Université du Québec à Montréal
Département de mathématiques
Marco A. Pérez B.
marco.perez@cirget.ca

THE DE RHAM'S THEOREM

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Contents

Introduction

The topological invariance of the de Rham groups suggests that there should be some purely topological way of computing them. In fact, there exists such a way and the connection between the de Rham groups and topology was first proved by Georges de Rham himself in the 1931.

In these notes we give a proof of de Rham's Theorem, that states there exists an isomorphism between de Rham and singular cohomology groups of a smooth manifold. In the first chapter we recall some notions of homological algebra, and then we summarize basic ideas of singular homology and cohomology ($H_p(\cdot)$ and $H^p(\cdot;\mathbb{R})$), such as the functoriality of $H_p(\cdot)$ and $H^p(\cdot;\mathbb{R})$ and the existence of Mayer-Vietoris sequences for both singular homology and cohomology.

In the second chapter, we recall the notion of de Rham cohomology and then we shall prove that in this cohomology there also exists a Mayer-Vietoris sequence. In order to find a connection between de Rham and singular cohomology, we need to know the Stokes' Theorem on a smooth manifold with boundary. Stokes' Theorem is an expression of duality between de Rham cohomology and the homology of chains. It says that the pairing of differential forms and chains, via integration, gives a homomorphism from de Rham cohomology to singular cohomology groups. We shall also see that this theorem is true on smooth manifolds with corners. Using this special case, we shall define explicitly a homomorphism between de Rham and singular cohomology. At the end of the chapter, we turn our attention to the de Rham's theorem. The proof we shall give is due to Glen E. Bredon.

Chapter 1

Elements of Homological Algebra and Singular Homology

In this chapter we recall some notions from basic homological algebra and algebraic topology. We define homology (and cohomology) groups of a chain complex (respectively, cochain complex) of modules over a ring R. Then we study some properties of a special type of homology defined for topological spaces.

1.1 Homology of chain complexes

Let R be a ring. A sequence $A = (A_p, d_A)$ of R-modules and homomorphisms

$$\cdots \longrightarrow A_{p+1} \xrightarrow{d_A} A_p \xrightarrow{d_A} A_{p-1} \longrightarrow \cdots$$

is called a **chain complex** if $d_A \circ d_A = 0$, i.e., $\mathrm{Im}(d_A) \subseteq \mathrm{Ker}(d_A)$. Thus, $\mathrm{Im}(d_A)$ is a submodule of $\mathrm{Ker}(d_A)$ and the p-th **homology group** of A is defined as the quotient module

$$H_p(A) = \frac{\mathrm{Ker}(d_A : A_p \longrightarrow A_{p-1})}{\mathrm{Im}(d_A : A_{p+1} \longrightarrow A_p)}.$$

Given two chain complexes $A = (A_p, d_A)$ and $B = (B_p, d_B)$, a **chain map** is a sequence of homomorphisms $f : A_p \longrightarrow B_p$ such that the square

$$\begin{tikzcd} A_p \arrow[r, "d_A"] \arrow[d, "f"'] & A_{p-1} \arrow[d, "f"] \\ B_p \arrow[r, "d_B"'] & B_{p-1} \end{tikzcd}$$

commutes, i.e., $f \circ d_A = d_B \circ f$.

Dually, a **cochain complex** is a sequence $A = (A_p, d_A)$ of R-modules and homomorphisms

$$\cdots \longrightarrow A_{p-1} \xrightarrow{d_A} A_p \xrightarrow{d_A} A_{p+1} \longrightarrow \cdots$$

such that $d_A \circ d_A = 0$. The p-th **cohomology group** of A is defined as

$$H^p(A) = \frac{\operatorname{Ker}(d_A : A_p \longrightarrow A_{p+1})}{\operatorname{Im}(d_A : A_{p-1} \longrightarrow A_p)}.$$

Given two cochain complexes $A = (A_p, d_A)$ and $B = (B_p, d_B)$, a **cochain map** is a sequence of homomorphisms $f : A_p \longrightarrow B_p$ such that the square

$$\begin{tikzcd} A_p \arrow[r, "d_A"] \arrow[d, "f"'] & A_{p+1} \arrow[d, "f"] \\ B_p \arrow[r, "d_B"'] & B_{p+1} \end{tikzcd}$$

commutes, i.e., $f \circ d_A = d_B \circ f$.

Proposition 1.1.1.

(1) Every chain map $f : A \longrightarrow B$ between chain complexes induces a homomorphism between homology groups.

(2) Every cochain map $f : A \longrightarrow B$ between cochain complexes induces a homomorphism between cohomology groups.

Proof: We only prove (2). Let $[a] \in H^p(A)$. Consider the commutative square

$$\begin{tikzcd} A_p \arrow[r, "d_A"] \arrow[d, "f"'] & A_{p+1} \arrow[d, "f"] \\ B_p \arrow[r, "d_B"'] & B_{p+1} \end{tikzcd}$$

We have $d_B(f(a)) = f(d_A(a)) = f(0) = 0$, since $a \in \operatorname{Ker}(d_A)$. Then $f(a) \in \operatorname{Ker}(d_B)$. Thus we set $f^* : H^p(A) \longrightarrow H^p(B)$ by $f^*([a]) = [f(a)]$. We must show that f^* is well defined. Let $a' \in \operatorname{Ker}(d_A)$ such that $[a] = [a']$. Then $a - a' = d_A(x)$, where $a \in A_{p-1}$. We have $f(a') - f(a) = f(a' - a) = f(d_A(x)) = d_B(f(x))$, where $f(x) \in B_{p-1}$. Then $[f(a)] = [f'(a)]$. It is only left to show that f^* is a group homomorphism. We have

$$\begin{aligned} f^*([a] + [a']) &= f^*([a + a']) = [f(a + a')] \\ &= [f(a) + f(a')] = [f(a)] + [f(a')] \\ &= f^*([a]) + f^*([a']), \end{aligned}$$

for every $[a], [a'] \in H^p(A)$. □

Lemma 1.1.1 (Five Lemma)**.** Given the following commutative diagram with exact rows

$$\begin{tikzcd} A_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "d_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "d_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r, "d'_1"'] & B_2 \arrow[r, "d'_2"'] & B_3 \arrow[r, "d'_3"'] & B_4 \arrow[r, "d'_4"'] & B_5 \end{tikzcd}$$

If f_1, f_2, f_4 and f_5 are isomorphisms, then so is f_3.

Proof: First, we prove that f_3 is injective. Let $x \in A_3$ such that $f_3(x) = 0$. Then $f_4(d_3(x)) = 0$ since the diagram commutes. We get $d_3(x) = 0$ since f_4 is injective. Thus, $x \in \mathrm{Ker}(d_3) = \mathrm{Im}(d_2)$. Then there exists $y \in A_2$ such that $x = d_2(y)$. Since the diagram commutes, we have $0 = f_3(x) = f_3(d_2(y)) = d'_2(f_2(y))$. It follows $f_2(y) \in \mathrm{Ker}(d'_2) = \mathrm{Im}(d'_1)$. Thus, there exists $z \in B_1$ such that $f_2(y) = d'_1(z)$. On the other hand, there exists $w \in A_1$ such that $z = f_1(w)$, since f_1 is surjective. So we get

$$f_2(y) = d'_1(z) = d'_1(f_1(w)) = f_2(d_1(w)).$$

It follows $y = d_1(w) \in \mathrm{Im}(d_1) = \mathrm{Ker}(d_2)$ since f_2 is injective. Hence $x = d_2(y) = 0$, i.e., f_3 is injective. Now we prove that f_3 is surjective. Let $y \in B_3$. Consider $d'_3(y) \in B_4 = \mathrm{Im}(f_4)$. There exists $x \in A_4$ such that $d'_3(y) = f_4(x)$. Since the diagram commutes, we have $f_5(d_4(x)) = d'_4(f_4(x)) = d'_4(d'_3(y)) = 0$. It follows $d_4(x) = 0$ since f_5 is injective. Thus $x \in \mathrm{Ker}(d_4) = \mathrm{Im}(d_3)$. Then there exists $z \in A_3$ such that $x = d_3(z)$. Consider $y - f_3(z) \in B_3$. We have

$$\begin{aligned} d'_3(y - f_3(z)) &= d'_3(y) - d'_3(f_3(z)) = f_4(x) - f_4(d_3(z)) \\ &= f_4(x - d_3(z)) = f_4(0) = 0. \end{aligned}$$

Then $y - f_3(z) \in \mathrm{Ker}(d'_3) = \mathrm{Im}(d'_2)$. Thus there exists $w \in B_2$ such that $y - f_3(z) = d'_2(w)$. Since f_2 is surjective, there exists $u \in A_2$ such that $w = f_2(u)$. So we have

$$y - f_3(z) = d'_2(w) = d'_2(f_2(u)) = f_3(d_2(u)).$$

Hence $y = f_3(z) + f_3(d_2(u)) = f_3(z + d_2(u))$ and f_3 is surjective. $\square$

1.2 Singular homology and cohomology

Let $\mathbb{R}^\infty$ have the standard basis $e_1, e_2, \ldots$, and let $e_0 = 0$. Then the **standard p-simplex** is defined as the set

$$\Delta_p = \left\{ x = \sum_{i=0}^{p} \lambda_i e_i : \sum_{i=0}^{p} \lambda_i = 1, 0 \leq \lambda_i \leq 1 \right\}.$$

Given points $v_0, \ldots, v_p$ in $\mathbb{R}^N$, let $[v_0, \ldots, v_p]$ denote the map $\Delta_p \longrightarrow \mathbb{R}^N$ taking

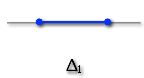
$$\sum_{i=0}^{p} \lambda_i e_i \longrightarrow \sum_{i=0}^{p} \lambda_i v_i.$$

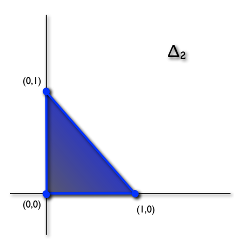
This is called an **affine singular p-simplex**. The expresion $[e_0, \ldots, \hat{e}_i, \ldots, e_p]$ denotes the affine singular $(p-1)$-simplex obtained by dropping the i-th vertex. Notice that its image is contained in Δ_p. The affine singular simplex

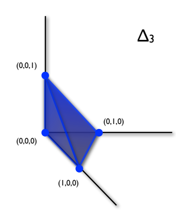
$$[e_0, \ldots, \hat{e}_i, \ldots, e_p] : \Delta_{p-1} \longrightarrow \Delta_p$$

is called the i-th **face map** and is denoted $F_{i,p}$.

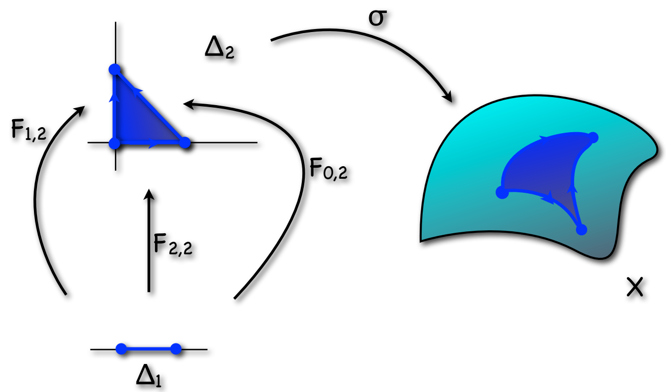
Example 1.2.1. The 0-simplex is the point 0, the 1-simplex is the line segment $[0,1]$, the 2-simplex is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ together with its interior, and the 3-simplex is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, together with its interior.







