$. Since $\|\cdot\| \leq\| \| \cdot\| \|$ we see that $\partial_{3}$ is also bounded. Let $\mathcal{D}_{m}:=\left\{\ell \in \mathrm{OP}^{-1} \quad \mid \partial_{3}(\ell) \in \mathcal{J}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)\right\}$. From the restriction Lemma of the previous subsection, Lemma 6.23, we know that $\left.\partial_{3}\right|_{\mathcal{D}_{m}}$ induces a closed derivation $\bar{\delta}_{3}$ with domain $\mathcal{D}_{m}$. This is clearly a closed extension of the derivation $\delta_{3}$ considered in Section 5.11.

DEfinition 6.27. If $m \geq 1$ we define $\mathcal{D}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ as $\operatorname{Dom} \bar{\delta}_{3}$ endowed with norm

$$
\begin{equation*}
\|\ell\|_{\mathcal{D}_{m}}:=\|\ell \mid\|+\left\|\left[\chi_{\mathrm{cy} 1}^{0}, \ell\right]\right\|_{\mathcal{J}_{m}} . \tag{6.28}
\end{equation*}
$$

We shall often simply write $\mathcal{D}_{m}$ instead of $\mathcal{D}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$.
Proposition 6.29. Let $m \geq 1$, then $\mathcal{D}_{m}$ is a Banach algebra with respect to (6.28) and, obviously, a subalgebra of $B^{*} \equiv B^{*}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$. Moreover, $\mathcal{D}_{m}$ is holomorphically closed in $B^{*}$.

Proof. From the results of the previous subsection, we know that $\operatorname{Dom}\left(\bar{\delta}_{3}\right)$, endowed with the graph norm, is a Banach algebra; since $\operatorname{Dom}\left(\bar{\delta}_{3}\right)$ is by definition $\mathcal{D}_{m}$, we have proved the first part of the Proposition. Finally, that $\mathcal{D}_{m} \equiv \operatorname{Dom}\left(\bar{\delta}_{3}\right)$ is holomorphically closed in $\mathrm{OP}^{-1}$ is a classic consequence of the fact that it is equal to the domain of a closed derivation. See [Roe88], p. 197 or [Con94], Lemma 2, p. 247. Since $\mathrm{OP}^{-1}$ is in turn holomorphically closed in $B^{*}$, see [MoN96] Theorem 3.3, we see that $\mathcal{D}_{m}$ is holomorphically closed in $B^{*}$ as required. The Proposition is proved.

The Banach algebra we have defined is still too large for the purpose of extending the eta cocycle. We shall first intersect it with another holomorphically closed Banach subalgebra of $B^{*}$.

Observe that there exists an action of $\mathbb{R}$ on $\Psi_{c}^{-1}\left(G_{\text {cyl }} / \mathbb{R}_{\Delta}\right) \subset \mathrm{OP}^{-1}(\operatorname{cyl}(Y)$, $\left.\mathcal{F}_{\text {cyl }}\right) \subset B^{*}$ defined by

$$
\begin{equation*}
\alpha_{t}(\ell):=e^{i t s} \ell e^{-i t s} \tag{6.30}
\end{equation*}
$$

with $t \in \mathbb{R}$, $s$ the variable along the cylinder and $\ell \in \Psi_{c}^{-1}\left(G_{\text {cyl }} / \mathbb{R}_{\Delta}\right)$. Note that $\alpha_{t}(\ell)$ is again $(\mathbb{R} \times \Gamma)$-equivariant; indeed $e^{i t s}$ is $\Gamma$-equivariant and moreover $T_{\lambda} \circ$ $\alpha_{t}(\ell) \circ T_{\lambda}^{-1}=\alpha_{t}(\ell), T_{\lambda}$ denoting the action induced by a translation on $\operatorname{cyl}(Y)$ by $\lambda \in \mathbb{R}$. It is clear that $\left\|\left\|\alpha_{t}(\ell)\right\|\right\|=\|\ell\| \|$; thus, by continuity, $\left\{\alpha_{t}\right\}_{t \in \mathbb{R}}$ yields a welldefined action, still denoted $\left\{\alpha_{t}\right\}_{t \in \mathbb{R}}$, of $\mathbb{R}$ on the Banach algebra $\mathrm{OP}^{-1}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$. Note that this action is only strongly continuous. Let $\partial_{\alpha}: \mathrm{OP}^{-1} \rightarrow \mathrm{OP}^{-1}$ be the unbounded derivation associated to $\left\{\alpha_{t}\right\}_{t \in \mathbb{R}}$

$$
\begin{equation*}
\partial_{\alpha}(\ell):=\lim _{t \rightarrow 0} \frac{\left(\alpha_{t}(\ell)-\ell\right)}{t} \tag{6.31}
\end{equation*}
$$

By definition

$$
\operatorname{Dom}\left(\partial_{\alpha}\right)=\left\{\ell \in \mathrm{OP}^{-1} \mid \partial_{\alpha}(\ell) \text { exists in } \mathrm{OP}^{-1}\right\}
$$

Proposition 6.32. The derivation $\partial_{\alpha}$ is closed.

Proof. Observe preliminary that if $A$ and $A^{\prime}$ are two closed operator on a Banach space $\mathcal{B}$ then their sum $A+A^{\prime}$ is also closed (with domain equal to the intersection of the two domains). The proof is elementary.

Next we claim that if $A$ is a densely defined operator and $A^{-1}: \mathcal{B} \rightarrow \operatorname{Dom}(A)$ exists and is bounded, then $A$ is closed. Indeed: suppose that $x_{j} \rightarrow x$ and $A x_{j} \rightarrow y$; we want to prove that $x \in \operatorname{Dom}(A)$ and $A x=y$. By hypothesis we know that $x_{j} \rightarrow A^{-1} y$. Thus $x=\lim _{j} x_{j}=A^{-1} y$. Since $A^{-1}$ is bijective, one has $x \in \operatorname{Dom}(A)$ and $A x=y$, as required.

Finally for each $\ell \in \mathrm{OP}^{-1}$ we consider the following Laplace transform $R(\ell):=$ $\int_{0}^{+\infty} d t e^{-t} \alpha_{t}(\ell)$. Since $\left\|\left\|\alpha_{t}(\ell)\right\|\right\|=\|\ell\| \|$, we see that the integral converges. Now, an elementary computation shows that $\left(I-\partial_{\alpha}\right) R=I$. Thus the previous statement, applied to $\left(I-\partial_{\alpha}\right)$, implies that $\left(I-\partial_{\alpha}\right)$ is a closed operator. Thus, by our first observation we get that $\partial_{\alpha}$ is closed. The Proposition is proved.

We endow $\operatorname{Dom}\left(\partial_{\alpha}\right)$ with the graph norm

$$
\begin{equation*}
\left\|\left|| \ell | \left\|+\left|\left\|\partial_{\alpha}(\ell) \mid\right\| .\right.\right.\right.\right. \tag{6.33}
\end{equation*}
$$

Proposition 6.34. $\operatorname{Dom}\left(\partial_{\alpha}\right)$ is a Banach algebra with respect to (6.33) and, obviously, a subalgebra of $B^{*} \equiv B^{*}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$; moreover it is holomorphically closed in $B^{*}$.

Proof. From the results of the previous subsection, we know that $\operatorname{Dom}\left(\partial_{\alpha}\right)$, endowed with the graph norm, is a Banach algebra. The first part of the Proposition is thus proved. That $\operatorname{Dom}\left(\partial_{\alpha}\right)$ is holomorphically closed in $\mathrm{OP}^{-1}$ is as before a consequence of the fact that it is equal to the domain of a closed derivation. Since, as before, $\mathrm{OP}^{-1}$ is in turn holomorphically closed in $B^{*}$, see [MoN96] Theorem 3.3, we see that $\operatorname{Dom}\left(\partial_{\alpha}\right)$ is holomorphically closed in $B^{*}$ as required. The Proposition is proved.

Let now $p \geq 1$ and consider $\mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$. Then $\alpha_{t}$ on $\mathrm{OP}^{-1}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ preserves the subspaces $\mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ and we therefore get a well-defined strongly continuous one-parameter group of automorphisms on each Banach algebra $\mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$. Let $\partial_{\alpha, p}$ be the associated derivation. Proceeding as in the proof of Proposition 6.32 we can check that this is a closed derivation with domain

$$
\operatorname{Dom}\left(\partial_{\alpha, p}\right)=\left\{\ell \in \mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right) \mid \lim _{t \rightarrow 0}\left(\alpha_{t}(\ell)-\ell\right) / t \text { exists in } \mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right)\right\}
$$

Similarly, proceeding as above, we can check that $\operatorname{Dom}\left(\partial_{\alpha, p}\right)$ is a Banach algebra with respect to the norm $\left|\left\|\ell\left|\left\|_{p}+\right\|\right|\left|\partial_{\alpha, p}(\ell)\right|\right\| \|_{p}\right.$.

Before going ahead we make a useful remark.
Remark 6.35. Multiplication in $B^{*}$ induces a bounded bilinear map

$$
\begin{equation*}
\operatorname{Dom}\left(\partial_{\alpha, p}\right) \times \operatorname{Dom}\left(\partial_{\alpha, q}\right) \longrightarrow \operatorname{Dom}\left(\partial_{\alpha, p+q}\right) \tag{6.36}
\end{equation*}
$$

The proof is an easy consequence of the derivation property and of the inequality $\left\|\left\|\ell^{\prime}\right\|\right\|_{p+q} \leq\|\ell \ell\|_{p}\left\|\ell^{\prime}\right\| \|_{q}$ for $\ell \in \mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ and $\ell^{\prime} \in \mathrm{OP}^{-q}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$.

We can now take the intersection of the Banach subalgebras $\mathcal{D}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ and $\operatorname{Dom}\left(\partial_{\alpha}\right)$ :

$$
\mathcal{D}_{m, \alpha}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right):=\mathcal{D}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right) \cap \operatorname{Dom}\left(\partial_{\alpha}\right)
$$

and we endow it with the norm

$$
\begin{equation*}
\|\ell\|_{m, \alpha}:=\|\ell\|\|+\|\left[\chi_{\mathrm{cy} 1}^{0}, \ell\right]\left\|_{\mathcal{J}_{m}}+\right\|\left\|\partial_{\alpha} \ell\right\| \| . \tag{6.37}
\end{equation*}
$$

Being the intersection of two holomorphically closed dense subalgebras, also $\mathcal{D}_{m, \alpha}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ enjoys this property.

We are finally ready to define the subalgebra we are interested in. Recall the function $f_{\mathrm{cyl}}(s, y)=\sqrt{1+s^{2}}$.

Definition 6.38. If $m \geq 1$ we define

$$
\begin{equation*}
\mathcal{B}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right):=\left\{\ell \in \mathcal{D}_{m, \alpha}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right) \mid[f, \ell] \text { and }[f,[f, \ell]] \text { are bounded }\right\} \tag{6.39}
\end{equation*}
$$

This will be endowed with norm

$$
\begin{aligned}
\|\ell\|_{\mathcal{B}_{m}} & :=\|\ell\|_{m, \alpha}+2\|[f, \ell]\|_{B^{*}}+\|[f,[f, \ell]]\|_{B^{*}} \\
& =\|\ell\|\|+\|\left[\chi_{\mathrm{cy} 1}^{0}, \ell\right]\left\|_{\mathcal{J}_{m}}+\right\| \mid \partial_{\alpha} \ell\| \|+2\|[f, \ell]\|_{B^{*}}+\|[f,[f, \ell]]\|_{B^{*}} .
\end{aligned}
$$

The appearance of the factor 2 will be clear from the proof of Lemma 6.43. Proceeding as in the proof of Proposition 6.15 one can prove that $\mathcal{B}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ is a holomorphically closed dense subalgebra of $B^{*}$. We shall often simply write $\mathcal{B}_{m}$ instead of $\mathcal{B}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right)$.

Let us go back to the foliated bundle with cylindrical end $(X, \mathcal{F})$. We now define

$$
\begin{equation*}
\mathcal{A}_{m}(X, \mathcal{F}):=\left\{k \in A^{*}(X, \mathcal{F}) ; \pi(k) \in \mathcal{B}_{m}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right), t(k) \in \mathcal{J}_{m}(X, \mathcal{F})\right\}( \tag{6.40}
\end{equation*}
$$

Now we observe that, as vector spaces,

$$
\begin{equation*}
\mathcal{A}_{m} \cong \mathcal{J}_{m} \oplus s\left(\mathcal{B}_{m}\right) \tag{6.41}
\end{equation*}
$$

In order to prove (6.41) we recall the $C^{*}$-sequence $0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*}(X, \mathcal{F}) \xrightarrow{\pi}$ $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow 0$ and the sections $s: B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow A^{*}(X, \mathcal{F})$ and $t: A^{*}(X, \mathcal{F}) \rightarrow C^{*}(X, \mathcal{F})$ defined in (4.8) and (4.13) respectively. Note that Ker $t=\operatorname{Im} s$ since $t(k)=k-s \circ \pi(k)$ and $\pi \circ s(\ell)=\ell$ for $k \in A^{*}(X, \mathcal{F})$ and $\ell \in B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. Moreover, we obviously have $\pi(a)=0$ and $t(a)=a$ for $a \in C^{*}(X, \mathcal{F})$.

Proof of (6.41) Define $\varphi: \mathcal{A}_{m} \rightarrow \mathcal{J}_{m} \oplus s\left(\mathcal{B}_{m}\right)$ by $\varphi(k)=(t(k), s \circ \pi(k))$. Define $\psi: \mathcal{J}_{m} \oplus s\left(\mathcal{B}_{m}\right) \rightarrow \mathcal{A}_{m}$ by $\psi(a, s(\ell))=a+s(\ell)$. Note that $\operatorname{Im} \psi \subset \mathcal{A}_{m}$ since $t(a+s(\ell))=a \in \mathcal{J}_{m}$ and $\pi(a+s(\ell))=\ell \in \mathcal{B}_{m}$. The maps $\varphi$ and $\psi$ are obviously linear. Then we have

$$
\begin{aligned}
\psi \circ \varphi(a, s(\ell))=( & t(a+s(\ell)), s \circ \pi(a+s(\ell)))=(a, s(\ell)), \\
& \varphi \circ \psi(k)=(k-s \circ \pi(k))+s \circ \pi(k)=k
\end{aligned}
$$

and we are done.
We endow $\mathcal{A}_{m}$ with the direct-sum norm:

$$
\begin{equation*}
\|k\|_{\mathcal{A}_{m}}:=\|t(k)\|_{\mathcal{J}_{m}}+\|\pi(k)\|_{\mathcal{B}_{m}} . \tag{6.42}
\end{equation*}
$$

Obviously $s$ induces a bounded linear map $\mathcal{B}_{m} \rightarrow \mathcal{A}_{m}$ of Banach spaces and similarly for $\pi$. Moreover, note that the restriction of the norm $\left\|\|_{\mathcal{A}_{m}}\right.$ to the subalgebra $\mathcal{J}_{m}$ is precisely the norm $\left\|\|_{\mathcal{J}_{m}}\right.$.

We shall prove momentarily that these algebras fits into a short exact sequence; before doing this we prove a useful Lemma. Remark that for a foliation $\left(Y, \mathcal{F}_{Y}\right)$ without boundary, $\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ is a foliation with cylindrical ends; for the latter $\mathcal{J}_{m}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ makes perfect sense.
Lemma 6.43. Recall the function $\chi^{0}$ on $X$ and $\chi_{\mathrm{cyl}}^{0}$ on the cylinder $\operatorname{cyl}(\partial X)$. One has:
(1) $\chi^{0} \mathcal{J}_{m} \subset \mathcal{J}_{m}$ and $\mathcal{J}_{m} \chi^{0} \subset \mathcal{J}_{m}$;
(2) $\quad \chi^{0} \mathcal{J}_{m}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \chi^{0} \subset \mathcal{J}_{m}(X, \mathcal{F})$;
(3) on $\operatorname{cyl}(Y)$, for example on $\operatorname{cyl}(\partial X)$, we have $\mathcal{J}_{m} \mathcal{B}_{m} \subset \mathcal{J}_{m}$ and $\mathcal{B}_{m} \mathcal{J}_{m} \subset \mathcal{J}_{m}$;
(4) $\left(\chi^{0} \mathcal{B}_{m} \chi^{0}\right) \mathcal{J}_{m}(X, \mathcal{F}) \subset \mathcal{J}_{m}(X, \mathcal{F})$ and $\mathcal{J}_{m}(X, \mathcal{F})\left(\chi^{0} \mathcal{B}_{m} \chi^{0}\right) \subset \mathcal{J}_{m}(X, \mathcal{F})$;
(5) $\left(\chi^{0} \mathcal{B}_{m} \chi^{0}\right)\left(\chi^{0} \mathcal{B}_{m} \chi^{0}\right) \subset \chi^{0} \mathcal{B}_{m} \chi^{0}+\mathcal{J}_{m}$.

Proof. (1) The operators $g \chi^{0} k=\chi^{0} g k$ and $\chi^{0} k g$ are bounded if $k \in \mathcal{J}_{m}$. Thus one has $\chi^{0} \mathcal{J}_{m} \subset \mathcal{J}_{m}$. Similarly we proceed for the other inclusion.
(2) The proof is similar to 1 ).
(3) Take $k \in \mathcal{J}_{m}$ and $\ell \in \mathcal{B}_{m}$. Obviously one has $k \ell$ and $\ell k \in \mathcal{I}_{m}$, given that $\mathcal{J}_{m} \subset \mathcal{I}_{m}$ and that $\mathcal{I}_{m}$ is an ideal. Moreover, $g_{\mathrm{cy1}} k \ell$ is bounded and so is

$$
\begin{aligned}
k \ell g_{\mathrm{cyl}} & =k \ell f_{\mathrm{cyl}}^{2}=k\left[\ell, f_{\mathrm{cyl}}\right] f_{\mathrm{cyl}}+k f_{\mathrm{cyl}} \ell f_{\mathrm{cyl}} \\
& \left.\left.=k\left[\ell, f_{\mathrm{cyl}}\right], f_{\mathrm{cyl}}\right]+2 k f_{\mathrm{cyl}} \ell, f_{\mathrm{cyl}}\right]+k g_{\mathrm{cyl}} \ell
\end{aligned}
$$

given that $\left[\left[\ell, f_{\text {cyl }}\right], f_{\text {cyl }}\right], k f_{\text {cyl }},\left[\ell, f_{\text {cyl }}\right]$ and $k g_{\text {cyl }}$ are all bounded. Thus $k \ell \in \mathcal{J}_{m}$. Similarly one proves that $\ell k \in \mathcal{J}_{m}$.
(4) The proof is analogous to the one of 3 ), let us see the details for the second inclusion:

$$
\begin{aligned}
k \chi^{0} \ell \chi^{0} g & =k \chi^{0} \ell g_{\mathrm{cyl}} \chi^{0}=k \chi^{0} \ell f_{\mathrm{cyl}}^{2} \chi^{0}=k \chi^{0}\left[\ell, f_{\mathrm{cyl}}\right] f_{\mathrm{cyl}} \chi^{0}+k \chi^{0} f_{\mathrm{cyl}} \ell f_{\mathrm{cyl}} \chi^{0} \\
& =k \chi^{0}\left[\left[\ell, f_{\mathrm{cyl}}\right], f_{\mathrm{cyl}}\right] \chi^{0}+2 k f \chi^{0}\left[\ell, f_{\mathrm{cyl}}\right] \chi^{0}+k g \chi^{0} \ell \chi^{0}
\end{aligned}
$$

which is easily seen to be bounded using the definitions of $\mathcal{J}_{m}$ and $\mathcal{B}_{m}$. The rest of the proof is similar but easier.
(5) Note that, on the cylinder, $\left[\chi_{\text {cyl }}^{0}, \ell\right] \in \mathcal{J}_{m}$ if $\ell \in \mathcal{B}_{m}$. Thus for $\ell, \ell^{\prime} \in \mathcal{B}_{m}$ we have that $\chi^{0} \ell\left(1-\chi_{\mathrm{cyl}}^{0}\right) \ell^{\prime} \chi^{0}=\chi^{0}\left[\chi_{\mathrm{cyl}}^{0}, \ell\right]\left(1-\chi_{\mathrm{cyl}}^{0}\right)\left[\ell^{\prime}, \chi_{\mathrm{cyl}}^{0}\right] \chi^{0}$ belongs to $\mathcal{J}_{m}$, due to 1 ). This implies that

$$
\chi^{0} \ell \chi^{0} \ell^{\prime} \chi^{0}=\chi^{0} \ell \ell^{\prime} \chi^{0}-\chi^{0} \ell\left(1-\chi^{0}\right) \ell^{\prime} \chi^{0} \in \chi^{0} \mathcal{B}_{m} \chi^{0}+\mathcal{J}_{m}
$$

Proposition 6.44. $\left(\mathcal{A}_{m},\| \|_{\mathcal{A}_{m}}\right)$ is a Banach subalgebra of $A^{*}$. Moreover, $\mathcal{J}_{m}$ is an ideal in $\mathcal{A}_{m}$ and there is a short exact sequence of Banach algebras:

$$
\begin{equation*}
0 \rightarrow \mathcal{J}_{m}(X, \mathcal{F}) \rightarrow \mathcal{A}_{m}(X, \mathcal{F}) \xrightarrow{\pi} \mathcal{B}_{m}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow 0 \tag{6.45}
\end{equation*}
$$

Finally, $t: A^{*}(X, \mathcal{F}) \rightarrow C^{*}(X, \mathcal{F})$ restricts to a bounded section $t: \mathcal{A}_{m}(X, \mathcal{F}) \rightarrow$ $\mathcal{J}_{m}(X, \mathcal{F})$

Proof. Write $k=a+\chi^{0} \ell_{k} \chi^{0}$, with $\pi(k)=\ell_{k}$. By definition $t(k)=k-\chi^{0} \ell_{k} \chi^{0}=a \in$ $\mathcal{J}_{m}(X, \mathcal{F})$. Similarly we write $k^{\prime}=a^{\prime}+\chi^{0} \ell_{k^{\prime}} \chi^{0}$. We thus have

$$
k k^{\prime}=\left(a+\chi^{0} \ell_{k} \chi^{0}\right)\left(a^{\prime}+\chi^{0} \ell_{k^{\prime}} \chi^{0}\right)
$$

Since $\rho$ is an injective homomorphism we check easily that $\ell_{k k^{\prime}}=\ell_{k} \ell_{k^{\prime}}$ We compute, with $\ell \equiv \ell_{k}$ and $\ell^{\prime}=\ell_{k^{\prime}}$,

$$
\begin{aligned}
k k^{\prime} & =\left(a+\chi^{0} \ell \chi^{0}\right)\left(a^{\prime}+\chi^{0} \ell^{\prime} \chi^{0}\right) \\
& =a a^{\prime}+a \chi^{0} \ell^{\prime} \chi^{0}+\chi^{0} \ell \chi^{0} a^{\prime}+\chi^{0} \ell \chi^{0} \chi^{0} \ell^{\prime} \chi^{0} \\
& =a a^{\prime}+a \chi^{0} \ell^{\prime} \chi^{0}+\chi^{0} \ell \chi^{0} a^{\prime}+\chi^{0} \ell\left(\chi_{\mathrm{cyl}}^{0}-1\right) \ell^{\prime} \chi^{0}+\chi^{0} \ell \ell^{\prime} \chi^{0} \\
& =a a^{\prime}+a \chi^{0} \ell^{\prime} \chi^{0}+\chi^{0} \ell \chi^{0} a^{\prime}+\chi^{0}\left[\chi_{\mathrm{cyl}}^{0}, \ell\right]\left[\ell^{\prime}, \chi_{\mathrm{cyl}}^{0},\right] \chi^{0}+\chi^{0} \ell \ell^{\prime} \chi^{0} .
\end{aligned}
$$

The first three terms belong to $\mathcal{J}_{m}(X, \mathcal{F})$ because $\mathcal{J}_{m}(X, \mathcal{F})$ is an algebra and because of property 4) in the Lemma ; we also know that, by the very definition of $\mathcal{B}_{m},\left[\chi_{\text {cyl }}^{0}, \ell\right]$ and $\left[\chi_{\text {cyl }}^{0}, \ell^{\prime}\right]$ are in $\mathcal{J}_{m}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ so that their product is in $\mathcal{J}_{m}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. Using this, the second item of the Lemma and the identity $\ell_{k k^{\prime}}=\ell_{k} \ell_{k^{\prime}}$, we finally see that $\mathcal{A}_{m}$ is a subalgebra.

Next we prove that $\mathcal{A}_{m}$ is a Banach algebra. Recall that if $a \in \mathcal{A}_{m}$ then $\|a\|_{J_{m}}=$ $\|a\|_{m}+\|a g\|_{C^{*}}+\|g a\|_{C^{*}} ;$ this clearly satisfies $\left\|a a^{\prime}\right\|_{\mathcal{J}_{m}} \leq\|a\|_{\mathcal{J}_{m}}\left\|a^{\prime}\right\|_{\mathcal{J}_{m}}$. We shall prove that

$$
\left\|a \chi^{0} \ell \chi^{0}\right\|_{\mathcal{J}_{m}} \leq\|a\|_{\mathcal{J}_{m}}\|\ell\|_{\mathcal{B}_{m}} \quad \text { and } \quad\left\|\chi^{0} \ell \chi^{0} a\right\|_{\mathcal{J}_{m}} \leq\|a\|_{\mathcal{J}_{m}}\|\ell\|_{\mathcal{B}_{m}} .
$$

Indeed one has

$$
\begin{aligned}
\left\|a \chi^{0} \ell \chi^{0}\right\|_{\mathcal{J}_{m}}= & \left\|a \chi^{0} \ell \chi^{0}\right\|_{m}+\left\|a \chi^{0} \ell \chi^{0} g\right\|_{C^{*}}+\left\|g a \chi^{0} \ell \chi^{0}\right\|_{C^{*}} \\
= & \left\|a \chi^{0} \ell \chi^{0}\right\|_{m}+\left\|2 a f \chi^{0}[\ell, f] \chi^{0}+a \chi^{0}[[\ell, f], f] \chi^{0}+a g \chi^{0} \ell \chi^{0}\right\|_{C^{*}} \\
& +\left\|g a \chi^{0} \ell \chi^{0}\right\|_{C^{*}}
\end{aligned}
$$

$$
\begin{aligned}
\leq & \|a\|_{m}\|\ell\|_{B^{*}}+2\|a f\|_{m}\|[\ell, f]\|_{m}+\|a\|_{m}\|[[\ell, f], f]\|_{m}+\|a g\|_{C^{*}}\|\ell\|_{B^{*}} \\
& +\|g a\|_{C^{*}}\|\ell\|_{B^{*}} \\
\leq & \|a\|_{\mathcal{J}_{m}}\|\ell\|_{\mathcal{B}_{m}}
\end{aligned}
$$

Similarly one proves the second inequality. Then we have

$$
\begin{aligned}
\left\|k k^{\prime}\right\|_{\mathcal{A}_{m}}= & \left\|a a^{\prime}+a \chi^{0} \ell^{\prime} \chi^{0}+\chi^{0} \ell \chi^{0} a^{\prime}+\chi^{0}\left[\chi_{\mathrm{cy} 1}^{0}, \ell\right]\left[\chi_{\chi_{\mathrm{cy1}}}^{0}, \ell^{\prime}\right] \chi^{0}\right\|_{\mathcal{J}_{m}}+\left\|\ell \ell^{\prime}\right\|_{\mathcal{B}_{m}} \\
\leq & \|a\|_{\mathcal{J}_{m}}\left\|a^{\prime}\right\|_{\mathcal{J}_{m}}+\|a\|_{\mathcal{J}_{m}}\left\|\ell^{\prime}\right\|_{\mathcal{B}_{m}}+\|\ell\|_{\mathcal{B}_{m}}\left\|a^{\prime}\right\|_{\mathcal{J}_{m}} \\
& +\left\|\left[\chi_{\mathrm{cy1}}^{0}, \ell\right]\right\|_{\mathcal{J}_{m}}\left\|\left[\chi_{\mathrm{cy1}}^{0}, \ell^{\prime}\right]\right\|_{\mathcal{J}_{m}}+\|\ell\|_{\mathcal{B}_{m}}\left\|\ell^{\prime}\right\|_{\mathcal{B}^{m}} \\
\leq & \|k\|_{\mathcal{A}_{m}}\left\|k^{\prime}\right\|_{\mathcal{A}_{m}} .
\end{aligned}
$$

Thus $\mathcal{A}_{m}$ is a Banach algebra. Since it is clear that the inclusion of $\mathcal{A}_{m}$ into $A^{*}$ is bounded, we see that $\mathcal{A}_{m}$ is a Banach subalgebra of $A^{*}$. The fact that we obtain a short exact sequence of Banach algebras is now clear. Finally, observe that $t(k)=$ $k-s(\pi(k))$; thus the boundedness of $s$ implies that of $t$.
6.5 Smooth subalgebras defined by the modular automorphisms. The short exact sequence of Banach algebras $0 \rightarrow \mathcal{J}_{m} \rightarrow \mathcal{A}_{m} \rightarrow \mathcal{B}_{m} \rightarrow 0$ does not involve in any way the modular function $\psi$ and the two derivations $\delta_{1}$ and $\delta_{2}$. Thus we cannot expect the two Godbillon-Vey cyclic 2-cocycles to extend to the cyclic cohomology groups of these algebras. For this reason we need to further decrease the size of these subalgebras, taking into account the derivations $\delta_{1}$ and $\delta_{2}$.
6.5.1 Closable derivations defined by commutators. Let $k$ be an element either in $J_{c}(X, \mathcal{F}), A_{c}(X, \mathcal{F})$ or $B_{c}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. We consider $k$ as a $\Gamma$-equivariant family of operators $k=(k(\theta))_{\theta \in T}$ acting on a family of Hilbert spaces $\mathcal{H}_{\theta}$ as in Sections 3 and 4.

We first work on $A_{c}(X, \mathcal{F})$ which we endow with a Banach norm $\left\|\|_{0}\right.$ and denote it as $A_{c}^{0}$. Next, we consider the bimodule $A_{c}^{1}$, as in the preceding subsections, i.e. the bimodule built out of $A_{c}$ by considering operators acting from sections of $E$ to sections of the bundle with new equivariant structure, $E^{\prime}$. We endow the bimodule $A_{c}^{1}$ with a norm $\left\|\|_{1}\right.$. We shall assume that both $\| \|_{0}$ and $\left\|\|_{1}\right.$ are stronger than the $C^{*}$-norm:

$$
\begin{equation*}
\|k\|_{i} \geq\|k\|_{C^{*}}, \quad i=0,1 \tag{6.46}
\end{equation*}
$$

Let $f$ be a smooth function on $\tilde{V} \times T$ and consider the bimodule derivation $\delta:\left(A_{c}^{0},\| \|_{0}\right) \rightarrow\left(A_{c}^{1},\| \|_{1}\right)$ given by $\delta k:=[f, k]$. We assume that $f$ has been chosen so that $[f, k]$ is a $\Gamma$-equivariant family of operators. Note that, then,

$$
(\delta k)(\theta) \xi_{\theta}=f(x, \theta) k(\theta) \xi_{\theta}-k(\theta)\left(f(x, \theta) \xi_{\theta}\right)
$$

for $\xi_{\theta} \in \mathcal{H}_{\theta}$. We don't assume that $f$ is $\Gamma$-invariant, nor we assume that $f$ is compactly supported or even bounded (this being a basic difference with the case of $\chi^{0}$ already considered).

Proposition 6.47. Under the above assumptions we have that $\delta$ is a closable derivation.

Proof. Because of the Lemma above it suffices to show that $\delta$ satisfies the following property:

$$
\text { if }\left\|k_{i}\right\|_{0} \rightarrow 0 \text { and }\left\|\delta k_{i}-k\right\|_{1} \rightarrow 0, \text { with } k_{i} \in A_{c}, \quad \text { then } k=0
$$

Take $\xi, \eta \in C_{c}^{\infty}(\tilde{V} \times T ; E)$; these induce elements $\xi_{\theta}, \eta_{\theta} \in \mathcal{H}_{\theta}$ once we restrict them to $\tilde{V} \times\{\theta\}$. Since, from (6.46) the operator norm $\left\|\left[f, k_{i}\right](\theta)-k(\theta)\right\|$ is less than or equal to $\left\|\left[f, k_{i}\right]-k\right\|_{1}$, which in turn goes to zero, one has

$$
\left\langle\left[f, k_{i}\right](\theta) \xi_{\theta}, \eta_{\theta}\right\rangle \longrightarrow\left\langle k(\theta) \xi_{\theta}, \eta_{\theta}\right\rangle
$$

where $\left\rangle\right.$ denotes the inner product on $\mathcal{H}_{\theta}$. On the other hand

$$
\begin{aligned}
\left|\left\langle\left[f, k_{i}\right](\theta) \xi_{\theta}, \eta_{\theta}\right\rangle\right| & \leq\left|\left\langle f(\cdot, \theta) k_{i}(\theta) \xi_{\theta}, \eta_{\theta}\right\rangle\right|+\left|\left\langle k_{i}(\theta) f(\cdot, \theta) \xi_{\theta}, \eta_{\theta}\right\rangle\right| \\
& =\left|\left\langle k_{i}(\theta) \xi_{\theta}, \bar{f}(\cdot, \theta) \eta_{\theta}\right\rangle\right|+\left|\left\langle f(\cdot, \theta) \xi_{\theta}, k_{i}(\theta)^{*} \eta_{\theta}\right\rangle\right| \\
& \leq\left\|k_{i}(\theta)\right\|\left\|\xi_{\theta}\right\|\left\|\bar{f}(\cdot, \theta) \eta_{\theta}\right\|+\left\|f(\cdot, \theta) \xi_{\theta}\right\|\left\|k_{i}(\theta)\right\|\left\|\eta_{\theta}\right\| \\
& \leq C\left\|k_{i}(\theta)\right\| \\
& \leq C\left\|k_{i}(\theta)\right\|_{0}
\end{aligned}
$$

where $C$ is a constant depending on $\xi, \eta$ and $f$ but independent of $k_{i}$. Note that $\bar{f}(\cdot, \theta) \eta_{\theta}$ and $f(\cdot, \theta) \xi_{\theta}$ are of compact support in $\tilde{V} \times\{\theta\}$ and thus their norms are finite. Thus we obtain

$$
\left|\left\langle\left[f, k_{i}\right](\theta) \xi_{\theta}, \eta_{\theta}\right\rangle\right| \longrightarrow 0 \quad \text { as } \quad i \rightarrow \infty, \quad \text { since } \quad\left\|k_{i}\right\|_{0} \rightarrow 0 .
$$

This implies that $\left\langle k(\theta) \xi_{\theta}, \eta_{\theta}\right\rangle=0$ for any $\xi, \eta \in C_{c}^{\infty}(\tilde{V} \times T ; E)$ and hence the family $(k(\theta))_{\theta \in T}$ is the zero operator. Thus we have proved that $\delta$ is closable.
6.5.2 The smooth subalgebra $\mathfrak{J}_{\mathrm{m}} \subset C^{*}(X, \mathcal{F})$. We apply the above general results to the two derivations $\delta_{1}$ and $\delta_{2}$ introduced in Section 5.10, namely $\delta_{1}:=[\dot{\phi}$, and $\delta_{2}:=[\phi$,$] , with \phi$ equal to the logarithm of the modular function.

Recall, see Section 4.2, the $C^{*}$-algebra $C_{\Gamma}^{*}(\mathcal{H}) \supset C^{*}(X, \mathcal{F})$; it is obtained, by definition, by closing up the subalgebra $C_{\Gamma, c}(\mathcal{H}) \subset \operatorname{End}_{\Gamma}(\mathcal{H})$ consisting of those elements that preserve the continuous field $C_{c}(\tilde{V} \times T, E)$. We set

$$
\operatorname{Dom}\left(\delta_{2}^{\max }\right)=\left\{k \in C_{\Gamma, c}(\mathcal{H}) \mid[\phi, k] \in C_{\Gamma}^{*}(\mathcal{H})\right\}
$$

and

$$
\delta_{2}^{\max }: \operatorname{Dom}\left(\delta_{2}^{\max }\right) \rightarrow C_{\Gamma}^{*}(\mathcal{H}), \quad \delta_{2}^{\max }(k):=[\phi, k] .
$$

The same proof as above establishes that $\delta_{2}^{\max }$ is closable. Similarly, with self-explanatory notation, the bimodule derivation

$$
\delta_{1}^{\max }: \operatorname{Dom}\left(\delta_{1}^{\max }\right) \rightarrow C_{\Gamma}^{*}\left(\mathcal{H}, \mathcal{H}^{\prime}\right), \quad \delta_{1}^{\max }(k):=[\dot{\phi}, k],
$$

with $\operatorname{Dom}\left(\delta_{1}^{\max }\right):=\left\{k \in C_{\Gamma, c}(\mathcal{H}) \mid[\dot{\phi}, k] \in C_{\Gamma}^{*}\left(\mathcal{H}, \mathcal{H}^{\prime}\right)\right\}$ is closable. Let $\bar{\delta}_{j}^{\max }$ be their respective closures; thus, for example,

$$
\bar{\delta}_{2}^{\max }: \operatorname{Dom} \bar{\delta}_{2}^{\max } \subset C_{\Gamma}^{*}(\mathcal{H}) \longrightarrow C_{\Gamma}^{*}(\mathcal{H})
$$

and similarly for $\delta_{1}^{\max }$. Define now

$$
\mathfrak{D}_{2}:=\left\{a \in \operatorname{Dom} \bar{\delta}_{2}^{\max } \cap \mathcal{J}_{m}(X, \mathcal{F}) \mid \bar{\delta}_{2}^{\max } a \in \mathcal{J}_{m}(X, \mathcal{F})\right\}
$$

and $\bar{\delta}_{2}: \mathfrak{D}_{2} \rightarrow \mathcal{J}_{m}(X, \mathcal{F})$ as the restriction of $\bar{\delta}_{2}^{\max }$ to $\mathfrak{D}_{2}$ with values in $\mathcal{J}_{m}(X, \mathcal{F})$. We know from Lemma 6.23 that $\bar{\delta}_{2}$ is a closed derivation. Define similarly $\mathfrak{D}_{1}$ and the closed derivation $\bar{\delta}_{1}$.

