6. *Recall the definition of the connected sum of two connected smooth manifolds M_1, M_2 of equal dimension n: pick points $p_i \in M_i$, and coordinate charts $f_i : U_i \to \mathbb{R}^n$ such that $f_i(p_i) = 0$, and let $M_1\#M_2$ be the quotient of the disjoint union $M_1\backslash\{p_1\} \sqcup M_2\backslash\{p_2\}$ by the equivalence relation $x_1 \sim x_2$ if $x_i \in U_i$ and $f_1(x_1) = \frac{f_2(x_2)}{|f_2(x_2)|^2}$.*

   (a) *Show that replacing the charts f_i in the definition with different charts f_i' around p_i gives the same manifold $M_1\#M_2$ up to diffeomorphism, provided that the transition functions $f_i' \circ f_i^{-1}$ are orientation-preserving.*

   WLOG $f_2' = f_2$. Suppose we have a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ which equals $f_1' \circ f_1^{-1}$ on $B(r)$ and id outside $B(R)$ (where $B(r) \subset \mathbb{R}^n$ is ball of radius r centred at origin). Then we can define a diffeomorphism $g : M_1\#M_2 \to M_1\#'M_2$ by imposing:

   - $g = id$ on $M_1 \setminus f_1^{-1}(B(R))$
   - $f_1 \circ g = \phi \circ f_1$ on $U_1 \setminus \{p_1\}$
   - $g = id$ on $M_2 \setminus f_2^{-1}(B(1/r))$

   These three sets cover $M_1\#M_2$, and by the condition on ϕ the definition of g agrees on the overlaps.

   For r_0 sufficiently small, $f_1' \circ f_1^{-1}$ restricts to a smooth map $h : B(r_0) \to \mathbb{R}^n$, with D_0h invertible. Let $\rho : [0, \infty) \to [0, 1]$ be a smooth function that equals 1 on $[0, 1]$, and 0 on $[2, \infty)$. For $m > 0$ small, the first derivatives of the function $x \mapsto \rho(m|x|)\,(h(x) - D_0h(x))$ can be bounded in terms of m and the Hessian of h at 0. Q13b implies that for m sufficiently small, $\phi(x) = D_0h(x) + \rho(m|x|)\,(h(x) - D_0h(x))$ is a diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$. It equals h on $B(mr_0)$, and D_0h on $B(2mr_0)$.

   We can use Q13b again together with the fact that $GL_+(n, \mathbb{R})$ is generated by a neighbourhood of id to show that for any $A \in GL_+(n, \mathbb{R})$ there is a diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ which equals A on $B(2mr_0)$ and id outside $B(R)$ for some large R.

   (b) *Show that up to diffeomorphism, $M_1\#M_2$ is independent of the choice of points p_i. To be precise, show that if M_i are both orientable then the diffeomorphism type of $M_1\#M_2$ depends only on the orientations of the charts f_i, so that given orientations on M_i we can define $M_1\#M_2$ uniquely (including an orientation); if either of M_i are non-orientable then $M_1\#M_2$ is independent of all the choices in the definition.*

   WLOG $p_2' = p_2$. By the previous part, it does not matter what chart f_1' around p_1' we use in the definition of the connected sum. If $p_1' \in U_1$ then we can easily define a chart $f_1' : U_1 \to \mathbb{R}^n$ to equal f_1 outside a compact set $K \subset U_1$ while mapping p_1' to 0. Then we can define a diffeomorphism $M_1\#M_2 \to M_1\#'M_2$ by requiring it to be id on $M_1\backslash K$, $(f_1')^{-1} \circ f_1$ on U_1 and id on $M_2\backslash\{p_2\}$.

   If M_1 is oriented then a pair of charts can be connected by a sequence of overlapping charts with orientation-preserving transition functions iff they induce the same orientation; if M_1 is non-orientable then any pair of charts can be connected this way (cf. Q4).

7. (a) *Let M_1, M_2 be smooth manifolds with boundary, and $f : \partial M_1 \to \partial M_2$ a diffeomorphism. Define a natural smooth structure on $M_1 \cup_f M_2 = M_1 \sqcup M_2/\sim$, where $x_1 \sim x_2$ if $x_i \in \partial M_i$ and $f(x_1) = x_2$.*

   This problem was not very well designed, since in general it is best dealt with using the collar neighbourhood theorem, which I have not yet covered in lectures: For any manifold M with boundary, ∂M has a neighbourhood diffeomorphic to $\partial M \times [0, \epsilon)$.

   (b) *Let $f : S^{n-1} \to S^{n-1}$ be a diffeomorphism. Show that $B^n \cup_f B^n$ is homeomorphic to S^n. We call this a* twisted sphere. *Show that if it is not diffeomorphic to S^n, then f is not smoothly isotopic to $id_{S^{n-1}}$.*

   Define a homeomorphism $B^n \cup_{id_{S^{n-1}}} B^n \to B^n \cup_f B^n$ as the identity on the first half, and $x \mapsto |x|f(\frac{x}{|x|})$ on the second half (this is not smooth at the origin of the second half).

   If $S^{n-1} \times [0, 1] \to S^{n-1},\ (x, t) \mapsto F_t(x)$ is an isotopy with $F_t = id$ for t in a neighbourhood of 0 and $F_1 = f$ then we can define a diffeomorphism $B^n \cup_{id_{S^{n-1}}} B^n \to B^n \cup_f B^n$ as the identity on the first half, and $x \mapsto |x|F_{|x|}(\frac{x}{|x|})$ on the second half.