If X is a topological space then a **singular p-simplex** of X is a continuous map $\sigma_p : \Delta_p \longrightarrow X$. The **singular p-chain group** $C_p(X)$ is the free abelian group generated by the singular p-simplices of X. Thus, a p-chain in X is a formal finite sum $c = \sum_\sigma n_\sigma \sigma$ of p-simplices σ with integer coefficients n_σ. If $\sigma : \Delta_p \longrightarrow X$ is a singular p-simplex, then the i-th **face** of σ is $\sigma \circ F_{i,p}$.



singular simplex and face maps

The **boundary** of σ is $\partial_p(\sigma) = \sum_{i=0}^{p}(-1)^i \sigma \circ F_{i,p}$, a $(p-1)$-chain. If $c = \sum_\sigma n_\sigma \sigma$ is a p-chain, then we set

$$\partial_p(c) = \partial_p\left(\sum_\sigma n_\sigma \sigma\right) = \sum_\sigma n_\sigma \partial_p(\sigma).$$

That is, ∂_p is extended to $C_p(X)$ and so we have a group homomorphism

$$\partial_p : C_p(X) \longrightarrow C_{p-1}(X).$$

Lemma 1.2.1. $\partial_p \circ \partial_{p+1} = 0$.

Proof: See [2, Lemma 16.1, page 412]. □

By the previous lemma, we have a chain complex

$$\cdots \longrightarrow C_{p+1}(X) \xrightarrow{\partial_{p+1}} C_p(X) \xrightarrow{\partial_p} C_{p-1}(X) \longrightarrow \cdots .$$

The p-th homology group of this complex $H_p(X) = \frac{\mathrm{Ker}(\partial_p)}{\mathrm{Im}(\partial_{p+1})}$ is called the p-th **singular homology group** of X. Now consider the dual sequence

$$\cdots \longrightarrow \mathrm{Hom}(C_{p-1}(X), \mathbb{R}) \xrightarrow{\delta_{p-1}} \mathrm{Hom}(C_p(X), \mathbb{R}) \xrightarrow{\delta_p} \mathrm{Hom}(C_{p+1}(X), \mathbb{R}) \longrightarrow \cdots ,$$

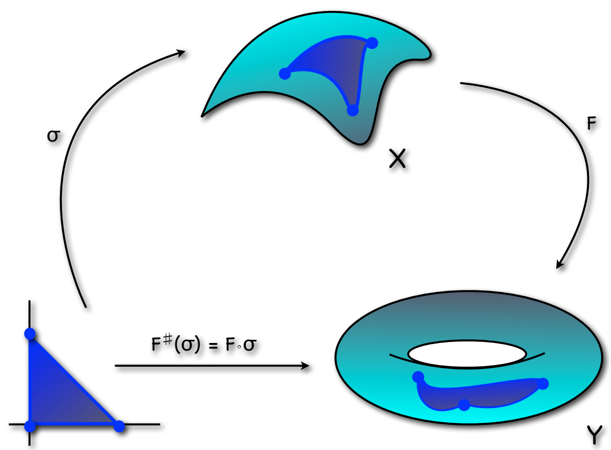
where each map δ_p, called the **coboundary**, is defined by $\delta_p(f) = f \circ \partial_{p+1}$ for every group homomorphism $f : C_p(X) \longrightarrow \mathbb{R}$. Notice that

$$\delta_{p+1} \circ \delta_p(f) = \delta_{p+1}(f \circ \partial_{p+1}) = f \circ (\partial_{p+1} \circ \partial_{p+2}) = f \circ 0 = 0.$$

So the previous sequence defines a cochain complex. The p-th cohomology group of this complex, denoted $H^p(X; \mathbb{R})$, is called the p-th **singular cohomology group** of X.

1.3 Induced homomorphisms

Any continuous map $F : X \longrightarrow Y$ induces a homomorphism $F_\# : C_p(X) \longrightarrow C_p(Y)$ by $F_\#(\sigma) = F \circ \sigma$, for every singular p-simplex σ.



The new homomorphism $F_\#$ induces a chain map, since

$$\begin{aligned} F_\# \circ \partial(\sigma) &= F_\# \left(\sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p} \right) = \sum_{i=0}^{p} (-1)^i F_\#(\sigma \circ F_{i,p}) \\ &= \sum_{i=0}^{p} (-1)^i (F \circ \sigma) \circ F_{i,p} = \partial(F \circ \sigma) = \partial \circ F_\#(\sigma). \end{aligned}$$

It follows $F_\sharp$ induces a homomorphism between homology groups $F_* : H_p(X) \longrightarrow H_p(Y)$, given by $F_*([\sigma]) = [F_\#(\sigma)]$, for every $[\sigma] \in H_p(X)$. Notice that $(G \circ F)_* = G_* \circ F_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{H_p(X)}$. Then, the p-th singular homology defines a covariant functor from the category **Top** of topological spaces and continuous functions, to the category **Ab** of abelian groups and homomorphisms. We shall also define a homomorphism $F^* : H^p(Y;\mathbb{R}) \longrightarrow H^p(X;\mathbb{R})$. Consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Hom}(C_{p-1}(Y),\mathbb{R}) \arrow[r, "\delta_Y"] \arrow[d, "F^\#"] & \mathrm{Hom}(C_p(Y),\mathbb{R}) \arrow[r, "\delta_Y"] \arrow[d, "F^\#"] & \mathrm{Hom}(C_{p+1}(Y),\mathbb{R}) \arrow[r] \arrow[d, "F^\#"] & \cdots \\ \cdots \arrow[r] & \mathrm{Hom}(C_{p-1}(X),\mathbb{R}) \arrow[r, "\delta_X"'] & \mathrm{Hom}(C_p(X),\mathbb{R}) \arrow[r, "\delta_X"'] & \mathrm{Hom}(C_{p+1}(X),\mathbb{R}) \arrow[r] & \cdots \end{tikzcd}$$

where $F^\#$ is the map $F^\#(f)(c) = f(F \circ c)$, for every homomorphism $f : C_p(Y) \longrightarrow \mathbb{R}$ and every singular p-chain c in X.

We prove the diagram above is commutative. Let σ be a singular p-simplex in X and $f \in \mathrm{Hom}(C_{p-1}(Y), \mathbb{R})$. We have

$$\begin{aligned}\delta_X(F^{\#}(f))(\sigma) &= F^{\#}(f) \circ \partial_X(\sigma) = F^{\#}(f)(\partial_X(\sigma)) = f(F \circ \partial_X(\sigma)) = f(F_{\#}(\partial_X(\sigma))) \\ &= f(\partial_Y(F_{\#}(\sigma))) = f(\partial_Y(F \circ \sigma)) = f \circ \partial_Y(F \circ \sigma) = \delta_Y(f)(F \circ \sigma) \\ &= F^{\#}(\delta_Y(f))(\sigma).\end{aligned}$$

So $F^{\#}$ defines a cochain map and hence it induces a homomorphism between cohomology groups $F^* : H^p(Y;\mathbb{R}) \longrightarrow H^p(X;\mathbb{R})$ given by $F^*([f]) = [F^{\#}(f)]$. Notice that $(G \circ F)^* = F^* \circ G^*$ and $(\mathrm{id_X})^* = \mathrm{id}_{H^p(X;\mathbb{R})}$. Then singular cohomology defines a contravariant functor from **Top** to **Ab**.

Let $\{X_j\}$ be any collection of topological spaces. Recall that the **disjoint union** of this family is the set

$$\coprod_j X_j = \bigcup_j \{(x, j) : x \in X_j\}.$$

Let $X = \coprod_j X_j$. For each j, let $i_j : X_j \longrightarrow X$ be the canonical injection defined by $i_j(x) = (x, j)$. The **disjoint union topology** on X is defined by the condition

$$\mathcal{U} \text{ is open in } X \text{ if and only if } i_j^{-1}(\mathcal{U}) \text{ is open in } X_j, \text{ for every } j.$$

Given two continuous maps $F, G : X \longrightarrow Y$, a **homotopy** between F and G is a continuous map $H : X \times [0, 1] \longrightarrow Y$ such that $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$. We denote $F \simeq G$ when there exists a homotopy between F and G. A continuous map $F : X \longrightarrow Y$ is a **homotopy equivalence** if there exists a continuous map $G : Y \longrightarrow X$ such that $G \circ F \simeq \mathrm{id}_X$ and $F \circ G \simeq \mathrm{id}_Y$. If $F : X \longrightarrow Y$ is a homotopy equivalence then we say that X and Y are **homotopy equivalent**.

Proposition 1.3.1 (Properties of singular cohomology)**.**

(1) For any one-point space $\{q\}$, $H^p(\{q\};\mathbb{R})$ is trivial except when $p = 0$, in which case it is 1-dimensional.

(2) If $\{X_j\}$ is any collection of topological spaces and $X = \coprod_j X_j$, then the inclusion maps $i_j : X_j \longrightarrow X$ induce an isomorphism from $H^p(X;\mathbb{R})$ to $\prod_j H^p(X_j;\mathbb{R})$.

(3) Homotopy equivalent spaces have isomorphic singular cohomology groups.

Proof: See [2, Proposition 16.4]. $\square$

1.4 The Mayer-Vietoris sequence for singular homology and cohomology

Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of X whose union is X. The commutative diagram

$$\begin{tikzcd}
& \mathcal{U} \arrow[dr, "k"] & \\
\mathcal{U}\cap\mathcal{V} \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\
& \mathcal{V} \arrow[ur, "l"'] &
\end{tikzcd}$$

in **Top** gives rise to the following commutative diagram in **Ab**, since H_p is a functor:

$$\begin{tikzcd}
& H_p(\mathcal{U}) \arrow[dr, "k_*"] & \\
H_p(\mathcal{U}\cap\mathcal{V}) \arrow[ur, "i_*"] \arrow[dr, "j_*"'] & & H_p(X) \\
& H_p(\mathcal{V}) \arrow[ur, "l_*"'] &
\end{tikzcd}$$

Theorem 1.4.1 (Mayer-Vietoris sequence for singular homology)**.** For each p there is a homomorphism

$$\partial_* : H_p(X) \longrightarrow H_{p-1}(\mathcal{U}\cap\mathcal{V}),$$

called the connecting homomorphism, such that the following sequence is exact:

$$\cdots \xrightarrow{\partial_*} H_p(\mathcal{U}\cap\mathcal{V}) \xrightarrow{\alpha} H_p(\mathcal{U}) \oplus H_p(\mathcal{V}) \xrightarrow{\beta} H_p(X) \xrightarrow{\partial_*} H_{p-1}(\mathcal{U}\cap\mathcal{V}) \xrightarrow{\alpha} \cdots,$$

where $\alpha([c]) = (i_*([c]), -j_*([c]))$, $\beta([c],[c']) = k_*([c]) + l_*([c'])$, and $\partial_*([e]) = [c]$ provided there exists $d \in C_p(\mathcal{U})$ and $d' \in C_p(\mathcal{V})$ such that $k_\#(d) + l_\#(d')$ is homologus to e and $(i_\#(c), -j_\#(c)) = (\partial d, \partial d')$.

Proof: See [2, Theorem 16.3, page 414]. $\square$

Similarly, we have the following commutative square in **Ab**:

$$\begin{tikzcd}
 & H^p(\mathcal{U};\mathbb{R}) \arrow[dr, "i^*"] & \\
H^p(X;\mathbb{R}) \arrow[ur, "k^*"] \arrow[dr, "l^*"'] & & H^p(\mathcal{U}\cap\mathcal{V};\mathbb{R}) \\
 & H^p(\mathcal{V};\mathbb{R}) \arrow[ur, "l^*"'] &
\end{tikzcd}$$

Theorem 1.4.2 (Mayer-Vietoris sequence for singular cohomology)**.** The following sequence is exact:

$$\cdots \xrightarrow{\partial^*} H^p(X;\mathbb{R}) \xrightarrow{k^*\oplus l^*} H^p(\mathcal{U};\mathbb{R}) \oplus H^p(\mathcal{V};\mathbb{R}) \xrightarrow{i^*-j^*} H^p(\mathcal{U}\cap\mathcal{V};\mathbb{R}) \xrightarrow{\partial^*} H^{p+1}(X;\mathbb{R}) \xrightarrow{k^*\oplus l^*} \cdots,$$

where $k^* \oplus l^*([f]) = ([k^\#(f)], [l^\#(g)])$, $(i^* - j^*)([f], [g]) = [i^\#(f)] - [j^\#(g)]$, and $\partial^*([f]) = [f] \circ \partial_*$.

Remark 1.4.1. The definition of ∂^* makes sense since $H^p(\mathcal{W};\mathbb{R}) \cong \mathrm{Hom}(H_p(\mathcal{W}), \mathbb{R})$, for every open subset $\mathcal{W}$ of X.