We set

$$
\begin{equation*}
\mathfrak{J}_{m}:=\mathcal{J}_{m} \cap \operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right) \equiv \mathcal{J}_{m} \cap \mathfrak{D}_{1} \cap \mathfrak{D}_{2} . \tag{6.48}
\end{equation*}
$$

We endow $\mathfrak{J}_{\mathrm{m}}$ with the norm

$$
\begin{equation*}
\|a\|_{\mathfrak{J}_{\mathrm{m}}}:=\|a\|_{m}+\left\|\bar{\delta}_{2} a\right\|_{m}+\left\|\bar{\delta}_{1} a\right\|_{m} . \tag{6.49}
\end{equation*}
$$

Proposition 6.50. $\mathfrak{J}_{\mathrm{m}}$ is holomorphically closed in $C^{*}(X, \mathcal{F})$.
Proof. We already know that the Banach algebra $\mathcal{J}_{m}$ is holomorphically closed in the $C^{*}$-algebra $C^{*}(X, \mathcal{F})$. On the other hand, we know [Roe88], p. 197 or [Con94], Lemma 2, p. 247, that $\operatorname{Dom}\left(\bar{\delta}_{1}\right)$ and $\operatorname{Dom}\left(\bar{\delta}_{2}\right)$ are holomorphically closed in $\mathcal{J}_{m}$ (since they are the domains of closed derivations). Thus $\mathfrak{J}_{\mathrm{m}}$ is holomorphically closed in $C^{*}(X, \mathcal{F})$ as required.
6.5.3 The smooth subalgebra $\mathfrak{B}_{\mathrm{m}} \subset B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. Consider $\mathcal{B}_{m}$; we consider the derivations $\delta_{1}:=\left[\dot{\phi}_{\partial},\right], \delta_{2}:=\left[\phi_{\partial},\right]$ on the cylinder $\mathbb{R} \times \partial X_{0}$; we have already encountered these derivations in Section 5.11, see more precisely Definition 5.46. Consider first $\delta_{2}$. Define a closed derivation $\bar{\partial}_{2}$ by taking the closure of the closable derivation $\Psi_{c}^{-1}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right) \xrightarrow{\partial_{2}} B^{*}$, with $\partial_{2}(\ell):=\left[\phi_{\partial}, \ell\right]$ and with $\Psi_{c}^{-1}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$ endowed with the norm $\left\|\|\cdot\| \mid\right.$. Then, from Lemma 6.23 , we know that $\left.\bar{\partial}_{2}\right|_{\mathfrak{D}_{2}}$, with

$$
\mathfrak{D}_{2}=\left\{b \in \operatorname{Dom}\left(\bar{\partial}_{2}\right) \mid \bar{\partial}_{2}(b) \in \mathcal{B}_{m}\right\}
$$

is a closed derivation with values in $\mathcal{B}_{m}$. We set $\bar{\delta}_{2}:=\left.\bar{\partial}_{2}\right|_{\mathfrak{D}_{2}}$; thus $\operatorname{Dom}\left(\bar{\delta}_{2}\right)=\mathfrak{D}_{2}$ and $\bar{\delta}_{2}:=\left.\bar{\partial}_{2}\right|_{\mathfrak{D}_{2}}$. A similarly definition of $\bar{\delta}_{1}$ and $\operatorname{Dom}\left(\bar{\delta}_{1}\right)$ can be given.

We set

$$
\begin{equation*}
\mathfrak{B}_{\mathrm{m}}:=\mathcal{B}_{m} \cap \operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right) \equiv \mathcal{B}_{m} \cap \mathfrak{D}_{1} \cap \mathfrak{D}_{2} . \tag{6.51}
\end{equation*}
$$

We endow $\mathfrak{B}_{\mathrm{m}}$ with the norm

$$
\begin{equation*}
\|\ell\|_{\mathfrak{B}_{\mathrm{m}}}:=\|\ell\|_{\mathcal{B}_{m}}+\left\|\bar{\delta}_{1} \ell\right\|_{\mathcal{B}_{m}}+\left\|\bar{\delta}_{2} \ell\right\|_{\mathcal{B}_{m}} . \tag{6.52}
\end{equation*}
$$

Proposition 6.53. $\mathfrak{B}_{\mathrm{m}}$ is holomorphically closed in $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$.

Proof. We already know that the Banach algebra $\mathcal{B}_{m}$ is holomorphically closed in the $C^{*}$-algebra
$B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. On the other hand, we know that $\operatorname{Dom}\left(\bar{\delta}_{1}\right)$ and $\operatorname{Dom}\left(\bar{\delta}_{2}\right)$ are holomorphically closed in $\mathcal{B}_{m}$. Thus $\mathfrak{B}_{\mathrm{m}}$ is holomorphically closed in $B^{*}(\operatorname{cyl}(\partial X)$, $\mathcal{F}_{\text {cyl }}$ ) as required.
6.5.4 The subalgebra $\mathfrak{A}_{\mathrm{m}} \subset A^{*}(X, \mathcal{F})$. Next we consider the Banach algebra $\mathcal{A}_{m}(X, \mathcal{F})$ which is certainly contained in $C_{\Gamma}^{*}(\mathcal{H})$, given that $A_{c}(X, \mathcal{F})$ is contained in $C_{\Gamma, c}(\mathcal{H})$. Consider again $\bar{\delta}_{j}^{\max }$ and restrict it to a derivation with values in $\mathcal{A}_{m}(X, F)$ :

$$
\bar{\delta}_{2}: \mathcal{D}_{2} \rightarrow \mathcal{A}_{m}(X, F)
$$

with $\mathcal{D}_{2}=\left\{a \in \operatorname{Dom} \bar{\delta}_{2}^{\max } \mid \bar{\delta}_{2}^{\max } a \in \mathcal{A}_{m}(X, F)\right\}$ and similarly for $\bar{\delta}_{1}$. We obtain in this way closed derivations $\bar{\delta}_{1}$ and $\bar{\delta}_{2}$ with domains $\operatorname{Dom} \bar{\delta}_{1}=\mathfrak{D}_{1}$ and $\operatorname{Dom} \bar{\delta}_{2}=\mathfrak{D}_{2}$. We set

$$
\begin{equation*}
\mathfrak{A}_{\mathrm{m}}:=\mathcal{A}_{m} \cap \operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right) \cap \pi^{-1}\left(\mathfrak{B}_{\mathrm{m}}\right) . \tag{6.54}
\end{equation*}
$$

We endow the algebra $\mathfrak{A}_{\mathrm{m}}$, which is a subalgebra of $A^{*}$, with the norm

$$
\begin{equation*}
\|k\|_{\mathfrak{A}_{\mathrm{m}}}:=\|k\|_{\mathcal{A}_{m}}+\left\|\bar{\delta}_{1} k\right\|_{\mathcal{A}_{m}}+\left\|\bar{\delta}_{2} k\right\|_{\mathcal{A}_{m}}+\|\pi(k)\|_{\mathfrak{B}_{\mathrm{m}}} . \tag{6.55}
\end{equation*}
$$

It is an easy exercise to show that $\mathfrak{A}_{\mathrm{m}}$ is a Banach algebra.
6.5.5 The modular Schatten extension We can finally state one of the basic results of this whole section:

Proposition 6.56. The map $\pi$ sends $\mathfrak{A}_{\mathrm{m}}$ into $\mathfrak{B}_{\mathbf{m}} ; \mathfrak{I}_{\mathrm{m}}$ is an ideal in $\mathfrak{A}_{\mathrm{m}}$ and we have a short exact sequence of Banach algebras

$$
\begin{equation*}
0 \rightarrow \mathfrak{J}_{\mathrm{m}} \rightarrow \mathfrak{A}_{\mathrm{m}} \xrightarrow{\pi} \mathfrak{B}_{\mathrm{m}} \rightarrow 0 \tag{6.56}
\end{equation*}
$$

The sections $s$ and $t$ restricts to bounded sections $s: \mathfrak{B}_{\mathrm{m}} \rightarrow \mathfrak{A}_{\mathrm{m}}$ and $t: \mathfrak{A}_{\mathrm{m}} \rightarrow \mathfrak{J}_{\mathrm{m}}$.
We give a proof of this Proposition in Section 10.3
6.6 Isomorphisms of K-groups. Let $0 \rightarrow J \rightarrow A \xrightarrow{\pi} B \rightarrow 0$ a short exact sequence of Banach algebras. Recall that $K_{0}(J):=K_{0}\left(J^{+}, J\right) \cong \operatorname{Ker}\left(K_{0}\left(J^{+}\right) \rightarrow \mathbb{Z}\right)$ and that $K\left(A^{+}, B^{+}\right)=K(A, B)$. For the definition of relative K-groups we refer, for example, to [Bla98, HR00,LMP09b]. Recall that a relative $K_{0}$-element for $A \xrightarrow{\pi} B$ with unital algebras $A, B$ is represented by a triple $\left(P, Q, p_{t}\right)$ with $P$ and $Q$ idempotents in $M_{k \times k}(A)$ and $p_{t} \in M_{k \times k}(B)$ a path of idempotents connecting $\pi(P)$ to $\pi(Q)$. The excision isomorphism

$$
\begin{equation*}
\alpha_{\mathrm{ex}}: K_{0}(J) \longrightarrow K_{0}(A, B) \tag{6.58}
\end{equation*}
$$

is given by $\alpha_{\mathrm{ex}}([(P, Q)])=[(P, Q, \mathbf{c})]$ with $\mathbf{c}$ denoting the constant path (this is not necessarily the 0-path, given that we are taking $\left.J^{+}\right)$. In particular, from the short
exact sequence given by the Wiener-Hopf extension of $B^{*} \equiv B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$, see (4.12), we obtain the isomorphism:

$$
\begin{equation*}
\alpha_{\mathrm{ex}}: K_{0}\left(C^{*}(X, \mathcal{F})\right) \xrightarrow{\simeq} K_{0}\left(A^{*}, B^{*}\right) \tag{6.59}
\end{equation*}
$$

whereas from the short exact sequence of subalgebras (6.56) we obtain the "smooth" excision isomorphism

$$
\begin{equation*}
\alpha_{\mathrm{ex}}^{s}: K_{0}\left(\mathfrak{J}_{\mathrm{m}}\right) \xrightarrow{\simeq} K_{0}\left(\mathfrak{A}_{\mathrm{m}}, \mathfrak{B}_{\mathrm{m}}\right) . \tag{6.60}
\end{equation*}
$$

On the other hand, since $\mathfrak{J}_{\mathrm{m}}$ is a smooth subalgebra of $C^{*}(X, \mathcal{F})$ (i.e. it is dense and holomorphically closed), we also have that the inclusion $\iota: \mathfrak{J}_{\mathrm{m}} \hookrightarrow C^{*}(X, \mathcal{F})$ induces an isomorphism $\iota_{*}: K_{0}\left(\mathfrak{J}_{\mathrm{m}}\right) \xrightarrow{\simeq} K_{0}\left(C^{*}(X, \mathcal{F})\right)$. Consider the homomorphism $\iota_{*}: K_{0}\left(\mathfrak{A}_{\mathbf{m}}, \mathfrak{B}_{\mathbf{m}}\right) \rightarrow K_{0}\left(A^{*}, B^{*}\right)$ induced by the inclusion. We have a commutative diagram

and since three of the four arrows are isomorphisms we conclude that $\iota_{*}$ : $K_{0}\left(\mathfrak{A}_{\mathrm{m}}, \mathfrak{B}_{\mathrm{m}}\right) \rightarrow K_{0}\left(A^{*}, B^{*}\right)$ is also an isomorphism. In particular,

$$
\begin{equation*}
K_{0}\left(A^{*}, B^{*}\right) \cong K_{0}\left(C^{*}(X, \mathcal{F})\right) \cong K_{0}\left(\mathfrak{J}_{\mathbf{m}}\right) \cong K_{0}\left(\mathfrak{A}_{\mathbf{m}}, \mathfrak{B}_{\mathbf{m}}\right) \tag{6.62}
\end{equation*}
$$

6.7 Notation. From now on we shall fix the dimension of the leaves, equal to $2 n$, and set

$$
\begin{equation*}
\mathfrak{J}:=\mathfrak{J}_{\mathrm{m}}, \quad \mathfrak{A}:=\mathfrak{A}_{\mathrm{m}} \quad \text { and } \quad \mathfrak{B}:=\mathfrak{B}_{\mathrm{m}} \tag{6.63}
\end{equation*}
$$

with $m=2 n+1$. The short exact sequence in (6.56), for such $m$, is denoted simply as

$$
\begin{equation*}
0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{A} \rightarrow \mathfrak{B} \rightarrow 0 . \tag{6.64}
\end{equation*}
$$

This is the intermediate subsequence, between $0 \rightarrow J_{c} \rightarrow A_{c} \rightarrow B_{c} \rightarrow 0$ and $0 \rightarrow$ $C^{*}(X, \mathcal{F}) \rightarrow A^{*}(X, \mathcal{F}) \rightarrow B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow 0$, that we have mentioned in the introductory remarks in Section 5.1.

## $7 \quad C^{*}$-index Classes: Excision

7.1 Geometric set-up and assumptions. Let $\left(X_{0}, \mathcal{F}_{0}\right), X_{0}=\tilde{M} \times_{\Gamma} T$, be a foliated bundle with boundary. Let $(X, \mathcal{F})$ be the associated foliated bundle with cylindrical ends. We assume that $\tilde{M}$ is of even dimension and consider the $\Gamma$-equivariant family of Dirac operators $D \equiv\left(D_{\theta}\right)_{\theta \in T}$ introduced in Section 3.2. Then $D$ splits into a direct sum $D^{+} \oplus D^{-}$. We denote as before by $D^{\partial} \equiv\left(D_{\theta}^{\partial}\right)_{\theta \in T}$ the boundary family obtained from $D^{+}$, and by $D^{\text {cyl }}$ the operator on the cylindrical foliated manifold $\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ induced by $D^{\partial} ; D^{\text {cyl }}$ is $\mathbb{R} \times \Gamma$-equivariant. From now on we shall make the following fundamental

Assumption: There exists $\epsilon>0$ such that $\forall \theta \in T$

$$
\begin{equation*}
L^{2}-\operatorname{spec}\left(D_{\theta}^{\partial}\right) \cap(-\epsilon, \epsilon)=\emptyset \tag{7.1}
\end{equation*}
$$

For specific examples where this assumption is satisfied, see [LP05].
7.2 Index classes in the closed case. Let $(Y, \mathcal{F}), Y=\tilde{N} \times_{\Gamma} T$, be a closed foliated bundle. We need to recall how in the closed case we can define an index class $\operatorname{Ind}(D) \in K_{*}\left(C^{*}(Y, \mathcal{F})\right)$.
7.2.1 The Connes-Skandalis projection. First recall that given vector bundles $E$ and $F$ on $Y$ with lifts $\widehat{E}, \widehat{F}$ on $\tilde{N} \times T$, we can define the space of $\Gamma$-compactly supported pseudodifferential operators of order $m$, denoted here $\Psi_{c}^{m}(G ; E, F)$. An element $P \in \Psi_{c}^{m}(G ; E, F)$ should be thought of as a $\Gamma$-equivariant family of pseudodifferential operators, $(P(\theta))_{\theta \in T}$ with Schwartz kernel $K_{P}$, a distribution on $G$, of compact support. See [MoN96] and [BP09] for more details.

The space $\Psi_{c}^{\infty}(G ; E, E):=\bigcup_{m \in \mathbb{Z}} \Psi_{c}^{m}(G ; E, E)$ is a filtered algebra. Moreover, assuming $E$ and $F$ to be hermitian and assigning to $P$ its formal adjoint $P^{*}=$ $\left(P_{\theta}^{*}\right)_{\theta \in T}$ gives $\Psi_{c}^{\infty}(G ; E, E)$ the structure of an involutive algebra; the formal adjoint of an element $P \in \Psi_{c}^{m}(G ; E, F)$ is in general an element in $\Psi_{c}^{m}(G ; F, E)$.

Consider now a $\mathbb{Z}_{2}$-graded odd Dirac operator $D=\left(D_{\theta}\right)_{\theta \in T} D_{\theta}=\left(\begin{array}{cc}0 & D_{\theta}^{-} \\ D_{\theta}^{+} & 0\end{array}\right)$, $\left(D_{\theta}^{-}\right)^{*}=D_{\theta}^{+}$acting on a $\mathbb{Z}_{2}$-graded vector bundle $E=E^{+} \oplus E^{-}$. Using the pseudodifferential calculus, one can prove that $D^{+}$admits parametrix $Q \in$ $\Psi_{c}^{-1}\left(G ; E^{-}, E^{+}\right):$

$$
\begin{equation*}
Q D^{+}=\operatorname{Id}-S_{+}, \quad D^{+} Q=\operatorname{Id}-S_{-} \tag{7.2}
\end{equation*}
$$

with remainders $S_{-}$and $S_{+}$that are in $C_{c}^{\infty}\left(G,\left(s^{*} E^{ \pm}\right)^{*} \otimes r^{*} E^{ \pm}\right) \equiv C_{c}^{\infty}\left(Y, \mathcal{F} ; E^{ \pm}\right)$.
All of this is carefully explained in [MoN96]; even more details are given in [BP09].
Consider the projection

$$
P_{Q}:=\left(\begin{array}{cc}
S_{+}^{2} & S_{+}\left(I+S_{+}\right) Q  \tag{7.3}\\
S_{-} D^{+} & I-S_{-}^{2}
\end{array}\right)
$$

See, for example, [Con94] (II.9. $\alpha$ ) and [CM98] (p. 353) for motivation. Set

$$
e_{0}:=\left(\begin{array}{ll}
1 & 0  \tag{7.4}\\
0 & 0
\end{array}\right), \quad e_{1}:=\left(\begin{array}{cc}
0 & 0 \\
0 & 1
\end{array}\right) .
$$

Also denote by $C_{c}^{\infty}(Y, \mathcal{F} ; E)^{++}$the algebra generated by $e_{0}, e_{1}$ and $C_{c}^{\infty}(Y, \mathcal{F} ; E)$. It is isomorphic to the direct sum $C_{c}^{\infty}(Y, \mathcal{F} ; E) \oplus \mathbb{C} e_{0} \oplus \mathbb{C} e_{1}$ as a linear space. Note that there exists a splitting short exact sequence: $0 \rightarrow C_{c}^{\infty}(Y, \mathcal{F} ; E) \rightarrow C_{c}^{\infty}(Y, \mathcal{F} ; E)^{++} \xrightarrow{\pi}$ $\mathbb{C} e_{0} \oplus \mathbb{C} e_{1} \rightarrow 0$, which naturally contains a subsequence $0 \rightarrow C_{c}^{\infty}(Y, \mathcal{F} ; E) \rightarrow$ $C_{c}^{\infty}(Y, \mathcal{F} ; E)^{+} \rightarrow \mathbb{C} \rightarrow 0$, where $C_{c}^{\infty}(Y, \mathcal{F} ; E)^{+}$is the algebra with unit $1=e_{0} \oplus e_{1}$
adjoined. Hence, comparing the induced exact sequences of $K_{0}$-groups, one has the following isomorphism:

$$
\begin{aligned}
K_{0}\left(C_{c}^{\infty}(Y, \mathcal{F} ; E)\right) & :=\operatorname{ker}\left[K\left(C_{c}^{\infty}(Y, \mathcal{F} ; E)^{+}\right) \rightarrow K_{0}(\mathbb{C})\right] \\
& \cong \operatorname{ker}\left[K\left(C_{c}^{\infty}(Y, \mathcal{F} ; E)^{++}\right) \rightarrow K_{0}\left(\mathbb{C} e_{0} \oplus \mathbb{C} e_{1}\right)\right]
\end{aligned}
$$

Now it is easy to verify that $P_{Q}$ and $e_{1}$ are idempotents in $C_{c}^{\infty}(Y, \mathcal{F} ; E)^{++}$. In fact they belong to $C_{c}^{\infty}(Y, \mathcal{F} ; E) \oplus \mathbb{C} e_{1} \subset C_{c}^{\infty}(X, \mathcal{F} ; E)^{++}$(but they are not in $\left.C_{c}^{\infty}(Y, \mathcal{F} ; E)^{+}\right) ;$moreover it is clear that $\pi\left(P_{Q}\right)=e_{1}=\pi\left(e_{1}\right)$. Thus we obtain a class $\left[P_{Q}\right]-\left[e_{1}\right] \in K_{0}\left(C_{c}^{\infty}(Y, \mathcal{F} ; E)\right)$. Notice that this class is well defined in $K_{0}\left(C_{c}^{\infty}(Y, \mathcal{F} ; E)\right)$, independent of the choice of the $\Gamma$-compactly supported parametrix. Recall now that there is an inclusion $C_{c}^{\infty}(Y, \mathcal{F} ; E) \hookrightarrow C^{*}(Y, \mathcal{F} ; E) \equiv \mathbb{K}(\mathcal{E})$; the Connes-Skandalis index class is the image of $\left[P_{Q}\right]-\left[e_{1}\right]$ under the induced homomorphism $K_{0}\left(C_{c}^{\infty}(Y, \mathcal{F} ; E)\right) \rightarrow K_{0}\left(C^{*}(Y, \mathcal{F} ; E)\right)$. Unless strictly necessary we don't introduce a new notation for the Connes-Skandalis index class in $K_{0}\left(C^{*}(Y, \mathcal{F} ; E)\right)$.
7.2.2 The graph projection. If we give up the requirement that the elements in our projection are of $\Gamma$-compact support then we have more representative for the index class. One which is particularly useful in computations of explicit index formulae is the index class defined by the family $e_{D}=\left(e_{D, \theta}\right)_{\theta \in T}$ of projections onto the graph (of the closure) of $D_{\theta}^{+}$. (With common abuse of notation we do not introduce a new symbol for closures.) The projection $e_{D}$ is explicitly given by

$$
e_{D}=\left(\begin{array}{cc}
\left(I+D^{-} D^{+}\right)^{-1} & \left(I+D^{-} D^{+}\right)^{-1} D^{-}  \tag{7.5}\\
D^{+}\left(I+D^{-} D^{+}\right)^{-1} & D^{+}\left(I+D^{-} D^{+}\right)^{-1} D^{-}
\end{array}\right) .
$$

Let $\mathfrak{s}$ be the grading operator on $E$. Define

$$
\begin{equation*}
\widehat{e}_{D}:=e_{D}-e_{1} . \tag{7.6}
\end{equation*}
$$

It is useful to point out, see [MoN96, p. 514], that

$$
\begin{equation*}
\widehat{e}_{D}=(\mathfrak{s}+D)^{-1} \tag{7.7}
\end{equation*}
$$

Notice that $(\mathfrak{s}+D)$ is invertible, indeed

$$
\begin{equation*}
(\mathfrak{s}+D)^{-1}=(\mathfrak{s}+D)\left(1+D^{2}\right)^{-1} \tag{7.8}
\end{equation*}
$$

One proves by finite propagation speed techniques that $\widehat{e}_{D}$ is in $C^{*}(Y, \mathcal{F} ; E)$, see [MoN96] (Section 7) for details; thus the following class is well defined

$$
\begin{equation*}
\left[e_{D}\right]-\left[e_{1}\right] \in K_{0}\left(C^{*}(Y, \mathcal{F} ; E)\right) \tag{7.9}
\end{equation*}
$$

Proposition 7.10. The Connes-Skandalis index class equals the class defined by the graph projection:

$$
\begin{equation*}
\left[P_{Q}\right]-\left[e_{1}\right]=\left[e_{D}\right]-\left[e_{1}\right] \quad \text { in } \quad K_{0}\left(C^{*}(Y, \mathcal{F} ; E)\right) \tag{7.11}
\end{equation*}
$$

For a proof see [MoN96], where two elements $u, v \in C^{*}(Y, \mathcal{F} ; E)^{++}$are explicitly defined such that $u v=P_{Q}$ and $v u=e_{D}$. Here $C^{*}(Y, \mathcal{F} ; E)^{++}$denotes as before the $C^{*}$-algebra generated by $e_{0}, e_{1}$ and $C^{*}(Y, \mathcal{F} ; E)$.

We define the index class associated to $D$, denoted $\operatorname{Ind}(D)$, as this common value, thus

$$
\operatorname{Ind}(D):=\left[P_{Q}\right]-\left[e_{1}\right]=\left[e_{D}\right]-\left[e_{1}\right] \in K_{0}\left(C^{*}(Y, \mathcal{F} ; E)\right.
$$

Remark. One could also introduce the Wassermann projection $W_{D}$, involving the heat kernel of the associated Laplacian, see [CM98]. One can prove that $\left[P_{Q}\right]-\left[e_{1}\right]=$ $\left[e_{D}\right]-\left[e_{1}\right]=\left[W_{D}\right]-\left[e_{1}\right]$ in $K_{0}\left(C^{*}(Y, \mathcal{F} ; E)\right)$.
7.3 The index class $\operatorname{Ind}(\boldsymbol{D})$. We now go back to our foliated bundle with boundary $\left(X_{0}, \mathcal{F}_{0}\right)$ and associated foliated bundle with cylindrical ends $(X, \mathcal{F})$. It is proved in [LP05] that given $D^{+}=\left(D_{\theta}^{+}\right)_{\theta \in T}$, a $\Gamma$-equivariant family with invertible boundary family $\left(D_{\theta}^{\partial}\right)_{\theta \in T}$, there exists a parametrix $Q$ for $D^{+}$with remainders $S_{-}$ and $S_{+}$in $C^{*}(X, \mathcal{F})$ :

$$
\begin{equation*}
Q D^{+}=\operatorname{Id}-S_{+}, \quad D^{+} Q=\operatorname{Id}-S_{-}, \quad S_{ \pm} \in \mathbb{K}(\mathcal{E}) \equiv C^{*}(X, \mathcal{F}) \tag{7.12}
\end{equation*}
$$

Thus, there is a well defined index class in $K_{0}\left(C^{*}(X, \mathcal{F})\right)$, fixed by the ConnesSkandalis projection $P_{Q}$. The construction explained in [LP05] is an extension to the foliated case of the parametrix construction of Melrose, using heavily $b$-calculus techniques; needless to say, all the complications in the foliated context go into dealing with the non-compactness of the leaves.

In Section 10.4 we sketch an elementary treatment of the parametrix construction for Dirac operators on manifolds with cylindrical ends, using one idea from the $b$-calculus but nothing more than the functional calculus on complete manifolds; in particular, we do not use any pseudodifferential calculus. In any case, either via the $b$-pseudodifferential calculus or using this elementary approach, we have the following fundamental result, valid for a Dirac operator on an even dimensional manifold with cylindrical ends with invertible boundary operator:

Theorem 7.13. Set $G=\left(I+D^{-} D^{+}\right)^{-1} D^{-}$and $G^{\prime}=-\chi\left(\left(D_{\text {cyl }}^{+}\right)^{-1}\left(I+D_{\text {cyl }}^{+} D_{\text {cyl }}^{-}\right)^{-1}\right) \chi$, with $\chi$ a smooth approximation of the characteristic function of $(-\infty, 0] \times \partial X_{0}$. Then the operator $Q=G-G^{\prime}$ is an inverse of $D^{+}$modulo $m$-Schatten class operators, with $m>\operatorname{dim} M$.

More generally one can prove the following:
Theorem 7.14. Let $D \equiv\left(D_{\theta}\right)_{\theta \in T}$ be a $\Gamma$-equivariant family of odd Dirac operators on a foliated bundle with cylindrical ends $(X, \mathcal{F}) \equiv\left(\tilde{V} \times_{\Gamma} T, \mathcal{F}\right)$. Assume (7.1). If $\operatorname{dim} M$ is even and $m>\operatorname{dim} M$, then there exists $Q \in \mathcal{L}(\mathcal{E}), S_{ \pm} \in \mathcal{I}_{m}(X, \mathcal{F})$ such that

$$
\begin{equation*}
I-Q D^{+}=S_{-}, \quad I-D^{+} Q=S_{+} \tag{7.15}
\end{equation*}
$$

We discuss the proof of these two Theorems in Section 10.4.
Definition 7.16. The index class associated to a Dirac operator $D=\left(D_{\theta}\right)_{\theta \in T}$ satisfying assumption (7.1) is the Connes-Skandalis index class $\left[P_{Q}\right]-\left[e_{1}\right]$ associated to the parametrix $Q$ appearing in (7.15). It is an element in $K_{0}\left(\mathcal{I}_{m}(X, \mathcal{F})\right) \cong$ $K_{0}\left(C^{*}(X, \mathcal{F})\right)$ for sufficiently large $m$ and denoted by $\operatorname{Ind}(D)$.
7.4 The relative index class $\operatorname{Ind}\left(D, D^{\partial}\right)$. Let $(X, \mathcal{F})$ be a foliated bundle with cylindrical ends. Let $\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ be the associated foliated cylinder and recall the Wiener-Hopf extension

$$
0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*}(X ; \mathcal{F}) \xrightarrow{\pi} B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right) \rightarrow 0
$$

of the $C^{*}$-algebra $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ of translation invariant operators. We shall be concerned with the K-group $K_{*}\left(C^{*}(X, \mathcal{F})\right)$ and the relative group $K_{*}\left(A^{*}(X ; \mathcal{F})\right.$, $\left.B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)\right)$, often denoted simply by $K_{*}\left(A^{*}, B^{*}\right)$. Recall that a relative element in $K_{0}\left(A^{*}, B^{*}\right)$ is represented by a triple ( $P, Q, p_{t}$ ) with $P$ and $Q$ idempotents in $M_{n \times n}\left(A^{*}\right)$ and $p_{t} \in M_{n \times n}\left(B^{*}\right)$ a path of idempotents connecting $\pi(P)$ to $\pi(Q)$.

Denote by $D^{\text {cyl }}$ the Dirac operator induced by $D^{\partial}$ on the cylinder. Consider the triple

$$
\left(e_{D}, e_{1}, p_{t}\right), \quad t \in[1,+\infty], \quad \text { with } p_{t}:= \begin{cases}e_{t D^{\mathrm{cy} 1}} & \text { if } t \in[1,+\infty)  \tag{7.17}\\ e_{1} & \text { if } t=\infty\end{cases}
$$

Proposition 7.18. Let $(X, \mathcal{F})$ be a foliated bundle with cylindrical end as above. Consider the Dirac operator on $X, D=\left(D_{\theta}\right)_{\theta \in T}$. Assume (7.1). Then the graph projection $e_{D}$ defines through (7.17) a relative class in $K_{0}\left(A^{*}, B^{*}\right)$.

We call the class defined above a relative index class and denote it by

$$
\operatorname{Ind}\left(D, D^{\partial}\right) \in K_{0}\left(A^{*}, B^{*}\right)
$$

Note that we could also employ the Wassermann projection in order to define this class; since we shall not need it we omit the (easy) details.

We shall give a proof of this Proposition in Section 10.5.
7.5 Excision for $C^{*}$-index classes. The main goal of this subsection is to state the following

Proposition 7.19. Let $D=\left(D_{\theta}\right)_{\theta \in T}$ be a $\Gamma$-equivariant family of Dirac operators on a foliated manifold with cylindrical ends $X=\tilde{V} \times_{\Gamma} T$. Assume that $\tilde{V}$ is even dimensional. Assume (7.1). Let $\alpha_{\mathrm{ex}}: K_{0}\left(C^{*}(X, \mathcal{F})\right) \rightarrow K_{0}\left(A^{*}, B^{*}\right)$ be the excision isomorphism for the short exact sequence

$$
0 \rightarrow C^{*}(X, \mathcal{F}) \rightarrow A^{*}(X, \mathcal{F}) \rightarrow B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}\right) \rightarrow 0
$$

Then

$$
\begin{equation*}
\alpha_{\mathrm{ex}}(\operatorname{Ind}(D))=\operatorname{Ind}\left(D, D^{\partial}\right) . \tag{7.20}
\end{equation*}
$$

We give a proof of this Proposition in Section 10.6.

## 8 Smooth Pairings

In the previous Section we have proved the existence of $C^{*}$-algebraic index classes. In this Section we shall prove that we can extend the cocycles $\tau_{G V}$ and ( $\left.\tau_{G V}^{r}, \sigma_{G V}\right)$ from $J_{c}$ and $A_{c} \xrightarrow{\pi_{c}} B_{c}$ to the smooth subalgebras $\mathfrak{J}$ and $\mathfrak{A} \xrightarrow{\pi} \mathfrak{B}$ and that we can simultaneously smooth-out our index classes and define them directly in $0 \rightarrow \mathfrak{J} \rightarrow$ $\mathfrak{A} \xrightarrow{\pi} \mathfrak{B} \rightarrow 0$. Once this will be achieved, we will be able to pair directly $\left[\tau_{G V}\right]$ with $\operatorname{Ind}(D)$ and $\left[\tau_{G V}^{r}, \sigma_{G V}\right]$ will $\operatorname{Ind}\left(D, D^{\partial}\right)$. This is, as often happens in higher index theory, a rather crucial point.

### 8.1 Smooth index classes.

Proposition 8.1. Let $D=\left(D_{\theta}\right)_{\theta \in T}$ and $X=\tilde{V} \times_{\Gamma} T$ as above; then the ConnesSkandalis projection $P_{Q}$ belongs to $\mathfrak{J}_{m} \oplus \mathbb{C} e_{1}$ with $m>\operatorname{dim} \tilde{V}$.

Proposition 8.2. Let $e_{D^{\text {cyl }}}$ be the graph projection for the translation invariant Dirac family $D^{\text {cyl }}=\left(D_{\theta}^{\text {cyl }}\right)_{\theta \in T}$ on the cylinder. Then $e_{D^{\text {cyl }}} \in \mathfrak{B}_{m} \oplus \mathbb{C} e_{1}$ with $m>\operatorname{dim} \tilde{V}$. More generally, $\forall s \geq 1$ we have $e_{s\left(D^{\mathrm{cy1}}\right)} \in \mathfrak{B}_{m} \oplus \mathbb{C} e_{1}$ with $m>\operatorname{dim} \tilde{V}$.

Proposition 8.3. Let $e_{D}$ be the graph projection on $X$. Then $e_{D} \in \mathfrak{A}_{m} \oplus \mathbb{C} e_{1}$ with $m>\operatorname{dim} \tilde{V}$.

We give a detailed proof of these three Propositions in Section 10.7.
As a consequence of these three statements we obtain easily the first two items of the following

Theorem 8.4. Consider the modular Schatten extension of Section 6.7, $0 \rightarrow \mathfrak{J} \rightarrow$ $\mathfrak{A} \rightarrow \mathfrak{B} \rightarrow 0$.
(1) The Connes-Skandalis projection defines a smooth index class $\operatorname{Ind}^{s}(D) \in$ $K_{0}(\mathfrak{J})$; moreover, if $\iota_{*}: K_{0}(\mathfrak{J}) \rightarrow K_{0}\left(C^{*}(X, \mathcal{F})\right)$ is the isomorphism induced by the inclusion $\iota$, then $\iota_{*}\left(\operatorname{Ind}^{s}(D)\right)=\operatorname{Ind}(D)$.
(2) The graph projections on $(X, \mathcal{F})$ and $\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ define a smooth relative index class $\operatorname{Ind}^{s}\left(D, D^{\partial}\right) \in K_{0}(\mathfrak{A}, \mathfrak{B})$; moreover, if $\iota_{*}: K_{0}(\mathfrak{A}, \mathfrak{B}) \rightarrow$ $K_{0}\left(A^{*}, B^{*}\right)$ is the isomorphism induced by the inclusion $\iota$, see (6.61), then $\iota_{*}\left(\operatorname{Ind}^{s}\left(D, D^{\partial}\right)\right)=\operatorname{Ind}\left(D, D^{\partial}\right)$.
(3) Finally, if $\alpha_{\mathrm{ex}}^{s}: K_{0}(\mathfrak{J}) \rightarrow K_{0}(\mathfrak{A}, \mathfrak{B})$ is the smooth excision isomorphism, then

$$
\begin{equation*}
\alpha_{\mathrm{ex}}^{s}\left(\operatorname{Ind}^{s}(D)\right)=\operatorname{Ind}^{s}\left(D, D^{\partial}\right) \quad \text { in } \quad K_{0}(\mathfrak{A}, \mathfrak{B}) \tag{8.5}
\end{equation*}
$$

Proof. The fact that the Connes-Skandalis projection $P_{Q}$ defines an index class $\operatorname{Ind}^{s}(D) \in K_{0}(\mathfrak{J})$ such that $\iota_{*}\left(\operatorname{Ind}^{s}(D)\right)=\operatorname{Ind}(D)$ in $K_{0}\left(C^{*}(X, \mathcal{F})\right)$, is a direct consequence of Proposition 8.1. Similarly, the fact that the graph projections on $(X, \mathcal{F})$ and $\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ define a smooth relative index class $\operatorname{Ind}^{s}\left(D, D^{\boldsymbol{\partial}}\right) \in K_{0}(\mathfrak{A}, \mathfrak{B})$ such that $\iota_{*}\left(\operatorname{Ind}^{s}\left(D, D^{\partial}\right)\right)=\operatorname{Ind}\left(D, D^{\partial}\right)$ in $K_{0}\left(A^{*}, B^{*}\right)$ is a direct consequence of Proposition 8.2 and Proposition 8.3. Regarding the third statement, namely that
$\alpha_{\text {ex }}^{s}\left(\operatorname{Ind}^{s}(D)\right)=\operatorname{Ind}^{s}\left(D, D^{\partial}\right)$, we argue as follows. Recall that we have a commutative diagram where all arrows are isomorphism:


Assume, by contradiction, that $\alpha_{\mathrm{ex}}^{s}\left(\operatorname{Ind}^{s}(D)\right)-\operatorname{Ind}^{s}\left(D, D^{\partial}\right) \neq 0$ in $K_{0}(\mathfrak{A}, \mathfrak{B})$. Then $\iota_{*}\left(\alpha_{\mathrm{ex}}^{s}\left(\operatorname{Ind}^{s}(D)\right)\right)-\iota_{*}\left(\operatorname{Ind}^{s}\left(D, D^{\partial}\right)\right) \neq 0$, given that $\iota_{*}$ is an isomorphism. By the commutativity of the diagram we thus have $\alpha_{\mathrm{ex}}\left(\iota_{*}\left(\operatorname{Ind}^{s}(D)\right)\right)-\iota_{*}\left(\operatorname{Ind}^{s}\left(D, D^{\partial}\right)\right) \neq 0$. Since we know that $\iota_{*}\left(\operatorname{Ind}^{s}(D)\right)=\operatorname{Ind}(D)$ and $\iota_{*}\left(\operatorname{Ind}^{s}\left(D, D^{\partial}\right)\right)=\operatorname{Ind}\left(D, D^{\partial}\right)$ we conclude that $\left.\alpha_{\mathrm{ex}}(\operatorname{Ind}(D))-\operatorname{Ind}\left(D, D^{\partial}\right)\right) \neq 0$ and this contradicts the excision formula (7.20) we have already proved.
8.2 Extended cocycles. We begin by recalling the definition of the pairing between $K$-groups and cyclic cohomology groups. First we state it in the absolute case, explaining the pairing between the $K_{0}$-group and the cyclic cohomology group of even degree. Here we shall follow the definition in [Con94] p. 224 rather than the one in [Con85] p.324; notice that the difference in these two definitions is only in the normalizing constants (and more precisely in powers of $2 \pi i$ ).

Let $A$ be an arbitrary Banach algebra with unit. Given a projection $e \in M_{n \times n}(A)$ and a continuous cyclic cocycle $\tau: A^{\otimes(2 p+1)} \rightarrow \mathbb{C}$ of degree $2 p$, the pairing $\langle$,$\rangle :$ $K_{0}(A) \times H C^{2 p}(A) \rightarrow \mathbb{C}$ is defined to be:

$$
\langle[e],[\tau]\rangle=\frac{1}{p!} \sum_{1 \leq i_{0}, i_{1}, \cdots, i_{2 p} \leq n} \tau\left(e_{i_{0} i_{1}}, e_{i_{1} i_{2}}, \cdots, e_{i_{2 p} i_{0}}\right),
$$

where $e_{i j}$ denotes the ( $i, j$ )-component of the idempotent $e$. In the sequel we denote the summation in the right hand side simply by $\tau(e, \ldots, e)$. This also satisfies

$$
\begin{equation*}
\langle[e],[\tau]\rangle=\langle[e],[S \tau]\rangle \tag{8.7}
\end{equation*}
$$

where $S \tau$ is the result of the $S$-operation in cyclic cohomology, see [Con85] and [Con94, p. 193] as well.

If $A$ is not unital, we take the algebra $A^{+}$with unit adjoined. We then extend $\tau$ to a multilinear map $\tau^{+}:\left(A^{+}\right)^{\otimes(2 p+1)} \rightarrow \mathbb{C}$ in such a way that $\tau^{+}\left(a_{0}, a_{1}, \ldots, a_{2 p}\right)=0$ if $a_{i} \in \mathbb{C} 1 \subset A^{+}$for some $0 \leq i \leq 2 p$. It is easily verified that $\tau^{+}$is again a cyclic cocycle on $A^{+}$. We shall often suppress the + in the notation of $\tau^{+}$and denote it simply by $\tau$. Given $\left[e_{1}\right]-\left[e_{0}\right] \in K_{0}(A)$ (note that $e_{i}(i=0,1)$ is a projection in a matrix algebra of $A^{+}$of a certain size), the pairing between $K_{0}(A)$ and $H C^{2 p}(A)$ is defined by the following formula:

$$
\left\langle\left[e_{1}\right]-\left[e_{0}\right],[\tau]\right\rangle=\frac{1}{p!}\left(\tau\left(e_{1}, \ldots, e_{1}\right)-\tau\left(e_{0}, \ldots, e_{0}\right)\right):=\frac{1}{p!}\left[\tau\left(e_{i}, \ldots, e_{i}\right)\right]_{0}^{1}
$$

Next, recall the definition of relative $K_{0}$-group: if $A$ and $B$ are unital Banach algebras and $\pi: A \rightarrow B$ denotes a unital bounded homomorphism, then the relative group $K_{0}(A, B)$ is the abelian group obtained from equivalence classes of triplets $\left(e_{1}, e_{0}, p_{t}\right)$ with $e_{0}$ and $e_{1}$ projections in a matrix algebra of $A$, say $e_{0}, e_{1} \in M_{n \times n}(A)$, and $p_{t}$ a continuous family of projections in $M_{n \times n}(B), t \in[0,1]$, satisfying $\pi\left(e_{i}\right)=p_{i}$ for $i=0,1$. Recall also from Section 5.3 that $(\tau, \sigma)$ is a relative cyclic cocycle of degree $2 p$ if $b \tau=\pi^{*} \sigma$ and $b \sigma=0$ with $\tau \in C_{\lambda}^{2 p}(A)$ and $\sigma \in C_{\lambda}^{2 p+1}(B)$. Then the pairing $K_{0}(A, B) \times H C^{2 p}(A, B) \rightarrow \mathbb{C}$ is defined by

$$
\begin{equation*}
\left\langle\left[\left(e_{1}, e_{0}, p_{t}\right)\right],[(\tau, \sigma)]\right\rangle=\frac{1}{p!}\left(\left[\tau\left(e_{i}, \ldots, e_{i}\right)\right]_{0}^{1}-(2 p+1) \int_{0}^{1} \sigma\left(\left[\dot{p}_{t}, p_{t}\right], p_{t}, \ldots, p_{t}\right) d t\right) . \tag{8.8}
\end{equation*}
$$

One can prove, thanks to the transgression formula of Connes-Moscovici [CM98, p. 354], that this formula is well defined. Notice that we take a piecewise $C^{1}$-family $p_{t}$ in the above formula. Here we need to make the following remark: the family $p_{t}$ in the triplet $\left(e_{1}, e_{0}, p_{t}\right)$ is, by definition, just a continuous family of projections; thus we need to replace it by a piecewise $C^{1}$-family in order to obtain a well-defined pairing. In fact, it is always possible to do this without changing the homotopy class of $p_{t}$ and, thus, the relative class of $\left(e_{1}, e_{0}, p_{t}\right)$. This follows from the following result: given a Banach algebra $\mathcal{A}$ and projections $p_{i}(i=0,1)$ in $\mathcal{A}$ with $\left\|p_{1}-p_{0}\right\|<1$, there exists a $C^{1}$-path $p_{t}$ of projections connecting $p_{0}$ with $p_{1}$ and such that the homotopy class of $p_{t}$ is uniquely determined. See [Bla98, section 4.6]. Taking this as granted, we divide a given continuous path of projections into the composition of small subpaths in such a way that the end points of each small subpath are close enough, namely the distance is less than 1 . Then we replace each small subpath by a $C^{1}$-path, thanks to the result just stated; the resulting homotopy class is the same as the one of the original continuous path. In such a way one can replace a continuous family of projections by a piecewise $C^{1}$-family without changing the relative class in $K$-theory.

Observe now that $\left[\tau_{G V}\right] \in H C^{2}\left(J_{c}\right)$ and $\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right] \in H C^{2}\left(A_{c}, B_{c}\right)$ can be paired with elements in $K_{0}\left(J_{c}\right)$ and $K_{0}\left(A_{c}, B_{c}\right)$ respectively. As in [MoN96], and with the pairing with the index classes in mind, we set

$$
\begin{equation*}
S^{p-1} \tau_{G V}:=\tau_{2 p} \quad \text { and } \quad\left(S^{p-1} \tau_{G V}^{r}, \frac{3}{2 p+1} S^{p-1} \sigma_{G V}\right):=\left(\tau_{2 p}^{r}, \sigma_{2 p+1}\right) \tag{8.9}
\end{equation*}
$$

with $S$ denoting the $S$-operation introduced in [Con85]. Recall the formula $b S \phi=$ $\frac{q+1}{q+3} S b \phi$ for a cyclic cochain of degree $q$ (see [Con85], p. 322). We then have

$$
b \tau_{2 p}^{r}=b S^{p-1} \tau_{G V}^{r}=\frac{3}{2 p+1} S^{p-1} b \tau_{G V}^{r}=\frac{3}{2 p+1} S^{p-1} \pi^{*} \sigma_{G V}=\pi^{*} \sigma_{2 p+1}
$$

and obtain in this way cyclic cohomology classes

$$
\begin{equation*}
\left[\tau_{2 p}\right] \in H C^{2 p}\left(J_{c}\right) \quad \text { and } \quad\left[\left(\tau_{2 p}^{r}, \sigma_{2 p+1}\right)\right] \in H C^{2 p}\left(A_{c}, B_{c}\right) \tag{8.10}
\end{equation*}
$$

Let $n, m$ be integers such as $2 n=$ dimension of $\tilde{V}$ and $m=2 n+1$. Thus $2 n$ is equal to the dimension of leaves in $X=\tilde{V} \times_{\Gamma} S^{1}$.

Proposition 8.11. Let $\mathfrak{J}:=\mathfrak{J}_{\mathbf{m}}$, Then the cocycle $\tau_{2 n}$ extends to a bounded cyclic cocycle on $\mathfrak{J}$.

Proof. By the definition of the $S$ operation in cyclic cohomology, we know that $\tau_{2 n}\left(k_{0}, \ldots, k_{2 n}\right)$ is expressed, up to a multiplicative constant, as the sum of elements of the following type

$$
\begin{array}{r}
\omega_{\Gamma}\left(k_{0} \cdots k_{i-1} \delta_{1}\left(k_{i}\right) k_{i+1} \cdots k_{j-1} \delta_{2}\left(k_{j}\right) k_{j+1} \cdots k_{2 n}\right) \\
-\omega_{\Gamma}\left(k_{0} \cdots k_{i-1} \delta_{2}\left(k_{i}\right) k_{i+1} \cdots k_{j-1} \delta_{1}\left(k_{j}\right) k_{j+1} \cdots k_{2 n}\right)
\end{array}
$$

We know, see Proposition 6.12, that $\omega_{\Gamma}$ is bounded with respect to the $\mathcal{I}_{1}$-norm; moreover, the product appearing in the above formula is bounded from $\mathfrak{J}_{\mathrm{m}}{ }^{\otimes m}$ to $\mathcal{I}_{1}$. This establishes the Proposition.

Proposition 8.12. The eta cocycle $\sigma_{m}$ extends to a bounded cyclic cocycle on $\mathfrak{B}_{m}$.
Proposition 8.13. Assume that $2 p=\operatorname{deg} S^{p-1} \tau_{G V}^{r}>q$ with $q=m(m-1)^{2}-2=$ $8 n^{3}+4 n^{2}-2$. Then the regularized Godbillon-Vey cochain $\tau_{2 p}^{r}=S^{p-1} \tau_{G V}^{r}$ extends to a bounded cyclic cochain on $\mathfrak{A}_{m}$.

We give a detailed proof of these two propositions in Section 10.8.
Fix $m=2 n+1$ with $2 n$ equal to dimension of the leaves and set as usual $\mathfrak{J}:=\mathfrak{J}_{\mathrm{m}}, \mathfrak{A}:=\mathfrak{A}_{\mathrm{m}}, \mathfrak{B}:=\mathfrak{B}_{\mathrm{m}}$. Using the above three Propositions we see that there are well defined classes

$$
\begin{equation*}
\left[\tau_{2 p}\right] \in H C^{2 p}(\mathfrak{J}) \quad \text { for } \quad p \geq n \quad \text { and } \quad\left[\left(\tau_{2 p}^{r}, \sigma_{2 p+1}\right)\right] \in H C^{2 p}(\mathfrak{A}, \mathfrak{B}) \quad \text { for } \quad 2 p>q . \tag{8.14}
\end{equation*}
$$

## 9 Index Theorems

9.1 The higher APS index formula for the Godbillon-Vey cocycle. We now have all the ingredients to state and prove a APS formula for the Godbillon-Vey cocycle. Let us summarize our geometric data.

Geometric data 9.1. We have a foliated bundle with boundary $\left(X_{0}, \mathcal{F}_{0}\right), X_{0}=$ $\tilde{M} \times_{\Gamma} T$. We assume that the dimension of $\tilde{M}$ is even and that all our geometric structures (metrics, connections, etc) are of product type near the boundary. We also consider $(X, \mathcal{F})$, the associated foliation with cylindrical ends. We are given a $\Gamma$-invariant $\mathbb{Z}_{2}$-graded hermitian bundle $\widehat{E}$ on the trivial fibration $\tilde{M} \times T$, endowed with a $\Gamma$-equivariant vertical Clifford structure along $\tilde{M}$. We have a resulting $\Gamma$-equivariant family of Dirac operators $D=\left(D_{\theta}\right)$.

We assume the boundary family to be invertible. Fix $m=2 n+1$ with $2 n$ the dimension of leaves and consider $\mathfrak{J}:=\mathfrak{J}_{\mathrm{m}}, \mathfrak{A}:=\mathfrak{A}_{\mathrm{m}}, \mathfrak{B}:=\mathfrak{B}_{\mathrm{m}}$. We have proved that there are well defined smooth index classes $\operatorname{Ind}^{s}(D) \in K_{0}(\mathfrak{J}), \operatorname{Ind}^{s}\left(D, D^{\boldsymbol{\partial}}\right) \in K_{0}(\mathfrak{A}, \mathfrak{B})$, where the first is given in terms of a parametrix $Q$ and the second given in term of the graph projections $e_{D}$ and $e_{D^{\text {cyl }}}$. Let $T=S^{1}$; consider $\tau_{2 p}:=S^{p-1} \tau_{G V}$ and $\left(\tau_{2 p}^{r}, \sigma_{2 p+1}\right):=\left(S^{p-1} \tau_{G V}^{r}, \frac{3}{2 p+1} S^{p-1} \sigma_{G V}\right)$. The following is a direct consequence of Section 8.2.

Proposition 9.2. There are well defined additive maps:

$$
\begin{gather*}
\left\langle\cdot,\left[\tau_{2 p}\right]\right\rangle: K_{0}(\mathfrak{J}) \rightarrow \mathbb{C}, \quad p \geq n  \tag{9.3}\\
\left\langle\cdot,\left[\left(\tau_{2 p}^{r}, \sigma_{2 p+1}\right)\right]\right\rangle: K_{0}(\mathfrak{A}, \mathfrak{B}) \rightarrow \mathbb{C}, \quad 2 p>q:=m(m-1)^{2}-2, \quad m=2 n+1 . \tag{9.4}
\end{gather*}
$$

Definition 9.5. Let $\left(X_{0}, \mathcal{F}_{0}\right), X_{0}=\tilde{M} \times_{\Gamma} S^{1}$, as above and assume (7.1). The Godbillon-Vey higher index is the number

$$
\begin{equation*}
\operatorname{Ind}_{G V}(D):=\left\langle\operatorname{Ind}^{s}(D),\left[\tau_{2 n}\right]\right\rangle \tag{9.6}
\end{equation*}
$$

with $2 n$ equal to the dimension of the leaves.
Notice that, in fact, $\operatorname{Ind}_{G V}(D):=\left\langle\operatorname{Ind}^{s}(D),\left[\tau_{2 p}\right]\right\rangle$ for each $p \geq n$, see (8.7).
The following theorem is the main result of this paper:
Theorem 9.7. Let $\left(X_{0}, \mathcal{F}_{0}\right)$, with $X_{0}=\tilde{M} \times_{\Gamma} S^{1}$, be a foliated bundle with boundary and let $D=\left(D_{\theta}\right)_{\theta \in S^{1}}$ denote a $\Gamma$-equivariant family of Dirac type operators as in the Geometric Data 9.1.. Assume (7.1) on the boundary family and fix an integer $p$ such that $2 p>q$. Then the following two equalities hold

$$
\begin{equation*}
\operatorname{Ind}_{G V}(D)=\left\langle\operatorname{Ind}^{s}\left(D, D^{\partial}\right),\left[\left(\tau_{2 p}^{r}, \sigma_{2 p+1}\right]\right\rangle=\int_{X_{0}} \mathrm{AS} \wedge \omega_{G V}-\eta_{G V}\right. \tag{9.8}
\end{equation*}
$$

with

$$
\begin{equation*}
\eta_{G V}:=\frac{(2 p+1)}{p!} \int_{0}^{\infty} \sigma_{2 p+1}\left(\left[\dot{p}_{t}, p_{t}\right], p_{t}, \ldots, p_{t}\right) d t, \quad p_{t}:=e_{t D^{\mathrm{cy1}}} \tag{9.9}
\end{equation*}
$$

defining the Godbillon-Vey eta invariant of the boundary family, $\omega_{G V}$ the GodbillonVey differential 3-form introduced in Section 2.5 and AS denoting the form induced on $X_{0}$ by the ( $\Gamma$-invariant) Atiyah-Singer form for the fibration $\tilde{M} \times S^{1} \rightarrow S^{1}$ and the Clifford bundle $\widehat{E}$.

Notice that using the Fourier transformation the Godbillon-Vey eta invariant $\eta_{G V}$ does depend only on the boundary family $D^{\partial} \equiv\left(D_{\theta}^{\partial}\right)_{\theta \in S^{1}}$.

Proof. For notational convenience we set $\tau_{2 p} \equiv \tau_{G V}, \tau_{2 p}^{r} \equiv \tau_{G V}^{r}$ and $\sigma_{2 p+1} \equiv \sigma_{G V}$. We also write $\alpha_{\mathrm{ex}}$ instead of $\alpha_{\mathrm{ex}}^{s}$. The left hand side of formula (9.8) is, by definition, the pairing $\left\langle\left[P_{Q}, e_{1}\right], \tau_{G V}\right\rangle$ with $P_{Q}$ the Connes-Skandalis projection. Recall that $\alpha_{\mathrm{ex}}\left(\left[P_{Q}, e_{1}\right]\right)$ is by definition $\left[P_{Q}, e_{1}, \mathbf{c}\right]$, with $\mathbf{c}$ the constant path with value $e_{1}$.

Since the derivative of the constant path is equal to zero and since $\left.\tau_{G V}^{r}\right|_{\mathfrak{J}}=\tau_{G V}$, using the obvious extension of (5.55), we obtain at once the crucial relation

$$
\begin{equation*}
\left\langle\alpha_{\mathrm{ex}}\left(\left[P_{Q}, e_{1}\right]\right),\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right]\right\rangle=\left\langle\left[P_{Q}, e_{1}\right],\left[\tau_{G V}\right]\right\rangle \tag{9.10}
\end{equation*}
$$

Now we use the excision formula, asserting that $\alpha_{\mathrm{ex}}\left(\left[P_{Q}, e_{1}\right]\right)$ is equal, as a relative class, to $\left[e_{D}, e_{1}, p_{t}\right]$ with $p_{t}:=e_{t D^{\text {cy1 }}}$. Thus

$$
\left\langle\left[e_{D}, e_{1}, p_{t}\right],\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right]\right\rangle=\left\langle\left[P_{Q}, e_{1}\right],\left[\tau_{G V}\right]\right\rangle
$$

which is the first equality in (9.8) (in reverse order). Using also the definition of the relative pairing we can summarize our results so far as follows:

$$
\begin{aligned}
\operatorname{Ind}_{G V}(D) & :=\left\langle\operatorname{Ind}^{s}(D),\left[\tau_{G V}\right]\right\rangle \\
& \equiv\left\langle\left[P_{Q}, e_{1}\right],\left[\tau_{G V}\right]\right\rangle \\
& =\left\langle\alpha_{\mathrm{ex}}\left(\left[P_{Q}, e_{1}\right]\right),\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right]\right\rangle \\
& =\left\langle\left[e_{D}, e_{1}, p_{t}\right],\left[\left(\tau_{G V}^{r}, \sigma_{G V}\right)\right]\right\rangle \\
& :=\frac{1}{p!} \tau_{G V}^{r}\left(e_{D}-e_{1}\right)+\frac{(2 p+1)}{p!} \int_{1}^{+\infty} \sigma_{G V}\left(\left[\dot{p}_{t}, p_{t}\right], p_{t}, \ldots, p_{t}\right) d t \\
& \equiv \frac{1}{p!} \tau_{G V}^{r}\left(\widehat{e}_{D}\right)+\frac{(2 p+1)}{p!} \int_{1}^{+\infty} \sigma_{G V}\left(\left[\dot{p}_{t}, p_{t}\right], p_{t}, \ldots, p_{t}\right) d t
\end{aligned}
$$

with $\widehat{e}_{D}=(D+\mathfrak{s})^{-1}$. Notice that the convergence at infinity of $\int_{1}^{+\infty} \sigma_{G V}\left(\left[\dot{p}_{t}, p_{t}\right]\right.$, $\left.p_{t}, \ldots, p_{t}\right) d t$ follows from the fact that the pairing is well defined. Replace $D$ by $u D, u>0$. We obtain, after a simple change of variable in the integral,

$$
\frac{(2 p+1)}{p!} \int_{u}^{+\infty} \sigma_{G V}\left(\left[\dot{p}_{t}, p_{t}\right], p_{t}, \ldots, p_{t}, p_{t}\right) d t=-\left\langle\operatorname{Ind}^{s}(u D),\left[\tau_{G V}\right]\right\rangle+\frac{1}{p!} \tau_{G V}^{r}\left(\widehat{e}_{u D}\right) .
$$

But the absolute pairing $\left\langle\operatorname{Ind}^{s}(u D),\left[\tau_{G V}\right]\right\rangle$ in independent of $u$ and of course equal to $\operatorname{Ind}_{G V}(D)$; thus

$$
\frac{(2 p+1)}{p!} \int_{u}^{+\infty} \sigma_{G V}\left(\left[\dot{p}_{t}, p_{t}\right], p_{t}, \ldots, p_{t}, p_{t}\right) d t=-\operatorname{Ind}_{G V}(D)+\frac{1}{p!} \tau_{G V}^{r}\left(\widehat{e}_{u D}\right)
$$

The second summand of the right hand side can be proved to converge as $u \downarrow 0$ to $\int_{X_{0}} \mathrm{AS} \wedge \omega_{G V}$ (this employs Getzler rescaling exactly as in [MoN96]). Thus the limit

$$
\frac{(2 p+1)}{p!} \lim _{u \downarrow 0} \int_{u}^{+\infty} \sigma_{G V}\left(\left[\dot{p}_{t}, p_{t}\right], p_{t}, \ldots, p_{t}, p_{t}\right) d t
$$

exists ${ }^{9}$ and is equal to $\int_{X_{0}} \mathrm{AS} \wedge \omega_{G V}-\operatorname{Ind}_{G V}(D)$. The theorem is proved

[^3]REmARK 9.11. The path $p_{t}=e_{t D^{\text {cyl }}}$ is a $C^{1}$-family of projections. In fact, one can easily obtain

$$
\begin{equation*}
\dot{p}_{t}=\frac{d}{d t}\left(\mathfrak{s}+t D^{\mathrm{cyl}}\right)^{-1}=-\left(\mathfrak{s}+t D^{\mathrm{cyl}}\right)^{-1} D^{\mathrm{cyl}}\left(\mathfrak{s}+t D^{\mathrm{cyl}}\right)^{-1} \tag{9.12}
\end{equation*}
$$

where we have used formula (7.7) for $D^{\text {cyl }}$.
9.2 The classic Atiyah-Patodi-Singer index theorem. The classic Ati-yah-Patodi-Singer index theorem on manifolds with cylindrical ends is obtained proceeding as above, but pairing the index class with the 0 -cocycle $\tau_{0}$ and the relative index class with the relative 0 -cocycle $\left(\tau_{0}^{r}, \sigma_{1}\right)$. (If we use the Wassermann projection we don't need to use the $S$ operation; if we use the graph projection then we need to consider $\tau_{2 n}:=S^{n} \tau_{0}$ and $\sigma_{2 n+1}:=S^{n} \sigma_{1}$ with $2 n$ equal to the dimension of the manifold.) Equating the absolute and the relative pairing, as above, we obtain an index theorem. It can be proved that this is precisely the Atiyah-Patodi-Singer index theorem on manifolds with cylindrical ends; in other words, the eta-term we obtain from the relative pairing is precisely the Atiyah-Patodi-Singer eta invariant for the boundary operator. In this computation the explicit formula for $\dot{p}_{t}$, given in (9.12), is employed. As we have pointed out in the Introduction this approach to the classic APS index theorem was announced by the first author in [Mor98].

REmARK. This approach to the classic APS index formula is also a Corollary of the main result of the December 2009 preprint of Lesch, Moscovici and Pflaum [LMP09a], that is, the computation of the Connes-Chern character of the relative homology cycle associated to a Dirac operator on a manifold with boundary in terms of local data and a higher eta cochain for the commutative algebra of smooth functions on the boundary (see also [Get93] and [Wu92]). Needless to say, the results in [LMP09a] go well beyond the computation of the index; however, they don't appear to have much in common with the non-commutative results presented in this paper.
9.3 Gluing formulae for Godbillon-Vey indices. A direct application of our formula is a gluing formula for Godbillon-Vey indices: if $Y:=\tilde{N} \times \Gamma$ is a closed foliated bundle and $\tilde{N}=\tilde{N}^{1} \cup_{H} \tilde{N}^{2}$ with $H$ a $\Gamma$-invariant hypersurfaces, then we obtain

$$
\tilde{N} \times_{\Gamma} T=: Y=X^{1} \cup_{Z} X^{2}:=\left(\tilde{N}^{1} \times_{\Gamma} T\right) \cup_{\left(H \times_{\Gamma} T\right)}\left(\tilde{N}^{2} \times_{\Gamma} T\right)
$$

Under the invertibility assumption (7.1) and assuming all geometric structures to be of product type near $H$, we have, with obvious notation,

$$
\operatorname{Ind}_{G V}(D)=\operatorname{Ind}_{G V}\left(D^{1}\right)+\operatorname{Ind}_{G V}\left(D^{2}\right)
$$

9.4 The Godbillon-Vey eta invariant. Let $Y=\tilde{N} \times_{\Gamma} T$ be a closed foliated bundle and let $D=\left(D_{\theta}\right)_{\theta \in T}$ be an equivariant Dirac family satisfying assumption (7.1). We do not assume that $Y$ is the boundary of a foliated bundle with boundary; in particular, we don't assume that $D$ arises a boundary family. Then, thanks to Proposition 8.12, we know that for $\epsilon>0$ the following integral is well defined $\frac{(2 n+1)}{n!} \int_{\epsilon}^{1 / \epsilon} \sigma_{2 n+1}\left(\left[\dot{p}_{t}, p_{t}\right], p_{t}, \ldots, p_{t}, p_{t}\right) d t$ with $2 n-1$ equal to the dimension of the leaves of $Y$.

If the integral converges as $\epsilon \downarrow 0$ then its value defines the Godbillon-Vey eta invariant of the foliated bundle $\tilde{N} \times_{\Gamma} T$. This is a $C^{*}$-algebraic invariant (precisely because we are assuming (7.1)).

One might speculate that there is a corresponding von Neumann invariant, defined in the same way, but without the assumption (7.1). This is indeed the situation for the von Neumann eta invariant of a measured foliation; it exists without any invertibility assumption on the operator.

## 10 Proofs

In this Section we have collected all long proofs. On the one hand this results in some repetitions leading to one or two additional pages; on the other hand in this way we were able to present the main ideas of this paper without long and technical interruptions.
10.1 Proof of Lemma 4.7. Recall that we want to prove that there exists a bounded linear map $s: B^{*} \rightarrow \mathcal{L}(\mathcal{E})$ extending $s_{c}: B_{c} \rightarrow \mathcal{L}(\mathcal{E}), s_{c}(\ell):=\chi^{0} \ell \chi^{0}$, and that the composition $\rho=\pi s$ induces an injective $C^{*}$-homomorphism $\rho: B^{*} \rightarrow \mathcal{Q}(\mathcal{E})$. Our first task is to make sense of the operators appearing in the statement of the Lemma. Thus consider the function $\chi^{0}$ and its lift to the covering $\tilde{X}:=\tilde{V} \times T$, which will be still denoted by $\chi^{0}$. Consider the family of operators induced by the multiplication operator by $\chi^{0}$. To be precise this consists of the multiplication operators on the Hilbert spaces $L^{2}(\tilde{V} \times\{\theta\})$, for $\theta \in T$, obtained by restriction of $\chi^{0}$ to $\tilde{V} \times\{\theta\}$. Call the resulting family of operators simply the multiplication operator by $\chi^{0}$ and still denote it by $\chi^{0}$. Similarly, we consider $\chi_{\text {cyl }}^{0}$ and the induced multiplication. Given a translation invariant operator $\ell \in B_{c}$, we can consider the compressed element $\chi^{0} \ell \chi^{0}$ as a $\Gamma$-equivariant family of operators acting on the Hilbert spaces $L^{2}(\tilde{V} \times\{\theta\})$; in order to define this element rigorously we decompose the family of Hilbert spaces $\mathcal{H}=\left\{L^{2}(\tilde{V} \times\{\theta\})\right\}_{\theta \in T}$ as follows: write $\mathcal{H}$ as the direct sum

$$
\begin{equation*}
\mathcal{H}=\mathcal{H}_{0} \oplus \mathcal{H}_{\mathrm{cyl}}^{-} \tag{10.1}
\end{equation*}
$$

of families of Hilbert spaces associated to the decomposition $(X, \mathcal{F})=\left(X_{0}\right.$, $\left.\mathcal{F}_{0}\right) \cup_{\left(\partial X_{0}, \mathcal{F}_{\partial}^{-}\right)}\left((-\infty, 0] \times \partial X_{0}, \mathcal{F}_{\text {cyl }}\right) ;$ accordingly $\chi^{0} \ell \chi^{0}$ is represented by a matrix as $\left(\begin{array}{cc}0 \\ 0 & \chi^{0} \ell \chi^{0}\end{array}\right)$. We shall prove below that $\chi^{0} \ell \chi^{0}$ belongs to $\mathcal{L}(\mathcal{E})$ and therefore defines a class in $\mathcal{Q}(\mathcal{E})$. Here observe that $\chi^{0} \ell \chi^{0}$ admits a $\Gamma$-equivariant kernel function on
$\tilde{V} \times \tilde{V} \times T$ for $\ell \in B_{c}$. Although it is not continuous, it is certainly a measurable function.

Sublemma 10.2. Let $\ell \in B_{c}$. Then the element $\chi^{0} \ell \chi^{0}$ belongs to $\mathcal{L}(\mathcal{E})$.
Proof. Let $\chi_{\epsilon}$ be the function introduced in (4.6) and set $\sigma_{\epsilon}=\chi^{0}-\chi_{\epsilon}$. We may assume that $\sigma_{\epsilon}(p)$ converges to zero for almost every $p \in X$ as $\epsilon \rightarrow 0$. We often suppress $\epsilon$ when it is clear from the context. Given $\ell \in B_{c}$, we have $\chi^{0} \ell \chi^{0}-\chi \ell \chi=$ $\sigma \ell \chi+\chi \ell \sigma+\sigma \ell \sigma$. Note that $\sigma \ell \chi, \chi \ell \sigma$ and $\sigma \ell \sigma$ admit kernel functions that have $\Gamma$-compact support (although, again, they are not continuous). For such a function $k$ the $\Gamma$-Hilbert-Schmidt norm $\left\|\|_{2}\right.$ will be defined in Definition 6.1 in Section 6.2. We have

$$
\left\|\sigma_{\epsilon} \ell \chi\right\|_{2}^{2} \leq \sup _{\theta \in T}\left(\int_{\tilde{V}_{\theta} \times \tilde{V}_{\theta}}\left|\chi_{\Gamma}(x) \sigma_{\epsilon}(x) \ell\left(x, x^{\prime}, \theta\right)\right|^{2} d x d x^{\prime}\right), \quad \text { with } \quad \tilde{V}_{\theta} \equiv \tilde{V} \times\{\theta\}
$$

which implies $\left\|\sigma_{\epsilon} \ell \chi\right\|_{2} \rightarrow 0$ as $\epsilon \rightarrow 0$ due to Lebesgue's dominated convergence theorem. A similar argument proves that $\left\|\chi \ell \sigma_{\epsilon}\right\|_{2}$ and $\left\|\sigma_{\epsilon} \ell \sigma_{\epsilon}\right\|_{2}$ also converge to zero. Now, if $k \in C_{c}(G)$ then, see Proposition 6.4, we know that

$$
\begin{equation*}
\|k\|_{C^{*}} \leq\|k\|_{2} \tag{10.3}
\end{equation*}
$$

This implies that $\left\|\chi^{0} \ell \chi^{0}-\chi_{\epsilon} \ell \chi_{\epsilon}\right\|_{C^{*}} \rightarrow 0$ as $\epsilon \rightarrow 0$. We thus obtain $\chi^{0} \ell \chi^{0} \in \mathcal{L}(\mathcal{E})$ for $\ell \in B_{c}$ since $\chi_{\epsilon} \ell \chi_{\epsilon} \in \mathcal{L}(\mathcal{E})$.

This completes the proof of Sublemma 10.2.
We go on establishing a result on the elements of $B_{c}$; it will be often used in the sequel.
Sublemma 10.4. Let $\ell \in B_{c}$. Then $\chi^{\lambda} \ell\left(1-\chi^{\lambda}\right),\left(1-\chi^{\lambda}\right) \ell \chi^{\lambda}$ and $\left[\chi^{\lambda}, \ell\right]$ are all of $\Gamma$-compact support on $\operatorname{cyl}(\partial X)$.

Proof. Recall first that by definition of $B_{c}$ the support of $\ell$ is compact on $(\operatorname{cyl}(\partial X) \times$ $\operatorname{cyl}(\partial X)) / \mathbb{R} \times \Gamma$; observe also that $\chi^{\lambda} \ell-\ell \chi^{\lambda}=\chi^{\lambda} \ell\left(1-\chi^{\lambda}\right)-\left(1-\chi^{\lambda}\right) \ell \chi^{\lambda}, \forall \ell \in B_{c}$. We can explicitly write down the kernels $k_{1}, k_{2}$ and $k$ corresponding to $\chi^{\lambda} \ell(1-$ $\left.\chi^{\lambda}\right),\left(1-\chi^{\lambda}\right) \ell \chi^{\lambda}$ and $\left[\chi^{\lambda}, \ell\right]$. The first two are given by:

$$
\begin{align*}
& k_{1}\left(y, s, y^{\prime}, s^{\prime}, \theta\right)= \begin{cases}\ell\left(y, y^{\prime}, s-s^{\prime}, \theta\right) & \text { if } s \leq-\lambda, s^{\prime} \geq-\lambda \\
0 & \text { otherwise }\end{cases}  \tag{10.5}\\
& k_{2}\left(y, s, y^{\prime}, s^{\prime}, \theta\right)= \begin{cases}\ell\left(y, y^{\prime}, s-s^{\prime}, \theta\right) & \text { if } s^{\prime} \leq-\lambda, \quad s \geq-\lambda \\
0 & \text { otherwise }\end{cases} \tag{10.6}
\end{align*}
$$

whereas the third is obviously given by the relation $\chi^{\lambda} \ell-\ell \chi^{\lambda}=\chi^{\lambda} \ell\left(1-\chi^{\lambda}\right)-(1-$ $\left.\chi^{\lambda}\right) \ell \chi^{\lambda}$, viz.

$$
k\left(y, s, y^{\prime}, s^{\prime}, \theta\right)= \begin{cases}\ell\left(y, y^{\prime}, s-s^{\prime}, \theta\right) & \text { if } s \leq-\lambda, \quad s^{\prime} \geq-\lambda  \tag{10.7}\\ -\ell\left(y, y^{\prime}, s-s^{\prime}, \theta\right) & \text { if } s^{\prime} \leq-\lambda, \quad s \geq-\lambda \\ 0 & \text { otherwise }\end{cases}
$$

In these formulae $y, y^{\prime} \in \partial \tilde{M}, s, s^{\prime} \in \mathbb{R}, \theta \in T$ and we have used the translation invariance of $\ell$ in order to write $\ell\left(s, y, s^{\prime}, y^{\prime}, \theta\right) \equiv \ell\left(y, y^{\prime}, s-s^{\prime}, \theta\right)$. These explicit formulae establish the sublemma; indeed since $\ell$ is of $\mathbb{R} \times \Gamma$-compact support it is immediate to check that the kernels appearing in (10.5), (10.6) and (10.7) are all of $\Gamma$-compact support.