Proof: See [2, Theorem 16.5, page 416]. $\square$

Chapter 2

De Rham's Theorem

2.1 De Rham cohomology

De Rham cohomology is defined constructing a chain complex of forms on a smooth manifold X. The groups of this complex are the sets $\Omega^k(X)$ of differential k-forms, and the connecting homomorphism is the exterior derivative. The existence of such a complex is given by the following result.

Theorem 2.1.1. If X is a smooth manifold, then there exists a unique linear map

$$d : \Omega^k(X) \longrightarrow \Omega^{k+1}(X)$$

such that:

(1) If $f \in \Omega^0(X) = C^\infty(X)$, then $df = Df$.

(2) $d \circ d = 0$

(3) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, where $\alpha \in \Omega^k(X)$.

Proof: See [3, Theorem 2.20, page 65]. □

The map d is called the **exterior derivative**. Locally, d is defined as follows: if $(\mathcal{U}, \varphi = (x_1, \dots, x_n))$ is a smooth chart, then every k-form on X can be written as

$$\alpha|_{\mathcal{U}} = \sum_{i_1 < \cdots < i_k} a_{i_1 \dots i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $a_{i_1 \dots i_k} \in C^\infty(X)$, thus we define $d\alpha|_\mathcal{U}$ by

$$d\alpha|_\mathcal{U} = \sum_{i_1 < \dots < i_k} da_{i_1 \dots i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Recall that $\Omega^p(X) = 0$ if $p > n = \dim(X)$ and that, by convention, $\Omega^p(X) = 0$ if $p < 0$. We have a cochain complex

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \xrightarrow{d} 0 \longrightarrow \cdots .$$

The p-th cohomology group of this complex defined by $H^p_{\mathrm{dR}}(X) = \frac{\mathrm{Ker}(d|_{\Omega^p(X)})}{\mathrm{Im}(d|_{\Omega^{p-1}(X)})}$, is called the p-th **de Rham cohomology group** of X. Smooth maps between manifolds $F : X \longrightarrow Y$ induce homomorphisms between cohomology groups, since the pullback map $F^* : \Omega^p(Y) \longrightarrow \Omega^p(X)$ commutes with the exterior derivative. We denote the induced map in cohomology by $F^* : H^p_{\mathrm{dR}}(Y) \longrightarrow H^p_{\mathrm{dR}}(X)$, which is defined as $F^*([\omega]_{\mathrm{dR}}) = [F^*(\omega)]_{\mathrm{dR}}$. If $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ are smooth maps, then $(G \circ F)^* = F^* \circ G^*$ and $(\mathrm{id}_X)^* = \mathrm{id}_{H^p_{\mathrm{dR}}(X)}$. Thus de Rham cohomology defines a contravariant functor from the category **Man** of smooth manifolds and smooth maps to the category **Ab**. Given two smooth maps $F, G : X \longrightarrow Y$, a **homotopy** between F and G is a smooth map $H : X \times [0,1] \longrightarrow Y$ such that $H(x,0) = F(x)$ and $H(x,1) = G(x)$.

Proposition 2.1.1 (Properties of de Rham cohomology)**.**

(1) Diffeomorphic manifolds have isomorphic de Rham cohomology groups.

(2) $H^p(X) = 0$ if $p > \dim(X)$.

(3) $H^0_{\mathrm{dR}}(X) \cong \mathbb{R}$

(4) If X and Y are homotopy equivalent, then $H^p_{\mathrm{dR}}(X) \cong H^p_{\mathrm{dR}}(Y)$ for each p.

Proof: (1) and (2) are trivial. For (3), there are no -1-forms, so $\mathrm{Im}(d|_{\Omega^{-1}(X)}) = \{0\}$. A closed 0-form is a smooth real valued function f such that $df = 0$. Since X is connected, we have that f must be constant. Thus $H^0_{\mathrm{dR}}(X) = \mathrm{Ker}(d|_{\Omega^0(X)}) = \{\text{constant functions}\} \cong \mathbb{R}$. You can see a proof for (4) in [2, Theorem 15.6, page 393]. $\square$

Proposition 2.1.2.

(1) Let $\{X_j\}$ be a countable collection of smooth manifolds, and let $X = \coprod_j X_j$. For each p, the inclusion maps $i_j : X_j \longrightarrow X$ induce an isomorphism from $H^p_{\mathrm{dR}}(X)$ to the direct product space $\prod_j H^p_{\mathrm{dR}}(X_j)$.

(2) $H^p_{dR}(\{q\}) = 0$ for $p > 0$.

(3) **Poncaré's Lemma:** Let $\mathcal{U}$ be a convex open subset of $\mathbb{R}^n$. Them $H^p_{\mathrm{dR}}(\mathcal{U}) = 0$ for $p > 0$.

Proof:

(1) It suffices to show that the pullback maps $i_j^* : \Omega^p(X) \longrightarrow \Omega^p(X_j)$ induce an isomorphism from $\Omega^p(X)$ to $\prod_j \Omega^p(X_j)$. Define $\varphi : \Omega^p(X) \longrightarrow \prod_j \Omega^p(X_j)$ by

$$\varphi(\omega) = (i_1^*(\omega), i_2^*(\omega), \dots) = (\omega|_{X_1}, \omega|_{X_2}, \dots).$$

This maps is clearly a homomorphism of $\mathbb{R}$-modules. Now if $\varphi(\omega) = 0$ then $\omega|_{X_j} = 0$ for every j. Since $X = \coprod_j X_j$, we have $\omega = 0$. Let $\omega_1, \omega_2, \dots$ be p-forms on X_1, X_2, ..., respectively. Define a smooth p-form ω on X by $\omega = \omega_j$ on X_j. We have $\varphi(\omega) = (\omega_1, \omega_2, \dots)$. Therefore, φ is an isomorphism.

(2) Since $\dim(\{q\}) = 0$, the equality follows.

(3) Fix $q \in \mathcal{U}$. For every $x \in \mathcal{U}$ there exists a line segment joining x and q. We can define a homotopy $H : \mathcal{U} \times [0,1] \longrightarrow \mathcal{U}$ by $H(x,t) = q + t(x-q)$. Consider the inclusion and identity maps $i : \{q\} \longrightarrow \mathcal{U}$ and $\mathrm{id}_{\mathcal{U}}$. We have that H is a homotopy between $\mathrm{id}_{\mathcal{U}}$ and the constant map $c : \mathcal{U} \longrightarrow \mathcal{U}$ ($x \mapsto q$). Then $i \circ \mathrm{id}_{\{q\}} = c \simeq \mathrm{id}_{\mathcal{U}}$ and $\mathrm{id}_{\{q\}} \circ i = \mathrm{id}_{\{q\}}$. Thus i is a homotopy equivalence and hence $H^p_{\mathrm{dR}}(\mathcal{U}) \cong H^p_{\mathrm{dR}}(\{q\}) = 0$ for $p > 0$.

$\square$

2.2 Mayer-Vietoris sequence for de Rham cohomology

Consider the commutative diagram in **Top**

$$\begin{tikzcd} & \mathcal{U} \arrow[dr, "k"] & \\ \mathcal{U}\cap\mathcal{V} \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\ & \mathcal{V} \arrow[ur, "l"'] & \end{tikzcd}$$

where the maps i, j, k and l are inclusions and $\mathcal{U}$ and $\mathcal{V}$ is an open covering of X.
Consider the pullback maps

$$\begin{aligned} i^* &: \Omega^p(\mathcal{U}) \longrightarrow \Omega^p(\mathcal{U}\cap\mathcal{V}) & i^*(\omega) &= \omega|_{\mathcal{U}\cap\mathcal{V}}, \\ j^* &: \Omega^p(\mathcal{V}) \longrightarrow \Omega^p(\mathcal{U}\cap\mathcal{V}) & j^*(\omega) &= \omega|_{\mathcal{U}\cap\mathcal{V}}, \\ k^* &: \Omega^p(X) \longrightarrow \Omega^p(\mathcal{U}) & k^*(\omega) &= \omega|_{\mathcal{U}}, \\ l^* &: \Omega^p(X) \longrightarrow \Omega^p(\mathcal{V}) & l^*(\omega) &= \omega|_{\mathcal{V}}. \end{aligned}$$

Lemma 2.2.1. The sequence

$$0 \longrightarrow \Omega^p(X) \xrightarrow{k^*\oplus l^*} \Omega^p(\mathcal{U}) \oplus \Omega^p(\mathcal{V}) \xrightarrow{i^*-j^*} \Omega^p(\mathcal{U}\cap\mathcal{V}) \longrightarrow 0$$

is exact, where $k^*\oplus l^*(\omega) = (\omega|_{\mathcal{U}}, \omega|_{\mathcal{V}})$ and $(i^*-j^*)(\omega,\omega') = \omega|_{\mathcal{U}\cap\mathcal{V}} - \omega'|_{\mathcal{U}\cap\mathcal{V}}$.

Proof:

(1) $k^*\oplus l^*$ is injective: Let ω be a p-form on $\mathcal{U}\cap\mathcal{V}$ such that $k^*\oplus l^*(\omega) = 0$. Then $\omega|_{\mathcal{U}} = 0$ and $\omega|_{\mathcal{V}}$. Hence $\omega = 0$ on X.

(2) $\mathrm{Ker}(i^*-j^*) = \mathrm{Im}(k^*\oplus l^*)$: Let $\omega \in \Omega^p(X)$. We have

$$\begin{aligned} (i^*-j^*)\circ(k^*\oplus l^*)(\omega) &= (i^*-j^*)(\omega|_{\mathcal{U}}, \omega|_{\mathcal{V}}) = (\omega|_{\mathcal{U}})|_{\mathcal{U}\cap\mathcal{V}} - (\omega|_{\mathcal{V}})|_{\mathcal{U}\cap\mathcal{V}} \\ &= \omega|_{\mathcal{U}\cap\mathcal{V}} - \omega|_{\mathcal{U}\cap\mathcal{V}} = 0 \end{aligned}$$

Then $\mathrm{Ker}(i^*-j^*) \supseteq \mathrm{Im}(k^*\oplus l^*)$. Now let $(\omega,\omega') \in \mathrm{Ker}(i^*-j^*)$. We have $\omega|_{\mathcal{U}\cap\mathcal{V}} = \omega'|_{\mathcal{U}\cap\mathcal{V}}$, i.e., ω and ω' agree on $\mathcal{U}\cap\mathcal{V}$. So we define $\eta \in \Omega^p(X)$ by setting

$$\eta = \begin{cases} \omega & \text{on } \mathcal{U}, \\ \omega' & \text{on } \mathcal{V}. \end{cases}$$

Clearly we have $k^*\oplus l^*(\eta) = (\omega,\omega')$. Hence the other inclusion follows.

(3) $i^* - j^*$ is surjective: Let ω be an arbitrary p-form on $\mathcal{U} \cap \mathcal{V}$ and let $\{\varphi, \psi\}$ be a partition of unity subordinate to the covering $\{\mathcal{U}, \mathcal{V}\}$. Define $\eta \in \Omega^p(\mathcal{U})$ by

$$\eta = \begin{cases} \psi \cdot \omega & \text{on } \mathcal{U} \cap \mathcal{V}, \\ 0 & \text{on } \mathcal{U} - \operatorname{supp}(\psi) \end{cases}$$

We verify that η is well defined. Let $x \in (\mathcal{U} \cap \mathcal{V}) \cap (\mathcal{U} - \operatorname{supp}(\psi)) = (\mathcal{U} \cap \mathcal{V}) - \operatorname{supp}(\psi)$. We have $\psi(x) \cdot \omega(x) = 0$ since $\omega(x) = 0$. Then η is a smooth p-form on $\mathcal{U}$. Similarly, we define the smooth p-form η' on $\mathcal{V}$ by

$$\eta' = \begin{cases} -\varphi \cdot \omega & \text{on } \mathcal{U} \cap \mathcal{V}, \\ 0 & \text{on } \mathcal{V} - \operatorname{supp}(\varphi). \end{cases}$$

Hence $(\eta, \eta') \in \Omega^p(\mathcal{U}) \oplus \Omega^p(\mathcal{V})$ and

$$(i^* - j^*)(\eta, \eta') = \eta|_{\mathcal{U}\cap\mathcal{V}} - \eta'|_{\mathcal{U}\cap\mathcal{V}} = \psi \cdot \omega + \varphi \cdot \omega = (\psi + \varphi) \cdot \omega = \omega$$

$\square$

Theorem 2.2.1. There exists a long exact sequence

$$\cdots \xrightarrow{\Delta} H^p_{\mathrm{dR}}(X) \xrightarrow{k^* \oplus l^*} H^p_{\mathrm{dR}}(\mathcal{U}) \oplus H^p_{\mathrm{dR}}(\mathcal{V}) \xrightarrow{i^* - j^*} H^p_{\mathrm{dR}}(\mathcal{U} \cap \mathcal{V}) \xrightarrow{\Delta} H^{p+1}_{\mathrm{dR}}(X) \longrightarrow \cdots$$

Proof: We shall use the following commutative diagram with exact rows in order to construct the connecting homomorphism Δ.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \Omega^{p-1}(X) \arrow[r] \arrow[d] & \Omega^{p-1}(\mathcal{U}) \oplus \Omega^{p-1}(\mathcal{V}) \arrow[r] \arrow[d] & \Omega^{p-1}(\mathcal{U} \cap \mathcal{V}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^{p}(X) \arrow[r] \arrow[d] & \Omega^{p}(\mathcal{U}) \oplus \Omega^{p}(\mathcal{V}) \arrow[r] \arrow[d] & \Omega^{p}(\mathcal{U} \cap \mathcal{V}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^{p+1}(X) \arrow[r] \arrow[d] & \Omega^{p+1}(\mathcal{U}) \oplus \Omega^{p+1}(\mathcal{V}) \arrow[r] \arrow[d] & \Omega^{p+1}(\mathcal{U} \cap \mathcal{V}) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Let ω be a closed p-form on $\mathcal{U} \cap \mathcal{V}$. Since $i^* - j^*$ is surjective, there exists $(\eta, \eta') \in \Omega^p(\mathcal{U}) \oplus \Omega^p(\mathcal{V})$ such that $\omega = (i^* - j^*)(\eta, \eta') = \eta|_{\mathcal{U}\cap\mathcal{V}} - \eta'|_{\mathcal{U}\cap\mathcal{V}}$. Since ω is closed we have $(i^* - j^*) \circ (d, d)(\eta, \eta') = (i^* - j^*)(d\eta, d\eta') = 0$. We get $(d\eta, d\eta') \in \operatorname{Ker}(i^* - j^*) = \operatorname{Im}(k^* \oplus l^*)$. Thus there exists $\alpha \in \Omega^{p+1}$ such that $(d\eta, d\eta') = k^* \oplus l^*(\alpha)$. On the other hand

$$k^* \oplus l^*(d\alpha) = (d, d)(k^* \oplus l^*(\alpha)) = (d, d)(d\eta, d\eta') = (dd\eta, dd\eta') = (0, 0).$$

Since $k^* \oplus l^*$ is injective, we get $d\alpha = 0$. Hence α is a closed $(p+1)$-form. It makes sense to define $\Delta([\omega]_{\mathrm{dR}}) = [\alpha]_{\mathrm{dR}}$. We verify that Δ is well defined. The class $[\alpha]_{\mathrm{dR}}$ does not depend on the choice of the pair (η, η'). Let $(\zeta, \zeta') \in \Omega^p(\mathcal{U}) \oplus \Omega^p(\mathcal{V})$ such that $w = (i^* - j^*)(\zeta, \zeta')$. Let α' be a $(p+1)$-form such that $k^* \oplus l^*(\alpha') = (d\zeta, d\zeta')$. Note that $(\eta, \eta') - (\zeta, \zeta') \in \operatorname{Ker}(i^* - j^*) = \operatorname{Im}(k^* \oplus l^*)$. Then there exists $\beta \in \Omega^p(X)$ such that $(\eta, \eta') - (\zeta, \zeta') = k^* \oplus l^*(\beta)$. We have

$$k^* \oplus l^*(\alpha - \alpha') = (d\eta - d\zeta, d\eta' - d\zeta') = d \circ (k^* \oplus l^*)(\beta) = (k^* \oplus l^*)(d\beta).$$

Since $k^* \oplus l^*$ is injective, we get $\alpha - \alpha' = d\beta$, i.e., $[\alpha]_{\mathrm{dR}} = [\alpha']_{\mathrm{dR}}$. Now we prove that the sequence

$$\cdots \xrightarrow{\Delta} H^p_{\mathrm{dR}}(X) \xrightarrow{k^* \oplus l^*} H^p_{\mathrm{dR}}(\mathcal{U}) \oplus H^p_{\mathrm{dR}}(\mathcal{V}) \xrightarrow{i^* - j^*} H^p_{\mathrm{dR}}(\mathcal{U} \cap \mathcal{V}) \xrightarrow{\Delta} H^{p+1}_{\mathrm{dR}}(X) \longrightarrow \cdots$$

is exact. Exactness at $H^p_{\mathrm{dR}}(\mathcal{U}) \oplus H^p_{\mathrm{dR}}(\mathcal{V})$ follows from the previous lemma.

(i) $\operatorname{Ker}(k^* \oplus l^*) = \operatorname{Im}(\Delta)$: Let $\Delta([\omega]_{\mathrm{dR}}) \in \operatorname{Im}(\Delta)$. We have

$$k^* \oplus l^*(\Delta([\omega]_{\mathrm{dR}})) = [k^* \oplus l^*(\alpha)]_{\mathrm{dR}} = [(d\eta, d\eta')]_{\mathrm{dR}} = ([d\eta]_{\mathrm{dR}}, [d\eta']_{\mathrm{dR}}) = 0.$$

So the inclusion $\supseteq$ holds. Now let $[\alpha]_{\mathrm{dR}} \in \operatorname{Ker}(k^* \oplus l^*)$. We have $[\alpha|_{\mathcal{U}}]_{\mathrm{dR}} = 0$ and $[\alpha|_{\mathcal{V}}]_{\mathrm{dR}} = 0$, i.e. $\alpha|_{\mathcal{U}} = d\beta$ for some $\beta \in \Omega^p(\mathcal{U})$, and $\alpha|_{\mathcal{V}} = d\gamma$ for some $\gamma \in \Omega^p(\mathcal{V})$. Let ω be the following p-form on $\mathcal{U} \cap \mathcal{V}$. We have $\omega = \beta|_{\mathcal{U}\cap\mathcal{V}} - \gamma|_{\mathcal{U}\cap\mathcal{V}} = (i^* - j^*)(\beta, \gamma)$. Then $d\omega = d\beta|_{\mathcal{U}\cap\mathcal{V}} - d\gamma|_{\mathcal{U}\cap\mathcal{V}} = \alpha - \alpha = 0$, i.e. ω is a closed p-form on $\mathcal{U} \cap \mathcal{V}$. Also, $k^* \oplus l^*(\alpha) = (\alpha|_{\mathcal{U}}, \alpha|_{\mathcal{V}}) = (d\beta, d\gamma)$. Hence $[\alpha]_{\mathrm{dR}} = \Delta([\omega]_{\mathrm{dR}})$ and the other inclusion also holds.

(ii) $\operatorname{Ker}(\Delta) = \operatorname{Im}(i^* - j^*)$: Let $(i^* - j^*)([\eta]_{\mathrm{dR}}, [\eta']_{\mathrm{dR}}) \in \operatorname{Im}(i^* - j^*)$, where η and η' are closed p-forms on $\mathcal{U}$ and $\mathcal{V}$, respectively. We have

$$\Delta \circ (i^* - j^*)([\eta]_{\mathrm{dR}}, [\eta']_{\mathrm{dR}}) = \Delta([\eta|_{\mathcal{U}\cap\mathcal{V}} - \eta'|_{\mathcal{U}\cap\mathcal{V}}]_{\mathrm{dR}}) = 0$$

since $d\eta = 0$ and $d\eta' = 0$. So we get the inclusion $\supseteq$. Now let $[\omega]_{\mathrm{dR}} \in \operatorname{Ker}(\Delta)$. Then $[\alpha]_{\mathrm{dR}} = 0$, i.e., there exists $\beta \in \Omega^p(X)$ such that $\alpha = d\beta$. Let $\eta \in \Omega^p(\mathcal{U})$ and $\eta' \in \Omega^p(\mathcal{V})$ satisfying $\alpha|_{\mathcal{U}} = d\eta$, $\alpha|_{\mathcal{V}} = d\eta'$ and $\omega = \eta|_{\mathcal{U}\cap\mathcal{V}} - \eta'|_{\mathcal{U}\cap\mathcal{V}}$. We have $d(\eta - \beta) = 0$ and $d(\eta' - \beta) = 0$. It follows $([\eta - \beta]_{\mathrm{dR}}, [\eta' - \beta]_{\mathrm{dR}}) \in H^p_{\mathrm{dR}}(\mathcal{U}) \oplus H^p_{\mathrm{dR}}(\mathcal{V})$. Hence $(i^* - j^*)([\eta - \beta]_{\mathrm{dR}}, [\eta' - \beta]_{\mathrm{dR}}) = [\eta|_{\mathcal{U}\cap\mathcal{V}} - \eta'|_{\mathcal{U}\cap\mathcal{V}}]_{\mathrm{dR}} = [\omega]_{\mathrm{dR}}$ and the other inclusion holds

$\square$

Corollary 2.2.1. The connecting homomorphism $\Delta : H^p_{\mathrm{dR}}(\mathcal{U} \cap \mathcal{V}) \longrightarrow H^{p+1}_{\mathrm{dR}}(X)$ is defined as follows: for each $\omega \in \mathrm{Ker}(d : \Omega^p(\mathcal{U} \cap \mathcal{V}) \longrightarrow \Omega^{p+1}(X))$, there are smooth p-forms $\eta \in \Omega^p(\mathcal{U})$ and $\eta' \in \Omega^p(\mathcal{V})$ such that $\omega = \eta|_{\mathcal{U}\cap\mathcal{V}} - \eta'|_{\mathcal{U}\cap\mathcal{V}}$, and then $\Delta([\omega]_{\mathrm{dR}}) = [d\eta]_{\mathrm{dR}}$, where η is extended by zero to all of X.

Proof: Let η and η' be the forms defined in Part (3) of Lemma 2.2.1's proof. Then $\omega = \eta - \eta'$ on $\mathcal{U} \cap \mathcal{V}$. Let σ be the p-form on X obtained by extending $d\eta$ to be zero outside $\mathcal{U} \cap \mathcal{V}$. We have $\sigma|_{\mathcal{U}\cap\mathcal{V}} = d\eta|_{\mathcal{U}\cap\mathcal{V}} = d(\omega + \eta')|_{\mathcal{U}\cap\mathcal{V}} = d\eta'|_{\mathcal{U}\cap\mathcal{V}}$, since ω is closed. It follows $\sigma|_{\mathcal{U}} = \sigma|_{\mathcal{V}} = d\eta = d\eta' = 0$ on $X - \mathcal{U} \cap \mathcal{V}$, and $\sigma|_{\mathcal{U}} = \sigma|_{\mathcal{U}\cap\mathcal{V}} = d\eta|_{\mathcal{U}\cap\mathcal{V}}$ and $\sigma|_{\mathcal{V}} = \sigma|_{\mathcal{U}\cap\mathcal{V}} = d\eta'|_{\mathcal{U}\cap\mathcal{V}}$ on $\mathcal{U} \cap \mathcal{V}$. Then $\Delta([\omega]_{\mathrm{dR}}) = [\sigma]_{\mathrm{dR}}$. $\square$

2.3 Stokes' Theorem

A **topological manifold with boundary** is a topological space X such that every point has a neighbourhood homeomorphic to an open subset of Euclidean half-space

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}.$$

Let X be a topological manifold with boundary. The **interior** of X, denoted $\mathrm{int}(X)$, is the set of points in X which have neighbourhoods homeomorphic to an open subset of $\mathrm{int}(\mathbb{H}^n)$. The **boundary** of X, denoted ∂X, is the complement of $\mathrm{int}(X)$ in X.