Consider now the map $s_{c}: B_{c} \rightarrow \mathcal{L}(\mathcal{E}), s_{c}(\ell)=\chi^{0} \ell \chi^{0}$, appearing in the statement of Lemma 4.7. The fact that the map $s_{c}$ extends to a bounded linear map $s: B^{*} \rightarrow \mathcal{L}(\mathcal{E})$ is clear; indeed we have

$$
\left\|s_{c}(\ell)\right\|_{C^{*}}=\left\|\chi^{0} \ell \chi^{0}\right\|_{C^{*}} \leq\|\ell\|_{C^{*}}
$$

It remains to show that $\rho:=\pi s$ is a injective and a $C^{*}$-algebra homomorphism. For the latter property observe that $\rho_{c}:=\pi s_{c}$ does satisfy $\rho_{c}\left(\ell \ell^{\prime}\right)=\rho_{c}(\ell) \rho_{c}\left(\ell^{\prime}\right)$ : indeed, if $\ell, \ell^{\prime} \in B_{c}$ then

$$
\begin{aligned}
\rho_{c}\left(\ell \ell^{\prime}\right) & =\pi\left(\chi^{0} \ell \ell^{\prime} \chi^{0}\right)=\pi\left(\left(\chi^{0} \ell \chi^{0} \ell^{\prime} \chi^{0}\right)+\left(\chi^{0} \ell\left(1-\chi^{0}\right) \ell^{\prime} \chi^{0}\right)\right) \\
& =\pi\left(\left(\chi^{0} \ell \chi^{0} \ell^{\prime} \chi^{0}\right)\right)+\pi\left(\left(\chi^{0} \ell\left(1-\chi^{0}\right) \ell^{\prime} \chi^{0}\right)\right)=\pi\left(\chi^{0} \ell \chi^{0} \chi^{0} \ell^{\prime} \chi^{0}\right) \\
& =\pi\left(\chi^{0} \ell \chi^{0}\right) \pi\left(\chi^{0} \ell^{\prime} \chi^{0}\right)=\rho_{c}(\ell) \rho_{c}\left(\ell^{\prime}\right)
\end{aligned}
$$

since $\pi\left(\left(\chi^{0} \ell\left(1-\chi^{0}\right) \ell^{\prime} \chi^{0}\right)\right)=0$ given that $\chi^{0} \ell\left(1-\chi^{0}\right)$ is of $\Gamma$-compact support (we have used Sublemma 4.10 here). By continuity it follows that $\rho\left(\ell \ell^{\prime}\right)=\rho(\ell) \rho\left(\ell^{\prime}\right)$ for $\ell, \ell^{\prime} \in B^{*}$. The fact that it is a $*$-homomorphism is clear.

Injectiveness is implied at once by the following:

$$
\begin{equation*}
s\left(B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}\right)\right) \cap C^{*}(X, \mathcal{F} ; E)=0 . \tag{10.8}
\end{equation*}
$$

Let us prove (10.8). First observe that, because of the translation invariance of the elements in $B_{c}$ we immediately have that $s_{c}\left(B_{c}\right) \cap C_{c}(X, \mathcal{F} ; E)=0$. Next we show that $s_{c}\left(B_{c}\right) \cap C^{*}(X, \mathcal{F} ; E)=0$. Suppose the contrary and let $a \in s_{c}\left(B_{c}\right) \cap$ $C^{*}(X, \mathcal{F} ; E), a \neq 0$. Then $a=\chi^{0} \ell \chi^{0}$ for $\ell \in B_{c}$ and $\exists a_{j} \in C_{c}(X, \mathcal{F} ; E)$ such that $\left\|a_{j}-a\right\|_{C^{*}} \rightarrow 0$ as $j \rightarrow \infty$. The first information tells us that there exists a $c \in \mathbb{R}^{+}$ and $y, y^{\prime} \in \partial \tilde{M}$ such that $a\left(y, t, y^{\prime}, t+c\right) \neq 0$ for each $t>0$. Take a bump-function $\delta(t)$ at $\left(y, t, y^{\prime}, t+c\right)$ with $\|\delta(t)\|_{L^{2}}=1$. Then, keeping the notation $a$ for the operator defined by $a$, we have that for some $\epsilon>0$ we have $\|a(\delta(t))\|_{L^{2}}>\epsilon>0 \forall t>0$. On the other hand, for each fixed $j$ we also have that $\left\|a_{j}(\delta(t))\right\|_{L^{2}} \rightarrow 0$ as $t \rightarrow+\infty$, given that $a_{j}$ is an element of $C_{c}(X, \mathcal{F} ; E)$. Write now $\|a(\delta(t))\|_{L^{2}} \leq\left\|\left(a-a_{j}\right)(\delta(t))\right\|_{L^{2}}+$ $\left\|a_{j}(\delta(t))\right\|_{L^{2}} \leq\left\|\left(a-a_{j}\right)\right\|_{C^{*}}+\left\|a_{j}(\delta(t))\right\|_{L^{2}}$. Then, choosing $j$ big enough we can make the first summand smaller than $\epsilon / 2$. For such a $j$ we can then choose $t$ big enough so that $\left\|a_{j}(\delta(t))\right\|_{L^{2}}$ is also smaller than $\epsilon / 2$. Summarizing, $\epsilon<\|a(\delta(t))\|_{L^{2}}<\epsilon$, a contradiction. Finally, we show that $s\left(B^{*}\right) \cap C^{*}(X, \mathcal{F} ; E)=0$. Assume the contrary and let $\kappa \in s\left(B^{*}\right) \cap C^{*}(X, \mathcal{F} ; E), \kappa \neq 0$. Then $\exists \ell \in B^{*}$ such that $\kappa=s(\ell)$. Choose $\ell_{j} \in B_{c}$ such that $\ell_{j} \rightarrow \ell$; clearly $s\left(\ell_{j}\right)=\chi^{0} \ell_{j} \chi^{0} \rightarrow \kappa$. Set $\kappa_{j}:=s\left(\ell_{j}\right)$, so that $\left\|\kappa_{j}-\kappa\right\|_{C^{*}} \rightarrow 0$. On the other hand there exists $a_{j} \in C_{c}(X, \mathcal{F} ; E)$ such that $\left\|a_{j}-\kappa\right\|_{C^{*}} \rightarrow 0$. Proceeding as above we have that there exists an $\epsilon>0$ such that $\left\|\kappa_{j}(\delta(t))\right\|_{L^{2}}>\epsilon$ for
each $t>0$. Observe now that $\left\|\kappa_{j}(\delta(t))\right\|_{L^{2}} \leq\left\|\kappa_{j}-\kappa\right\|_{C^{*}}+\left\|a_{j}-\kappa\right\|_{C^{*}}+\left\|a_{j}(\delta(t))\right\|_{L^{2}}$ and the right hand side can be made smaller than $\epsilon$ by choosing $j$ and $t$ suitably. Thus, there exists $j$ and $t$ such that $\epsilon<\left\|\kappa_{j}(\delta(t))\right\|_{L^{2}}<\epsilon$, a contradiction.

The proof of Lemma 4.7 is complete.

### 10.2 Proof of Proposition 5.52: $\left(\tau_{G V}^{r}, \sigma_{G V}\right)$ is a relative cyclic 2-cocycle.

We shall in fact directly prove the more general Proposition 5.59 and then show how this Proposition immediately gives a proof of Proposition 5.52.

We are considering $\Omega_{B}:=B_{c}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }} ; E_{\mathrm{cyl}}\right) \oplus B_{c}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}} ; E_{\mathrm{cyl}}, E_{\text {cyl }}^{\prime}\right)$ with the algebra structure given as in Lemma 5.29. Similarly we consider $\Omega_{A}:=$ $A_{c}(X, \mathcal{F} ; E) \oplus A_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right)$ and $\Omega_{J}:=J_{c}(X, \mathcal{F} ; E) \oplus J_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right)$ with the algebra structure given as in Lemma 5.29; the homomorphism $\pi_{c}: A_{c} \rightarrow B_{c}$ induces an algebra homomorphism $\pi_{\Omega}: \Omega_{A} \rightarrow \Omega_{B}$ and a short exact sequence $0 \rightarrow \Omega_{J} \rightarrow$ $\Omega_{A} \xrightarrow{\pi_{\Omega}} \Omega_{B} \rightarrow 0$. Recall the bimodule trace $\omega_{\Gamma}^{\text {cyl }}$ on $B_{c}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }} ; E_{\text {cyl }}, E_{\text {cyl }}^{\prime}\right)$ and that it induces a trace $\tau_{\Gamma}^{\text {cyl }}$ on $\Omega_{B}$. We also have a bimodule trace $\omega_{\Gamma}$ on $J_{C}\left(X, \mathcal{F} ; E, E^{\prime}\right)$ inducing a trace $\tau_{\Gamma}$ on $\Omega_{J}$. Finally, let $\tau_{\Gamma}^{r}$ be the functional on $\Omega_{A}$ induced by $\omega_{\Gamma}^{r}$. In other words, we employ the weight $\omega_{\Gamma}^{r}$ on the algebra $A_{c}(X, \mathcal{F} ; E)$ in order to define a map, still denoted $\omega_{\Gamma}^{r}$, on the bimodule $A_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right)$; then we set

$$
\begin{equation*}
\left.\tau_{\Gamma}^{r}\right|_{A_{c}(X, \mathcal{F} ; E)}:=0,\left.\quad \tau_{\Gamma}^{r}\right|_{A_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right)}:=\omega_{\Gamma}^{r} . \tag{10.9}
\end{equation*}
$$

We know that $\tau_{\Gamma}^{r}$ is not a trace map on the algebra $\Omega_{A}$ since the bimodule regularized trace $\omega_{\Gamma}^{r}$ does not satisfy the tracial property. Remark however that by using Melrose' formula for the $b$-trace of a commutator followed by (5.26), one can show that

$$
\begin{equation*}
\omega_{\Gamma}^{r}\left(k k^{\prime}-k^{\prime} k\right)=\omega_{\Gamma}^{\mathrm{cyl}}\left(\ell\left[\chi^{0}, \ell^{\prime}\right]\right) \tag{10.10}
\end{equation*}
$$

if $k \in A_{c}(X, \mathcal{F} ; E), k^{\prime} \in A_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right), \pi_{c}(k)=\ell, \pi_{c}\left(k^{\prime}\right)=\ell^{\prime}$. Notice that Melrose' proof extend to the regularized weight $\omega_{\Gamma}^{r}$ (even though $\omega_{\Gamma}$ is a weight and not, in general, a trace). Alternatively, we can simply adapt the alternative proof of Proposition 5.19, which works here for the linear functional $\omega_{\Gamma}^{r}: A_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right) \rightarrow \mathbb{C}$; namely we write, using Proposition 5.54, and for $k \in A_{c}(X, \mathcal{F} ; E), k^{\prime} \in A_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right)$,

$$
\begin{aligned}
\omega_{\Gamma}^{r}\left(k k^{\prime}-k^{\prime} k\right)= & \omega_{\Gamma}\left(t\left(k k^{\prime}-k^{\prime} k\right)\right) \\
= & \omega_{\Gamma}\left(\left[a, a^{\prime}\right]+\left[\chi^{\mu} \ell \chi^{\mu}, a^{\prime}\right]+\left[a, \chi^{\mu} \ell^{\prime} \chi^{\mu}\right]-\chi^{\mu} \ell\left(1-\chi^{\mu}\right) \ell^{\prime} \chi^{\mu}\right. \\
& \left.+\chi^{\mu} \ell^{\prime}\left(1-\chi^{\mu}\right) \ell \chi^{\mu}\right) \\
= & \omega_{\Gamma}\left(-\chi^{\mu} \ell\left(1-\chi^{\mu}\right) \ell^{\prime} \chi^{\mu}+\chi^{\mu} \ell^{\prime}\left(1-\chi^{\mu}\right) \ell \chi^{\mu}\right)=\omega_{\Gamma}\left(\ell\left[\chi^{0}, \ell^{\prime}\right]\right) \\
\equiv & -\omega_{\Gamma}\left(\left[\chi^{0}, \ell\right] \ell^{\prime}\right)
\end{aligned}
$$

where we have used the bimodule-trace property for $\omega_{\Gamma}$ in order to justify the third equality. Note also that, with obvious notation, $a \in J_{c}(X, \mathcal{F} ; E), k \in$ $A_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right) \Rightarrow a k \in J_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right) ; a \in J_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right), k \in A_{c}(X, \mathcal{F} ; E) \Rightarrow$
$a k \in J_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right)$, and similarly for $k a$. Thus, in any case, by using (10.10) we obtain immediately that

$$
\begin{equation*}
\tau_{\Gamma}^{r}\left(\kappa \kappa^{\prime}-\kappa^{\prime} \kappa\right)=\tau_{\Gamma}^{\mathrm{cyl}}\left(\lambda\left[\chi^{0}, \lambda^{\prime}\right]\right) \tag{10.11}
\end{equation*}
$$

for $\kappa, \kappa^{\prime} \in \Omega_{A}$ with $\pi_{\Omega}(\kappa)=\lambda \in \Omega_{B}$ and $\pi_{\Omega}\left(\kappa^{\prime}\right)=\lambda^{\prime} \in \Omega_{B}$.
We pause here in order to remark that we have now checked that the algebras and the (regularized) trace functionals we have been considering for a foliated bundle with cylindrical ends do satisfy all the requirements that we had abstracted in the discussion in Section 5.13 leading to Proposition 5.59.

Let $\left\{e_{1}, \ldots, e_{n}\right\}$ be the standard orthonormal basis of $\mathbb{R}^{n}$ and $\Lambda^{*} \mathbb{R}^{n}$ denote the exterior algebra endowed with the induced basis. We shall use standard multi-index notation; thus a generic element of the basis in $\Lambda^{*} \mathbb{R}^{n}$ will be denoted by $e_{\mathbf{J}}$. Then $\Omega_{A} \otimes \Lambda^{*} \mathbb{R}^{n}$ becomes a graded algebra with respect to the multiplication $\left(\kappa \otimes e_{\mathbf{J}}\right)\left(\kappa^{\prime} \otimes\right.$ $\left.e_{\mathbf{I}}\right)=\kappa \kappa^{\prime} \otimes e_{\mathbf{J}} \wedge e_{\mathbf{I}}$ and the grading in $\Lambda^{*} \mathbb{R}^{n}$. (Here we forget the grading originally defined on $\Omega_{A}$.)

Recall that we are assuming the following conditions:

- There exist derivations $\delta_{i}^{A}$ on $\Omega_{A}$ with $i=1, \ldots, n$, which are pairwise commuting and preserve the subalgebra $\Omega_{J}$;
- There exist a derivation $\delta_{i}^{B}$ on $\Omega_{B}$ with $i=1, \ldots, n$ that are compatible with $\delta_{i}^{A}$ on $\Omega_{A}$, namely, they satisfy that $\pi_{\Omega} \delta_{i}^{A}=\delta_{i}^{B} \pi_{\Omega}$;
- the derivations satisfy Stokes' formulas: $\tau_{\Gamma}^{r}\left(\delta_{i}^{A} \kappa\right)=0$ and $\tau_{\Gamma}^{\mathrm{cyl}}\left(\delta_{i}^{B} \lambda\right)=0$ for $\kappa \in \Omega_{A}$ and $\lambda \in \Omega_{B}$ with $i=1, \ldots, n$.
Note that $\delta_{i}^{B}$ are also pairwise commuting since $\delta_{i}^{A}$ are. In the sequel, we often suppress the suffix and simply denote them by $\delta_{i}$.


## Notation:

- Given an element $a \otimes e_{\mathbf{J}} \in \Omega_{A} \otimes \Lambda^{n} \mathbb{R}^{n}$, we set

$$
\begin{equation*}
\left\langle a \otimes e_{\mathbf{J}}\right\rangle_{r}:=\tau_{\Gamma}^{r}(a)\left\langle e_{\mathbf{J}}, e_{1} \wedge \cdots \wedge e_{n}\right\rangle, \tag{10.12}
\end{equation*}
$$

where $\langle$,$\rangle denotes the induced inner product on \Lambda^{*} \mathbb{R}^{n}$.

- We define $D: \Omega_{A} \otimes \Lambda^{*} \mathbb{R}^{n} \rightarrow \Omega_{A} \otimes \Lambda^{*} \mathbb{R}^{n}$ to be $D\left(a \otimes e_{\mathbf{J}}\right):=\sum_{i=1}^{n} \delta_{i} a \otimes e_{i} \wedge e_{\mathbf{J}}$ for $a \otimes e_{\mathbf{J}} \in \Omega_{A} \otimes \Lambda^{*} \mathbb{R}^{n}$.

Lemma 10.13.
(1) $D$ is a skew-derivation on $\Omega_{A} \otimes \Lambda^{*} \mathbb{R}^{n}$ and one has $D^{2}=0$;
(2) Stokes formula holds: $\langle D \kappa\rangle_{r}=0$ for $\kappa \in \Omega_{A} \otimes \Lambda^{n-1} \mathbb{R}^{n}$.

Proof. It is straightforward to see that $D$ is a skew-derivation. Next $D^{2}\left(\kappa \otimes e_{\mathbf{J}}\right)=$ $D\left(\sum_{j=1}^{n} \delta_{j} \kappa \otimes e_{j} \wedge e_{\mathbf{J}}\right)=\sum_{i, j=1}^{n} \delta_{i} \delta_{j} \kappa \otimes e_{i} \wedge e_{j} \wedge e_{\mathbf{J}}=0$ since $\left[\delta_{i}, \delta_{j}\right]=0$ and $e_{i} \wedge e_{j}+e_{j} \wedge e_{i}=0$. The second property is obvious from $\tau_{\Gamma}^{r}\left(\delta_{i} \kappa\right)=0$.

Recall that our starting point is the cyclic $n$-cocycle on $J_{c}$ given by

$$
\tau_{n}\left(a_{0}, \ldots, a_{n}\right)=\frac{1}{n!} \sum_{\alpha \in \mathfrak{S}_{n}} \operatorname{sign}(\alpha) \tau_{\Gamma}\left(a_{0} \delta_{\alpha(1)} a_{1} \ldots \delta_{\alpha(n)} a_{n}\right) \quad a_{i} \in J_{c}
$$

Let us take the multilinear map

$$
\psi\left(k_{0}, \ldots, k_{n}\right):=\left\langle k_{0} D k_{1} \ldots D k_{n}\right\rangle_{r} / n!=\frac{1}{n!} \sum_{\alpha \in \mathfrak{S}_{n}} \operatorname{sign}(\alpha)\left\langle k_{0} \delta_{\alpha(1)} k_{1} \ldots \delta_{\alpha(n)} k_{n}\right\rangle_{r}
$$

with $k_{i} \in A_{c}$ and set $\psi^{(i)}\left(k_{0}, \ldots, k_{n}\right)=\left\langle k_{i} D k_{i+1} \ldots D k_{n} D k_{0} \ldots D k_{i-1}\right\rangle_{r} / n$ ! and $\psi^{(0)}=\psi$ for $i=1 \ldots, n$. Due to Lemma 10.13, we have

$$
\begin{aligned}
\psi^{(i)}\left(k_{0}, \ldots, k_{n}\right)= & (-1)^{n-i}\left\{\left\langle D\left(k_{i} D k_{i+1} \ldots D k_{n}\right) k_{0}\left(D k_{1} \ldots D k_{i-1}\right)\right\rangle_{r}\right. \\
& \left.-\left\langle\left(D k_{i} \ldots D k_{n}\right) k_{0}\left(D k_{1} \ldots D k_{i-1}\right)\right\rangle_{r}\right\} / n! \\
= & (-1)^{n-i+1}\left\langle\left(D k_{i} \ldots D k_{n}\right) k_{0}\left(D k_{1} \ldots D k_{i-1}\right)\right\rangle_{r} / n!
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
b \psi^{(i)}\left(k_{0}, \ldots, k_{n}\right)= & (-1)^{n-i+1}\left\{(-1)^{i-1}\left\langle\left(D k_{i+1} \ldots D k_{n+1}\right) k_{0}\left(D k_{1} \ldots D k_{i-1}\right) k_{i}\right\rangle_{r}\right. \\
& \left.+(-1)^{i}\left\langle k_{i}\left(D k_{i+1} \ldots D k_{n+1}\right) k_{0}\left(D k_{1} \ldots D k_{i-1}\right)\right\rangle_{r}\right\} / n! \\
= & (-1)^{n}\left\langle\left[\left(D k_{i+1} \ldots D k_{n+1}\right) k_{0}\left(D k_{1} \ldots D k_{i-1}\right), k_{i}\right]\right\rangle_{r} / n!
\end{aligned}
$$

and

$$
\begin{aligned}
b \psi\left(k_{0}, \ldots, k_{n}\right) & =(-1)^{n}\left\langle k_{0}\left(D k_{1} \ldots D k_{n}\right) k_{n+1}\right\rangle_{r} / n!+(-1)^{n+1}\left\langle k_{n+1} k_{0}\left(D k_{1} \ldots D k_{n}\right)\right\rangle_{r} / n! \\
& =(-1)^{n}\left\langle\left[k_{0}\left(D k_{1} \ldots D k_{n}\right), k_{n+1}\right]\right\rangle_{r} / n!
\end{aligned}
$$

Set

$$
\begin{equation*}
\tau_{n}^{r}\left(k_{0}, \ldots, k_{n}\right)=\frac{1}{n+1} \sum_{i=0}^{n}(-1)^{n(i+1)} \psi^{(i)}\left(k_{0}, \ldots, k_{n}\right) \tag{10.14}
\end{equation*}
$$

Obviously it is a cyclic $n$-cochain. Recall the $(n+1)$-eta cocycle $\sigma_{n+1}$ associated to $\tau_{n}$. Recall that our goal is to prove the relative cocycle condition $b \tau_{n}^{r}=\left(\pi_{c}\right)^{*} \sigma_{n+1}$. We compute, using the above results,
$b \tau_{n}^{r}\left(k_{0}, \ldots, k_{n+1}\right)=\frac{1}{(n+1)!} \sum_{i=1}^{n+1}(-1)^{n i}\left\langle\left[\left(D k_{i+1} \ldots D k_{n+1}\right) k_{0}\left(D k_{1} \ldots D k_{i-1}\right), k_{i}\right]\right\rangle_{r}$.

Due to (10.11), formula (10.15) is equal to:

$$
\begin{align*}
& \frac{1}{(n+1)!} \sum_{i=1}^{n+1}(-1)^{n i} \sum_{\alpha \in \mathfrak{S}_{n}} \tau_{\Gamma}^{r}\left(\left[\left(\delta_{\alpha(i+1)} k_{i+1} \ldots \delta_{\alpha(n+1)} k_{n+1}\right) k_{0}\left(\delta_{\alpha(1)} k_{1} \ldots \delta_{\alpha(i-1)} k_{i-1}, k_{i}\right]\right)\right. \\
& \quad \times\left\langle e_{\alpha}, e_{1} \wedge \cdots \wedge e_{n}\right\rangle \\
& =\frac{1}{(n+1)!} \sum_{i=1}^{n+1}(-1)^{n i} \sum_{\alpha \in \mathfrak{S}_{n}} \tau_{\Gamma}^{\mathrm{cyl}}\left(\left(\delta_{\alpha(i+1)} \ell_{i+1} \ldots \delta_{\alpha(n+1)} \ell_{n+1}\right)\right. \\
& \quad \times \ell_{0}\left(\delta_{\alpha(1)} \ell_{1} \ldots \delta_{\alpha(i-1)} \ell_{i-1} \delta_{n+1} \ell_{i}\right)\left\langle e_{\alpha}, e_{1} \wedge \cdots \wedge e_{n}\right\rangle, \tag{10.16}
\end{align*}
$$

where we write $\ell_{i}:=\pi_{c}\left(k_{i}\right), e_{\alpha}:=e_{\alpha(i+1)} \wedge \cdots \wedge e_{\alpha(n+1)} \wedge e_{\alpha(1)} \wedge \cdots \wedge e_{\alpha(i-1)}$ and $\delta_{n+1} \ell_{i}:=\left[\chi^{0}, \ell_{i}\right]$. To be precise here, $\alpha$ in the summation above is considered as a bijective mapping from $\{1, \ldots, i-1, i+1, \ldots, n+1\}$ to $\{1, \ldots, n\}$ rather than a permutation of $\{1, \ldots, n\}$. To such an $\alpha$ we shall assign another permutation $\beta \in \mathfrak{S}_{n+1}$ by setting $\beta(i)=n+1$ and $\beta(j)=\alpha(j)$ for $j \neq i$. The signature of these permutations are related as follows:

$$
\begin{aligned}
&\left\langle e_{\alpha}\right.\left., e_{1} \wedge \cdots \wedge e_{n}\right\rangle \\
& \quad=(-1)^{(i-1)(n-i+1)}\left\langle e_{\alpha(1)} \wedge \cdots \wedge e_{\alpha(i-1)} \wedge e_{\alpha(i+1)} \wedge \ldots e_{\alpha(n+1)}, e_{1} \wedge \cdots \wedge e_{n}\right\rangle \\
& \quad=(-1)^{(i-1)(n-i+1)} \operatorname{sign}(\alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{sign}(\alpha) & =\left\langle e_{\beta(1)} \wedge \cdots \wedge e_{\beta(i-1)} \wedge e_{\beta(i+1)} \wedge \cdots \wedge e_{\beta(n+1)}, e_{1} \wedge \cdots \wedge e_{n}\right\rangle \\
& =(-1)^{n-i+1}\left\langle e_{\beta(1)} \wedge \cdots \wedge e_{\beta(n+1)}, e_{1} \wedge \cdots \wedge e_{n+1}\right\rangle=(-1)^{n-i+1} \operatorname{sign}(\beta)
\end{aligned}
$$

Now observing that $\tau_{\Gamma}^{\text {cyl }}$ is a trace map on $B_{c}$ and that $\mathfrak{S}_{n+1}=\cup_{i=1}^{n+1}\{\beta \in$ $\left.\mathfrak{S}_{n+1} \mid \beta(i)=n+1\right\}$, the formula (10.16) turns out to be:

$$
\begin{aligned}
& \frac{1}{(n+1)!} \sum_{i=1}^{n+1}(-1)^{n i} \sum_{\alpha \in \mathfrak{S}_{n}} \tau_{\Gamma}^{\mathrm{cyl}}\left(\ell_{0} \delta_{\beta(1)} \ell_{1} \ldots \delta_{\beta(n+1)} \ell_{n+1}\right)\left\langle e_{\alpha}, e_{1} \wedge \cdots \wedge e_{n}\right\rangle \\
& =\frac{1}{(n+1)!} \sum_{i=1}^{n+1} \sum_{\beta(i)=n+1}(-1)^{n-i+1} \operatorname{sign}(\alpha) \tau_{\Gamma}^{\mathrm{cyl}}\left(\ell_{0} \delta_{\beta(1)} \ell_{1} \ldots \delta_{\beta(n+1)} \ell_{n+1}\right) \\
& =\frac{1}{(n+1)!} \sum_{\beta \in \mathfrak{S}_{n+1}} \operatorname{sign}(\beta) \tau_{\Gamma}^{\mathrm{cyl}}\left(\ell_{0} \delta_{\beta(1)} \ell_{1} \ldots \delta_{\beta(n+1)} \ell_{n+1}\right) .
\end{aligned}
$$

This proves the fundamental equation $b \tau_{n}^{r}=\left(\pi_{c}\right)^{*} \sigma_{n+1}$. The above arguments prove Proposition 5.59 in the case considered in this paper. The more general statement is just an abstraction of this particular case.

We shall apply Proposition 5.59 in order to prove the equation $b \tau_{G V}^{r}=\left(\pi_{c}\right)^{*} \sigma_{G V}$. In order to do this, we simply need to verify the assumption we have made on the derivations $\delta_{i}^{A}, \delta_{l}^{B}$. Recall the situation in Sections 5.10 and 5.11. There exist derivations $\delta_{j}: \Omega_{A} \rightarrow \Omega_{A}$ for $j=1,2$ with $\delta_{1} \kappa=[\dot{\phi}, \kappa], \delta_{2} \kappa=[\phi, \kappa]$, which are pairwise
commuting. These are defined in the same way as in (5.45). There also exist derivations on $\Omega_{B}$ defined in the same way as in (5.48): $\delta_{3} \lambda:=\left[\chi^{0}, \lambda\right], \delta_{2} \lambda:=\left[\phi_{\partial}, \lambda\right]$ and $\delta_{1} \lambda:=\left[\dot{\phi}_{\partial}, \lambda\right]$ for $\lambda \in \Omega_{B}$ (here we denote the corresponding derivations by the same letters). It is straightforward from the definition to verify that $\delta_{1}$ and $\delta_{2}$ are compatible on $\Omega_{A}$ and $\Omega_{B}$. Thus the remaining part is to prove the Stokes formulas. With respect to $\tau_{\Gamma}^{\text {cyl }}$ the formula is already verified in the proof of Proposition 5.49. As far as the regularized trace is concerned, we have

Lemma 10.17. One has $\tau_{\Gamma}^{r}\left(\delta_{1} k\right)=0=\tau_{\Gamma}^{r}\left(\delta_{2} k\right) \quad \forall k \in \Omega_{A}$. Put it differently, $\langle D \alpha\rangle_{r}=0 \forall \alpha \in \Omega_{A} \otimes \Lambda^{1} \mathbb{R}^{2}$.

Proof. Since $\tau_{\Gamma}^{r}$ is an extension of $\omega_{\Gamma}^{r}$, it suffices to show that $\omega_{\Gamma}^{r}\left(\delta_{1} k\right)=0 \quad \forall k \in$ $A_{c}(X, \mathcal{F} ; E)$ and $\omega_{\Gamma}^{r}\left(\delta_{2} k\right) \quad \forall k \in A_{c}\left(X, \mathcal{F} ; E, E^{\prime}\right)$ Recall the definition of $\omega_{\Gamma}^{r}$ given in (5.50). Remark that $[\phi, k]$, which is by definition $\delta_{1}(k)$, is given explicitly at $\left(x, x^{\prime}, \theta\right) \in \tilde{V} \times \tilde{V} \times T$ by $\left(\phi(x, \theta)-\phi\left(x^{\prime}, \theta\right)\right) k\left(x, x^{\prime}, \theta\right)$. Next, from the definition of $\phi$ (it is the logarithm of the Radon-Nikodym derivative of measures that are constant in the normal direction near the boundary), we see that $\pi_{c}([\phi, k])=\left[\phi_{\partial}, \ell\right]$ with $\pi_{c}(k)=\ell$ and with $\phi_{\partial}$ the restriction of $\phi$ to $\partial X_{0}$ (extended to be constant along the cylinder). Thus the value of $\left[\phi_{\partial}, \ell\right]$ at $\left(y, t, y^{\prime}, t^{\prime}, \theta\right)$ is equal to $\left(\phi_{\partial}(y, \theta)-\right.$ $\left.\phi_{\partial}\left(y^{\prime}, \theta\right)\right) \ell\left(y, y^{\prime}, t-t^{\prime}, \theta\right)$. In any case, by applying the definition of $\omega_{\Gamma}^{r}$ (see again (5.50)), which involves $[\phi, k](x, x, \theta)$ and $\left[\phi_{\partial}, \ell\right](y, t, y, t, \theta)$, we immediately get that $\omega_{\Gamma}^{r}\left(\delta_{1} k\right)=0$. Similarly one proves that $\omega_{\Gamma}^{r}\left(\delta_{2} k\right)=0$.

Now all the requirements needed in order to apply Proposition 5.59 are verified for $\tau_{G V}^{r}$ and $\sigma_{G V}$. Thus the proof of the equation $b \tau_{G V}^{r}=\left(\pi_{c}\right)^{*} \sigma_{G V}$ is completed.
10.3 The modular Schatten extension: proof of Proposition 6.56. Recall that we want to show that there is a short exact sequence of Banach algebras $0 \rightarrow$ $\mathfrak{J}_{\mathrm{m}} \rightarrow \mathfrak{A}_{\mathrm{m}} \xrightarrow{\pi} \mathfrak{B}_{\mathrm{m}} \rightarrow 0$ Moreover, the sections $s$ and $t$ restricts to bounded sections $s: \mathfrak{B}_{\mathrm{m}} \rightarrow \mathfrak{A}_{\mathrm{m}}$ and $t: \mathfrak{A}_{\mathrm{m}} \rightarrow \mathfrak{J}_{\mathrm{m}}$.

We begin with two Sublemmas.
Sublemma 10.18. Let us set $B_{c}^{\prime}:=\Psi_{c}^{-1}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$. If $\ell_{0}$ is an element in $B_{c}^{\prime}$, then $\chi^{0} \ell_{0} \chi^{0}$ belongs to $\operatorname{Dom}\left(\bar{\delta}_{j}^{\max }\right)$ for $j=1,2$ and it follows that

$$
\bar{\delta}_{2}^{\max }\left(\chi^{0} \ell_{0} \chi^{0}\right)=\chi^{0}\left[\phi_{\partial}, \ell_{0}\right] \chi^{0} \text { and } \bar{\delta}_{1}^{\max }\left(\chi^{0} \ell_{0} \chi^{0}\right)=\chi^{0}\left[\dot{\phi}_{\partial}, \ell_{0}\right] \chi^{0}
$$

Proof. We shall work on $\delta_{2}$ first. Let $\chi_{\epsilon}$ be a smooth approximation of the function induced by $\chi^{0}$ on $\tilde{V} \times T$. It is easily verified that $\chi^{0} \ell_{0} \chi^{0}$ preserves the continuous field $C_{c}^{\infty}(\tilde{V} \times T)$ and that $\left[\phi, \chi_{\epsilon} \ell_{0} \chi_{\epsilon}\right]=\chi_{\epsilon}\left[\phi_{\partial}, \ell_{0}\right] \chi_{\epsilon}$ belongs to $C_{\Gamma}^{*}(\mathcal{H})$, since $\left[\phi_{\partial}, \ell_{0}\right]$ is again a compactly supported pseudodifferential operator of order -1 . Thus one has $\chi_{\epsilon} \ell_{0} \chi_{\epsilon} \in \operatorname{Dom}\left(\delta_{2}^{\max }\right)$ and $\delta_{2}^{\max }\left(\chi_{\epsilon} \ell_{0} \chi_{\epsilon}\right)=\chi_{\epsilon}\left[\phi_{\partial}, \ell_{0}\right] \chi_{\epsilon}$. Next we observe that $\left\|\chi_{\epsilon} b \chi_{\epsilon}-\chi^{0} b \chi^{0}\right\|_{C^{*}} \longrightarrow 0$ as $\epsilon \rightarrow 0$ for any $b \in B_{c}^{\prime}$. Indeed, according to Lemma 6.26 we can choose an approximating sequence $\left\{b_{i}\right\}$ in $B_{c}$ such that $\left\|b_{i}-b\right\|_{C^{*}} \rightarrow 0$; then one has
$\left\|\chi_{\epsilon} b \chi_{\epsilon}-\chi^{0} b \chi^{0}\right\|_{C^{*}} \leq\left\|\chi_{\epsilon}\left(b-b_{i}\right) \chi_{\epsilon}\right\|_{C^{*}}+\left\|\chi_{\epsilon} b_{i} \chi_{\epsilon}-\chi^{0} b_{i} \chi^{0}\right\|_{C^{*}}+\left\|\chi^{0}\left(b_{i}-b\right) \chi^{0}\right\|_{C^{*}} \longrightarrow 0$
since $\left\|\chi_{\epsilon} b_{i} \chi_{\epsilon}-\chi^{0} b_{i} \chi^{0}\right\|_{C^{*}} \longrightarrow 0$ for $b_{i} \in B_{c}$ due to Sublemma 10.2. This implies that $\left\|\chi_{\epsilon} \ell_{0} \chi_{\epsilon}-\chi^{0} \ell_{0} \chi^{0}\right\|_{C^{*}} \longrightarrow 0$ and that

$$
\left\|\delta_{2}^{\max }\left(\chi_{\epsilon} \ell_{0} \chi_{\epsilon}\right)-\chi^{0}\left[\phi_{\partial}, \ell_{0}\right] \chi^{0}\right\|=\left\|\chi_{\epsilon}\left[\phi_{\partial}, \ell_{0}\right] \chi_{\epsilon}-\chi^{0}\left[\phi_{\partial}, \ell_{0}\right] \chi^{0}\right\|_{C^{*}} \rightarrow 0
$$

as $\epsilon \downarrow 0$. Since $\chi_{\epsilon} \ell_{0} \chi_{\epsilon} \in C_{\Gamma, c}(\mathcal{H})$ this proves that $\chi^{0} \ell_{0} \chi^{0}$ belongs to $\operatorname{Dom}\left(\bar{\delta}_{2}^{\max }\right)$ and that $\bar{\delta}_{2}^{\max }\left(\chi^{0} \ell_{0} \chi^{0}\right)=\chi^{0}\left[\phi_{\partial}, \ell_{0}\right] \chi^{0}$ as required. We can apply a similar argument to the second derivation and prove that $\chi^{0} \ell_{0} \chi^{0}$ belongs to $\operatorname{Dom}\left(\bar{\delta}_{1}^{\max }\right)$ and that $\bar{\delta}_{1}^{\max }\left(\chi^{0} \ell_{0} \chi^{0}\right)=\chi^{0}\left[\dot{\phi}_{\partial}, \ell_{0}\right] \chi^{0}$.

Sublemma 10.19. Assume that $\ell \in \mathcal{B}_{m} \cap \operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right)$. Then $s(\ell) \in$ $\operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right)$ and $\bar{\delta}_{j}(s(\ell))=s\left(\bar{\delta}_{j} \ell\right)$ for $j=1,2$.

Proof. Notice that we employ the same notation for the derivations on the cylinder $\operatorname{cyl}(\partial X)$ and on $X$; this should not cause confusion here. Let $\ell$ be an element $\operatorname{Dom}\left(\bar{\delta}_{2}\right)$. Then, by definition, there exists a sequence $\left\{\ell_{i}\right\} \in B_{c}^{\prime}$ such that $\left\|\left|\left|\ell_{i}-\ell\right| \| \rightarrow 0\right.\right.$ and $\left[\phi_{\partial}, \ell_{i}\right]$ converges in $C^{*}$-norm as $i \rightarrow+\infty$. Thus, there exists an element $\bar{\delta}_{2} \ell \in \mathcal{B}_{m}$. We then obtain $\left\|s\left(\ell_{i}\right)-s(\ell)\right\|_{C^{*}} \rightarrow 0$ and $\left\|s\left(\left[\phi_{\partial}, \ell_{i}\right]\right)-s\left(\bar{\delta}_{2} \ell\right)\right\|_{C^{*}} \rightarrow 0$, since we certainly have $\|s(\ell)\|_{C^{*}} \leq\|\ell\|_{C^{*}} \leq\|\ell\| \|$ for $\ell \in B^{*}$. Using the previous sublemma we have $s\left(\left[\phi_{\partial}, \ell_{i}\right]\right):=\chi^{0}\left[\phi_{\partial}, \ell_{i}\right] \chi^{0}=\bar{\delta}_{2}^{\max }\left(\chi^{0} \ell_{i} \chi^{0}\right)=\bar{\delta}_{2}^{\max }{ }_{s}\left(\ell_{i}\right)$. Hence we obtain

$$
\left\|\bar{\delta}_{2}^{\max }\left(s\left(\ell_{i}\right)\right)-s\left(\bar{\delta}_{2}(\ell)\right)\right\|_{C^{*}} \rightarrow 0
$$

Since $\bar{\delta}_{2}^{\max }$ is a closed derivation, this proves that $\bar{\delta}_{2}^{\max }(s(\ell))=s\left(\bar{\delta}_{2}(\ell)\right)$. Now recall that that $\mathcal{A}_{m} \cong \mathcal{J}_{m} \oplus s\left(\mathcal{B}_{m}\right)$, see (6.41). Then one has $\bar{\delta}_{2}^{\max }(s(\ell))=s\left(\bar{\delta}_{2}(\ell)\right) \in$ $s\left(\mathcal{B}_{m}\right) \subset \mathcal{A}_{m}$, since $\bar{\delta}_{2}(\ell) \in \mathcal{B}_{m}$. This implies that $s(\ell) \in \operatorname{Dom}\left(\bar{\delta}_{2}\right)$ by the definition of domain for $\overline{\delta_{2}}$ and thus yields $\bar{\delta}_{2}(s(\ell))=\bar{\delta}_{2}^{\max }(s(\ell))=s\left(\bar{\delta}_{2}(\ell)\right)$. A similar argument will work for $\overline{\delta_{1}}$. The proof of this second Sublemma is completed.

We now go back to the proof of Proposition 6.56. First we show that $\mathfrak{A}_{\mathrm{m}}$ is isomorphic as Banach space to the direct sum $\mathfrak{J}_{\mathbf{m}} \oplus s\left(\mathfrak{B}_{\mathbf{m}}\right)$, in a way compatible with the identification $\psi: \mathcal{J}_{m} \oplus s\left(\mathcal{B}_{m}\right) \rightarrow \mathcal{A}_{m}$ sending $(k, s(\ell))$ to $k+s(\ell)$ explained in (6.41). Let $\ell$ be an element in $\mathfrak{B}_{\mathbf{m}}$, which is by definition $\mathcal{B}_{m} \cap \operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right)$. Using the last Sublemma we then see that $s(\ell) \in \mathcal{A}_{m} \cap \operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right)$ and hence that $s(\ell) \in$ $\mathfrak{A}_{\mathbf{m}}$, given that $\pi \circ s(\ell)=\ell \in \mathfrak{B}_{\mathbf{m}}$. Moreover, if $a \in \mathfrak{J}_{\mathbf{m}}:=\mathcal{J}_{m} \cap \operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right)$, then we certainly have $a \in \mathfrak{A}_{\mathbf{m}}$ since $\pi(a)=0 \in \mathfrak{B}_{\mathbf{m}}$. This proves that $\mathfrak{J}_{\mathbf{m}} \oplus s\left(\mathfrak{B}_{\mathbf{m}}\right)$ is sent into $\mathfrak{A}_{\mathrm{m}}$ by $\psi$. Conversely, given $k \in \mathcal{A}_{m}$ we can write $k=a+s(\ell)$, with $a \in \mathcal{J}_{m}$ and $\ell \in \mathcal{B}_{m}$. If $k \in \mathfrak{A}_{\mathbf{m}}$, then $\pi(k)=\pi(a)+\pi(s(\ell))=\ell \in \mathfrak{B}_{\mathbf{m}}$ by definition of $\mathfrak{A}_{\mathbf{m}}$. This implies in turn that $a=k-s(\ell) \in \mathfrak{A}_{\mathbf{m}}$ because $k$ and $s(\ell)$ belong to $\mathfrak{A}_{\mathbf{m}}$. We have proved above that $\ell \in \mathfrak{B}_{\mathbf{m}} \Rightarrow s(\ell) \in \mathfrak{A}_{\mathbf{m}}$; thus $a \in \mathfrak{A}_{\mathrm{m}} \cap \mathcal{J}_{m}$ which is nothing but $\mathfrak{J}_{\mathbf{m}}$ by definition. This proves that $k=a+s(\ell)$ belongs to the image of $\mathfrak{J}_{\mathbf{m}} \oplus s\left(\mathfrak{B}_{\mathbf{m}}\right)$ through (6.41). Thus we have established that $\mathfrak{A}_{\mathbf{m}}$ is isomorphic to the direct sum $\mathfrak{J}_{\mathbf{m}} \oplus s\left(\mathfrak{B}_{\mathbf{m}}\right)$. Now it is clear that the sequence (6.56) $0 \rightarrow \mathfrak{J}_{\mathrm{m}} \rightarrow \mathfrak{A}_{\mathrm{m}} \xrightarrow{\pi} \mathfrak{B}_{\mathrm{m}} \rightarrow 0$ is exact, since $\pi \circ s=\mathrm{Id}$ on $\mathfrak{B}_{\mathbf{m}}$. Moreover, one has

$$
\bar{\delta}_{j}(k)=\bar{\delta}_{j}(a)+\bar{\delta}_{j}(s(\ell))=\bar{\delta}_{j}(a)+s\left(\bar{\delta}_{j}(\ell)\right) .
$$

This proves that $\bar{\delta}_{j}$ commutes with $\pi: \mathfrak{A}_{\mathbf{m}} \rightarrow \mathfrak{B}_{\mathrm{m}}$ as well as with $s: \mathfrak{B}_{\mathbf{m}} \rightarrow \mathfrak{A}_{\mathrm{m}}$. This implies that $\pi$ and $s$ are bounded linear maps. The boundedness of $t$ follows from that of $s$. Finally, it is obvious that $\pi$ is a homomorphism and that $\mathfrak{J}_{\mathrm{m}}=\operatorname{Ker} \pi$ is an ideal in $\mathfrak{A}_{\mathrm{m}}$.
10.4 The index class: an elementary approach to the parametrix construction. In this Subsection we sketch a proof of Theorem 7.13 and Theorem 7.14. We first recall some elementary results for a Dirac operator $D$ on an even dimensional manifold $X$ with cylindrical end obtained from a Riemannian manifold $\left(X_{0}, g\right)$ with boundary $\partial X_{0}=Y$ and with $g$ a product metric near the boundary. As usual we denote the infinite cylinder $\mathbb{R} \times \partial X_{0} \equiv \mathbb{R} \times Y$ by the simple notation $\operatorname{cyl}(Y)$. Finally, we denote by $\mathfrak{s}$ the grading operator on the $\mathbb{Z}_{2}$-graded bundle $E$ on which $D$ acts; we shall employ the same symbol for the grading on the induced bundle on the cylinder. The following lemmas are elementary.

Lemma 10.20. Let $f \in C^{\infty}(X)$. We assume that $f$ and $d f$ are bounded. Then we have the following equality of $L^{2}$-bounded operators $\left[(D+\mathfrak{s})^{-1}, f\right]=$ $-(D+\mathfrak{s})^{-1} \operatorname{cl}(d f)(\mathfrak{s}+D)^{-1}$.

Lemma 10.21. Let $\chi$ be a smooth approximation of the characteristic function of $(-\infty, 0] \times Y$ in $\operatorname{cyl}(Y)$. Consider $\chi$ as a multiplication operator from $C_{c}^{\infty}\left(\operatorname{cyl}(Y), E_{\text {cyl }}\right)$ to $C_{c}^{\infty}(X, E)$. Similarly consider the operator given by Clifford multiplication $\operatorname{cl}(d \chi)$. Then $D \chi=\chi D_{\text {cyl }}+\operatorname{cl}(d \chi)$ as operators $C_{c}^{\infty}\left(\operatorname{cyl}(Y), E_{\text {cyl }}\right) \rightarrow C_{c}^{\infty}(X, E)$

Lemma 10.22. Let $\varphi_{1}, \varphi_{2} \in C_{c}^{\infty}(X)$. Then as a bounded operator on $L^{2}(X, E)$ the operator $\varphi_{1}(D+\mathfrak{s})^{-1} \varphi_{2}$ belongs to $\mathcal{I}_{m}$, the $m$-Schatten ideal.
Let $\varphi \in C_{c}^{\infty}(X)$. Then as bounded operators on $L^{2}(X, E)$ the operators $\varphi(D+\mathfrak{s})^{-1}$ and $(D+\mathfrak{s})^{-1} \varphi$ belongs to $\mathcal{I}_{m}$.

As an application of these Lemmas, with simple algebraic manipulations, one can prove the following
Proposition 10.23. The difference $(\mathfrak{s}+D)^{-1}-\chi\left(\mathfrak{s}+D^{\text {cyl }}\right)^{-1} \chi$ is a $m$-Schatten operator, with $m>\operatorname{dim} M$.

We shall now construct a parametrix for $D^{+}$; in fact we shall construct an inverse of $D^{+}$modulo $m$-Schatten class operators, with $m>\operatorname{dim} X$. We introduce the following useful notation: if $L$ and $M$ are two bounded operators on a Hilbert space and if $m \in[1,+\infty)$ then

$$
\begin{equation*}
L \sim_{m} K \quad \text { if } \quad L-M \in \mathcal{I}_{m} \tag{10.24}
\end{equation*}
$$

Consider the operator

$$
\begin{equation*}
G=\left(I+D^{-} D^{+}\right)^{-1} D^{-} . \tag{10.25}
\end{equation*}
$$

Using elementary properties of the functional calculus for Dirac operators on complete manifolds, we certainly have that $I-G D^{+}=\left(I+D^{-} D^{+}\right)^{-1}, I-D^{+} G=$
$\left(I+D^{+} D^{-}\right)^{-1}$. The operator $G$, as well as the two remainders, do not have Schwartz kernels that are localized near the diagonal; still they are perfectly defined and they are all bounded on $L^{2}$. For notational convenience we set $\left(D^{ \pm}\right)_{\text {cyl }}=: D_{\text {cyl }}^{ \pm}$. Recall that up to standard identifications $D_{\text {cyl }}^{ \pm}= \pm \partial_{x}+D^{\partial}$, acting on the restriction of $E^{+}$to the boundary, extended in the obvious way to the cylinder. We consider the operator

$$
\begin{equation*}
G^{\prime}:=-\chi\left(\left(D_{\text {cyl }}^{+}\right)^{-1}\left(I+D_{\text {cyl }}^{+} D_{\text {cyl }}^{-}\right)^{-1}\right) \chi . \tag{10.26}
\end{equation*}
$$

Then, a simple computation proves that

$$
\begin{align*}
& G^{\prime} D^{+}=-\chi\left(I+D_{\mathrm{cyl}}^{-} D_{\mathrm{cyl}}^{+}\right)^{-1} \chi+\chi\left(D_{\mathrm{cyl}}^{+}\right)^{-1}\left(I+D_{\mathrm{cyl}}^{+} D_{\mathrm{cyl}}^{-}\right)^{-1} \mathrm{cl}(d \chi)  \tag{10.27}\\
& D^{+} G^{\prime}=-\chi\left(I+D_{\mathrm{cyl}}^{+} D_{\mathrm{cyl}}^{-}\right)^{-1} \chi-\operatorname{cl}(d \chi)\left(D_{\mathrm{cyl}}^{+}\right)^{-1}\left(I+D_{\mathrm{cyl}}^{+} D_{\mathrm{cyl}}^{-}\right)^{-1} \chi . \tag{10.28}
\end{align*}
$$

Inspired by the $b$-calculus we set $Q:=G-G^{\prime} . Q$ is clearly bounded on $L^{2}$. For the benefit of reader we restate the theorem we wish to prove (Theorem 7.13):

Theorem 10.29. The operator $Q$ is an inverse of $D^{+}$modulo m-Schatten class operators, with $m>\operatorname{dim} M$.

Proof. First we observe that $\left(I+D^{2}\right)^{-1}=(\mathfrak{s}+D)^{-2}$. Using this we check that $\left(I+D^{2}\right)^{-1}-\chi\left(I+D_{\text {cyl }}^{2}\right)^{-1} \chi$ can be expressed as

$$
\begin{aligned}
& (\mathfrak{s}+D)^{-1}\left((\mathfrak{s}+D)^{-1}-\chi\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-1} \chi\right)+\left((\mathfrak{s}+D)^{-1}-\chi\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-1} \chi\right) \chi\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-1} \chi \\
& \quad+\chi\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-1}\left(\chi^{2}-1\right)\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-1} \chi .
\end{aligned}
$$

Since this term is $m$-Schatten, wee see that $\left(I+D^{2}\right)^{-1} \sim_{m} \chi\left(I+D_{\text {cyl }}^{2}\right)^{-1} \chi$. Now, from (10.27), we have

$$
G^{\prime} D^{+} \sim_{m}-\chi\left(I+D_{\text {cyl }}^{-} D_{\mathrm{cyl}}^{+}\right)^{-1} \chi, \quad D^{+} G^{\prime} \sim_{m}-\chi\left(I+D_{\text {cyl }}^{+} D_{\mathrm{cyl}}^{-}\right)^{-1} \chi
$$

so that, if we define $S_{+}:=I-Q D^{+}, S_{-}:=I-D^{+} Q$ and recall that $Q=G-G^{\prime}$, we obtain

$$
\begin{aligned}
S_{+}= & I-\left(G-G^{\prime}\right) D^{+}=\left(I+D^{-} D^{+}\right)^{-1}+G^{\prime} D^{+} \sim_{m}\left(I+D^{-} D^{+}\right)^{-1} \\
& -\chi\left(I+D_{\mathrm{cyl}}^{-} D_{\mathrm{cyl}}^{+}\right)^{-1} \chi \sim_{m} 0 .
\end{aligned}
$$

Thus the remainder $S_{+}$is of $m$-Schatten class. Similarly we proceed for $S_{-}$. The theorem is proved.

We have presented the parametrix construction in the case $T=$ point, $\Gamma=\{1\}$. However, a similar proof applies to the general case of a foliated bundle with cylindrical ends $(X, \mathcal{F}) \equiv\left(\tilde{V} \times_{\Gamma} T, \mathcal{F}\right)$ with $\tilde{V}$ of even dimension. ${ }^{10}$ It will suffice to apply to the $\Gamma$-equivariant family $\left(D_{\theta}\right)_{\theta \in T}$ the functional calculus along the fibers of the

[^4]trivial fibration $\tilde{V} \times T \rightarrow T$ (obtaining, of course, $\Gamma$-equivariant families). All our argument apply verbatim once we observe that given compactly supported smooth functions $\varphi, \psi$ on $X$, the family $\left(\varphi\left(D_{\theta}+\mathfrak{s}\right)^{-1} \psi\right)_{\theta}$ defines an element in $\mathbb{K}(\mathcal{E})$, the compacts of the Hilbert module $\mathcal{E}$. In fact, once we observe that such an element is in fact in $\mathcal{I}_{m}(X, \mathcal{F})$, if $m>\operatorname{dim} \tilde{V}$, we can finally conclude that Theorem 7.14 holds.
10.5 Proof of the existence of the relative index class. In this subsection we give a proof of Proposition 7.18. Denote by $D^{\text {cyl }}$ the Dirac operator induced by $D^{\partial}$ on the cylinder. Consider the triple
\[

\left(e_{D}, e_{1}, p_{t}\right), \quad t \in[1,+\infty], \quad with p_{t}:= $$
\begin{cases}e_{\left(t D^{\mathrm{cyl}}\right)} & \text { if } t \in[1,+\infty)  \tag{10.30}\\ e_{1} & \text { if } t=\infty\end{cases}
$$
\]

First, we need to justify the fact that the relevant elements here are in the right algebras. Thus we need to show that $e_{D}$ is in $A^{*}(X ; \mathcal{F})$ and that $e_{\left(t D^{\text {cyl }}\right)}$ is in $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. We start with the latter. Fix for simplicity $t=1$. We need to show that there exists a sequence of elements $k_{j} \in B_{c}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ such that $\left\|e_{\left(D^{\text {cyl }}\right)}-k_{j}\right\| \longrightarrow 0$ as $j \rightarrow+\infty$, with the norm denoting the $C^{*}$-norm of Section 4.3. We use the fact that $D^{\text {cyl }}$ is an $\mathbb{R} \times \Gamma$-equivariant family. (Strictly speaking we are taking the closure of the operators in this family.) Proceeding precisely as in [MoN96], Section 7, thus following ideas of Roe, we are reduced to the following remark: if $f$ is a rapidly decreasing function on $\mathbb{R}$ with compactly supported Fourier transform, then $f\left(D^{\text {cyl }}\right)$ is given by (the family of integral operators induced by) an element in $B_{c}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. The proof of the last assertion is an easy generalization of the well known results by Roe, see for example [Roe87] or the detailed discussion in [Roe88]. Since the functions as $f$ are dense in $C_{0}(\mathbb{R})$ the assertion follows.

Next we show that $e_{D} \in A^{*}(X ; \mathcal{F})$. First of all, we need to show that $e_{D} \in \mathcal{L}(\mathcal{E})$. This is the same proof as in [MoN96].

Now we need to show that the image of $e_{D}$ in $\mathcal{Q}(\mathcal{E})$ is in the image of $\rho$. Write $e_{D}=\left(e_{D}-\chi^{0} e_{\left(D^{\text {cyl }}\right)} \chi^{0}\right)+\chi^{0} e_{\left(D^{\text {cyl }}\right)} \chi^{0}$. Since we have proved that $e_{D^{\text {cyl }}}$ is in $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$, it suffices to show that

$$
\begin{equation*}
e_{D}-\chi^{0} e_{\left(D^{\text {cyl }}\right)} \chi^{0} \in \mathbb{K}(\mathcal{E}) \tag{10.31}
\end{equation*}
$$

In order to prove (10.31) we first show that $e_{D}-\chi e_{\left(D^{\text {cyl }}\right)} \chi \in \mathbb{K}(\mathcal{E})$, with $\chi$ a smooth approximation of $\chi^{0}$. Using (7.7) we reduce ourselves to establishing that $(\mathfrak{s}+D)^{-1}-$ $\chi\left(\mathfrak{s}+D^{\text {cyl }}\right)^{-1} \chi$, which we already know. As far as $(\mathfrak{s}+D)^{-1}-\chi^{0}\left(\mathfrak{s}+D^{\text {cyl }}\right)^{-1} \chi^{0}$ is concerned, we simply choose a sequence of smooth functions $\chi_{j}$ converging to $\chi^{0}$ in $L^{2}$ and we use the fact that $\mathbb{K}(\mathcal{E})$ is closed in $\mathcal{L}(\mathcal{E})$; we have already used this argument in the proof of Sublemma 10.2. The proof of (10.31) is complete.

Finally, we need to prove that $p_{t}$ is a continuous path in $B^{*}$ joining $\pi\left(e_{D}\right)$ to $e_{1}$ Now, the above argument shows that for $t \in[1,+\infty) \pi\left(e_{t D}\right)=e_{t\left(D^{\text {cyl }}\right)}=p_{t}$, so we only need to show that $e_{\left(t D^{\text {cyl }}\right)}$ converges to $e_{1}$ in the $C^{*}$-norm of $B^{*}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ as $t \rightarrow \infty$; however, using assumption (7.1) this follows easily.

The proof of Proposition 7.18 is complete
10.6 Proof of the excision formula (7.20). Let $Q \in \mathcal{L}\left(\mathcal{E}^{-}, \mathcal{E}^{+}\right)$be the parametrix for $D^{+}$obtained as in Theorem 7.13.

We consider

$$
e\left(D^{+}, Q\right):=\binom{I}{D^{+}}\left(\begin{array}{ll}
S_{+} & Q
\end{array}\right)=\left(\begin{array}{ll}
S_{+} & Q  \tag{10.32}\\
D^{+} S_{+} & D^{+} Q
\end{array}\right) .
$$

The following Lemma is elementary to check
Lemma 10.33. $e\left(D^{+}, Q\right)$ is an idempotent in $A^{*}(X, \mathcal{F}) \equiv A^{*}$. Moreover, if $P_{Q}$ denotes, as usual, the Connes-Skandalis projection associated to $Q$, then

$$
P_{Q}=\left(\begin{array}{cc}
I & Q  \tag{10.34}\\
0 & I
\end{array}\right)^{-1} e\left(D^{+}, Q\right)\left(\begin{array}{cc}
I & Q \\
0 & I
\end{array}\right)
$$

The path obtained substituting $s Q, s \in[0,1]$, to $Q$ in the first and third matrix appearing on the right hand side of (10.34) is a path of projections in $A^{*}$ and connects the projection $P_{Q} \in C^{*}(X, \mathcal{F}) \subset A^{*}$ with the projection $e\left(D^{+}, Q\right)$. On the other hand, another direct computation shows that if $G=\left(I+D^{-} D^{+}\right)^{-1} D^{-}$, then $e\left(D^{+}, G\right)=e_{D}$, the graph projection. Recall that $Q=G-G^{\prime}$, with $G^{\prime}$ given by (10.26); by composing the path of projections

$$
\left(\begin{array}{ll}
I & s Q \\
0 & I
\end{array}\right)^{-1} e\left(D^{+}, Q\right)\left(\begin{array}{ll}
I & s Q \\
0 & I
\end{array}\right)
$$

with the path of projections $e\left(D^{+}, G-\tau G^{\prime}\right), \tau \in[0,1]$, we obtain a path of projections $H(t)$ in $A^{*}$ joining $P_{Q}=H(1)$ to $e_{D}=H(0)$. Consider now

$$
\begin{equation*}
D_{\mu}^{+}:=\mu D^{+}, \quad G_{\mu}:=\left(I+D_{\mu}^{-} D_{\mu}^{+}\right)^{-1} D_{\mu}^{-}, \quad Q(\mu, \tau):=G_{\mu}-\tau G_{\mu}^{\prime} \tag{10.35}
\end{equation*}
$$

with $G_{\mu}^{\prime}$ as in (10.26) but defined in terms of $D_{\mu}^{+}$. We have then

$$
\begin{equation*}
D_{\mu}^{+} Q(\mu, \tau)=I-S_{-}(\mu, \tau), \quad Q(\mu, \tau) D_{\mu}^{+}=I-S_{+}(\mu, \tau) \tag{10.36}
\end{equation*}
$$

In this notation the above path, $H(t)$, first joins $P_{Q(1,1)}$ to $e\left(D^{+}, Q(1,1)\right)$ and then joins $e\left(D^{+}, Q(1,1)\right)$ to $e\left(D^{+}, Q(1,0)\right)$, which is $e_{D}$. We write

$$
P_{Q} \equiv P_{Q(1,1)} \curvearrowright e\left(D^{+}, Q(1,1)\right) \curvearrowright e\left(D^{+}, Q(1,0)\right) \equiv e_{D}
$$

Similarly, we can consider $P_{Q(\mu, 1)} \curvearrowright e\left(D^{+}, Q(\mu, 1)\right) \curvearrowright e\left(D^{+}, Q(\mu, 0)\right) \equiv e_{\mu D}$ with the second homotopy provided by $e\left(D^{+}, Q(\mu, \tau)\right), \tau \in[0,1]$. Let $H(\mu, t)$ be this homotopy, connecting $P_{Q(\mu, 1)}$ to $e_{\mu D}$. We set $p(\mu, t):=\pi(H(\mu, t))$, where $\mu \in[1,+\infty), t \in[0,1]$. We also set

$$
p(\infty, t):=\left(\begin{array}{cc}
0 & 0  \tag{10.37}\\
0 & I
\end{array}\right), \quad \forall t \in[0,1] .
$$

Assume we could prove that the above defined function $p(\mu, t)$ is continuous on $[1,+\infty]_{\mu} \times[0,1]_{t}$. Then from the above discussion we obtain that
$\left(H(t),\left(\begin{array}{ll}0 & 0 \\ 0 & I\end{array}\right), p(\mu, \tau)\right)$ joins $\left(H(1),\left(\begin{array}{ll}0 & 0 \\ 0 & I\end{array}\right), p(\mu, 1)\right)$ to $\left(H(0),\left(\begin{array}{ll}0 & 0 \\ 0 & I\end{array}\right), p(\mu, 0)\right)$.
But, as already remarked, $H(1)=P_{Q}$ and $H(0)=e_{D}$; moreover $p(\mu, 1)$ is the constant path, indeed $p(\mu, 1):=\pi(H(\mu, 1))=\pi\left(P_{Q(\mu, 1)}\right)=\left(\begin{array}{cc}0 & 0 \\ 0 & I\end{array}\right)$, given that $P_{Q(\mu, 1)}$ is a true Connes-Skandalis projection, thus with the property that $P_{Q(\mu, 1)}-\left(\begin{array}{ll}0 & 0 \\ 0 & I\end{array}\right) \in$ $C^{*}(X, \mathcal{F})$; finally, $H(\mu, 0)=e_{\mu D}$, so that $p(\mu, 0)=e_{\mu D^{\text {cy1 }} ;}$; thus, taking into account (10.37), we see that $p(\mu, 0)$ is precisely the path of projections appearing in the definition of the relative index class. Summarizing, if we could prove that $p(\mu, t)$ is continuous on $[1,+\infty]_{\mu} \times[0,1]_{t}$ then

$$
\left[P_{Q},\left(\begin{array}{ll}
0 & 0 \\
0 & I
\end{array}\right), \text { const }\right]=\left[e_{D},\left(\begin{array}{ll}
0 & 0 \\
0 & I
\end{array}\right), p_{\mu}\right]
$$

which is what we need to prove in order to conclude. Now, $p(\mu, t)$ is certainly continuous in $[1,+\infty) \times[0,1]$; we end the proof by showing that, in the $C^{*}$-norm,

$$
\lim _{\mu \rightarrow+\infty} p(\mu, t)=\left(\begin{array}{ll}
0 & 0 \\
0 & I
\end{array}\right)
$$

uniformly in $t \in[0,1]$.
We begin with the projection of the first homotopy, that connecting $P_{Q(\mu, 1)}$ to $e\left(\mu D^{+}, Q(\mu, 1)\right)$. This is

$$
\pi\left(\left(\begin{array}{cc}
I & s Q(\mu, 1)  \tag{10.38}\\
0 & I
\end{array}\right)^{-1} e\left(D_{\mu}^{+}, Q(\mu, 1)\right)\left(\begin{array}{cc}
I & s Q(\mu, 1) \\
0 & I
\end{array}\right)\right), \quad s \in[0,1]
$$

which is easily seen to be equal to

$$
\left(\begin{array}{cc}
0 & (1-s) \pi(Q(\mu, 1)) \\
0 & 1
\end{array}\right)
$$

Set $D_{\text {cyl }}^{ \pm}:=\left(D^{ \pm}\right)^{\text {cyl }}$. Now we write explicitly:

$$
\pi(Q(\mu, 1))=\mu D_{\text {cyl }}^{-}\left(I+D_{\text {cyl }}^{-} D_{\text {cyl }}^{+} \mu^{2}\right)^{-1}-\frac{1}{\mu}\left(D_{\text {cyl }}^{+}\right)^{-1}\left(I+D_{\text {cyl }}^{-} D_{\text {cyl }}^{+} \mu^{2}\right)^{-1}
$$

which does converge to 0 in the $C^{*}$-norm as $\mu \rightarrow+\infty$. Thus (10.38) converges to $\left(\begin{array}{ll}0 & 0 \\ 0 & I\end{array}\right)$ uniformly in $s$, as required. Next we look at the second path, connecting $e\left(\mu D^{+}, Q(\mu, 1)\right)$ to $e\left(\mu D^{+}, Q(\mu, 0)\right)$. We need to compute explicitly
$\pi\left(e\left(\mu D^{+}, Q(\mu, \tau)\right)\right.$ and show that it goes to 0 uniformly in $\tau$. An explicit and elementary computation shows that

$$
\begin{aligned}
\pi\left(e\left(\mu D^{+}, Q(\mu, \tau)\right)=\right. & \left(\begin{array}{cc}
\left(I+\mu^{2} D_{\text {cyl }}^{-} D_{\text {cyl }}^{+}\right)^{-1} & \left.\left(I+\mu^{2} D_{\text {cyl }}^{-} D_{\text {cyl }}^{+}\right)^{-1} \mu D_{\text {cyl }}^{-}\right) \\
\mu D_{\text {cyl }}^{+}\left(I+\mu^{2} D_{\text {cyl }}^{-} D_{\text {cyl }}^{+}\right)^{-1} & I-\left(I+\mu^{2} D_{\text {cyl }}^{+} D_{\text {cyl }}^{-}\right)^{-1}
\end{array}\right) \\
& +\left(I+\mu^{2}\left(D^{\text {cyl }}\right)^{2}\right)^{-1}\left(\begin{array}{cc}
-\tau & \tau\left(\mu D_{\text {cyl }}^{-}\right)^{-1} \\
-\tau \mu D_{\text {cyl }}^{+} & \tau
\end{array}\right) .
\end{aligned}
$$

The second summand converges uniformly to 0 in the $C^{*}$-norm, whereas the first summand converges uniformly to $\left(\begin{array}{ll}0 & 0 \\ 0 & I\end{array}\right)$ in the $C^{*}$-norm. This ends the proof.

### 10.7 Proof of the existence of smooth index classes.

10.7.1 Proof of Proposition 8.1. Recall the Connes-Skandalis projection

$$
P_{Q}:=\left(\begin{array}{ll}
S_{+}^{2} & S_{+}\left(I+S_{+}\right) Q \\
S_{-} D^{+} & I-S_{-}^{2}
\end{array}\right)
$$

Let

$$
\widehat{P}_{Q}:=\left(\begin{array}{ll}
S_{+}^{2} & S_{+}\left(I+S_{+}\right) Q \\
S_{-} D^{+} & -S_{-}^{2}
\end{array}\right)
$$

We want to show that

$$
\widehat{P}_{Q} \in \mathcal{J}_{m}(X, \mathcal{F}) \cap \operatorname{Dom} \bar{\delta}_{1} \cap \operatorname{Dom} \bar{\delta}_{2},
$$

with $m>2 n$ and $2 n$ equal to the dimension of the leaves of $(X, \mathcal{F})$. We fix such an $m$. We set, as usual, $D_{\text {cyl }}^{ \pm}:=\left(D^{ \pm}\right)^{\text {cyl }}$. We begin by showing that the Connes-Skandalis matrix $\widehat{P}_{Q}$ is in $\mathcal{J}_{m}(X, \mathcal{F})$. First we show that it belongs to $\mathcal{I}_{m}$. Recall our parametrix $Q=G-G^{\prime}$ with $G=\left(I+D^{-} D^{+}\right)^{-1} D^{-}$and $G^{\prime}:=\chi\left(\left(D_{\text {cyl }}^{+}\right)^{-1}\left(I+D_{\text {cyl }}^{+} D_{\text {cyl }}^{-}\right)^{-1}\right) \chi$. We know that $S_{+}:=I-Q D^{+}$and $S_{-}:=I-D^{+} Q$ are elements in $\mathcal{I}_{m}(X, \mathcal{F})$ for $m>\operatorname{dim} \tilde{V}$; hence, obviously, so they are $\left(S_{ \pm}\right)^{2}$ and $\left(S_{+}\left(I+S_{+}\right)\right) Q$. Thus we only need to show that $S_{-} D^{+}$belongs to $\mathcal{I}_{m}(X, \mathcal{F})$ for $m>\operatorname{dim} \tilde{V}$. Recall that $S_{-}=\left(I+D^{+} D^{-}\right)^{-1}+D^{+} G^{\prime}$; thus $S_{-} D^{+}=\left(I+D^{+} D^{-}\right)^{-1} D^{+}+D^{+} G^{\prime} D^{+}$. Now, with elementary algebraic manipulations, we can express the last term as

$$
\begin{aligned}
& \left(\left(I+D^{+} D^{-}\right)^{-1} D^{+}-\chi\left(I+D_{\mathrm{cyl}}^{+} D_{\mathrm{cyl}}^{-}\right)^{-1} D_{\mathrm{cyl}}^{+} \chi\right) \\
& \quad+\left(\chi\left(I+D_{\mathrm{cyl}}^{+} D_{\mathrm{cyl}}^{-}\right)^{-1} \operatorname{cl}(d \chi)-\operatorname{cl}(d \chi)\left(I+D_{\mathrm{cyl}}^{+} D_{\mathrm{cyl}}^{-}\right)^{-1} \chi\right. \\
& \left.\quad+\operatorname{cl}(d \chi)\left(D_{\mathrm{cyl}}^{+}\right)^{-1}\left(I+D_{\mathrm{cyl}}^{+} D_{\mathrm{cyl}}^{-}\right)^{-1} \operatorname{cl}(d \chi)\right)
\end{aligned}
$$

with $d$ denoting the differential along $\tilde{V}$ in the product $\tilde{V} \times T$. See formula (2.3). Employing the usual reasoning, the first term is easily seen to be in $\mathcal{I}_{m}(X, \mathcal{F})$ for $m>\operatorname{dim} \tilde{V}$; we have already proved that the same is true for the second term. Thus $S_{-} D^{+}$is in $\mathcal{I}_{m}(X, \mathcal{F})$ for $m>\operatorname{dim} \tilde{V}$.

Thus, we have proved that $\widehat{P}_{Q} \in \mathcal{I}_{m}(X, \mathcal{F})$ for $m>\operatorname{dim} \tilde{V}$.
Next we show that $\widehat{P}_{Q} \in \mathcal{J}_{m}$. Consider for example

$$
S_{+}=\left(I+D^{-} D^{+}\right)^{-1}-\chi\left(I+D_{\mathrm{cyl}}^{-} D_{\mathrm{cyl}}^{+}\right)^{-1} \chi+\chi\left(D_{\mathrm{cyl}}^{+}\right)^{-1}\left(I+D_{\mathrm{cyl}}^{+} D_{\mathrm{cyl}}^{-}\right)^{-1} \mathrm{cl}(d \chi)
$$

We want to show that $g S_{+}$is bounded. However, from the explicit expression we have just written this is readily checked by hand using (variants of) the following

Lemma 10.39. The operator $g\left(1+D^{2}\right)^{-1}$ is bounded.
Proof. Write $g\left(1+D^{2}\right)^{-1}=f f(D+\mathfrak{s})^{-1}(D+\mathfrak{s})^{-1}$ and write the last term as $f\left[f,(D+\mathfrak{s})^{-1}\right](D+\mathfrak{s})^{-1}+f(D+\mathfrak{s})^{-1} f(D+\mathfrak{s})^{-1}$ which is in turn equal to $f(D+$ $\mathfrak{s})^{-1} \operatorname{cl}(d f)(D+\mathfrak{s})^{-1}(D+\mathfrak{s})^{-1}+f(D+\mathfrak{s})^{-1} f(D+\mathfrak{s})^{-1}$. Thus it suffices to show that $f(D+\mathfrak{s})^{-1}$ and $(D+\mathfrak{s})^{-1} f$ are bounded. This is easily proved using the Sublemma 10.49 below. The Lemma is proved.

Next we show that $\widehat{P}_{Q} \in \operatorname{Dom} \bar{\delta}_{1} \cap \operatorname{Dom} \bar{\delta}_{2}$. First of all, we have the following
Lemma 10.40. Under assumption (7.1) we have that

$$
D_{\mathrm{cyl}}^{-1} \in \operatorname{Dom}\left(\bar{\delta}_{\mathrm{cyl}, 1}^{\max }\right) \cap \operatorname{Dom}\left(\bar{\delta}_{\mathrm{cyl}, 2}^{\max }\right)
$$

with $\delta_{\mathrm{cyl}, 2}:=\left[\phi_{\partial},\right]$ and $\delta_{\mathrm{cyl}, 1}:=\left[\dot{\phi}_{\partial},\right]$
Proof. Consider a smooth function $h \in C^{\infty}(\mathbb{R})$ such that $h(x)=1 / x$ for $|x|>\epsilon$, with $\epsilon$ as in our invertibility assumption (7.1). Clearly $h\left(D_{\text {cyl }}\right)=D_{\text {cyl }}^{-1}$. We can find a sequence of functions $\left\{\beta_{\lambda}\right\}_{\lambda>0}$ with the following properties:
(1) $\widehat{\beta}_{\lambda}$ is compactly supported;
(2) $\left\{\beta_{\lambda}\right\}_{\lambda>0}$ is a Cauchy sequence in $W^{2}(\mathbb{R})$-norm;
(3) $\beta_{\lambda} \longrightarrow h$ in sup-norm as $\lambda \rightarrow+\infty$.