To see how to define a smooth structure on a topological manifold with boundary, recall that a smooth map from an arbitrary subset $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^k$ is defined to be a map that admits a smooth extension to an open neighbourhood of each point of A. Thus if $\mathcal{U}$ is an open subset of $\mathbb{H}^n$, a map $F : \mathcal{U} \longrightarrow \mathbb{R}^k$ is smooth if for each $x \in \mathcal{U}$, there exists an open set $\mathcal{V} \subseteq \mathbb{R}^n$ and a smooth map $\tilde{F} : \mathcal{V} \longrightarrow \mathbb{R}^k$ that agrees with F on $\mathcal{U}$. If F is such a map, the restriction of F to $\mathcal{U} \cap \mathrm{int}(\mathbb{H}^n)$ is smooth in the usual sense. By continuity, all the partial derivatives of F at points of $\mathcal{U} \cap \partial\mathbb{H}^n$ are determined by their values in $\mathrm{int}(\mathbb{H}^n)$, and therefore in particular are independent of the choice of the extension.

Now let X be a topological manifold with boundary. A smooth structure for X is defined to be a maximal smooth atlas. With such a structure X is called a **smooth manifold with boundary**. A point $p \in X$ is called a **boundary point** if its image under some smooth chart is in

$$\partial(\mathbb{H}^n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\},$$

and an **interior point** if its image under some smooth chart is in

$$\mathrm{int}(\mathbb{H}^n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

The boundary ∂X is an $(n-1)$-dimensional manifold. If X is oriented, then it induces an orientation on ∂X (see [2, Proposition 13.17, page 339]).

Theorem 2.3.1 (Stokes)**.** Let X be a smooth and oriented n-dimensional manifold with boundary, and let ω be an $(n-1)$-form on X having compact support. Then

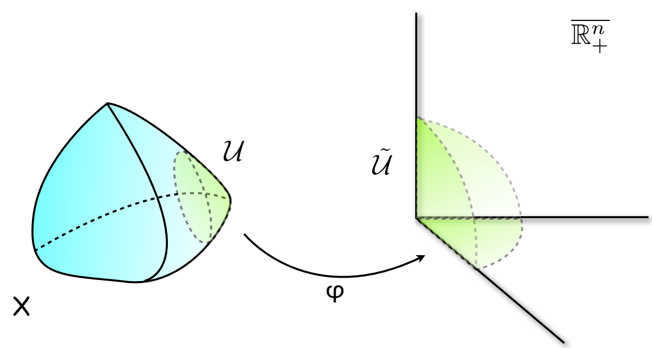
$$\int_X d\omega = \int_{\partial X} \omega.$$

Proof: See [2, Theorem 14.9, page 358]. $\square$

Now we study the notion of a manifold with corners. Consider the set

$$\overline{\mathbb{R}^n_+} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1, \ldots, x_n \geq 0\}.$$

Suppose X is a topological manifold with boundary. A **chart with corners** for X is a pair $(\mathcal{U}, \varphi)$, where $\mathcal{U}$ is an open subset of X and φ is a homeomorphism from $\mathcal{U}$ to a (relatively) open set $\tilde{U} \subseteq \overline{\mathbb{R}^n_+}$.



Two charts with corners $(\mathcal{U}, \varphi)$, $(\mathcal{V}, \psi)$ are said to be **smoothly compatible** if the composite map $\varphi \circ \psi^{-1} : \psi(\mathcal{U} \cap \mathcal{V}) \longrightarrow \varphi(\mathcal{U} \cap \mathcal{V})$ is smooth. A **smooth structure with corners** on a topological manifold with boundary is a maximal collection of smoothly compatible charts with corners whose domains cover the entire set X. A topological manifold with boundary together with a smooth structure with corners is called a **smooth manifold with corners**.

Theorem 2.3.2 (Stokes' Theorem on Manifolds with Corners)**.** Let X be a smooth and oriented n-dimensional manifold with corners, and let ω be an $(n-1)$-form on X having compact support. Then

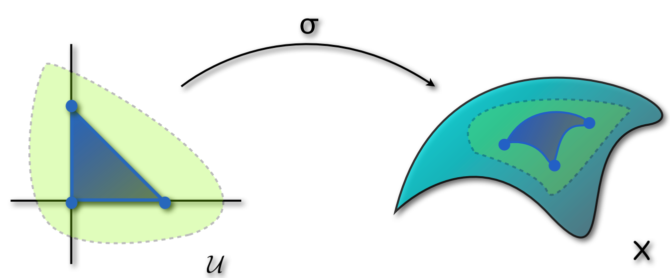
$$\int_X d\omega = \int_{\partial X} \omega.$$

Proof: See [2, Theorem 14.20, page 367]. $\square$

2.4 Smooth singular cohomolohy and de Rham homomorphisms

The de Rham's Theorem states that there exists an isomorphism from $H^p_{\mathrm{dR}}(X)$ to $H^p(X;\mathbb{R})$, for every smooth manifold X. A good way to construct such a homomorphism is defining a cochain map $\psi : \Omega^p(X) \longrightarrow \mathrm{Hom}(C_p(X), \mathbb{R})$ and then take the induced homomorphism in cohomology. Let ω be a p-form on X and σ be a singular simplex. We define a map $C_p(X) \longrightarrow \mathbb{R}$ by pulling ω back by σ. However, the problem with this procedure is that σ need not be smooth. So we are going to consider smooth singular simplices in order to pull ω back by σ.

A **smooth singular p-simplex** in X is a smooth map $\sigma : \Delta_p \longrightarrow X$. Recall that smoothness is defined for maps whose domain is an open set. Thus by σ smooth on Δ_p we mean there is an open set $\mathcal{U} \supseteq \Delta_p$ and a smooth map $F : \mathcal{U} \longrightarrow X$ such that $F|_{\mathcal{U}} = \sigma$.



smooth singular simplex

The subgroup of $C_p(X)$ generated by all smooth singular p-simplices is called the **smooth chain group** in dimension p and is denoted $C^\infty_p(X)$. An element $c \in C^\infty_p(X)$ is called a **smooth chain**. Consider the boundary map $\partial : C_p(X) \longrightarrow C_{p-1}(X)$ given by $\partial\sigma = \sum_{i=0}^{p}(-1)^i \sigma \circ F_{i,p}$, where $F_{i,p} : \Delta_{p-1} \longrightarrow \Delta_p$ is the i-th face map. Restricting ∂ to $C^\infty_p(X)$ we have that $\partial\sigma$ is a smooth singular $p-1$ simplex. Recall that $\partial \circ \partial = 0$, so we get a complex

$$\cdots \longrightarrow C^\infty_{p+1}(X) \xrightarrow{\partial} C^\infty_p(X) \xrightarrow{\partial} C^\infty_{p-1}(X) \longrightarrow \cdots .$$

Applying the contravariant functor $\mathrm{Hom}(-, \mathbb{R})$ we obtain the associated dual complex

$$\cdots \longrightarrow \mathrm{Hom}(C^\infty_{p-1}(X), \mathbb{R}) \xrightarrow{\delta} \mathrm{Hom}(C^\infty_p(X), \mathbb{R}) \xrightarrow{\delta} \mathrm{Hom}(C^\infty_{p+1}(X), \mathbb{R}) \longrightarrow \cdots .$$

The p-th cohomology group of the previous cochain complex

$$H^p_\infty(X; \mathbb{R}) = \frac{\mathrm{Ker}(\delta : \mathrm{Hom}(C^\infty_p(X), \mathbb{R}) \longrightarrow \mathrm{Hom}(C^\infty_{p+1}(X), \mathbb{R}))}{\mathrm{Im}(\delta : \mathrm{Hom}(C^\infty_{p-1}(X), \mathbb{R}) \longrightarrow \mathrm{Hom}(C^\infty_p(X), \mathbb{R}))},$$

is called the p-th **smooth singular cohomology group** of X. If $F : X \longrightarrow Y$ is a smooth map and c is a smooth chain in X, then $F \circ c$ is a smooth chain in Y. So we have that $F^\sharp(f) \in \mathrm{Hom}(C^\infty_p(X), \mathbb{R})$ for every $c \in \mathrm{Hom}(C^\infty_p(Y), \mathbb{R})$. Hence, smooth maps $F : X \longrightarrow Y$ induce homomorphisms between smooth cohomology groups $F^* : H^p_\infty(Y; \mathbb{R}) \longrightarrow H^p_\infty(X; \mathbb{R})$.

Given a smooth singular p-simplex $\sigma : \Delta_p \longrightarrow X$ and a p-form on X, we integrate the pullback $\sigma^*\omega$ on Δ_p and set

$$\int_\sigma \omega := \int_{\Delta_p} \sigma^*\omega. \qquad (1)$$

There is a problem when we consider the previous integral. Recall that integration of forms is defined over oriented manifold. The standard p-simplex Δ_p is an example of a manifold with corners whose boundary is $\partial\Delta_p$. Also, we can give to Δ_p an orientation as follows: take the positive orientation on the 0-simplex Δ_0 and, if an orientation has been chosen for Δ_{p-1}, choose the one on $\partial\Delta_p$ making the face map $F_{0,p} : \Delta_{p-1} \longrightarrow \partial\Delta_p$ orientation preserving. Hence, it makes sense to define (1). Now we define $\int_c \omega$ for any smooth chain $c = \sum_\sigma n_\sigma \sigma$ in $C^\infty_p(X)$ extending (1) linearly, i.e.,

$$\int_c \omega := \sum_\sigma n_\sigma \int_\sigma \omega.$$

This provides a homomorphism $\Psi(\omega) : C^\infty_p(X) \longrightarrow \mathbb{R}$ given by $\Psi(\omega)(c) = \int_c \omega$. Thus, for each p-form ω on X we have a homomorphism $\Psi(\omega) \in \mathrm{Hom}(C^\infty_p(X), \mathbb{R})$. So we get a $\mathbb{R}$-linear map of vector spaces $\Psi : \Omega^p(X) \longrightarrow \mathrm{Hom}(C^\infty_p(X), \mathbb{R})$.

Proposition 2.4.1. The diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{p-1}(X) \arrow[r, "d"] \arrow[d, "\Psi"] & \Omega^p(X) \arrow[r, "d"] \arrow[d, "\Psi"] & \Omega^{p+1}(X) \arrow[r] \arrow[d, "\Psi"] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}(C^\infty_{p-1}(X), \mathbb{R}) \arrow[r, "\delta"'] & \mathrm{Hom}(C^\infty_p(X), \mathbb{R}) \arrow[r, "\delta"'] & \mathrm{Hom}(C^\infty_{p+1}, \mathbb{R}) \arrow[r] & \cdots
\end{tikzcd}$$

commutes, i.e., the map $\Psi : \Omega^p(X) \longrightarrow \mathrm{Hom}(C^\infty_p(X), \mathbb{R})$ is a cochain map

Proof: Let ω be a $(p-1)$-form on X and σ a smooth singular p-simplex in X. We have

$$\begin{aligned}\delta(\Psi(\omega))(\sigma) = \Psi(\omega)(\partial\sigma) = \int_{\partial\sigma} \omega = \int_{\sum_{i=0}^{p}(-1)^i \sigma\circ F_{i,p}} \omega = \sum_{i=0}^{p} (-1)^i \int_{\sigma\circ F_{i,p}} \omega \\ = \sum_{i=0}^{p}(-1)^i \int_{\Delta_{p-1}} (\sigma\circ F_{i,p})^*\omega = \sum_{i=0}^{p}(-1)^i \int_{\Delta_{p-1}} F_{i,p}^* \circ \sigma^*\omega\end{aligned}$$

On the other hand

$$\int_{\Delta_{p-1}} F_{i,p}^* \circ \sigma^*\omega = \begin{cases} \int_{F_{i,p}(\Delta_{p-1})} \sigma^*\omega & \text{if } F_{i,p} \text{ is orientation preserving,} \\ -\int_{F_{i,p}(\Delta_{p-1})} \sigma^*\omega & \text{if } F_{i,p} \text{ is orientation reversing.} \end{cases}$$

Let π be the map sending $e_0 \mapsto e_i$, $e_1 \mapsto e_0$, $\ldots$, $e_i \mapsto e_{i-1}$, $e_{i+1} \mapsto e_{i+1}$, $\mapsto$, $e_p \mapsto e_p$. We shall see that $F_{i,p} = \pi \circ F_{0,p}$.

$$\begin{aligned} \pi\circ F_{0,p}(e_0) &= \pi(e_1) = e_0 \\ \pi\circ F_{0,p}(e_1) &= \pi(e_2) = e_1 \\ &\vdots \\ \pi\circ F_{0,p}(e_{i-1}) &= \pi(e_i) = e_{i-1} \\ \pi\circ F_{0,p}(e_i) &= \pi(e_{i+1}) = e_{i+1} \\ &\vdots \\ \pi\circ F_{0,p}(e_{p-1}) &= \pi(e_p) = e_p. \end{aligned}$$

Hence $F_{i,p} = \pi \circ F_{0.p}$. Recall that $F_{0,p}$ is orientation preserving. So $F_{i,p}$ is orientation preserving if and only if π is. On the other hand, π is the permutation

$$(0, 1, \ldots, i, i+1, \ldots, p) \mapsto (i, 0, \ldots, i-1, i+1, \cdots, p)$$

whose sign is $(-1)^i$. It follows $F_{i,p}$ is orientation preserving if and only if i is even.