The function $\beta_{\lambda}$ such that $\widehat{\beta}_{\lambda}=\rho_{\lambda} \widehat{h}$, with $\rho_{\lambda}$ as in [MoN96] p. 515, does satisfy these three properties. We assume this for the time being and we conclude the proof of the Lemma. First, from the very definition of $\beta_{\lambda}$ and from finite propagation techniques we have that $\beta_{\lambda}\left(D_{\text {cyl }}\right)$ is a $(-1)$-order pseudodifferential operator of compact $\mathbb{R} \times \Gamma$-support. Next, from property (3), we see that $\beta_{\lambda}\left(D_{\text {cyl }}\right) \longrightarrow h\left(D_{\text {cyl }}\right)=D_{\text {cyl }}^{-1}$ in $C^{*}$-norm when $\lambda \rightarrow+\infty$. Finally, from Duhamel formula we have:

$$
\delta_{2}\left(\beta_{\lambda}\left(D_{\text {cyl }}\right)=\left[\phi_{\partial}, \beta_{\lambda}\left(D_{\text {cyl }}\right)\right]=\int_{\mathbb{R}} d s \int_{0}^{1} d t \sqrt{-1} s \widehat{\beta}_{\lambda}(s) e^{\sqrt{-1} s t D_{\text {cyl }}}\left[\phi_{\partial}, D_{\text {cyl }}\right] e^{\sqrt{-1} s(1-t) D_{\text {cyl }}}\right.
$$

Moreover, as explained in [MoN96], p. 520, we have

$$
\left\|\left[\phi_{\partial}, \beta_{\lambda}\left(D_{\mathrm{cyl}}\right)\right]-\left[\phi_{\partial}, \beta_{\mu}\left(D_{\mathrm{cyl}}\right)\right]\right\| \leq C \int_{\mathbb{R}}\left|\widehat{\beta}_{\lambda}(s)-\widehat{\beta}_{\mu}(s) \| s\right| d s
$$

Now:

$$
\begin{aligned}
\int_{\mathbb{R}}\left|\widehat{\beta}_{\lambda}(s)-\widehat{\beta}_{\mu}(s) \| s\right| d s & =\int_{\mathbb{R}}\left|\widehat{\beta}_{\lambda}(s)-\widehat{\beta}_{\mu}(s)\right|\left|s \sqrt{1+s^{2}}\right| \frac{1}{\sqrt{1+s^{2}}} d s \\
& \leq\left\|\left(\widehat{\beta}_{\lambda}-\widehat{\beta}_{\mu}\right)\left|s \sqrt{1+s^{2}}\right|\right\|_{L^{2}(\mathbb{R})}\left\|\frac{1}{\sqrt{1+s^{2}}}\right\|_{L^{2}(\mathbb{R})} \\
& \leq C\left\|D_{\mathrm{cyl}}\left(1+D_{\mathrm{cyl}}^{2}\right)^{\frac{1}{2}}\left(\beta_{\lambda}-\beta_{\mu}\right)\right\|_{L^{2}(\mathbb{R})} \leq C^{\prime}\left\|\beta_{\lambda}-\beta_{\mu}\right\|_{W^{2}(\mathbb{R})} .
\end{aligned}
$$

Thus, from property (2), we infer that $\left[\phi_{\partial}, \beta_{\lambda}\left(D_{\text {cyl }}\right)\right]$ is a Cauchy sequence in $C^{*}$ norm. This means that $h\left(D_{\text {cyl }}\right)$, which is $D_{\text {cyl }}^{-1}$, is in the domain of the closure $\bar{\delta}_{\text {cyl }, 2}^{\max }$. Similarly we proceed for $\delta_{1}$.

It remains to prove that with our definition of $\beta_{\lambda}$ we can satisfy the three properties. The first one is obvious from the definition. For the second property we estimate, with $D:=\frac{1}{\sqrt{-1}} \frac{d}{d x}$ on $\mathbb{R}$ :

$$
\begin{aligned}
\left\|\beta_{\lambda}-\beta_{\mu}\right\|_{W^{2}(\mathbb{R})} & =\left\|\left(1+D^{2}\right)\left(\beta_{\lambda}-\beta_{\mu}\right)\right\|_{L^{2}(\mathbb{R})} \\
& =\left\|\left(1+s^{2}\right)\left(\widehat{\beta}_{\lambda}-\widehat{\beta}_{\mu}\right)\right\|_{L^{2}(\mathbb{R})}=\left\|\left(1+s^{2}\right)\left(\rho_{\lambda} \widehat{h}-\rho_{\mu} \widehat{h}\right)\right\|_{L^{2}(\mathbb{R})} \\
& =\left\|\left(1+s^{2}\right)^{2} \widehat{h}\left(\rho_{\lambda}-\rho_{\mu}\right) \frac{1}{1+s^{2}}\right\|_{L^{2}(\mathbb{R})} \\
& \leq\left\|\left(1+s^{2}\right)^{2} \widehat{h}\right\|_{L^{2}(\mathbb{R})}\left\|\left(\rho_{\lambda}-\rho_{\mu}\right) \frac{1}{1+s^{2}}\right\|_{L^{\infty}(\mathbb{R})} .
\end{aligned}
$$

In the last term, the first factor can be estimated directly and shown to be finite, using the equality

$$
\left\|\left(1+s^{2}\right)^{2} \widehat{h}\right\|_{L^{2}(\mathbb{R})}=\left\|\left(1+D^{2}\right)^{2} h\right\|_{L^{2}(\mathbb{R})}
$$

and the very definition of $h$ (namely, that it is equal to $1 / x$ for $|x|>\epsilon$ ); the second factor, on the other hand, is clearly Cauchy (from the definition of $\rho_{\lambda}$ ). Thus we have established (2). Finally, we tackle (3). Recall that the Fourier transformation extends to a bounded map from $L^{1}(\mathbb{R})$ to $C_{0}(\mathbb{R})$. Thus

$$
\begin{aligned}
\left\|\beta_{\lambda}-h\right\|_{C_{0}(\mathbb{R})} & \leq\left\|\widehat{\beta}_{\lambda}-\widehat{h}\right\|_{L^{1}(\mathbb{R})}=\left\|\left(\rho_{\lambda}-1\right) \widehat{h}\right\|_{L^{1}(\mathbb{R})} \\
& =\left\|\left(\rho_{\lambda}-1\right) \frac{1}{1+s^{2}}\left(1+s^{2}\right) \widehat{h}\right\|_{L^{1}(\mathbb{R})} \\
& \leq\left\|\left(\rho_{\lambda}-1\right) \frac{1}{1+s^{2}}\right\|_{L^{2}(\mathbb{R})}\left\|\left(1+s^{2}\right) \widehat{h}\right\|_{L^{2}(\mathbb{R})} .
\end{aligned}
$$

The second factor can be estimated as above and shown to be finite; the first factor goes to zero using Lebesgue dominated convergence theorem. The Lemma is now completely proved.

We go back to our goal, proving that $\widehat{P}_{Q}$ is in $\operatorname{Dom} \bar{\delta}_{1} \cap \operatorname{Dom} \bar{\delta}_{2}$. This means that for $j=1,2$ and $m>\operatorname{dim} \tilde{V}$ we have:

$$
\widehat{P}_{Q} \in \operatorname{Dom} \bar{\delta}_{j}^{\max } \cap \mathcal{J}_{m}(X, \mathcal{F}) \quad \text { and } \quad \bar{\delta}_{j}^{\max }\left(\widehat{P}_{Q}\right) \in \mathcal{J}_{m}(X, \mathcal{F})
$$

First, we establish the fact that $\widehat{P}_{Q} \in \operatorname{Dom} \bar{\delta}_{j}^{\max }$ (we already proved that $\widehat{P}_{Q} \in$ $\left.\mathcal{J}_{m}(X, \mathcal{F})\right)$. We concentrate on $\bar{\delta}_{2}^{\max }$; similar arguments will work for $\bar{\delta}_{1}^{\max }$. Recall that

$$
\widehat{P}_{Q}:=\left(\begin{array}{ll}
S_{+}^{2} & S_{+}\left(I+S_{+}\right) Q \\
S_{-} D^{+} & -S_{-}^{2}
\end{array}\right)
$$

Let us concentrate on each single entry of this matrix. For the sake of brevity, let us give all the details for the $(1,1)$-entry, $S_{+}^{2}$. It suffices to show that $S_{+} \in \operatorname{Dom} \bar{\delta}_{2}^{\max }$ and that $\bar{\delta}_{2}^{\max } S_{+} \in \mathcal{J}_{m}(X, \mathcal{F})$.

For notational convenience we set, for this proof only,

$$
\bar{\delta}_{2}^{\max }=: \Theta, \quad \bar{\delta}_{\mathrm{cyl}, 2}^{\max }=: \Theta_{\mathrm{cyl}} .
$$

We observe preliminarily that proceeding exactly as in [MoN96] we can prove that $(\mathfrak{s}+D)^{-1}$ is in $\operatorname{Dom} \Theta$; hence so is $(\mathfrak{s}+D)^{-2}$ which is equal to $\left(1+D^{2}\right)^{-1}$. The same proof establishes the corresponding result on the cylinder, for $\left(\mathfrak{s}+D_{\text {cyl }}\right)^{-1}$ and $\left(\mathfrak{s}+D_{\text {cyl }}\right)^{-2}=\left(1+D_{\text {cyl }}^{2}\right)^{-1}$. This, together with the last Lemma, shows also that $D_{\text {cyl }}^{-1}\left(1+D_{\text {cyl }}^{2}\right)^{-1}$ belongs to the domain of $\Theta_{\text {cyl }}$. Recall now that

$$
S_{+}=\left(I+D^{-} D^{+}\right)^{-1}-\chi\left(I+D_{\text {cyl }}^{-} D_{\text {cyl }}^{+}\right)^{-1} \chi+\chi\left(D_{\text {cyl }}^{+}\right)^{-1}\left(I+D_{\text {cyl }}^{+} D_{\text {cyl }}^{-}\right)^{-1} \operatorname{cl}(d \chi)
$$

The first summand is in $\operatorname{Dom} \Theta$, as we have already remarked. The second summand, $-\chi\left(I+D_{\text {cyl }}^{+} D_{\text {cyl }}^{-}\right)^{-1} \chi$, is obtained by grafting through pre-multiplication and post-multiplication by $\chi$ an element which is the domain of $\Theta_{\text {cyl }}$. Such a grafted element is easily seen to belong to $\operatorname{Dom} \Theta$, since we can simply choose as an approximating sequence the one obtained by grafting the approximating sequence for $\left(I+D_{\text {cyl }}^{+} D_{\text {cyl }}^{-}\right)^{-1}$. In the (easy) proof we use

$$
\phi \chi=\chi \phi_{\partial}, \quad \chi \phi=\chi \phi_{\partial}, \quad\left[\phi_{\partial}, \chi\right]=0 .
$$

(They all follow from the fact that the modular function is independent of the normal variable in a neighbourhood of the boundary of $X_{0}$.) Similarly, the third summand is in $\operatorname{Dom} \Theta$, given that, as we have observed above, $D_{\text {cyl }}^{-1}\left(1+D_{\text {cyl }}^{2}\right)^{-1}$ belongs to the domain of $\Theta_{\text {cyl }}$. Summarizing: $S_{+}$is an element in $\operatorname{Dom} \Theta$. Next we need to show that $\Theta\left(S_{+}\right)$belongs to $\mathcal{J}_{m}(X, \mathcal{F})$. First we prove that it is in $\mathcal{I}_{m}$. We first observe that $S_{+}$is the $(1,1)$-entry of the $2 \times 2$-matrix

$$
\begin{aligned}
& (\mathfrak{s}+D)^{-2}-\left(\begin{array}{ll}
\chi & 0 \\
0 & \chi
\end{array}\right)\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-2}\left(\begin{array}{ll}
\chi & 0 \\
0 & \chi
\end{array}\right) \\
& \quad+\left(\begin{array}{ll}
\chi & 0 \\
0 & \chi
\end{array}\right)\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-2} D_{\mathrm{cyl}}^{-1}\left(\begin{array}{ll}
0 & \operatorname{cl}(d \chi) \\
\operatorname{cl}(d \chi) & 0
\end{array}\right)
\end{aligned}
$$

We compute $\Theta$ of this term, finding

$$
\begin{aligned}
& \Theta\left((\mathfrak{s}+D)^{-2}\right)-\left(\begin{array}{ll}
\chi & 0 \\
0 & \chi
\end{array}\right) \Theta_{\mathrm{cyl}}\left(\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-2}\right)\left(\begin{array}{ll}
\chi & 0 \\
0 & \chi
\end{array}\right) \\
& \quad+\left(\begin{array}{ll}
\chi & 0 \\
0 & \chi
\end{array}\right) \Theta_{\mathrm{cyl}}\left(\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-2} D_{\mathrm{cyl}}^{-1}\right)\left(\begin{array}{ll}
0 & \operatorname{cl}(d \chi) \\
\operatorname{cl}(d \chi) & 0
\end{array}\right) .
\end{aligned}
$$

The last summand is certainly in $\mathcal{I}_{m}(X, \mathcal{F})$, since $d \chi$ is of compact support. It is clear that this last term is also in $\mathcal{J}_{m}$, i.e. it is bounded if it is multiplied on the
right and on the left by the function $g$. Thus we are left with the task of proving that

$$
\Theta\left((\mathfrak{s}+D)^{-2}\right)-\chi \Theta_{\text {cyl }}\left(\left(\mathfrak{s}+D_{\text {cyl }}\right)^{-2}\right) \chi
$$

is in $\mathcal{J}_{m}(X, \mathcal{F})$. We first show that this term is in $\mathcal{I}_{m}$. Remark that the above difference can be computed explicitly, using [MoN96]; we get

$$
\begin{aligned}
& (\mathfrak{s}+D)^{-1} \operatorname{cl}(d \phi)(\mathfrak{s}+D)^{-2}-\chi\left(\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-1} \operatorname{cl}\left(d \phi_{\partial}\right)\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-2}\right) \chi \\
& \quad+(\mathfrak{s}+D)^{-2} \operatorname{cl}(d \phi)(\mathfrak{s}+D)^{-1}-\chi\left(\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-2} \operatorname{cl}\left(d \phi_{\partial}\right)\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-1}\right) \chi .
\end{aligned}
$$

Now, proceeding as in the discussion on the parametrix given in Section 7.3, we can prove that each of these two differences is in $\mathcal{I}_{m}(X, \mathcal{F})$ Let us see the details; we concentrate on the first difference

$$
\begin{equation*}
(\mathfrak{s}+D)^{-1} \operatorname{cl}(d \phi)(\mathfrak{s}+D)^{-2}-\chi\left(\left(\mathfrak{s}+D_{\text {cyl }}\right)^{-1} \operatorname{cl}\left(d \phi_{\partial}\right)\left(\mathfrak{s}+D_{\text {cyl }}\right)^{-2}\right) \chi ; \tag{10.41}
\end{equation*}
$$

we shall analyze the second difference, namely

$$
\begin{equation*}
(\mathfrak{s}+D)^{-2} \operatorname{cl}(d \phi)(\mathfrak{s}+D)^{-1}-\chi\left(\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-2} \operatorname{cl}\left(d \phi_{\partial}\right)\left(\mathfrak{s}+D_{\mathrm{cyl}}\right)^{-1}\right) \chi \tag{10.42}
\end{equation*}
$$

later.
For notational convenience we set $A=(\mathfrak{s}+D)$ and $B=\left(\mathfrak{s}+D_{\text {cyl }}\right)$. Recall that $A^{-1} \sim_{m} \chi B^{-1} \chi$, see Proposition 10.23. There we also remarked that $f A^{-1} \sim_{m}$ $0, A^{-1} f \sim_{m} 0, g B^{-1} \sim_{m} 0$ and $B^{-1} g \sim_{m} 0$ if $f$ and $g$ are compactly supported. Rewrite the difference (10.41) as $A^{-1} \operatorname{cl}(d \phi) A^{-2}-\chi B^{-1} \operatorname{cl}\left(d \phi_{\partial}\right) B^{-2} \chi$. Using $A^{-1} \sim_{m}$ $\chi B^{-1} \chi$ we see that the difference is $\sim_{m}$-equivalent to

$$
\chi B^{-1} \chi \operatorname{cl}(d \phi) \chi B^{-1} \chi^{2} B^{-1} \chi-\chi B^{-1} \operatorname{cl}\left(d \phi_{\partial}\right) B^{-2} \chi
$$

Now add and subtract $\chi B^{-1} \chi \operatorname{cl}(d \phi) \chi B^{-2} \chi$ to the first summand; obtaining, for this first summand,

$$
\chi B^{-1} \chi \operatorname{cl}(d \phi) \chi B^{-2} \chi+\chi B^{-1} \chi \operatorname{cl}(d \phi) \chi B^{-1}\left(\chi^{2}-1\right) B^{-1}
$$

which, by our second remark, is $\sim_{m}$-equivalent to $\chi B^{-1} \chi \operatorname{cl}(d \phi) \chi B^{-2} \chi$. Here we have used that $\chi^{2}-1$ is compactly supported. Thus (10.41) is $\sim_{m}$-equivalent to $\chi B^{-1}\left(\chi \operatorname{cl}(d \phi) \chi-\operatorname{cl}\left(d \phi_{\partial}\right)\right) B^{-2} \chi$ which is equal to $\chi B^{-1}\left(\chi \operatorname{cl}\left(d \phi_{\partial}\right) \chi-\operatorname{cl}\left(d \phi_{\partial}\right)\right) B^{-2} \chi$, given that $\chi \operatorname{cl}(d \phi) \chi=\chi \operatorname{cl}\left(d \phi_{\partial}\right) \chi$. We can rewrite this last term as

$$
\chi B^{-1}\left(\chi \operatorname{cl}\left(d \phi_{\partial}\right) \chi-(\chi+(1-\chi)) \operatorname{cl}\left(d \phi_{\partial}\right)(\chi+(1-\chi))\right) B^{-2} \chi
$$

which is in turn equal to

$$
\begin{aligned}
& -\chi B^{-1}(1-\chi) \operatorname{cl}\left(d \phi_{\partial}\right)(1-\chi) B^{-2} \chi-\chi B^{-1} \chi \operatorname{cl}\left(d \phi_{\partial}\right)(1-\chi) B^{-2} \chi \\
& \quad-\chi B^{-1}(1-\chi) \operatorname{cl}\left(d \phi_{\partial}\right) \chi B^{-2} \chi .
\end{aligned}
$$

The last two summands are $\sim_{m}$-equivalent to 0 because $\chi(1-\chi)$ has compact support. Regarding the term $\chi B^{-1}(1-\chi) \operatorname{cl}\left(d \phi_{\partial}\right)(1-\chi) B^{-2} \chi$; we rewrite it as

$$
\chi(1-\chi) B^{-1} \operatorname{cl}\left(d \phi_{\partial}\right)(1-\chi) B^{-2} \chi+\chi\left[B^{-1},(1-\chi)\right] \operatorname{cl}\left(d \phi_{\partial}\right)(1-\chi) B^{-2} \chi
$$

and this is certainly $\sim_{m}$-equivalent to $\chi\left[B^{-1},(1-\chi)\right] \operatorname{cl}\left(d \phi_{\partial}\right)(1-\chi) B^{-2} \chi$. The latter term is in turn equal, up to a sign, to

$$
\chi B^{-1} \operatorname{cl}(d \chi) B^{-1} \operatorname{cl}\left(d \phi_{\partial}\right)(1-\chi) B^{-2} \chi
$$

which is $\sim_{m}$-equivalent to 0 ( $d \chi$ is compactly supported). Thus (10.41) is in $\mathcal{I}_{m}(X, \mathcal{F})$; similarly one proves that the second difference (10.42), viz. $(\mathfrak{s}+$ $D)^{-2} \operatorname{cl}(d \phi)(\mathfrak{s}+D)^{-1}-\chi\left(\left(\mathfrak{s}+D_{\text {cyl }}\right)^{-2} \operatorname{cl}\left(d \phi_{\partial}\right)\left(\mathfrak{s}+D_{\text {cyl }}\right)^{-1}\right) \chi$ is in $\mathcal{I}_{m}(X, \mathcal{F})$. Now, by direct inspection we also see that both the first difference (10.41) and the second difference $(10.42)$ are in $\mathcal{J}_{m}$, i.e. they are bounded if they are multiplied on the right and on the left by the function $g$. Thus, we have proved that $S_{+}$is in $\mathcal{J}_{m}(X, \mathcal{F}) \cap \operatorname{Dom} \bar{\delta}_{2}$. Similarly one proves that $S_{+} \in \operatorname{Dom} \bar{\delta}_{1}$, proving finally that

$$
S_{+} \in \mathcal{J}_{m}(X, \mathcal{F}) \cap \operatorname{Dom} \bar{\delta}_{1} \cap \operatorname{Dom} \bar{\delta}_{2} \equiv \mathfrak{J}_{\mathbf{m}}, \quad m>\operatorname{dim} \tilde{V}
$$

The reasoning for the other entries in the Connes-Skandalis projection is analogous and hence omitted. The proof of Proposition 8.1 is now complete.
10.7.2 Proof of Proposition 8.2. We shall first concentrate on the larger algebra $\mathcal{B}_{m}$; thus we begin by establishing Proposition 8.2. in this context, namely, we prove that $e_{\left(D^{\text {cyl }}\right)} \in \mathcal{B}_{m} \oplus \mathbb{C} e_{1}$ with $m$ greater than $2 n$, which is the dimension of the leaves of $(X, \mathcal{F})$.

Lemma 10.43. For the translation invariant Dirac family $D^{\text {cyl }}=\left(D_{\theta}^{\text {cyl }}\right)_{\theta \in T}$ on the cylinder we have:

$$
\begin{equation*}
\left[\chi^{0},\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\right] \in \mathcal{I}_{m}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}\right) \tag{10.44}
\end{equation*}
$$

with $\chi^{0}$ denoting as usual the function induced on the cylinder by $\chi_{\mathbb{R}}^{0}$, the characteristic function of $(-\infty, 0]$, and $\mathfrak{s}$ the grading operator.

Proof. We shall prove that $\left[\chi^{0},\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}\right]$ has finite Schatten $m$-norm. We shall denote by $t$ the variable along the $\mathbb{R}$-factor in the cylinder; we shall omit the vector bundles from the notation. First we observe that $\chi^{0}$ is bounded and only depends on the cylindrical variable. Observe next that $\left[D_{\tilde{N}}{ }^{\text {cyl }}, \chi^{0}\right]$ defines a family of bounded operators from $W^{1}((\partial \tilde{M} \times\{\theta\}) \times \mathbb{R}) \rightarrow W^{-1}((\partial \tilde{M} \times\{\theta\}) \times \mathbb{R})$ and the same is true for $\left[D^{\partial}, \chi^{0}\right]$; it is then elementary to check that $\left[D^{\partial}, \chi^{0}\right]=0$, as an operator from $W^{1}$ to $W^{-1}$. Similarly, the operator $\left[\partial_{t}, \chi^{0}\right]$ induces bounded maps $W^{1}((\partial \tilde{M} \times\{\theta\}) \times \mathbb{R}) \rightarrow$ $W^{-1}((\partial \tilde{M} \times\{\theta\}) \times \mathbb{R})$. We then have the following equality of bounded operators:

$$
\begin{aligned}
{\left[\chi^{0},\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\right] } & =\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\left[D^{\mathrm{cyl}}, \chi^{0}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} \\
& =\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\left[\partial_{t}, \chi^{0}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}
\end{aligned}
$$

Thus we can write

$$
\left[\chi^{0},\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\right]=\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)\left(I+\left(D^{\mathrm{cyl}}\right)^{2}\right)^{-1}\left[\partial_{t}, \chi^{0}\right]\left(I+\left(D^{\mathrm{cyl}}\right)^{2}\right)^{-1}\left(D^{\mathrm{cyl}}+\mathfrak{s}\right) .
$$

This means that it suffices to prove that $\left(I+\left(D^{\text {cyl }}\right)^{2}\right)^{-1 / 2}\left[\partial_{t}, \chi^{0}\right]\left(I+\left(D^{\text {cyl }}\right)^{2}\right)^{-1 / 2} \in$ $\mathcal{J}_{m}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$. We conjugate this operator with Fourier transform and obtain the operator

$$
T:=\left(I+t^{2}+\left(D^{\partial}\right)^{2}\right)^{-1 / 2}[\mathcal{H}, i t]\left(I+t^{2}+\left(D^{\partial}\right)^{2}\right)^{-1 / 2}
$$

with $\mathcal{H}$ denoting the Hilbert transform on $L^{2}(\mathbb{R})$.
Note that $[\mathcal{H}, t] \xi(t)=i / \pi \int_{\mathbb{R}} \xi(s) d s$. Thus, $T=\left(T_{\theta}\right)_{\theta \in T}$ and each $T_{\theta}$ is the composite $\iota_{\theta} \circ \pi_{\theta}$, with

$$
\begin{aligned}
\iota_{\theta} & : L^{2}(\partial \tilde{M} \times\{\theta\}) \rightarrow L^{2}((\partial \tilde{M} \times\{\theta\}) \times \mathbb{R}), \quad \iota_{\theta}(\eta)=\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \eta \\
& \pi_{\theta}: L^{2}((\partial \tilde{M} \times\{\theta\}) \times \mathbb{R}) \rightarrow L^{2}(\partial \tilde{M} \times\{\theta\}), \pi_{\theta}(\xi)(y) \\
& =\int_{\mathbb{R}}\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \xi(y, t) d t
\end{aligned}
$$

where, as before, we are omitting the vector bundles from the notation. Thus

$$
T_{\theta}^{m}=\iota_{\theta} \circ\left(\pi_{\theta} \circ \iota_{\theta}\right)^{m-1} \circ \pi_{\theta}
$$

On the other hand,

$$
\begin{aligned}
\pi_{\theta} \circ \iota_{\theta}(\eta) & =\int_{\mathbb{R}} d t\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2}\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \eta \\
& =\int_{\mathbb{R}} d t\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1} \eta=C\left(I+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \eta(y),
\end{aligned}
$$

with $C=\pi$ The last step can be justified as follows. Observe that $\forall a>0$ and $k \geq 0$

$$
\int_{\mathbb{R}} \frac{d t}{\left(t^{2}+a^{2}\right)^{k+1}}=\frac{\pi}{2^{2 k}} \frac{(2 k)!}{(k!)^{2}} a^{-2 k-1}
$$

Using this we can show that, in the strong topology,

$$
\int_{\mathbb{R}}\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-\frac{p+1}{2}}=\frac{\pi}{2^{p-1}} \frac{(p-1)!}{\left(\frac{p-1}{2}!\right)^{2}}\left(1+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-\frac{p}{2}}
$$

where $p=2 k+1$. Thus, for $p=1$ we have, in the strong topology,

$$
\int_{\mathbb{R}}\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1}=C\left(I+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-\frac{1}{2}}, \text { with } C=\pi
$$

which is what we had to justify. We thus obtain: $T_{\theta}^{m} \xi=C^{p}\left(\iota_{\theta} \circ\left(I+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-\frac{m-1}{2}} \circ \pi_{\theta} \xi\right)$ and we are left with the task of proving that $\pi_{\theta}$ and $\iota_{\theta}$ are bounded on $L^{2}$ (indeed, it is well known, see [MoN96], that $\left(I+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-\frac{1}{2}}$ has finite Schatten $(m-1)$-norm for $m-1>\operatorname{dim} \partial \tilde{M}$, which is the case here since $m>\operatorname{dim} \tilde{M})$.

Sublemma 10.45. The map $\iota_{\theta}: L^{2}(\partial \tilde{M} \times\{\theta\}) \rightarrow L^{2}((\partial \tilde{M} \times\{\theta\}) \times \mathbb{R}), \iota_{\theta}(\eta)=$ $\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \eta$ is bounded.
Proof. We set $\Delta_{\theta}:=\left(D_{\theta}^{\partial}\right)^{2}$ and compute:

$$
\begin{aligned}
\left\|\iota_{\theta}(\eta)\right\|_{L_{\theta}^{2}}^{2} & =\int_{(\partial \tilde{M} \times\{\theta\}) \times \mathbb{R}} d y d t\left|\left(I+t^{2}+\Delta_{\theta}\right)^{-\frac{1}{2}} \eta(y)\right|^{2} \\
& =\int_{\mathbb{R}} d t\left\|\left(I+t^{2}+\Delta_{\theta}\right)^{-\frac{1}{2}} \eta\right\|_{L^{2}(\partial \tilde{M} \times\{\theta\})}^{2} \\
& \leq\|\eta\|_{L^{2}(\partial \tilde{M} \times\{\theta\})} \int_{\mathbb{R}} d t\left\|\left(I+t^{2}+\Delta_{\theta}\right)^{-\frac{1}{2}}\right\|^{2}
\end{aligned}
$$

where $\left\|\left(I+t^{2}+\Delta_{\theta}\right)^{-\frac{1}{2}}\right\|^{2}$ is the operator norm and it is considered as a function of $t$. We are left with the task of proving that $\left\|\left(I+t^{2}+\Delta_{\theta}\right)^{-\frac{1}{2}}\right\|$ is in $L^{2}\left(\mathbb{R}_{t}\right)$, namely

$$
\begin{equation*}
\int_{\mathbb{R}}\left\|\left(I+t^{2}+\Delta_{\theta}\right)^{-\frac{1}{2}}\right\|^{2}<\infty \tag{10.46}
\end{equation*}
$$

In order to establish (10.46) we write

$$
\left\|\left(I+t^{2}+\Delta_{\theta}\right)^{-\frac{1}{2}}\right\|=\left(1+t^{2}\right)^{-\frac{1}{2}}\left\|\left(I+\frac{\Delta_{\theta}}{1+t^{2}}\right)^{-\frac{1}{2}}\right\|=\left(1+t^{2}\right)^{-\frac{1}{2}}\left\|f\left(D_{\theta}^{\partial}\right)\right\|
$$

with $f(x):=\left(1+\frac{x^{2}}{1+t^{2}}\right)^{-\frac{1}{2}}$. Now, the sup-norm of $f(x)$ is equal to 1 : thus $\left\|f\left(D_{\theta}^{\partial}\right)\right\| \leq 1$ from which (10.46) follows.
Sublemma 10.47. The map $\pi_{\theta}: L^{2}((\partial \tilde{M} \times\{\theta\}) \times \mathbb{R}) \rightarrow L^{2}(\partial \tilde{M} \times\{\theta\}), \pi_{\theta}(\xi)(y)=$ $\int_{\mathbb{R}}\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \xi(y, t) d t$ is bounded.
Proof. We can consider a decomposable element $\xi(y, t)=\eta(y) f(t)$, with $\eta \in$ $L^{2}(\partial \tilde{M} \times\{\theta\})$ and $f(t) \in L^{2}\left(\mathbb{R}_{t}\right)$. Then

$$
\begin{aligned}
\|\pi(\xi)\|_{L^{2}(\partial \tilde{M} \times\{\theta\})} & =\left.\int_{\partial \tilde{M} \times\{\theta\}} d y \int_{\mathbb{R}} d t\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \eta(y) f(t)\right|^{2} \\
& \leq \int_{\partial \tilde{M} \times\{\theta\}} d y\left(\int_{\mathbb{R}} d t\left|\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \eta(y) f(t)\right|^{2}\right)^{2} \\
& \leq \int_{\partial \tilde{M} \times\{\theta\}} d y\left(\int_{\mathbb{R}} d t|f(t)|^{2} \cdot \int_{\mathbb{R}} d t\left|\left(1+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \eta(y)\right|^{2}\right) \\
& =\|f\|_{L^{2}(\mathbb{R})}^{2} \int_{(\partial \tilde{M} \times\{\theta\}) \times \mathbb{R}} d y d t\left|\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \eta(y)\right|^{2} \\
& =\|f\|_{L^{2}(\mathbb{R})}^{2} \int_{\mathbb{R}} d t\left\|\left(I+t^{2}+\left(D_{\theta}^{\partial}\right)^{2}\right)^{-1 / 2} \eta(y)\right\|_{L^{2}(\partial \tilde{M} \times\{\theta\})}^{2} \\
& \leq\|f\|_{L^{2}(\mathbb{R})}^{2}\|\eta\|_{L^{2}(\partial \tilde{M} \times\{\theta\})}^{2} \int_{\mathbb{R}} d t\left\|\left(I+t^{2}+\Delta_{\theta}\right)^{-\frac{1}{2}}\right\|^{2} \\
& \leq\|\xi\|_{L^{2}((\partial \tilde{M} \times\{\theta\}) \times \mathbb{R})}^{2} \int_{\mathbb{R}} d t\left\|\left(I+t^{2}+\Delta_{\theta}\right)^{-\frac{1}{2}}\right\|^{2}
\end{aligned}
$$

where in the last term we are taking the operator norm considered as a function of $t$. Using (10.46) we finish the proof.

The proof of Lemma 10.43 is now complete.
Going back to the proof of Proposition 8.2, we observe that $\widehat{e}_{D^{\text {cyl }}} \in \mathrm{OP}^{-1}$ by the results of [MoN96]. Thus, using the Lemma we have just proved, we conclude that
with $m$ greater than the dimension of the leaves of $(X, \mathcal{F})$. Now we need to prove that, in fact, $\left[\chi^{0}, \widehat{e}_{D^{\text {cyl }}}\right]$ is in $\mathcal{J}_{m}$, that is $g_{\text {cyl }}\left[\chi^{0}, \widehat{e}_{D^{\text {cyl }}}\right]$ and $\left[\chi^{0}, \widehat{e}_{D^{\text {cyl }}}\right] g_{\text {cyl }}$ are bounded; this will ensure that $\widehat{e}_{D^{\text {cy1 }}} \in \mathcal{D}_{m}$.

Lemma 10.48. The operators $g_{\mathrm{cyl}}\left[\chi^{0}, \widehat{e}_{D^{\text {cy } 1}}\right]$ and $\left[\chi^{0}, \widehat{e}_{D^{\text {cy1 }}}\right] g_{\text {cyl }}$ are bounded.
Proof. Consider $g_{\mathrm{cyl}}\left[\chi^{0}, \widehat{e}_{D^{\text {cyl }}}\right]$; this is equal to

$$
\left[g_{\mathrm{cyl}},\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\right]\left[\partial_{t}, \chi^{0}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}+\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} g_{\mathrm{cyl}}\left[\partial_{t}, \chi^{0}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}
$$

which is in turn equal to

$$
\begin{aligned}
& \left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} 2 f_{\mathrm{cyl}} \mathrm{cl}\left(d f_{\mathrm{cyl}}\right)\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\left[\partial_{t}, \chi^{0}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} \\
& \quad+\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} g_{\mathrm{cyl}}\left[\partial_{t}, \chi^{0}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} .
\end{aligned}
$$

Sublemma 10.49. The operator $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1} f_{\text {cyl }}$ and $f_{\text {cyl }}\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}$ are bounded.
Assuming the sublemma for a moment we see that in the last displayed formula the term $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1} 2 f_{\text {cyl }}$ appearing in the first summand is bounded; thus so is $\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} 2 f_{\mathrm{cyl}} \mathrm{cl}\left(d f_{\mathrm{cyl}}\right)$ since $\operatorname{cl}\left(d f_{\mathrm{cyl}}\right)$ is itself bounded (here we use the very definition of $f_{\text {cyl }}$ ); the term $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}\left[\partial_{t}, \chi^{0}\right]\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}$ is known to be bounded (just see the proof of Lemma 10.43); next, we consider the term $\left(D^{\text {cyl }}+\right.$ $\mathfrak{s})^{-1} g_{\text {cyl }}\left[\partial_{t}, \chi^{0}\right]\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}$. We shall prove that with with $\Lambda:=\left(1+\left(D^{\text {cyl }}\right)^{2}\right)^{-\frac{1}{2}}$

$$
\Lambda g_{\mathrm{cyl}}\left[\partial_{t}, \chi^{0}\right] \Lambda:=T_{g}=T:=\Lambda\left[\partial_{t}, \chi^{0}\right] \Lambda
$$

and since the latter is bounded by Lemma 10.43, we will be able to conclude that $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1} g_{\text {cyl }}\left[\partial_{t}, \chi^{0}\right]\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}$ is bounded and that the Lemma holds.
For $\xi, \eta$ in $L^{2}$ we have:

$$
\begin{aligned}
\left\langle T_{\theta} \xi, \eta\right\rangle_{L^{2}} & =\int_{\mathbb{R} \times \tilde{N}} d t d y\left(\Lambda_{\theta} g_{\mathrm{cyl}}\left[\partial_{t}, \chi^{0}\right] \Lambda_{\theta} \xi\right)(t, y) \bar{\eta}(t, y) \\
& =\int_{\tilde{N}} d y\left(\int_{-\infty}^{0} d t\left(\Lambda_{\theta} \xi\right)(t, y) \partial_{t}\left(g \Lambda_{\theta} \bar{\eta}\right)(t, y)-\int_{-\infty}^{0} d t\left(\partial_{t} \Lambda_{\theta} \xi\right)(t, y)\left(g \Lambda_{\theta} \bar{\eta}\right)(t, y)\right) \\
& =-\int_{\tilde{N}}\left[\left(\Lambda_{\theta} \xi\right)(t, y) g(t)\left(\Lambda_{\theta} \bar{\eta}\right)(t, y)\right]_{t=0} \\
& =\int_{\tilde{N}}\left(\Lambda_{\theta} \xi\right)(0, y)\left(\Lambda_{\theta} \bar{\eta}\right)(0, y) \quad \text { since } \quad g(0)=1 \\
& =\left\langle\left(1+\left(D_{\theta}^{\mathrm{cyl}}\right)^{2}\right)^{-\frac{1}{2}}\left[\partial_{t}, \chi^{0}\right]\left(1+\left(D_{\theta}^{\mathrm{cyl}}\right)^{2}\right)^{-\frac{1}{2}} \xi, \eta\right\rangle_{L^{2}}
\end{aligned}
$$

where for the last equality we use again the computation done in the preceding four equalities.

We are left with the task of proving the Sublemma. To this end we observe that, with $\partial_{t}:=\frac{1}{i} \frac{d}{d t}$ we have $\left(D^{\text {cyl }}\right)^{2}=\partial_{t}^{2}+D_{\partial X}^{2}$; we also know that $\partial_{t}^{2}$ and $D_{\partial X}^{2}$ commute. It is easy to see that the (unique) self-adjoint extensions of $\left(1+\partial_{t}^{2}\right)$ and $\left(1+D_{\partial X}^{2}\right)$ are invertible and that the following two equalities hold:

$$
\begin{aligned}
& \left(1+\partial_{t}^{2}\right)^{-1}-\left(1+\left(D^{\mathrm{cyl}}\right)^{2}\right)^{-1}=\left(1+\partial_{t}^{2}\right)^{-1}\left(\left(1+\left(D^{\mathrm{cyl}}\right)^{2}\right)-\left(1+\partial_{t}^{2}\right)\right)\left(1+\left(D^{\mathrm{cyl}}\right)^{2}\right)^{-1} \\
& \quad=\left(1+\partial_{t}^{2}\right)^{-1} D_{\partial X}^{2}\left(1+\left(D^{\mathrm{cyl}}\right)^{2}\right)^{-1}
\end{aligned}
$$

Moreover the last operator is non-negative; thus $\left(1+\partial_{t}^{2}\right)^{-1} \geq\left(1+\left(D^{\text {cyl }}\right)^{2}\right)^{-1}$ and thus $\left.\left(1+\partial_{t}^{2}\right)^{-1 / 2} \geq\left(1+\left(D^{\text {cyl }}\right)^{2}\right)\right)^{-1 / 2}$. Then with $f_{\text {cyl }}=\sqrt{1+t^{2}}$ as usual, we have $\left\|\left(1+\left(D^{\mathrm{cyl}}\right)^{2}\right)^{-1 / 2} f_{\mathrm{cyl}} \xi\right\|_{L^{2}} \leq\left\|\left(1+\partial_{t}^{2}\right)^{-1 / 2} f_{\mathrm{cyl}} \xi\right\|$ with $\xi \in \mathcal{C}_{c}^{\infty}$. This means that if $\left(1+\partial_{t}^{2}\right)^{-1 / 2} f_{\text {cyl }}$ is bounded, then $\left(1+\left(D^{\text {cyl }}\right)^{2}\right)^{-1 / 2} f_{\text {cyl }}$ is also bounded. Now remark that $\left(1+\partial_{t}^{2}\right)^{-1 / 2} f_{\text {cyl }}$ has Schwartz kernel equal to $k(s, t)=e^{-|s-t|} f(s) \equiv$ $e^{-|s-t|} \sqrt{1+s^{2}}$ and that this is an $L^{2}$-function on $\mathbb{R} \times \mathbb{R}$. Thus the integral operator defined on $L^{2}(\mathbb{R})$ by $k(s, t)$ is Hilbert-Schmidt and thus, in particular, bounded. This implies that our operator, which is really $\left(1+\partial_{t}^{2}\right)^{-1 / 2} f_{\mathrm{cyl}} \otimes \operatorname{Id}_{\partial X}$ is also bounded. Summarizing, $\left(1+\left(D^{\text {cyl }}\right)^{2}\right)^{-1 / 2} f_{\text {cyl }}$ is bounded. Thus $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1} f_{\text {cyl }}$ is also bounded, since it can be written as $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}\left(1+\left(D^{\text {cyl }}\right)^{2}\right)^{1 / 2}\left(1+\left(D^{\text {cyl }}\right)^{2}\right)^{-1 / 2} f_{\text {cyl }}$. Finally notice that $f_{\text {cyl }}\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}=\left[f_{\text {cyl }},\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}\right]+\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1} f_{\text {cyl }}$ and we know that both summands on the right hand sides are bounded. The Sublemma (and thus the Lemma) is proved.

Thus we have proved that $\widehat{e}_{D^{\text {cyl }}} \in \mathcal{D}_{m}$. On the other hand, see Definition 6.38,

$$
\mathcal{B}_{m}:=\left\{\ell \in \mathcal{D}_{m} \cap \operatorname{Dom}\left(\partial_{\alpha}\right) \mid\left[f_{\mathrm{cyl}}, \ell\right],\left[f_{\mathrm{cyl}},\left[f_{\mathrm{cyl}}, \ell\right]\right] \text { is bounded }\right\} .
$$

Thus we need to show, first of all, that it is also true that $\widehat{e}_{D^{\text {cyl }}} \in \operatorname{Dom}\left(\partial_{\alpha}\right)$. We need to prove that the limit

$$
\lim _{t \rightarrow 0} \frac{1}{t}\left(\alpha_{t}\left(\widehat{e}_{D^{\mathrm{cy} 1}}\right)-\widehat{e}_{D^{\mathrm{cy} 1}}\right)
$$

exists in $\mathrm{OP}^{-1}$. We compute

$$
\begin{aligned}
& \frac{1}{t}( \left(\alpha_{t}\left(\widehat{e}_{\left.D^{\mathrm{cyl}}\right)}\right)-\widehat{e}_{D^{\mathrm{cyl}}}\right) \\
&=\frac{1}{t}\left[e^{i t s},\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\right] e^{-i t s}=\frac{1}{t}\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\left[D^{\mathrm{cyl}}, e^{i t s}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} e^{-i t s} \\
&=\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} i \operatorname{cl}(d s) e^{i t s}\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} e^{-i t s} \\
&=\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} i \operatorname{cl}(d s)\left[e^{i t s},\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\right] e^{-i t s}+\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} i \operatorname{cl}(d s)\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} \\
&=\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} i \operatorname{cl}(d s)\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\left[D^{\mathrm{cyl}}, e^{i t s}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} e^{-i t s} \\
&+\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} i \operatorname{cl}(d s)\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} \\
&=\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} i \operatorname{cl}(d s)\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} i \operatorname{cl}(d s) i t e^{i t s}\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} e^{-i t s} \\
&+\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} i \operatorname{cl}(d s)\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}
\end{aligned}
$$

and the last term converges to $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1} i \operatorname{cl}(d s)\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}$ as $t \rightarrow 0$. Thus $\widehat{e}_{D^{\text {cyl }}} \in \operatorname{Dom}\left(\partial_{\alpha}\right)$. Thus, we have proved that $\widehat{e}_{D^{\text {cyl }}} \in \mathcal{D}_{m, \alpha}:=\mathcal{D}_{m} \cap \operatorname{Dom}\left(\partial_{\alpha}\right)$.

Next we need to show that $\left[f_{\text {cyl }}, \widehat{e}_{D^{\text {cyl }}}\right]$ and $\left[f_{\text {cyl }},\left[f_{\text {cyl }}, \widehat{e}_{D^{\text {cyl }}}\right]\right]$ are bounded. However, this is elementary at this point: for example $\left[f_{\text {cyl }}, \widehat{e}_{D^{\text {cyl }}}\right]$ is nothing but $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1} i \operatorname{cl}\left(d f_{\text {cyl }}\right)\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}$ which is indeed bounded. Similarly we proceed for $\left[f_{\mathrm{cyl}},\left[f_{\mathrm{cyl}}, \widehat{e}_{D^{\mathrm{cy1}}}\right]\right]$.

Next we prove that $\widehat{e}_{D^{\text {cyl }}} \equiv\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1} \in \operatorname{Dom}\left(\bar{\delta}_{j}\right), j=1,2$. We only do it for $\bar{\delta}_{2}$, the arguments for $\bar{\delta}_{1}$ are similar. It suffices to show that $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1} \in \operatorname{Dom}\left(\bar{\partial}_{2}\right)$ and $\bar{\partial}_{2}\left(\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}\right) \in \mathcal{B}_{\underline{m}}$. Using [MoN96] Proposition 7.17, we know that $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}$ does belong to $\operatorname{Dom}\left(\bar{\partial}_{2}\right)$ and moreover

$$
\bar{\partial}_{2}\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}=\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\left[D^{\mathrm{cyl}}, \phi_{\partial}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} .
$$

In order to see that the right hand side of this formula belongs to $\mathcal{B}_{m}$ we show separately that it belongs to $\mathcal{D}_{m}$ and $\operatorname{Dom}\left(\partial_{\alpha}\right)$ and that, in addition, it is such that its commutator and its double-commutator with $f_{\text {cyl }}$ is bounded. First of all we employ Lemma 10.43 , which shows that $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1} \in \mathcal{D}_{m}=\operatorname{Dom}\left(\bar{\delta}_{3}\right)$. Since, by the same arguments, $\left[D^{\text {cyl }}, \phi_{\partial}\right]\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}$, which is the composition of Clifford multiplication by $d \phi_{\partial}$ with $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}$, also belongs to $\operatorname{Dom}\left(\bar{\delta}_{3}\right)$ we conclude that their product is in $\operatorname{Dom}\left(\bar{\delta}_{3}\right)$ i.e. in $\mathcal{D}_{m}$. Here we have used the fact that the domain of a closed derivation is a Banach algebra. Exactly the same argument, together with the above proof that $\left(D^{\text {cyl }}+\mathfrak{s}\right)^{-1}$ belongs to $\operatorname{Dom}\left(\partial_{\alpha}\right)$, establishes that

$$
\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\left[D^{\mathrm{cyl}}, \phi_{\partial}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} \in \operatorname{Dom}\left(\partial_{\alpha}\right)
$$

Finally it is clear that the above term is such that its commutator with $f_{\text {cyl }}$ is bounded; similarly we proceed with its double-commutator. This completes the proof of the Proposition 8.2.
10.7.3 Proof of Proposition 8.3. We need to show that $\widehat{e}_{D} \in \mathfrak{A}_{\mathrm{m}}:=\mathcal{A}_{m} \cap$ $\operatorname{Dom}\left(\bar{\delta}_{1}\right) \cap \operatorname{Dom}\left(\bar{\delta}_{2}\right) \cap \pi^{-1}\left(\mathfrak{B}_{\mathbf{m}}\right)$, for $m>\operatorname{dim} \tilde{V}$. First of all we prove that $\widehat{e}_{D} \in \mathcal{A}_{m}$. Thus we need to show that $t\left(\widehat{e}_{D}\right) \in \mathcal{J}_{m}$ and that $\pi\left(\widehat{e}_{D}\right) \in \mathcal{B}_{m}$. However, this is clear from our previous results. Indeed, $\pi\left(\widehat{e}_{D}\right)=\widehat{e}_{D_{\text {cy1 }}}$ and we have proved in the previous proposition that the right hand side is in $\mathcal{B}_{m}$. Similarly, if $\chi$ is a smooth approximation of $\chi^{0}$, we can write:

$$
\begin{equation*}
t\left(\widehat{e}_{D}\right):=\widehat{e}_{D}-\chi^{0} \widehat{e}_{D_{\mathrm{cy} 1}} \chi^{0}=\left(\widehat{e}_{D}-\chi \widehat{e}_{D_{\mathrm{cy} 1}} \chi\right)+\left(\chi \widehat{e}_{D_{\mathrm{cy} 1}} \chi-\chi^{0} \widehat{e}_{D_{\mathrm{cy} 1}} \chi^{0}\right) . \tag{10.50}
\end{equation*}
$$

We have already proved that the first summand $\widehat{e}_{D}-\chi \widehat{e}_{D_{\text {cyl }}} \chi$ is in $\mathcal{I}_{m}$. We now proceed to show that it is indeed in $\mathcal{J}_{m}$. We need to show that $\varphi(D+\mathfrak{s})^{-1}$ and $(D+\mathfrak{s})^{-1} \varphi$ are not only in $\mathcal{I}_{m}$ but in fact in $\mathcal{J}_{m}$ provided that $\varphi$ is compactly supported. However, this is can be proved as follows. First, $g \varphi(D+\mathfrak{s})^{-1}$ is clearly bounded, given that it is in $\mathcal{I}_{m}$ (indeed $g \varphi$ is compactly supported). As far as $\varphi(D+\mathfrak{s})^{-1} g$, we rewrite it as $\varphi\left[(D+\mathfrak{s})^{-1}, g\right]+\varphi g(D+\mathfrak{s})^{-1}$. The latter is equal to $-\varphi(D+\mathfrak{s})^{-1} 2 \operatorname{cl}(d f) f(D+\mathfrak{s})^{-1}+\varphi g(D+\mathfrak{s})^{-1}$. This is bounded using the above reasoning and Sublemma 10.49. Summarizing, we have proved that $\widehat{e}_{D}-\chi \widehat{e}_{D_{\text {cyl }}} \chi \in$ $\mathcal{J}_{m}$.

As far as the second term in (10.50) is concerned we simply observe that it can be rewritten as $\left(\chi \widehat{e}_{D}\left(\chi-\chi^{0}\right)\right)+\left(\chi-\chi^{0}\right) \widehat{e}_{D_{\text {cy1 }}} \chi^{0}$ and both these terms are $m$-Schatten class if $m>\operatorname{dim} \tilde{V}$, given that $\left(\chi-\chi^{0}\right)$ is compactly supported. This trick can be also used in order to show that if we multiply by $g$ on the left and on the right we get a bounded operator, according to what we have observed above. Notice that since $\pi\left(\widehat{e}_{D}\right)=\widehat{e}_{D_{\text {cyl }}}$ we also have, directly from the previous Subsubsection, that $\widehat{e}_{D} \in \pi^{-1}\left(\mathfrak{B}_{\mathbf{m}}\right)$. Next we need to show that $\widehat{e}_{D}$ is in the domain of the two closed derivations introduced in Section 6.5.4. Consider, for example, $\bar{\delta}_{2}$. Recall that $\bar{\delta}_{2}: \operatorname{Dom} \bar{\delta}_{2} \rightarrow \mathcal{A}_{m}(X, F)$ with

$$
\operatorname{Dom} \bar{\delta}_{2}=\left\{a \in \operatorname{Dom} \bar{\delta}_{2}^{\max } \mid \bar{\delta}_{2}^{\max } a \in \mathcal{A}_{m}(X, F)\right\}
$$

The fact that $\widehat{e}_{D} \equiv(\mathfrak{s}+D)^{-1}$ is in $\operatorname{Dom} \bar{\delta}_{2}^{\max }$ is proved in [MoN96] where it is also proved that

$$
\begin{equation*}
\bar{\delta}_{2}^{\max }\left((\mathfrak{s}+D)^{-1}\right)=(\mathfrak{s}+D)^{-1}[D, \phi](\mathfrak{s}+D)^{-1} \tag{10.51}
\end{equation*}
$$

Thus we only need to show that the right hand side belongs to $\mathcal{A}_{m}(X, F)$, where we recall that $\mathcal{A}_{m}(X, F)=\left\{a \in A^{*}\right.$ such that $\pi(a) \in \mathcal{B}_{m}\left(\operatorname{cyl}(\partial X), F_{\text {cyl }}\right)$ and $t(a) \in$ $\left.\mathcal{J}_{m}(X, F)\right\}$. But the image under $\pi$ of the right hand side of (10.51) is precisely $\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\left[D^{\mathrm{cyl}}, \phi_{\partial}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}$ which was shown to belong to $\mathcal{B}_{m}$ at the end of the proof of the previous Proposition. Thus we are left with the task of proving that

$$
t\left((\mathfrak{s}+D)^{-1}[D, \phi](\mathfrak{s}+D)^{-1}\right) \in \mathcal{J}_{m}(X, F)
$$

By definition of $t$ this means that

$$
(\mathfrak{s}+D)^{-1}[D, \phi](\mathfrak{s}+D)^{-1}-\chi^{0}\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\left[D^{\mathrm{cyl}}, \phi_{\partial}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} \chi^{0} \in \mathcal{J}_{m}(X, F) .
$$

However, this can be proved by first reducing to $\chi$, a smooth approximation of $\chi^{0}$, using the same reasoning as in (10.50); then, in order to show that

$$
(\mathfrak{s}+D)^{-1}[D, \phi](\mathfrak{s}+D)^{-1}-\chi\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1}\left[D^{\mathrm{cyl}}, \phi_{\partial}\right]\left(D^{\mathrm{cyl}}+\mathfrak{s}\right)^{-1} \chi \in \mathcal{J}_{m}(X, F)
$$

we employ the same arguments used in order to establish that (10.41) is in $\mathcal{J}_{m}(X, F)$. The proof of Proposition 8.3 is now complete.

### 10.8 Proofs for the extension of the relative GV cyclic cocycle.

10.8.1 Further properties of the modular Schatten extension. This Subsection is devoted to the statement of some technical properties of the algebras $\mathfrak{B}_{m}$. In the next Subsection we shall use these properties in order to show that the GodbillonVey eta cocycle extends from $B_{c}$ to $\mathfrak{B}_{2 n+1}$ with $2 n$ equal to the dimension of the leaves.

We begin with the Banach algebra $\mathrm{OP}^{-1}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$. Here $Y=\tilde{N} \times_{\Gamma} T$ is a foliated bundle without boundary.

We have proved in Proposition 6.25 that $\mathrm{OP}^{-1}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ is a subalgebra of $B^{*}$. The latter was proved to be isomorphic to $C^{*} \mathbb{R} \otimes C^{*}(Y, \mathcal{F})$, see Proposition 4.3. Thus, the Fourier transformation applied to $B^{*}$ has values in $C_{0}\left(\mathbb{R}, C^{*}(Y, \mathcal{F})\right)$. In particular, the image of $\mathrm{OP}^{-1}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ under Fourier transformation is a subalgebra of $C_{0}\left(\mathbb{R}, C^{*}(Y, \mathcal{F})\right)$. Put it differently, the Fourier transform $\hat{\ell}$ of an element $\ell \in \mathrm{OP}^{-1}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ is a continuous function $\hat{\ell}: \mathbb{R} \rightarrow C^{*}(Y, \mathcal{F})$ vanishing at infinity.

Recall also the Banach algebras $\mathrm{OP}^{-p}(\operatorname{cyl}(Y), \mathcal{F})$, see Proposition 6.25. Similarly, we can define $\mathrm{OP}^{-p}(Y, \mathcal{F})$ for a closed foliated bundle $(Y, \mathcal{F})$. Thinking to $\mathrm{OP}^{-p}(Y, \mathcal{F})$ as a subalgebra of the von Neumann algebra of the foliation $(Y, \mathcal{F})$, we see that an element $b$ in $\mathrm{OP}^{-p}(Y, \mathcal{F})$ is given by a family of operators $\left(b_{\theta}\right)_{\theta \in T}$, with $b_{\theta}$ acting on Sobolev spaces along $N \times\{\theta\}$. Let $f: \mathbb{R} \rightarrow \mathrm{OP}^{-p}(Y, \mathcal{F})$ be a measurable $\mathrm{OP}^{-p}(Y, \mathcal{F})$-valued function. This means that $f=\left(f_{\theta}\right)_{\theta \in T}$ with $f_{\theta}$ a measurable function with values in the bounded operators of order $-p$ on Sobolev spaces on $N \times\{\theta\}$. We define a norm $\left.\|f\|_{L^{2}(\mathbb{R}, \mathrm{OP}}{ }^{-p}(Y, \mathcal{F})\right)$ by setting

$$
\begin{equation*}
\|f\|_{L^{2}\left(\mathbb{R}, \mathrm{OP}^{-p}(Y, \mathcal{F})\right)}=\sup _{\theta \in T}\left(\int_{\mathbb{R}} \mid\left\|f_{\theta}(x)\right\| \|_{p}^{2} d x\right)^{\frac{1}{2}} . \tag{10.52}
\end{equation*}
$$

Moreover, let $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathrm{OP}^{-p}(Y, \mathcal{F})$ be a measurable $\mathrm{OP}^{-p}(Y, \mathcal{F})$-valued function; it is also considered as a family $\left(g_{\theta}\right)_{\theta \in T}$, with $g_{\theta}$ a measurable function on $\mathbb{R} \times \mathbb{R}$ with values in the bounded operators of order $-p$ on Sobolev spaces on $N \times\{\theta\}$. We define a norm $\|g\|_{L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-p}(Y, \mathcal{F})\right)}$ by the analogue of (10.52). It is easily verified that $L^{2}\left(\mathbb{R} \times \mathbb{R}, \operatorname{OP}^{-p}(Y, \mathcal{F})\right)$ is a Banach algebra with the convolution product $g * h$ given by the family $\left(g_{\theta} * h_{\theta}\right)_{\theta \in T}$, with

$$
g_{\theta} * h_{\theta}(x, z)=\int_{\mathbb{R}} d y g_{\theta}(x, y) h_{\theta}(y, z)
$$

for $g=\left(g_{\theta}\right)_{\theta \in T}, h=\left(h_{\theta}\right)_{\theta \in T} \in L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-p}(Y, \mathcal{F})\right)$. Notice that the product in the integrand involves the algebra structure of $\mathrm{OP}^{-p}(Y, \mathcal{F})$. Remark that if $p+q>\operatorname{dim} N$ then there exists a bounded pairing

$$
\begin{equation*}
\langle,\rangle: L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-p}(Y, \mathcal{F})\right) \times L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right) \rightarrow \mathbb{C} \tag{10.53}
\end{equation*}
$$

with

$$
\langle g, h\rangle=\int_{\mathbb{R} \times \mathbb{R}} d x d y \omega_{\Gamma}^{Y}\left(g(x, y) h(x, y)^{*}\right) .
$$

Indeed, given $g, h \in C_{c}^{\infty}\left(G_{\text {cyl }}\right)$, considered as elements in $L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-p}(Y, \mathcal{F})\right)$ and $L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)$ respectively, we have

$$
\langle g, h\rangle \leq \int_{\mathbb{R} \times \mathbb{R}} d x d y\left|\omega_{\Gamma}^{Y}\left(g(x, y) h(x, y)^{*}\right)\right|
$$

$$
\begin{aligned}
& \leq C \int_{\mathbb{R} \times \mathbb{R}} d x d y\left\|g(x, y) h(x, y)^{*} \mid\right\|_{p+q} \\
& \leq C \int_{\mathbb{R} \times \mathbb{R}} d x d y\|g(x, y)\|\left\|_{p}\right\|\left|\|(x, y) \mid\|_{q}\right. \\
& \leq\|g\|_{L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-p}(Y, \mathcal{F})\right)}\|h\|_{L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)}
\end{aligned}
$$

In the second step we have used [MoN96], p. 508, Cor. 6.11. Thus (10.53) is bounded. Observe now that $\omega_{\Gamma}^{\text {cyl }}(g h)=\left\langle g, h^{*}\right\rangle$; thus the above inequality also implies that

$$
\begin{equation*}
L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-p}(Y, \mathcal{F})\right) \times L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right) \ni(g, h) \rightarrow \omega_{\Gamma}^{\mathrm{cyl}}(g h) \in \mathbb{C} \tag{10.54}
\end{equation*}
$$

is bounded.
The following Lemma will play a crucial role:
Lemma 10.55. (Key Lemma) (1) If $\ell \in \mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ and $0 \leq q<p-1 / 2$ then for each $x \in \mathbb{R}$

$$
\begin{equation*}
\|\hat{\ell}(x)\|\left\|_{q} \leq\right\| \ell \|_{p} \tag{10.56}
\end{equation*}
$$

so that, in particular, $\hat{\ell}(x)$ is an element of $\mathrm{OP}^{-q}(Y, \mathcal{F})$ for each $x \in \mathbb{R}$ and for each $0 \leq q<p-1 / 2$. Moreover there is a constant $C>0$ such that

$$
\begin{equation*}
\|\hat{\ell}\|_{L^{2}\left(\mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)} \leq C\|\ell\| \|_{p} \quad \text { for } \quad 0 \leq q<p-1 / 2 \tag{10.57}
\end{equation*}
$$

(2) If $\ell \in \mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right) \cap \operatorname{Dom} \partial_{\alpha, p}$ then $\hat{\ell}$ is differentiable as a function from $\mathbb{R}$ with values in the Banach algebra $\mathrm{OP}^{-q}(Y, \mathcal{F}), 0 \leq q<p-1 / 2$. Moreover there is a constant $C>0$ such that

$$
\left.\left\|\frac{d \hat{\ell}}{d x}\right\|_{L^{2}\left(\mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)} \leq C \right\rvert\,\left\|\partial_{\alpha} \ell\right\|_{p} \quad \text { for } \quad 0 \leq q<p-1 / 2
$$

(3) Given $\ell \in \mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right) \cap \operatorname{Dom} \partial_{\alpha, p}$, the commutator $\left[\chi^{0}, \ell\right]$ admits a kernel function $k: \mathbb{R} \times \mathbb{R} \rightarrow \operatorname{OP}^{-q}(Y, \mathcal{F})$ and there exists a constant $C$ such that

$$
\begin{equation*}
\|k\|_{L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)} \leq C\left(\| \|\| \|_{p}+\left\|\partial_{\alpha} \ell\right\| \|_{p}\right) \quad \text { for } \quad 0 \leq q<p-1 / 2 . \tag{10.58}
\end{equation*}
$$

Proof. For the first item we need to show that given $\ell \in \mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right), \hat{\ell}(x)$ is in $\mathrm{OP}^{-q}(Y, \mathcal{F})$ and there is a constant $C>0$ such that

$$
\|\hat{\ell}\|_{L^{2}\left(\mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)} \leq C\| \| \ell \|_{p} \quad \text { for } \quad 0 \leq q<p-1 / 2
$$

We consider $\ell \in \mathrm{OP}^{-p}(\operatorname{cyl}(Y))$ as a family of operators $\left(\ell_{\theta}\right)_{\theta \in T}$ with $\ell_{\theta}: L^{2}(\operatorname{cyl}(\tilde{N}) \times$ $\{\theta\}) \rightarrow L^{2}(\operatorname{cyl}(\tilde{N}) \times\{\theta\})$. By definition one has

$$
\left\|\ell_{\theta}\right\|_{n+p, n}=\left\|(1+\Delta)^{(n+p) / 2} \ell_{\theta}(1+\Delta)^{-n / 2}\right\|_{C^{*}}
$$

with $\Delta$ denoting the Laplacian on the cylinder. Applying the Fourier transformation along the cylindrical variable we obtain:

$$
\begin{aligned}
\|\ell\|_{n+p, n} & =\sup _{x \in \mathbb{R}}\left\|\left(1+x^{2}+\Delta_{N}\right)^{(n+p) / 2} \hat{\ell}_{\theta}(x)\left(1+x^{2}+\Delta_{N}\right)^{-n / 2}\right\|_{C^{*}} \\
& \geq \sup _{x \in \mathbb{R}}\left\|\left(1+x^{2}\right)^{(p-q) / 2}\left(1+\Delta_{N}\right)^{(n+q) / 2} \hat{\ell}_{\theta}(x)\left(1+x^{2}+\Delta_{N}\right)^{-n / 2}\right\|_{C^{*}} \\
& \geq\left(1+x^{2}\right)^{(p-q) / 2}\left\|\left(1+\Delta_{N}\right)^{(n+q) / 2} \hat{\ell}_{\theta}(x)\left(1+x^{2}+\Delta_{N}\right)^{-n / 2}\right\|_{C^{*}}
\end{aligned}
$$

where we have used the fact that

$$
\left\|\left(1+x^{2}+\Delta_{N}\right)^{r} \xi\right\| \geq\left\|\left(1+x^{2}\right)^{r} \xi\right\| \quad \text { and } \quad\left\|\left(1+x^{2}+\Delta_{N}\right)^{r} \xi\right\| \geq\left\|\left(1+\Delta_{N}\right)^{r} \xi\right\| .
$$

Taking adjoints we also obtain:

$$
\begin{aligned}
\left\|\ell_{\theta}\right\|_{n+p, n} & \geq\left(1+x^{2}\right)^{(p-q) / 2}\left\|\left(1+x^{2}+\Delta_{N}\right)^{-n / 2}\left(\hat{\ell}_{\theta}(x)\right)^{*}\left(1+\Delta_{N}\right)^{(n+q) / 2}\right\|_{C^{*}} \\
& \geq\left(1+x^{2}\right)^{(p-q) / 2}\left\|\left(1+\Delta_{N}\right)^{-n / 2}\left(\hat{\ell}_{\theta}(x)\right)^{(n}\left(1+\Delta_{N}\right)^{(n+q) / 2}\right\|_{C^{*}} \\
& =\left(1+x^{2}\right)^{(p-q) / 2}\left\|\left(1+\Delta_{N}\right)^{(n+q) / 2} \hat{\ell}_{\theta}(x)\left(1+\Delta_{N}\right)^{-n / 2}\right\|_{C^{*}} \\
& =\left(1+x^{2}\right)^{(p-q) / 2}\left\|\hat{\ell}_{\theta}(x)\right\|_{n+q, n} .
\end{aligned}
$$

Applying the same argument we prove also that $\left\|\ell_{\theta}\right\|_{-n,-n-p} \geq\left(1+x^{2}\right)^{(p-q) / 2}$ $\left\|\hat{\ell}_{\theta}(x)\right\|_{-n,-n-q}$. These inequalities imply that

$$
\left\|\left\|\ell_{\theta}\right\|_{p} \geq\left(1+x^{2}\right)^{(p-q) / 2}\right\| \hat{\ell}_{\theta}(x)\| \|_{q} .
$$

Note now that $\left(1+x^{2}\right)^{-(p-q) / 2}$ is a $L^{2}$-function if $q<p-1 / 2$; let $C$ be the $L^{2}$-norm of $\left(1+x^{2}\right)^{-(p-q) / 2}$. We thus obtain

$$
C^{2}\| \| \ell_{\theta}\| \|_{p}^{2} \geq \int_{\mathbb{R}}\| \| \hat{\ell}_{\theta}(x)\| \|_{q}^{2}
$$

which implies that

$$
C \mid\|\ell\|_{p} \geq \sup _{\theta \in T}\left(\int_{\mathbb{R}}\| \| \hat{\ell}_{\theta}(x)\| \|_{q}^{2}\right)^{1 / 2}=\|\hat{\ell}\|_{L^{2}\left(\mathbb{R}, \mathrm{OP}^{-q}(Y)\right)}
$$

The first part of the Lemma is proved.
Next we tackle the second item. We first show that if, in addition, $\ell \in \operatorname{Dom} \partial_{\alpha, p}$, then $\hat{\ell}$ is differentiable as a function $\mathbb{R} \rightarrow \mathrm{OP}^{-q}$. Consider $\partial_{\alpha, p}(\ell)$, an element in $\mathrm{OP}^{-p}$ by hypothesis. Remark that the automorphism $\alpha_{t}$ appearing in the definition of $\partial_{\alpha}$, see (6.31), corresponds to the translation operator by $t$ under Fourier transformation. Thus we have, using item 1),

$$
\left\|\frac{\hat{\ell}(x+t)-\hat{\ell}(x)}{t}-\widehat{\partial_{\alpha, p}(\ell)}(x)\right\|_{\mathrm{OP}^{-q}} \leq\left\|\frac{\alpha_{t}(\ell)-\ell}{t}-\partial_{\alpha, p}(\ell)\right\|_{\mathrm{OP}^{-p}}
$$

and we know by hypothesis that the right hand side converges to 0 as $t \rightarrow 0$. Thus the limit

$$
\lim _{t \rightarrow 0} \frac{\hat{\ell}(x+t)-\hat{\ell}(x)}{t}
$$

exists in $\operatorname{OP}^{-q}(Y, \mathcal{F})$ for each $x \in \mathbb{R}$ and it is equal to $\widehat{\partial_{\alpha, p}(\ell)}(x)$. This proves the differentiability of $\hat{\ell}$. The estimate in this second item is now a consequence of the one in the first item.

Finally, we tackle the third item of the Lemma. We must show that given $\ell \in$ $\mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right) \cap \operatorname{Dom}_{\alpha, p}$, the commutator $\left[\chi^{0}, \ell\right]$ admits a kernel function $k$ : $\mathbb{R} \times \mathbb{R} \rightarrow \mathrm{OP}^{-q}(Y, \mathcal{F})$ and there exists a constant $C$ such that

$$
\|k\|_{L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)} \leq C\left(\|\ell\|\left\|_{p}+\right\| \partial_{\alpha} \ell \mid \|_{p}\right) \quad \text { for } \quad 0 \leq q<p-1 / 2
$$

Let $\ell$ be an element in $\mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right) \cap \operatorname{Dom} \partial_{\alpha, p}$. We know that $\lim _{t \rightarrow 0}(\hat{\ell}(x+$ $t)-\hat{\ell}(x)) / t$ exists in $\mathrm{OP}^{-q}$ for each $x \in \mathbb{R}$. Set

$$
\omega(u, v)=\frac{\hat{\ell}(u)-\hat{\ell}(v)}{u-v}
$$

The above argument shows that $\omega$ is a continuous function from $\mathbb{R} \times \mathbb{R}$ into $\mathrm{OP}^{-q}(Y, \mathcal{F})$. Recall the Hilbert transformation $\mathcal{H}: L^{2}(\mathbb{R}) \rightarrow L^{2}(\mathbb{R})$, see the proof of Proposition 5.25. It can also be defined on $L^{2}\left(\mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)$. Here we recall that for $\ell \in \mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ we have proved that $\hat{\ell} \in L^{2}\left(\mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)$. We know that the Hilbert transformation corresponds to the multiplication operator by $2 \chi^{0}-1$ under Fourier transformation $F$ (by this we mean that $\left.F\left(2 \chi^{0}-1\right) F^{-1}=\mathcal{H}\right)$. Thus $\left[\chi^{0}, \ell\right]$ corresponds to $[\mathcal{H}, \hat{\ell}] / 2$ under Fourier transform. As already remarked from the very definition of $\mathcal{H}$ we know that $[\mathcal{H}, \hat{\ell}]$ is the integral operator with kernel function equal to $-i / \pi \omega(u, v)$. This proves the first part of the statement in item 3) but for the operator $F \circ\left[\chi^{0}, \ell\right] \circ F^{-1}$. We now establish the estimate claimed in item 3) but for $\omega$; we thus estimate $\|\omega\|_{L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)}$, with $0 \leq q<p-1 / 2$.

Let $(u, v)$ be a point in $\mathbb{R} \times \mathbb{R}$, with $|u-v| \geq 1$. Setting $t=u-v$ we get

$$
\||\omega(u, v)|\|_{q} \leq\left(\left\|\left|| \hat { \ell } ( u ) | \left\|_{q}+\left|\|\hat{\ell}(v) \mid\|_{q}\right) /|t|\right.\right.\right.\right.
$$

which implies that

$$
\begin{gathered}
\int_{|u-v| \geq 1} d u d v\left|\|\omega(u, v) \mid\|_{q}^{2} \leq \int_{|t| \geq 1} \int_{\mathbb{R}} d t d v\left(\left\|\hat{\ell}(v+t)\left|\left\|_{q}+\right\| \hat{\ell}(v)\right|\right\|_{q}\right)^{2} / t^{2}\right. \\
=\int_{|t| \geq 1} \frac{d t}{t^{2}} \int_{\mathbb{R}} d v\left(\left\|| | \hat { \ell } ( v + t ) | \| _ { q } ^ { 2 } + \| | \hat { \ell } ( v ) \left|\left\|_{q}^{2}+2\left|\|\hat{\ell}(v)\|\left\|_{q} \mid\right\| \hat{\ell}(v+t)\| \|_{q}\right)\right.\right.\right.\right.
\end{gathered}
$$

which is bounded by a constant times $\left\|\|\ell\|_{p}^{2}\right.$ given that $\hat{\ell} \in L^{2}\left(\mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)$. In the region $|u-v|<1$ we have

$$
\hat{\ell}(u)-\hat{\ell}(v)=\int_{u}^{v} d s \frac{d \hat{\ell}}{d s}(s)
$$

which gives $\omega(u, v)=\frac{1}{u-v} \int_{u}^{v} d s \frac{d \hat{\ell}}{d s}(s)$. It then follows that

$$
\begin{aligned}
\int_{|u-v|<1} d u d v| ||\omega(u, v)| \|_{q}^{2} & \leq \int_{|t|<1} d t \int_{\mathbb{R}} d v \frac{1}{t^{2}}\left(\int_{v}^{v+t} d s\left|\left\|\left.\frac{d \hat{\ell}}{d s}(s) \right\rvert\,\right\|_{q}\right)^{2}\right. \\
& \leq \int_{|t|<1} d t \int_{\mathbb{R}} d v \frac{1}{t^{2}}\left|\int_{v}^{v+t} d s\right|\left|\int_{v}^{v+t} d s\right|\left\|\frac{d \hat{\ell}}{d s}(s)\left|\|_{q}^{2}\right|\right. \\
& =\int_{|t|<1} \frac{d t}{|t|} \int_{\mathbb{R}} d v\left|\int_{0}^{t} d r\right|\left\|\frac{d \hat{\ell}}{d s}(v+r)\left|\|_{q}^{2}\right|\right. \\
& \left.=\int_{|t|<1} \frac{d t}{|t|}\left|\int_{0}^{t} d r \int_{\mathbb{R}} d v\right|\left\|\frac{d \hat{\ell}}{d s}(v+r)\right\|_{q}^{2} \right\rvert\, \\
& =\int_{|t|<1} d t \int_{\mathbb{R}} d v\left|\left\|\left.\frac{d \hat{\ell}}{d s}(v) \right\rvert\,\right\|_{q}^{2}\right.
\end{aligned}
$$

which is bounded by a constant times $\left\|\left|\partial_{\alpha} \ell \|\right|_{p}^{2}\right.$ (by item 2)). The third item of the Key Lemma is thus proved for the operator $F \circ\left[\chi^{0}, \ell\right] \circ F^{-1}$. Observe now that conjugation by Fourier transformation $F$ defines an isometry on $L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-q}\right)$; thus $\left[\chi^{0}, \ell\right]$ admits a kernel function which is in $L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-q}\right)$. This means that

$$
\int_{\mathbb{R} \times \mathbb{R}} d u d v\| \| \omega(u, v)\| \|_{q}^{2}=\int_{\mathbb{R} \times \mathbb{R}} d u d v\| \| k(u, v) \|_{q}^{2}
$$

and this implies the estimate we wanted to prove.
10.8.2 Proof of Proposition 8.12 (extension of the eta cocycle). We want to show that if $m=2 n+1$, with $2 n$ equal to the dimension of leaves, then the eta cocycle $\sigma_{m}$ extends to a bounded cyclic cocycle on $\mathfrak{B}_{m}$.