Thus we have

$$\begin{aligned}
\delta(\Psi(\omega))(\sigma) &= \sum_{i=0}^{p}(-1)^i \int_{\Delta_{p-1}} F_{i,p}^* \circ \sigma^*\omega = \sum_{i=0}^{p}(-1)^i(-1)^i \int_{F_{i,p}(\Delta_{p-1})} \sigma^*\omega \\
&= \sum_{i=0}^{p} \int_{F_{i,p}(\Delta_{p-1})} \sigma^*\omega \\
&= \int_{\bigcup_{i=0}^{p} F_{i,p}(\Delta_{p-1})} \sigma^*\omega \text{ since the integral is additive} \\
&= \int_{\partial\Delta_p} \sigma^*\omega \\
&= \int_{\Delta_p} d(\sigma^*\omega) \text{ by the Stokes' Theorem on manifolds with corners} \\
&= \int_{\Delta_p} \sigma^*(d\omega) = \int_\sigma d\omega = \Psi(d\omega)(\sigma)
\end{aligned}$$

□

By the previous proposition we have that Ψ induces a homomorphism between cohomology groups $\Psi^* : H^p_{\mathrm{dR}}(X) \longrightarrow H^p_\infty(X;\mathbb{R})$ given by $\Psi^*([\omega]_{\mathrm{dR}}) = [\Psi(\omega)]$. Now we see the relation between the induced homomorphisms Ψ^* and F^*.

Proposition 2.4.2. If $F : X \longrightarrow Y$ is a smooth map then the following diagram commutes:

$$\begin{tikzcd}
H^p_{\mathrm{dR}}(Y) \arrow[r, "F^*"] \arrow[d, "\Psi^*"'] & H^p_{\mathrm{dR}}(X) \arrow[d, "\Psi^*"] \\
H^p_{\mathrm{dR}}(Y;\mathbb{R}) \arrow[r, "F^*"'] & H^p_\infty(X;\mathbb{R})
\end{tikzcd}$$

Proof: Let ω be a closed p-form on Y and σ a smooth singular p-simplex in X. We have

$$\Psi(F^*\omega)(\sigma) = \int_\sigma F^*\omega = \int_{\Delta_p} \sigma^* F^*\omega = \int_{\Delta_p} (F\circ\sigma)^*\omega = \int_{F\circ\sigma} \omega = \Psi(\omega)(F\circ\sigma) = F^\sharp(\Psi(\omega))(\sigma).$$

Then $\Psi(F^*\omega) = F^\sharp(\Psi\omega)$ for every closed p-form ω on Y, and hence

$$\Psi^* \circ F^*([\omega]_{\mathrm{dR}}) = F^* \circ \Psi^*([\omega]_{\mathrm{dR}}).$$

□

A smooth manifold X is said to be a **de Rham manifold** if for each p the homomorphism $\Psi^* : H^p_{\mathrm{dR}}(X) \longrightarrow H^p_\infty(X; \mathbb{R})$ is an isomorphism.

Corollary 2.4.1. If $F : X \longrightarrow Y$ is a diffeomorphism and X is a de Rham manifold, then Y is also de Rham.

Proof: The induced homomorphisms F^* in both de Rham and smooth singular cohomology are isomorphisms since $F : X \longrightarrow Y$ is a diffeomorphism. By the previous proposition, we can write $\Psi^*_Y : H^p_{\mathrm{dR}}(Y) \longrightarrow H^p_\infty(Y; \mathbb{R})$ as the composition $\Psi^* = (F^*)^{-1} \circ \Psi^*_X \circ F^*$, where $(F^*)^{-1}$, Ψ^*_X and F^* are isomorphisms. Hence Ψ^*_Y is an isomorphism, i.e., Y is a de Rham manifold. $\square$

2.5 De Rham's Theorem

Using the last definition we gave in the previous section, we can formulate de Rham's Theorem as follows:

Theorem 2.5.1. Every smooth manifold is de Rham.

Before giving the proof, we introduce some definitions in order to simplify the proof. If X is a smooth manifold, an open cover $\{\mathcal{U}_i\}$ of X is called a **de Rham cover** if each open set $\mathcal{U}_i$ is a de Rham manifold, and every finite intersection $\mathcal{U}_{i_1} \cap \cdots \cap \mathcal{U}_{i_k}$ is de Rham. A de Rham cover that is also a basis fir the topology of X is called a **de Rham basis** for X. First, we are going to prove the theorem for five particular cases:

Case 1. If $\{X_j\}$ is any countable collection of de Rham manifolds, then their disjoint union is de Rham.

Proof: Let $X = \coprod_j X_j$. We know the inclusion maps $i_j : X_j \longrightarrow X$ induce isomorphisms $\varphi : H^p_{\mathrm{dR}}(X) \longrightarrow \prod_j H^p_{\mathrm{dR}}(X_j)$ and $\varphi' : H^p_\infty(X; \mathbb{R}) \longrightarrow \prod_j H^p_\infty(X_j; \mathbb{R})$. Recall that $\varphi([\omega]_{\mathrm{dR}}) = ([\omega|_{X_j}]_{\mathrm{dR}})_j$. On the other hand, if $f : C^p_\infty(X) \longrightarrow \mathbb{R}$ then $\varphi'([f]) = ([f_j])_j$, where $f_j : C^p_\infty(X_j) \longrightarrow \mathbb{R}$ is the linear map given by $f_j(\sigma) = f(i_j \circ \sigma)$, for every smooth singular p-simplex $\sigma : \Delta_p \longrightarrow X_j$. For each j, we have an isomorphism

$$\Psi^*_j : H^p_{\mathrm{dR}}(X_j) \longrightarrow H^p_\infty(X_j; \mathbb{R}).$$

So $\prod_j \Psi_j^* : \prod_j H^p_{\mathrm{dR}}(X_j) \longrightarrow \prod_j H^p_\infty(X_j;\mathbb{R})$ is an isomorphism. Let ω be a closed p-form on X. We have

$$\prod_j \Psi_j^* \circ \varphi([\omega]_{\mathrm{dR}}) = \prod_j \Psi_j^*(([\omega|_{X_j}])_j) = (\Psi_j^*([\omega|_{X_j}]_{\mathrm{dR}}))_j$$
$$\varphi' \circ \Psi^*([\omega]_{\mathrm{dR}}) = \varphi'([\Psi(\omega)]) = ([(\Psi(\omega))_j])_j$$

Let σ_j be a p-simplex on X_j.

$$\Psi_j(\omega|_{X_j})(\sigma_j) = \int_{\sigma_j} \omega|_{X_j} = \int_{\Delta_p} \sigma_j^*\omega|_{X_j},$$
$$(\Psi(\omega))_j(\sigma_j) = \Psi(\omega)(i_j \circ \sigma_j) = \int_{i_j\circ\sigma_j} \omega = \int_{\Delta_p} \sigma_j^* i_j^* \omega = \int_{\Delta_p} \sigma_j^*\omega|_{X_j}.$$

Hence $\prod_j \Psi_j^* \circ \varphi([\omega]_{\mathrm{dR}}) = \varphi' \circ \Psi^*([\omega]_{\mathrm{dR}})$. So we have $\prod_j \psi_j^* \circ \varphi = \varphi' \circ \Psi^*$. Since φ' is an isomorphism, we get $\Psi^* = (\varphi')^{-1} \circ \prod_j \psi_j^* \circ \varphi$. It follows that Ψ^* is an isomorphism. $\square$

Case 2. Every convex open subset of $\mathbb{R}^n$ is de Rham.

Proof: Let $\mathcal{U}$ be a convex open subset of $\mathbb{R}^n$. By Poincaré's Lemma, $H^p_{\mathrm{dR}}(\mathcal{U}) = H^p_{\mathrm{dR}}(\{q\})$, where q is some fixed point in $\mathcal{U}$. By Proposition 1.3.1, we also have $H^p_\infty(\mathcal{U};\mathbb{R}) = H^p_\infty(\{q\};\mathbb{R})$. If $p \geq 1$, we have $H^p_{\mathrm{dR}}(\{q\}) = 0$ and $H^p_\infty(\{q\};\mathbb{R}) = 0$. In this case, Ψ^* is clearly an isomorphism. If $p = 0$ then $H^0_{\mathrm{dR}}(\{q\}) = \mathbb{R}$ and $H^0_\infty(\{q\};\mathbb{R}) = \mathbb{R}$. So it is only left to show that Ψ^* is non-zero, since it is a group homomorphism. Let σ be a smooth 0-simplex and f the constant function 1, then

$$\Psi(f)(\sigma) = \int_\sigma f = \int_{\Delta_0} \sigma^* f = (f \circ \sigma)(0) = 1.$$

We have $\Psi(1) \equiv 1$ and then $[\Psi(f)] \neq 0$. Hence Ψ^* is not the zero map, i.e., ψ^* is an isomorphism if $p = 0$. $\square$

Case 3. If X has a finite de Rham cover, then X is de Rham.

Proof: We use induction. Let $\{\mathcal{U}, \mathcal{V}\}$ be a finite de Rham cover of X. Then $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{U} \cap \mathcal{V}$ are de Rham manifolds. We shall prove that $X = \mathcal{U} \cup \mathcal{V}$ is de Rham. Considering the Mayer-Vietoris sequences for both de Rham and smooth singular cohomology, we have the following diagram

$$\begin{tikzcd}
H^{p-1}_{\mathrm{dR}}(\mathcal{U}) \oplus H^{p-1}_{\mathrm{dR}}(\mathcal{V}) \arrow[r] \arrow[d, "(\Psi^*_{\mathcal{U}}, \Psi^*_{\mathcal{V}})"] & H^{p-1}_{\mathrm{dR}}(\mathcal{U} \cap \mathcal{V}) \arrow[r, "\Delta"] \arrow[d, "\Psi^*_{\mathcal{U}\cap\mathcal{V}}"] & H^{p}_{\mathrm{dR}}(X) \arrow[r] \arrow[d, "\Psi^*"] & {} \\
H^{p-1}_{\infty}(\mathcal{U};\mathbb{R}) \oplus H^{p-1}_{\infty}(\mathcal{V};\mathbb{R}) \arrow[r] & H^{p-1}_{\infty}(\mathcal{U} \cap \mathcal{V};\mathbb{R}) \arrow[r, "\partial^*"'] & H^{p}_{\infty}(X;\mathbb{R}) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^{p}_{\mathrm{dR}}(X) \arrow[r] \arrow[d, "\Psi^*"] & H^{p}_{\mathrm{dR}}(\mathcal{U}) \oplus H^{p}_{\mathrm{dR}}(\mathcal{V}) \arrow[r] \arrow[d, "(\Psi^*_{\mathcal{U}}, \Psi^*_{\mathcal{V}})"] & H^{p}_{\mathrm{dR}}(\mathcal{U} \cap \mathcal{V}) \arrow[d, "\Psi^*_{\mathcal{U}\cap\mathcal{V}}"] \\
{} \arrow[r] & H^{p}_{\infty}(X;\mathbb{R}) \arrow[r] & H^{p}_{\infty}(\mathcal{U};\mathbb{R}) \oplus H^{p}_{\infty}(\mathcal{V};\mathbb{R}) \arrow[r] & H^{p}_{\infty}(\mathcal{U} \cap \mathcal{V};\mathbb{R})
\end{tikzcd}$$