Proof. We begin by observing that from its very definition $\sigma_{2 n+1}$ is the sum of elements of the following type $\omega_{\Gamma}\left(b\left[\chi^{0}, \ell\right] b^{\prime}\right)$ where
$-b$ is a product of $p$ elements in $\mathrm{OP}^{-1}$;
$-b^{\prime}$ is a product of $s$ elements in $\mathrm{OP}^{-1}$;

- $m=2 n+1$ is equal to $p+s$.

Decompose $b$ according to the analogue of the direct sum decomposition explained around formula (10.1). Then $b=\binom{b_{00} b_{01}}{b_{10} b_{11}}$ with $b_{00}=\chi^{0} b \chi^{0}, b_{01}=\chi^{0} b\left(1-\chi^{0}\right)$, $b_{10}=\left(1-\chi^{0}\right) b \chi^{0}, b_{11}=\left(1-\chi^{0}\right) b\left(1-\chi^{0}\right)$. Remark for later use that $b_{01}=\chi^{0}\left[\chi^{0}, b\right](1-$ $\left.\chi^{0}\right), b_{10}=\left(1-\chi^{0}\right)\left[b, \chi^{0}\right] \chi^{0}$. Remark also that $\left[\chi^{0}, b\right]=\left(\begin{array}{cc}0 & b_{01} \\ -b_{10} & 0\end{array}\right)$. Thus

$$
\left(b\left[\chi^{0}, \ell\right] b^{\prime}\right)_{00}=b_{00}\left[\chi^{0}, \ell\right]_{01} b_{10}^{\prime}+b_{01}\left[\chi^{0}, \ell\right]_{10} b_{00}^{\prime}
$$

and similarly

$$
\left(b\left[\chi^{0}, \ell\right] b^{\prime}\right)_{11}=b_{10}\left[\chi^{0}, \ell\right]_{01} b_{11}^{\prime}+b_{11}\left[\chi^{0}, \ell\right]_{10} b_{01}^{\prime} .
$$

We then have that

$$
\omega_{\Gamma}\left(b\left[\chi^{0}, \ell\right] b^{\prime}\right)=\omega_{\Gamma}\left(\left(b\left[\chi^{0}, \ell\right] b^{\prime}\right)_{00}\right)+\omega_{\Gamma}\left(\left(b\left[\chi^{0}, \ell\right] b^{\prime}\right)_{11}\right)
$$

Here we are using the fact that the intersection of the diagonal and the support of the kernels defined by the off-diagonal terms in the above decomposition have measure zero. We shall work on the term $\omega_{\Gamma}\left(b_{00}\left[\chi^{0}, \ell\right]_{01} b_{10}^{\prime}\right)=\omega_{\Gamma}\left(b_{00}\left[\chi^{0}, \ell\right]\left[b^{\prime}, \chi^{0}\right] \chi^{0}\right)$ that appears in $\omega_{\Gamma}\left(b\left[\chi^{0}, \ell\right] b^{\prime}\right)$ (it is the first term in the first summand on the right hand side). Due to the key Lemma 10.55 one has $\left[\chi^{0}, \ell\right] \in L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-u}(Y, \mathcal{F})\right)$ and $\left[\chi^{0}, b^{\prime}\right] \in L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-t}(Y, \mathcal{F})\right)$ with $u<1 / 2$ and $t<s-1 / 2$. Given $b \in$ $\mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$ and $k \in L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-u}(Y, \mathcal{F})\right)$, we observe that the product $b k$ induces a bounded linear map

$$
\mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right) \times L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-u}(Y, \mathcal{F})\right) \longrightarrow L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-r}(Y, \mathcal{F})\right)
$$

with $r<p$. This is proved as follows. Let $F$ denote the Fourier transformation with respect to $\mathbb{R}$ on the family of Hilbert spaces $\left(L^{2}(\mathbb{R} \times N \times\{\theta\})\right)_{\theta \in T}$ and consider $F \circ k \circ F^{-1}$. It is obvious that $F \circ k \circ F^{-1} \in L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-u}(Y, \mathcal{F})\right)$. It is easy to see that $\hat{b}=F \circ b \circ F^{-1}$ with $\hat{b}$ equal to the Fourier transform of $b$ already defined before the key Lemma; thus one has $F \circ(b k) \circ F^{-1}=\hat{b} \circ F \circ k \circ F^{-1}$. Now we apply the key Lemma and see that $\hat{b}$ belongs to $L^{2}\left(\mathbb{R}, \mathrm{OP}^{-q}(Y, \mathcal{F})\right)$ with $q<p-1 / 2$; moreover $\hat{b} \circ F \circ k \circ F^{-1} \in L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathcal{B}^{-(q+u)}(Y, \mathcal{F})\right)$ since $\|\hat{b}(x)\| \|_{q}<+\infty$. Thus, thanks to the above formulas, $F \circ(b k) \circ F^{-1}$ is an element of $L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-r}(Y, \mathcal{F})\right)$, with $r:=q+u<p$, which implies that $b k$ also belongs to $L^{2}\left(\mathbb{R} \times \mathbb{R}, \operatorname{OP}^{-r}(Y, \mathcal{F})\right)$ with $r<p$. Now, using the key Lemma again we have a bounded linear map

$$
\begin{aligned}
& \mathrm{OP}^{-p}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right) \otimes \mathrm{OP}^{-1}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right) \otimes \mathrm{OP}^{-s}\left(\operatorname{cyl}(Y), \mathcal{F}_{\mathrm{cyl}}\right) \\
& \quad \rightarrow L^{2}\left(\mathbb{R} \times \mathbb{R}, B^{-r}(Y, \mathcal{F})\right) \otimes L^{2}\left(\mathbb{R} \times \mathbb{R}, \mathrm{OP}^{-t}(Y, \mathcal{F})\right)
\end{aligned}
$$

defined by

$$
b \otimes \ell \otimes b^{\prime} \rightarrow b_{00}\left[\chi^{0}, \ell\right] \otimes\left[b^{\prime}, \chi^{0}\right] \chi^{0}
$$

with $r<p, t<s-1 / 2$. Thus one has $r+t<p+s-1 / 2$ and hence can take $r+t>\operatorname{dim} N$. Thus we conclude, see (10.54), that $\omega_{\Gamma}\left(b_{00}\left[\chi^{0}, \ell\right]\left[b^{\prime}, \chi^{0}\right] \chi^{0}\right)$ is a bounded linear functional with respect to $b, \ell$ and $b^{\prime}$. A similar argument can be applied to the remaining terms

$$
\begin{aligned}
& \omega_{\Gamma}\left(b_{01}\left[\chi^{0}, \ell\right]_{10} b_{00}^{\prime}\right)=\omega_{\Gamma}\left(\chi^{0}\left[\chi^{0}, b\right]\left[\chi^{0}, \ell\right] b_{00}^{\prime}\right) \\
& \omega_{\Gamma}\left(b_{10}\left[\chi^{0}, \ell\right]_{01} b_{11}^{\prime}\right)=\omega_{\Gamma}\left(\left(1-\chi^{0}\right)\left[\chi^{0}, b\right]\left[\chi^{0}, \ell\right] b_{11}^{\prime}\right) \\
& \omega_{\Gamma}\left(b_{11}\left[\chi^{0}, \ell\right]_{10} b_{01}^{\prime}\right)=\omega_{\Gamma}\left(b_{11}\left[\chi^{0}, \ell\right]\left[\chi^{0}, b^{\prime}\right]\left(1-\chi^{0}\right)\right)
\end{aligned}
$$

to conclude that they are also bounded with respect to $b$. $\ell$ and $b^{\prime}$. Thus we have proved that $\omega_{\Gamma}\left(b\left[\chi^{0}, \ell\right] b^{\prime}\right)$ is bounded with respect $b, \ell$ and $b^{\prime}$. Observing that the derivations $\bar{\delta}_{j}, j=1,2$, are bounded from $\mathfrak{B}_{m}$ to $\mathrm{OP}^{-1}$ we finally see that the eta cocycle $\sigma_{m}$ extends to a continuous cyclic cocycle on $\mathfrak{B}_{m}$. This completes the proof.
10.8.3 Proof of Proposition 8.13 (extension of the regularized $G V$ cyclic cochain). Recall that we want to show that if $\operatorname{deg} S^{p-1} \tau_{G V}^{r}=2 p>m(m-1)^{2}-2=$ $m^{3}-2 m^{2}+m-2$, with $m=2 n+1$ and $2 n$ equal to the dimension of the leaves in $(X, \mathcal{F})$, then the regularized Godbillon-Vey cochain $S^{p-1} \tau_{G V}^{r}$ extends to a bounded cyclic cochain on $\mathfrak{A}_{m}$.

Proof. Recall the Banach space decomposition $\mathcal{A}_{m}=\mathcal{J}_{m} \oplus \chi^{0} \mathcal{B}_{m} \chi^{0}$. We consider elements in $\mathcal{A}_{m}$ of the the following type:

- $k^{\alpha}=k_{1} \cdots k_{n_{\alpha}}$, the product of $n_{\alpha}$ elements in $\mathcal{J}_{m}$
- $b^{\beta}=\chi^{0} \ell_{1} \chi^{0} \ell_{2} \chi^{0} \cdots \chi^{0} \ell_{n_{\beta}} \chi^{0}$, the product of $n_{\beta}$ elements in $\chi^{0} \mathcal{B}_{m} \chi^{0}$.

We call $n_{\alpha}$ and $n_{\beta}$ the length of $k^{\alpha}$ and $b^{\beta}$ respectively.
Let $a_{j}=k_{j}+\chi^{0} \ell_{j} \chi^{0} \in \mathcal{A}_{m}, j=1, \ldots, r$ and consider the product $a:=a_{1} \cdots a_{r}$. We write $a=\sum_{\gamma} a^{\gamma}$ with $a^{\gamma}$ a product of a certain number of elements of type $k^{\alpha}$ and of type $b^{\beta}$.

Lemma 10.59. Suppose that $r>s(t-1)+s-1$. Then for $a=a_{1} \cdots a_{r}, a=\sum_{\gamma} a^{\gamma}$, at least one of the following will occur for each $a^{\gamma}$.
(1) $a^{\gamma}$ contains at least $s$ elements in $\mathcal{J}_{m}$;
(2) $a^{\gamma}$ contains one element of the form $b^{\beta}$ whose length is at least $t$.

Proof. The proof of the Lemma is elementary. Fix $r=s(t-1)+s-1$. Then the generic element $a^{\gamma}$ in the statement of the Lemma will satisfy at least one of the two above conditions or will be of the form

$$
b^{\gamma_{1}} k_{1} b^{\gamma_{2}} k_{2} b^{\gamma_{3}} \cdots b^{\gamma_{s-1}} k_{s-1} b^{\gamma_{s}}
$$

where the length of each $b^{\gamma_{i}}$ is $t-1$ and the total length is $r$. It is then easy to see that if now $r$ is strictly larger than $s(t-1)+s-1$ then one of the above two conditions must necessarily occur.

Observe now that $\chi^{0} \ell_{1} \chi^{0} \ell_{2} \chi^{0}-\chi^{0} \ell_{1} \ell_{2} \chi^{0}=\chi^{0}\left[\chi^{0}, \ell_{1}\right]\left[\chi^{0}, \ell_{2}\right] \chi^{0}$. This simple observation is at the basis of the following
Lemma 10.60. Let $b^{\beta}$ be an element of length $t$, namely $b^{\beta}=\prod_{j=1}^{t} \chi^{0} \ell_{j} \chi^{0}$ with $\ell_{j} \in \mathcal{B}_{m}$. Then one has

$$
\begin{equation*}
b^{\beta}=\chi^{0}\left(\prod_{j=1}^{t} \ell_{j}\right) \chi^{0}+\chi^{0} c \chi^{0} \tag{10.61}
\end{equation*}
$$

where $c$ is a linear combination of $c_{k}$ and $c_{k}$ is the product of $t_{1}$ elements of type $\left[\chi^{0}, \ell_{i}\right]$ and $t_{2}$ elements of type $\ell_{i}$ with $t_{1}+t_{2}=t$ and $t_{1} \geq 1$. Moreover the number of such $c_{k}$ is at most $2^{t-1}-1$

The proof of Lemma 10.60 is based on an elementary induction argument.
Consider now the product, $a$, of $r$ elements $a_{i} \in \mathcal{A}_{m}$ and write $a=\sum_{\gamma} a^{\gamma}$ as above. Then, obviously, either one of the following will apply to each $a^{\gamma}$ :
(a) $a^{\gamma}$ is of the form $b^{\beta}$ introduced after the statement of the Proposition, namely $a^{\gamma}=\prod_{i=1}^{r} \chi^{0} \ell_{i} \chi^{0} ;$
(b) $a^{\gamma}$ contains at least one $k_{j} \in \mathcal{J}_{m}$.

Suppose now that $r>m(m-1)+m-1=m^{2}-2 m$. Recall the definition of the map $t: \mathcal{A}_{m} \rightarrow \mathcal{J}_{m}$, see (4.13). Clearly, by definition, in case b) we have that $t\left(a^{\gamma}\right)=a^{\gamma}$, since $a^{\gamma} \in \mathcal{J}_{m}$ given that $\mathcal{J}_{m}$ is an ideal in $\mathcal{A}_{m}$. In case a) we can write

$$
t\left(a^{\gamma}\right)=\prod_{i=1}^{r} \chi^{0} \ell_{i} \chi^{0}-\chi^{0}\left(\prod_{i=1}^{r} \ell_{i}\right) \chi^{0}=\sum_{j} \chi^{0} c_{j} \chi^{0}
$$

according to Lemma 10.60. Here $c_{j}$ is a product of $r$ elements out of $\left[\chi^{0}, \ell_{i}\right]$ and $\ell_{i}$. Then at least one of the following will occur:
(a-1) $c_{j}$ contains at least $m$ elements of the form $\left[\chi^{0}, \ell_{j}\right]$;
(a-2) $c_{j}$ contains a consecutive product of at least $m$ elements in $\mathcal{B}_{m}$.

The latter claim is proved by the same reasoning in the proof of Lemma 10.59. Now, in the case a-1) one has $c_{j} \in \mathcal{I}_{1}$, given that $\left[\chi^{0}, \ell\right] \in \mathcal{J}_{m}$. In case a-2) we apply the following Lemma, Lemma 10.62, in order to see that $c_{j} \in \mathcal{I}_{1}$, observing that $c_{j}$ contains a consecutive product of at least $m$ elements $\ell_{j}$ and, according to Lemma 10.60, at least one $\left[\chi^{0}, \ell_{i}\right]$ (which belongs to $\mathcal{J}_{m}$ by definition). All things considered we have shown that in case a) the element $c_{j}$ and thus $t\left(a^{\gamma}\right)$ belongs to $\mathcal{I}_{1}$ for $a=a_{1} \cdots a_{r}$ and $r>m^{2}-2 m$.

Lemma 10.62. Recall the Banach algebra $\mathrm{OP}^{-p}$ on $\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)$, defined as the closure of $\Psi_{c}^{-p}\left(G_{\mathrm{cyl}} / \mathbb{R}_{\Delta}\right)$ with respect to the norm $\left\|\left\|\left\|\|_{p}\right.\right.\right.$. If $p$ is greater than the dimension of the leaves, then for each natural number $\nu \geq 1$

$$
\begin{equation*}
\mathcal{J}_{\nu} \mathrm{OP}^{-p} \subset \mathcal{I}_{1} \quad \text { and } \quad \mathrm{OP}^{-p} \mathcal{J}_{\nu} \subset \mathcal{I}_{1} . \tag{10.63}
\end{equation*}
$$

Moreover if $k \in \mathcal{J}_{\nu}$ and $\ell \in \mathrm{OP}^{-p}$ then

$$
\begin{equation*}
\|k b\|_{\mathcal{I}_{1}} \leq C\|k\|_{\mathcal{J}_{\nu}}\||b|\|_{p} \quad \text { and } \quad\|b k\|_{\mathcal{I}_{1}} \leq C\| \| b \mid\| \|_{p}\|k\|_{\mathcal{J}_{\nu}} \tag{10.64}
\end{equation*}
$$

with $C$ is a constant depending only on the Dirac operator $D$ on $\left(Y, \mathcal{F}_{Y}\right)$.
We remark that it is precisely for the validity of this Lemma that the extra condition involving $g(s, y)=1+s^{2}$ was added in the definition of $\mathcal{B}_{k}$ and $\mathcal{J}_{k}$.
Proof. Let $k \in \mathcal{J}_{\nu}$ and let $\ell \in \mathrm{OP}^{-p}$. One can write $k \ell=k g g^{-1}\left(1+D^{2}\right)^{-p / 2}(1+$ $\left.D^{2}\right)^{p / 2} \ell$. Note that $k g$ and $\left(1+D^{2}\right)^{p / 2} \ell$ are bounded since $k \in \mathcal{J}_{\nu}$ and $\ell \in \mathrm{OP}^{-p}$. Next we prove that $g^{-1}\left(1+D^{2}\right)^{-p / 2} \in \mathcal{I}_{1}$. It suffices to show that $g^{-1 / 2}\left(1+D^{2}\right)^{-p / 4} \in \mathcal{I}_{2}$; equivalently, using that $g=(s+i)(s-i)$ we can prove that $(s \pm i)^{-1}\left(1+D^{2}\right)^{-p / 4} \in \mathcal{I}_{2}$. Let us fix the plus sign, for example. We want to show that $A:=(s+i)^{-1}(1+$ $\left.D^{2}\right)^{-q / 2} \in \mathcal{I}_{2}$ if $2 q$ is greater than the dimension of the leaves. First, we conjugate $A$
with the Fourier transformation in the cylindrical direction, obtaining $F \circ A \circ F^{-1}=$ $\left(i+\left(\frac{1}{i} \frac{d}{d t}\right)\right)^{-1}\left(1+t^{2}+D_{Y}\right)^{-q / 2}$. For fixed $t \in \mathbb{R}$ let $K_{\theta, t}$ the Schwartz kernel of $\left(1+t^{2}+D_{Y}\right)^{-q / 2}$ along $\tilde{N} \times\{\theta\}$. Using elementary properties of the Fourier transformation one can check that $F \circ A_{\theta} \circ F^{-1}$ has Schwartz kernel $L_{\theta}\left(t, s, y, y^{\prime}\right)$ given, up to a multiplicative constant, by

$$
L_{\theta}\left(s, t, y, y^{\prime}\right)=u(s-t) e^{-|s-t|} K_{\theta, t}\left(y, y^{\prime}\right)
$$

with $u(x)=\chi_{[0,+\infty)}$. Now we estimate

$$
\begin{aligned}
\left\|F \circ A \circ F^{-1}\right\|_{\mathcal{I}_{2}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)} & =\sup _{\theta \in T}\left(\int_{\mathbb{R} \times \mathbb{R}} d s d t \int_{\tilde{N} \times \tilde{N}} d y d y^{\prime} \chi_{\Gamma}\left|L_{\theta}\left(s, t, y, y^{\prime}\right)\right|^{2}\right) \\
& \leq \sup _{\theta \in T}\left(\int_{\mathbb{R} \times \mathbb{R}} d s d t e^{-2|s-t|} \int_{\tilde{N} \times \tilde{N}} d y d y^{\prime} \chi_{\Gamma}\left|K_{\theta, t}\left(y, y^{\prime}\right)\right|^{2}\right) \\
& \leq \int d s d t e^{-2|s-t|} \frac{1}{1+t^{2}}\left\|\left(1+D_{Y}^{2}\right)^{-(q-1) / 2}\right\|_{\mathcal{I}_{2}\left(Y, \mathcal{F}_{Y}\right)}^{2}<+\infty
\end{aligned}
$$

Here we have used the characteristic function $\chi_{\Gamma}$ for a fundamental domain for the action of $\Gamma$ on $\tilde{N}$. We have also used the inequality of positive self-adjoint elements $\left(1+t^{2}+D_{Y}^{2}\right)^{-q / 2} \leq\left(1+t^{2}\right)^{-1 / 2}\left(1+D_{Y}^{2}\right)^{-(q-1) / 2}$ (we have already used this inequality in the proof of the key Lemma). This implies that

$$
\begin{aligned}
\sup _{\theta \in T}\left(\int_{\tilde{N} \times \tilde{N}} d y d y^{\prime} \chi_{\Gamma}\left|K_{\theta, t}\left(y, y^{\prime}\right)\right|^{2}\right) & =\left\|\left(1+t^{2}+D_{Y}^{2}\right)^{-q / 2}\right\|_{\mathcal{I}_{2}\left(Y, \mathcal{F}_{Y}\right)}^{2} \\
& \leq \frac{1}{1+t^{2}}\left\|\left(1+D_{Y}^{2}\right)^{-(q-1) / 2}\right\|_{\mathcal{I}_{2}\left(Y, \mathcal{F}_{Y}\right)}
\end{aligned}
$$

which is what is used above. Since we have proved that $\left\|F \circ A \circ F^{-1}\right\|_{\mathcal{I}_{2}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)}$ is finite, we conclude that $\|A\|_{\mathcal{I}_{2}\left(\operatorname{cyl}(Y), \mathcal{F}_{\text {cyl }}\right)}$ is also finite. Thus $g^{-1}\left(1+D^{2}\right)^{-p / 2}$ is in $\mathcal{I}_{1}$ if $p$ is greater than the dimension of the leaves. Now it is obvious that $k \chi^{0} \ell \chi^{0} \in \mathcal{I}_{1}$ from the ideal property of $\mathcal{I}_{1}$. Similarly $\chi^{0} \ell \chi^{0} k \in \mathcal{I}_{1}$. In order to get the estimate in 10.64 we use standard properties:

$$
\begin{aligned}
\|k b\|_{\mathcal{I}_{1}} & =\left\|(k g)\left(g^{-1}\left(1+D^{2}\right)^{-p / 2}\right)\left(\left(1+D^{2}\right)^{p / 2} b\right)\right\|_{\mathcal{I}_{1}} \\
& \leq\left\|g^{-1}\left(1+D^{2}\right)^{-p / 2}\right\|_{\mathcal{I}_{1}}\|k g\|_{C^{*}}\left\|\left(1+D^{2}\right)^{p / 2} b\right\|_{C^{*}} \\
& \leq C\|k g\|_{C^{*}}\left\|\left(1+D^{2}\right)^{p / 2} b\right\|_{C^{*}} \leq C\|k\|_{\mathcal{J}_{\nu}}\|b\|_{p^{*}} .
\end{aligned}
$$

The Lemma is proved.
Now we consider the case (b), namely $a^{\gamma}$ contains at least one $k_{j} \in \mathcal{J}_{m}$. Applying the same argument as in Lemma 10.59 we see that at least one of the following will occur if $r>m(t-1)+m-1$ :
(b-1) $a^{\gamma}$ contains at least $m$ elements in $\mathcal{J}_{m}$;
(b-2) $a^{\gamma}$ contains a $b^{\beta}$ of length $t$.

In the case b-1) one has $a^{\gamma} \in \mathcal{I}_{1}$ in an obvious way. In case b-2), we apply Lemma 10.60 in order to see that $b^{\beta}$ has the form $\chi^{0}\left(\prod_{i=1}^{t} \ell_{i}\right) \chi^{0}+\sum \chi^{0} c_{j} \chi^{0}$. The first term belongs to $\chi^{0} \mathrm{OP}^{-t} \chi^{0}$. Then, the corresponding term in $a^{\gamma}$ will be in $\mathcal{I}_{1}$ if $t \geq m$, since we can apply Lemma 10.62 once we recall that $a^{\gamma}$ contains at least one $k_{j} \in \mathcal{J}_{m}$. Here we are using a small extension of Lemma 10.62:
if $p$ is greater than the dimension of the leaves, then

$$
\begin{aligned}
& \mathcal{J}_{\nu}(X, \mathcal{F})\left(\chi^{0} \mathrm{OP}^{-p}\left(\operatorname{cyl} \partial X, \mathcal{F}_{\text {cyl }}\right) \chi^{0}\right) \subset \mathcal{I}_{1}(X, \mathcal{F}) \quad \text { and } \\
& \left(\chi^{0} \mathrm{OP}^{-p}\left(\operatorname{cyl} \partial X, \mathcal{F}_{\text {cyl }}\right) \chi^{0}\right) \mathcal{J}_{\nu}(X, \mathcal{F}) \subset \mathcal{I}_{1}(X, \mathcal{F})
\end{aligned}
$$

Now recall that $t\left(a^{\gamma}\right) \in \mathcal{I}_{1}$ for $a=a_{1} \cdots a_{r}$ if $r>m^{2}-2 m$. Applying the same reasoning to $b^{\beta}$ with length $t$ we obtain $c_{j} \in \mathcal{I}_{1}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ if $t>m^{2}-2 m$. Then the corresponding term in $a^{\gamma}$ also belongs to $\mathcal{I}_{1}(X, \mathcal{F})$. Thus, if $r>m(t-1)+(m-1)$, with $t-1=m^{2}-2 m$, namely $r>m(m-1)^{2}-1$, then we can conclude that $c_{j} \in \mathcal{I}_{1}\left(\operatorname{cyl}(\partial X), \mathcal{F}_{\text {cyl }}\right)$ in both cases b-1) and b-2) ; consequently $a^{\gamma}$, which in this case is $t\left(a^{\gamma}\right)$, belongs to $\mathcal{I}_{1}(X, \mathcal{F})$.

Now we put everything together and we show that the regularized cochain $\tau_{2 p}^{r}$ extends to a continuous cochain on $\mathfrak{A}_{m}$, with $m=2 n+1$. Recall that the regularized cochain is defined through the regularized weight which was shown to be equal to $\omega_{\Gamma} \circ t$ on $A_{c} \subset A^{*}$. See Proposition 5.54. Here we recall for later use that $\omega_{\Gamma}$ extends continuously to $\mathcal{I}_{1}$. For an element such as

$$
a_{0} \cdots a_{i-1} \delta_{1}\left(a_{i}\right) a_{i+1} \cdots a_{j-1} \delta_{2}\left(a_{j}\right) a_{j+1} \cdots a_{2 p}
$$

with $a_{k} \in A_{c} \subset \mathfrak{A}_{m}$, we need to prove that

$$
\begin{equation*}
\left|\omega_{\Gamma}\left(t\left(a_{0} \cdots a_{i-1} \delta_{1}\left(a_{i}\right) a_{i+1} \cdots a_{j-1} \delta_{2}\left(a_{j}\right) a_{j+1} \cdots a_{2 p}\right)\right)\right| \leq C \prod_{j=1}^{2 p}\left\|a_{j}\right\|_{\mathfrak{A}_{m}} \tag{10.65}
\end{equation*}
$$

We shall prove a stronger statement, namely that the left hand side of (10.65) makes already sense for $a_{j} \in \mathfrak{A}_{m}$ and for the closures $\bar{\delta}_{j}$ and that the estimate in (10.65) holds.

Let $a=a_{1} a_{2} \cdots a_{r}$, with $a_{j} \in \mathcal{A}_{m}$, and write, as above, $a=\sum_{\gamma} a^{\gamma}$. Suppose that $r=2 p+1,2 p>m(m-1)^{2}-2$, so that $r>m(m-1)^{2}-1$. We have proved that $t\left(a^{\gamma}\right) \in \mathcal{I}_{1}$ for each $\gamma$. We will now estimate the norm $\left\|t\left(a^{\gamma}\right)\right\|$ in terms of the norms $\left\|a_{j}\right\|_{\mathcal{A}_{m}}$. We shall analyze one by one the terms appearing in cases (a-1), (a-2), (b-1), (b-2). To this end recall that if $q>\operatorname{dim} V$ then for $k \in \mathcal{J}_{\nu}$ and $\ell \in \mathrm{OP}^{-q}$ the following estimate holds

$$
\begin{equation*}
\|k b\|_{\mathcal{I}_{1}} \leq C\|k\|_{\mathcal{J}_{\nu}}\||b|\|_{q} \quad \text { and } \quad\|b k\|_{\mathcal{I}_{1}} \leq C\| \| b\| \|_{q}\|k\|_{\mathcal{J}_{\nu}} \tag{10.66}
\end{equation*}
$$

with $C$ depending only on the Dirac operator on $\left(Y, \mathcal{F}_{Y}\right)$.
Consider first the case a); then $t\left(a^{\gamma}\right)=\sum_{j} \chi^{0} c_{j} \chi^{0}$. In case a-1) $c_{j}$ contains at least $m$ elements in $\mathcal{J}_{m}$, say $k_{1}=\left[\chi^{0}, \ell_{1}\right], \ldots, k_{t}=\left[\chi^{0}, \ell_{t}\right]$ with $t \geq m$; thus, without
loss of generality, we can assume that $c_{j}=\left[\chi^{0}, \ell_{1}\right] \cdots\left[\chi^{0}, \ell_{t}\right] \ell_{t+1} \cdots \ell_{r}$. Then

$$
\begin{aligned}
\left\|\chi^{0} c_{j} \chi^{0}\right\|_{\mathcal{I}_{1}(X, \mathcal{F})} & \leq\left\|c_{j}\right\|_{\mathcal{I}_{1}\left(\operatorname{cyl} \partial X, \mathcal{F}_{\text {cy } 1}\right)} \leq \prod_{i=1}^{t}\left\|\left[\chi^{0}, \ell_{i}\right]\right\|_{\mathcal{I}_{m}} \prod_{i=t+1}^{r}\left\|\ell_{i}\right\|_{B^{*}} \\
& \leq \prod_{i=1}^{t}\left\|\left[\chi^{0}, \ell_{i}\right]\right\|_{\mathcal{J}_{m}} \prod_{i=t+1}^{r}\left\|\ell_{i}\right\|_{B^{*}} \leq \prod_{i=1}^{r}\left\|a_{i}\right\|_{\mathcal{A}_{m}}
\end{aligned}
$$

where we recall that if $a=\chi^{0} \ell \chi^{0}+k$ then $\|a\|_{\mathcal{A}_{m}}:=\|\ell\|_{\mathcal{B}_{m}}+\|k\|_{\mathcal{J}_{m}}$ and that

$$
\|\ell\|_{\mathcal{B}_{m}}:=\|\ell \ell\|+\left\|\left[\chi_{\mathrm{cy1}}^{0}, \ell\right]\right\|_{\mathcal{J}_{m}}+\| \| \partial_{\alpha} \ell\| \|+2\|[f, \ell]\|_{B^{*}}+\|[f,[f, \ell]]\|_{B^{*}}
$$

so that, clearly,

$$
\|a\|_{\mathcal{A}_{m}} \geq\|\ell\|_{\mathcal{B}_{m}} \geq\| \| \ell \mid\|+\|\left[\chi^{0}, \ell\right]\left\|_{\mathcal{J}_{m}} \geq\right\| \ell\left\|_{B^{*}}+\right\|\left[\chi^{0}, \ell\right] \|_{\mathcal{J}_{m}} .
$$

Next we tackle the case (a-2). Then we can assume without loss of generality that $c_{j}$ is of the form $b\left(\ell_{1} \cdots \ell_{t}\right) b^{\prime}$ with $t \geq m$ and $b$ and $b^{\prime}$ are certain products of $\ell_{i}$ and $\left[\chi^{0}, \ell_{j}\right]$ and either $b$ or $b^{\prime}$ contains at least one $\left[\chi^{0}, \ell_{k}\right]$. Say that it is $b$ that contains $\left[\chi^{0}, \ell_{k}\right]$. Then, using (10.66) for $q=t$ and $\nu=m$ we get

$$
\begin{aligned}
\left\|\chi^{0} c_{j} \chi^{0}\right\|_{\mathcal{I}_{1}(X, \mathcal{F})} & \leq\left\|c_{j}\right\|_{\mathcal{I}_{1}\left(\operatorname{cyl} \partial X, \mathcal{F}_{\text {cyl }}\right)} \leq C\|b\|_{\mathcal{J}_{m}}\left\|b^{\prime}\right\|_{B^{*}}\left\|\ell_{1} \cdots \ell_{t}\right\|_{t} \\
& \leq C\|b\|_{\mathcal{J}_{m}}\left\|b^{\prime}\right\|_{B^{*}} \prod_{i=1}^{t}\left\|\ell_{i}\right\| \mid \leq C \prod_{i=1}^{r}\left\|a_{i}\right\|_{\mathcal{A}_{m}} .
\end{aligned}
$$

Thus, in case (a) we have proved that $\left\|t\left(a^{\gamma}\right)\right\|_{\mathcal{I}_{1}} \leq C 2^{r} \prod_{i=1}^{r}\left\|a_{i}\right\|_{\mathcal{A}_{m}}$ since the number of $c_{j}$ is at most $2^{r}$.

Now we consider the case b). In the case b-1) we have that $a^{\gamma}$ is a product of $\left[\chi^{0}, \ell_{1}\right], \ldots,\left[\chi^{0}, \ell_{t}\right], t \geq m$ and of $\ell_{t+1}, \ldots, \ell_{r}$. Then, as already remarked, $a^{\gamma} \in$ $\mathcal{I}_{1}, t\left(a^{\gamma}\right)=a^{\gamma}$, and moreover, from standard estimates we have

$$
\left\|a^{\gamma}\right\|_{\mathcal{I}_{1}(X, \mathcal{F})} \leq \prod_{i=1}^{t}\left\|\left[\chi^{0}, \ell_{i}\right]\right\|_{\mathcal{I}_{m}} \prod_{i=t+1}^{r}\left\|\ell_{i}\right\|_{B^{*}} \leq \prod_{i=1}^{t}\left\|\left[\chi^{0}, \ell_{i}\right]\right\|_{\mathcal{J}_{m}} \prod_{i=t+1}^{r}\left\|\ell_{i}\right\|_{B^{*}} \leq \prod_{i=1}^{r}\left\|a_{i}\right\|_{\mathcal{A}_{m}} .
$$

In case (b-2) we can write $a^{\gamma}=c b^{\beta} c^{\prime}$ with $b^{\beta}=\prod_{i=1}^{t} \chi^{0} \ell_{i} \chi^{0}$ and $t \geq m$, and $c, c^{\prime}$ are certain products of $k_{i} \in \mathcal{J}_{m}$ and $\chi^{0} \ell_{i} \chi^{0}$ for $a_{i}=k_{i}+\chi^{0} \ell_{i} \chi^{0}, i=1, \ldots, r$. Here we know that at least one $k_{i} \in \mathcal{J}_{m}$ will appear in $c$ or $c^{\prime}$. Say that it is $c$ that contains such $k_{i}$. Then we can apply the same argument in case (a) and conclude that

$$
\begin{aligned}
\left\|a^{\gamma}\right\|_{\mathcal{I}_{1}(X, \mathcal{F})} & \leq\left\|c b^{\beta}\right\|_{\mathcal{I}_{1}}\left\|c^{\prime}\right\|_{C^{*}} \leq C\|c\|\left\|_{\mathcal{J}_{m}}\right\| \prod_{i=1}^{t} \ell_{i}\| \|_{t}\left\|c^{\prime}\right\|_{C^{*}} \leq C\|c\|_{\mathcal{J}_{m}} \prod_{i=1}^{t}\| \| \ell_{i}\| \| c^{\prime} \|_{C^{*}} \\
& \leq C \prod_{i=1}^{r}\left\|a_{i}\right\|_{\mathcal{A}_{m}}
\end{aligned}
$$

with (10.66) used in order to justify the second estimate. We finish the proof by observing that the two closed derivations $\bar{\delta}_{1}$ and $\bar{\delta}_{2}$ are bounded from $\mathfrak{A}_{m}$ to $\mathcal{A}_{m}$ and that the inclusion $\mathfrak{A}_{m} \subset \mathcal{A}_{m}$ is bounded; this proves that (10.65) holds for $a_{j} \in \mathfrak{A}_{m}$ which is what is needed in order to conclude.

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[^0]:    ${ }^{1}$ In the Morita-equivalent picture we would be considering the small algebra $C(T) \rtimes_{\text {alg }} \Gamma \subset$ $C(T) \rtimes_{r} \Gamma$.

[^1]:    ${ }^{4}$ Observe that this family is denoted $\tilde{D}$ both in [MoN96] and [BP09], the symbol $D$ being employed for the longitudinal operator induced on the quotient $Y=\tilde{N} \times_{\Gamma} T$. However, in this paper we shall work exclusively with the $\Gamma$-equivariant picture, which is why we don't use the tilde notation.
    ${ }_{5}$ The abuse consists in omitting the hats in the notations; in other words this space should be really denoted by $\Psi_{c}^{m}(G ; \widehat{E}, \widehat{F})$. It should be added here that the notation for this space of operators is not unique. In [MoN96] $\Psi_{c}^{m}(G ; E, F)$ is simply denoted as $\Psi_{\Gamma}^{m}(\widehat{E}, \widehat{F})$ whereas it is denoted $\Psi_{x, c}^{m}(\tilde{N} \times T ; \widehat{E}, \widehat{F})$ in [LP05] with $\rtimes$ denoting equivariance and $c$ denoting again of $\Gamma$-compact support.

[^2]:    8 This will not be a weight, given that on a bimodule there is no notion of positive element.

[^3]:    ${ }^{9}$ The situation here is similar to the one for the eta invariant in the seminal paper of Atiyah-Patodi-Singer; the regularity there is a consequence of their index theorem.

[^4]:    10 Similar arguments establish the analogues in odd dimension.