We prove this diagram commutes:

(1) $\Psi^*_{\mathcal{U}\cap\mathcal{V}} \circ (i^* - j^*) = (i^* - j^*) \circ (\Psi^*_{\mathcal{U}}, \Psi^*_{\mathcal{V}})$: Let $([\omega]_{\mathrm{dR}}, [\omega']_{\mathrm{dR}}) \in H^{p-1}_{\mathrm{dR}}(\mathcal{U}) \oplus H^{p-1}_{\mathrm{dR}}(\mathcal{V})$. We have

$$\begin{aligned}
\Psi^*_{\mathcal{U}\cap\mathcal{V}} \circ (i^* - j^*)([\omega]_{\mathrm{dR}}, [\omega']_{\mathrm{dR}}) &= \Psi^*_{\mathcal{U}\cap\mathcal{V}}([\omega|_{\mathcal{U}\cap\mathcal{V}}]_{\mathrm{dR}} - [\omega'|_{\mathcal{U}\cap\mathcal{V}}]_{\mathrm{dR}}) \\
&= \Psi^*_{\mathcal{U}\cap\mathcal{V}}([\omega|_{\mathcal{U}\cap\mathcal{V}}]_{\mathrm{dR}}) - \Psi^*_{\mathcal{U}\cap\mathcal{V}}([\omega'|_{\mathcal{U}\cap\mathcal{V}}]_{\mathrm{dR}}) \\
&= [\Psi_{\mathcal{U}\cap\mathcal{V}}(\omega|_{\mathcal{U}\cap\mathcal{V}})] - [\Psi_{\mathcal{U}\cap\mathcal{V}}(\omega'|_{\mathcal{U}\cap\mathcal{V}})] \\
&= (i^* - j^*)([\Psi_{\mathcal{U}}(\omega)], [\Psi_{\mathcal{V}}(\omega')]) \\
&= (i^* - j^*) \circ (\Psi^*_{\mathcal{U}}, \Psi^*_{\mathcal{V}})([\omega]_{\mathrm{dR}}, [\omega']_{\mathrm{dR}}).
\end{aligned}$$

(2) $(\Psi^*_{\mathcal{U}}, \Psi^*_{\mathcal{V}}) \circ (k^* \oplus l^*) = (k^* \oplus l^*) \circ \Psi^*$: Let $[\omega] \in H^{p}_{\mathrm{dR}}(\mathcal{U} \cap \mathcal{V})$.

$$\begin{aligned}
(\Psi^*_{\mathcal{U}}, \Psi^*_{\mathcal{V}}) \circ (k^* \oplus l^*)([\omega]_{\mathrm{dR}}) &= (\Psi^*_{\mathcal{U}}, \Psi^*_{\mathcal{V}})([\omega|_{\mathcal{U}}]_{\mathrm{dR}}, [\omega|_{\mathcal{V}}]_{\mathrm{dR}}) \\
&= (\Psi^*_{\mathcal{U}}([\omega|_{\mathcal{U}}]_{\mathrm{dR}}), \Psi^*_{\mathcal{V}}([\omega|_{\mathcal{V}}]_{\mathrm{dR}})) \\
&= (\Psi^*([\omega|_{\mathcal{U}}]_{\mathrm{dR}}), \Psi^*([\omega|_{\mathcal{V}}]_{\mathrm{dR}})) = ([\Psi_{\mathcal{U}}(\omega|_{\mathcal{U}})], [\Psi_{\mathcal{V}}(\omega|_{\mathcal{V}})]) \\
&= (k^*([\Psi(\omega)]), l^*([\Psi(\omega)])) = (k^* \oplus l^*)([\Psi(\omega)]) \\
&= (k^* \oplus l^*) \circ \Psi^*([\omega]_{\mathrm{dR}})
\end{aligned}$$

(3) $\Psi^* \circ \Delta = \partial^* \circ \Psi^*|_{\mathcal{U}\cap\mathcal{V}}$: Using the identification $H^p_{\infty}(\mathcal{U} \cap \mathcal{V};\mathbb{R}) \cong \mathrm{Hom}(H_p^{\infty}(\mathcal{U} \cap \mathcal{V});\mathbb{R})$, where $H_p^{\infty}(\mathcal{U} \cap \mathcal{V})$ denote the smooth singular homology. Let $[\omega]_{\mathrm{dR}} \in H^{p-1}_{\mathrm{dR}}(\mathcal{U} \cap \mathcal{V})$ and $[e] \in H^p_{\infty}(\mathcal{U} \cup \mathcal{V};\mathbb{R})$. We have to show

$$\Psi^*(\Delta([\omega]_{\mathrm{dR}}))([e]) = \partial^*(\Psi^*|_{\mathcal{U}\cap\mathcal{V}}([\omega]_{\mathrm{dR}}))([e]),$$

where $\partial^* : \mathrm{Hom}(H_{p-1}^\infty(\mathcal{U}\cap\mathcal{V}),\mathbb{R}) \longrightarrow \mathrm{Hom}(H_p^\infty(\mathcal{U}\cup\mathcal{V}),\mathbb{R})$ is the dual map of $\partial_* : H_p(\mathcal{U}\cup\mathcal{V}) \longrightarrow H_{p-1}(\mathcal{U}\cap\mathcal{V})$, the connecting homomorphism in the Mayer-Vietoris for singular homology. We have $\partial_*([e]) = [c]$, provided there exist $d \in C_p^\infty(\mathcal{U})$ and $d' \in C_p^\infty(\mathcal{V})$ such that $[k_\sharp d + l_\sharp d'] = [e]$ and $(i_\sharp c - j_\sharp c) = (\partial d, \partial d')$. Then

$$\partial^*(\Psi^*|_{\mathcal{U}\cap\mathcal{V}}([\omega]_{\mathrm{dR}}))([e]) = \Psi^*|_{\mathcal{U}\cap\mathcal{V}}([\omega]_{\mathrm{dR}})([e]) = \Psi^*|_{\mathcal{U}\cap\mathcal{V}}([\omega]_{\mathrm{dR}})([c]) = \int_c \omega,$$

$$\Psi^*(\Delta([\omega]_{\mathrm{dR}}))([e]) = \int_e \sigma, \text{ where } \sigma \text{ is a smooth } p\text{-form representing } \Delta([\omega]_{\mathrm{dR}}).$$

Thus, it is only left to show that $\int_e \sigma = \int_c \omega$. By Corollary 2.2.1, we can choose $\sigma = d\eta$ (extended by zero to all of $\mathcal{U}\cup\mathcal{V}$), where $\eta \in \Omega^{p-1}(\mathcal{U})$ and $\eta' \in \Omega^{p-1}(\mathcal{V})$ are smooth forms such that $\omega = \eta|_{\mathcal{U}\cap\mathcal{V}} - \eta'|_{\mathcal{U}\cap\mathcal{V}}$. On the other hand, $c = \partial d$, where d and d' are smooth p-chains in $\mathcal{U}$ and $\mathcal{V}$, respectively, such that $d + d'$ represents the same homology class of e. Since $\partial d + \partial d' = \partial e = 0$ and $d\eta|_{\mathcal{U}\cap\mathcal{V}} - d\eta'|_{\mathcal{U}\cap\mathcal{V}} = d\omega = 0$, we have

$$\begin{aligned}
\int_c \omega = \int_{\partial d} \omega = \int_{\partial d} \eta - \int_{\partial d} \eta' &= \int_{\partial d} \eta - \int_{-\partial d'} \eta' \\
&= \int_{\partial d} \eta + \int_{\partial d'} \eta' = \int_d d\eta + \int_{d'} d\eta' \\
&= \int_d d\eta + \int_{d'} d\eta = \int_d \sigma + \int_{d'} \sigma \\
&= \int_e \sigma.
\end{aligned}$$

Therefore, $\Psi^* \circ \Delta = \partial^* \circ \Psi^*|_{\mathcal{U}\cap\mathcal{V}}$.

We have that the previous diagram commutes. Moreover, $(\Psi^*_{\mathcal{U}}, \Psi^*_{\mathcal{V}})$ and $\Psi^*_{\mathcal{U}\cap\mathcal{V}}$ are isomorphisms. It follows by the Five Lemma that Ψ^* is an isomorphism.

Now assume the statement is true for smooth manifolds admitting a de Rham cover with $k \geq 2$ sets, and suppose that $\{\mathcal{U}_1, \ldots, \mathcal{U}_{k+1}\}$ is a de Rham cover of X. Put

$$\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k \quad \text{and} \quad \mathcal{V} = \mathcal{U}_{k+1}.$$

We have that $\mathcal{U}$ and $\mathcal{V}$ are de Rham manifolds. Note that

$$\mathcal{U}\cap\mathcal{V} = (\mathcal{U}_1 \cap \mathcal{U}_{k+1}) \cup \cdots \cup (\mathcal{U}_k \cap \mathcal{U}_{k+1}),$$

where each $\mathcal{U}_i \cap \mathcal{U}_{k+1}$ is de Rham and every finite intersection of $\mathcal{U}_i \cap \mathcal{U}_{k+1}$'s is de Rham. It follows by induction hypothesis that $\mathcal{U}\cap\mathcal{V}$ is de Rham. Applying the same argument above, we obtain $X = \mathcal{U}\cup\mathcal{V}$ is de Rham. $\square$

In Case 4 we shall prove that every smooth manifold having a de Rham basis is de Rham. Before giving a proof of this fact, we need to show the existence of exhaustion functions for any smooth manifold. If X is a smooth manifold, an **exhaustion function** for X is a continuous function $f : X \longrightarrow \mathbb{R}$ with the property that the set $f^{-1}((-\infty, c])$ is compact, for every $c \in \mathbb{R}$. A subset K of X is said to be **precompact** in X if $\overline{K}$ is compact in X.

Lemma 2.5.1. Every smooth manifold has a countable basis of precompact coordinate balls.

Proof: Let X be an n-dimensional smooth manifold. First, suppose that X is covered by a single chart. Let $\varphi : X \longrightarrow \varphi(X) \subseteq \mathbb{R}^n$ be a global coordinate map, and let $\mathcal{B}$ be the collection of all open balls $B(x, r) \subseteq \mathbb{R}^n$, where r is rational, x has rational coordinates and $\overline{B(x, r)} \subseteq \varphi(X)$. Clearly each $B(x, r)$ is precompact in $\varphi(X)$, and that $\mathcal{B}$ is a countable basis for the relative topology of $\varphi(X)$. Since $\varphi(X)$ is a homeomorphism, we have that $\varphi^{-1} : \varphi(X) \longrightarrow X$ is continuous and so the collection $\{\varphi^{-1}(B(x, r)) : B(x, r) \in \mathcal{B}\}$ is a countable basis for X. Also, $\overline{\varphi^{-1}(B(x, r))} = \varphi^{-1}(\overline{B(x, r)})$ is compact in X. Hence each $\varphi^{-1}(B(x, r))$ is a precompact coordinate ball.

Now suppose that X is an arbitrary n-manifold. Let $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}$ be an atlas of X. Since X is second countable, X is covered by countable many charts $\{(\mathcal{U}_i, \varphi_i)\}$. By the argument of the previous case, each $\mathcal{U}_i$ has a countable basis of precompact coordinate balls, and the union of all these countable bases is a countable basis for the topology of X. We have to show that each $V \subseteq \mathcal{U}_i$ is precompact in X, where V is a precompact ball of the basis of $\mathcal{U}_i$. Denote $\overline{V}_{\mathcal{U}_i}$ the closure of V with respect to $\mathcal{U}_i$. We have that $\overline{V}_{\mathcal{U}_i}$ is compact in $\mathcal{U}_i$ and then closed in X. So we have that $\overline{V} = \overline{V}_{\mathcal{U}_i}$ and V is precompact in X. □

Proposition 2.5.1. Every smooth manifold admits a smooth positive exhaustion function.

Proof: By the previous lemma, X has a countable cover $\{V_i\}$ of precompact open sets. Let $\{f_i\}$ be a partition of unity subordinate to the covering $\{V_i\}$. Define $f : X \longrightarrow \mathbb{R}$ by $f(x) = \sum_{i=1}^{\infty} i \cdot f_i(x)$. The function f is smooth because only finitely many terms are non-zero in a neighbourhood of any point. It is also positive since $f(x) \geq \sum_{i=1}^{\infty} f_i(x) = 1$. Let N be a positive integer, we prove $f(x) \leq N \Longrightarrow x \in \bigcup_{j=1}^{N} \overline{V}_j$. Suppose $x \notin \bigcup_{j=1}^{N} \overline{V}_j$. Then $f_j(x) = 0$ if $1 \leq j \leq N$ since $\operatorname{supp}(f_j) \subseteq \overline{V}_j$. Thus

$$f(x) = \sum_{j=N+1}^{\infty} j \cdot f_j(x) > \sum_{j=N+1}^{\infty} N \cdot f_j(x) = N \cdot \sum_{j=N+1}^{\infty} f_j(x),$$

so $f(x) > N$. Let $c \in \mathbb{R}$. There exists $N \in \mathbb{N}$ such that $c \leq N$. Then

$$f^{-1}((-\infty, c]) \subseteq f^{-1}((-\infty, N]) \subseteq \bigcup_{j=1}^{N} \overline{V}_j$$

and hence $f^{-1}((-\infty, c])$ is a closed subset of the compact set $\bigcup_{j=1}^{N} \overline{V}_j$. It follows $f^{-1}((-\infty, c])$ is compact. $\square$

Case 4. If X has a de Rham basis, then X is de Rham.

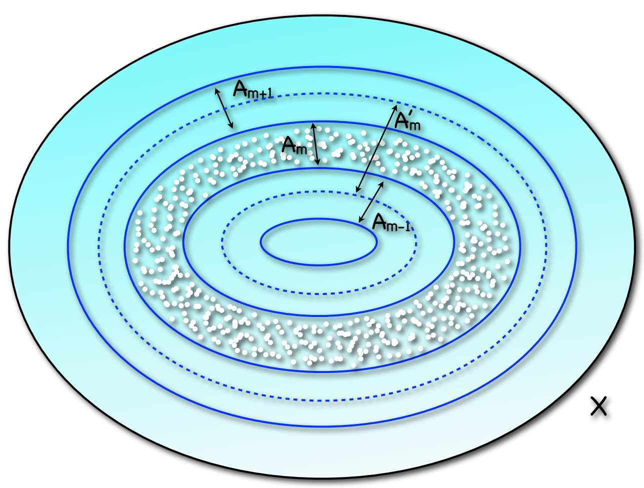
Proof: Let $f : X \longrightarrow \mathbb{R}$ be an exhaustion function. For each $m \in \mathbb{N}$ the set $f^{-1}((-\infty, m])$ is compact in X. On the other hand, the set

$$A_m = \{x \in X : m \leq f(x) \leq m+1\}$$

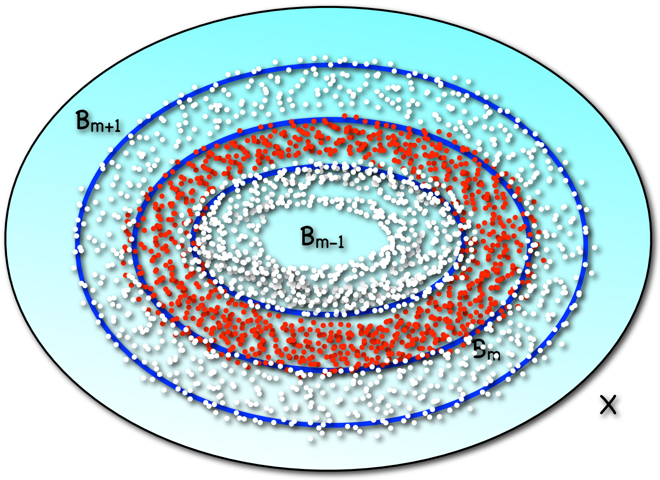
is a closed subset of the compact set $f^{-1}((-\infty, m+1])$, and hence it is compact. The set

$$A'_m = \left\{x \in X : m - \frac{1}{2} < f(x) < m + \frac{3}{2}\right\}$$

is an open set containing the set A_m, then for each $x \in A_m$ there exists a de Rham basis element $\mathcal{U}_m^x$ such that $x \in \mathcal{U}_m^x \subseteq A'_m$.



The collection $\{\mathcal{U}_m^x\}$ forms an open cover of A_m. Since A_m is compact, there is a finite subcover of A_m. Let $B_m = \bigcup_{i=1}^n \mathcal{U}_m^i$, then $\{\mathcal{U}_m^i\}$ is a finite de Rham cover of B_m. It follows by Case 3 that B_m is de Rham. Note that $B_m \subseteq A'_m$, so $B_m \cap B_n \neq \emptyset$ iff $n = m-1, m, m+1$.



Hence $\mathcal{U} = \cup_{m \text{ odd}} B_m$ is the disjoint union of de Rham sets. It follows by Case 1 that $\mathcal{U}$ is de Rham. Similarly, $\mathcal{V} = \cup_{n \text{ even}} B_n$ is also de Rham. Also, $X = \mathcal{U} \cup \mathcal{V}$. It is only left to show that $\mathcal{U} \cap \mathcal{V}$ is de Rham in order to prove that $\{\mathcal{U}, \mathcal{V}\}$ is a finite de Rham cover of X. We have

$$\begin{aligned} \mathcal{U} \cap \mathcal{V} &= \left(\bigcup_{m \text{ odd}} B_m \right) \bigcap \left(\bigcup_{n \text{ even}} B_n \right) = \bigcup B_m \cap B_n \\ &= \left(\bigcup_{m \text{ odd}} B_m \cap B_{m-1} \right) \bigcup \left(\bigcup_{m \text{ odd}} B_m \cap B_{m+1} \right). \end{aligned}$$

This is a disjoint union. In fact,

(i) m and n are odd: If $(B_m \cap B_{m-1}) \cap (B_n \cap B_{n-1}) \neq \emptyset$ then $n = m, m-1, m+1$. If $n = m-1$ then $\emptyset \neq (B_m \cap B_{m-1}) \cap (B_{m-1} \cap B_{m-2}) = \emptyset$. We get a similar contradiction if $n = m+1$. Then $n = m$.

(ii) Similarly, $(B_m \cap B_{m+1}) \cap (B_n \cap B_{n+1}) \neq \emptyset \Longrightarrow n = m$.

(iii) If $(B_m \cap B_{m-1}) \cap (B_n \cap B_{n+1}) \neq \emptyset$ then $B_{m-1} \cap B_{n+1} \neq \emptyset$. It follows $n+1 = m-1, m, m-2$.

   (a) If $n+1 = m-1$ then $\emptyset \neq (B_m \cap B_{m-1}) \cap (B_{m-2} \cap B_{m-1}) = \emptyset$.

   (b) The case $n+1 = m$ is not possible since n and m are odd.

(c) If $n+1 = m-2$ then $\emptyset \neq (B_m \cap B_{m-1}) \cap (B_{m-3} \cap B_{m-2}) = \emptyset$.

Hence $(B_m \cap B_{m-1}) \cap (B_n \cap B_{n+1}) = \emptyset$.

So we have that $\mathcal{U} \cap \mathcal{V}$ is the disjoint union of sets of the form $B_m \cap B_{m-1}$ and $B_m \cap B_{m+1}$, with m odd. Note that

$$B_m \cap B_{m-1} = \left(\bigcup_{j=1}^{p} \mathcal{U}_m^j \right) \bigcap \left(\bigcup_{i=1}^{q} \mathcal{U}_{m-1}^i \right) = \bigcup_{i,j} (\mathcal{U}_m^j \cap \mathcal{U}_{m-1}^i),$$

where each $\mathcal{U}_m^j \cap \mathcal{U}_{m-1}^i$ and each finite intersection of the $\mathcal{U}_m^j \cap \mathcal{U}_{m-1}^i$'s is de Rham since $\{U_\alpha\}$ is a de Rham basis. So $\{\mathcal{U}_m^j \cap \mathcal{U}_{m-1}^i\}$ is a finite de Rham cover of $B_m \cap B_{m-1}$. It follows by Case 3 that $B_m \cap B_{m-1}$ is de Rham. Similarly, $B_m \cap B_{m+1}$ is de Rham. Hence, $\mathcal{U} \cap \mathcal{V}$ is the disjoint union of de Rham sets, then $\mathcal{U} \cap \mathcal{V}$ is de Rham by Case 1. □

Case 5. Any open subset of $\mathbb{R}^n$ is de Rham.

Proof: Let $\mathcal{U}$ be an open set of $\mathbb{R}^n$. Then $\mathcal{U}$ has a cover of open balls. Since each open ball is convex, it follows by Case 2 that it is de Rham. Also, every finite intersection of open balls is convex and so de Rham. Hence $\mathcal{U}$ has a de Rham basis of open balls. By Case 4, $\mathcal{U}$ is de Rham. □

Now we are ready to proof the de Rham's Theorem

Proof of de Rham's Theorem". Let $\mathcal{A} = \{(\mathcal{U}_\alpha, \varphi_\alpha)\}$ be an atlas of X. Each $\mathcal{U}_\alpha$ is diffeomorphic to an open set $\mathcal{V}_\alpha$ of $\mathbb{R}^n$, which is de Rham by Case 5. It follows by Corollary 2.4.1 that $\mathcal{U}_\alpha$ is de Rham. By a similar argument one can prove that each finite intersection $\mathcal{U}_{\alpha_1} \cap \cdots \cap \mathcal{U}_{\alpha_n}$ is de Rham. Hence X has a de Rham basis and so it is de Rham, by Case 4. □

The proof we just gave here is due to Glen E. Bredon, and it dates to 1962 when he gave it in a course on Lie groups. In fact, Bredon proved a more general result, namely

Lemma 2.5.2 (Bredon)**.** Let X be a smooth n-dimensional manifold. Suppose that $P(\mathcal{U})$ is a statement about open subsets of X, satisfying the following three properties:

(1) $P(\mathcal{U})$ is true for $\mathcal{U}$ diffeomorphic to an open convex subset of $\mathbb{R}^n$;

(2) $P(\mathcal{U})$, $P(\mathcal{V})$ and $P(\mathcal{U}\cap\mathcal{V}) \implies P(\mathcal{U}\cup\mathcal{V})$; and

(3) $\{U_\alpha\}$ disjoint, and $P(\mathcal{U}_\alpha)$ for all $\alpha \implies P(\cup_\alpha \mathcal{U}_\alpha)$.

Then $P(X)$ is true.

Note that the proof given in these notes is a particular case of this lemma, putting

$$P(\mathcal{U}) = \ \Psi^* : H^p_{\mathrm{dR}}(\mathcal{U}) \longrightarrow H^p_\infty(\mathcal{U};\mathbb{R}) \text{ is an isomorphism.}$$

In fact, (1), (2) and (3) are Case 2, Case 3 and Case 1, respectively. The previous lemma can be proven using the same argument of the proof given in Case 4.

We have proven the de Rham's Theorem for smooth singular cohomology. We would like to replace that by ordinary singular cohomology. The inclusion $C^\infty_p(X) \longrightarrow C_p(X)$ induces, via $\mathrm{Hom}(\cdot,\mathbb{R})$, the cochain map $\mathrm{Hom}(C_p(X),\mathbb{R}) \longrightarrow \mathrm{Hom}(C^\infty_p(X),\mathbb{R})$. Exactly the same proof as the above use of Lemma 2.5.2 shows that the induced map $H^p(X;\mathbb{R}) \longrightarrow H^p_\infty(X;\mathbb{R})$ is an isomorphism for all smooth manifolds X.

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