Ph.D. course *Hecke algebras*

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Exercises

Unless otherwise specified we denote by (W, S) a Coxeter system and by $\mathcal{H}$ the corresponding Hecke algebra.

Exercise 1. Let G be a group and R be a commutative ring with 1. Recall the definition of the group algebra RG and the identification of RG as a subset of R^G, which is the set of all functions $G \to R$. Prove that the multiplication in RG corresponds in R^G to the convolution defined as

$$(f_1 * f_2)(g) = \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

with $f_i \in RG$ for all i.

Exercise 2. Let $\mathbb{F}$ be any field, let B be the set of upper triangular invertible 2×2 matrices with coefficients in $\mathbb{F}$, and set

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Prove that

$$\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \beta \right) \mapsto \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot s \cdot \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

is a bijection $B \times \mathbb{F} \to BsB$.

Exercise 3. Let (W, S) be a Coxeter system with graph of type A_{n-1}. Prove that W is isomorphic to the symmetric group S_n if $n \in \{2, 3\}$.

Exercise 4. Let n be a positive integer and choose two numbers $i < j \in \{1, \dots, n\}$. Find an expression of the transposition $(i \quad j)$ as a product of transpositions of the form $(k \quad k+1)$ with exactly $2(j-i)-1$ factors.

Exercise 5. Prove that any dihedral group is a Coxeter group of rank 2 with generators $S = \{\sigma, \sigma\rho\}$ where σ is a reflection and ρ is a rotation.

Exercise 6. Prove the Proposition on page 16 of the handwritten notes.

Exercise 7. Consider a dihedral group presented as a Coxeter group of rank 2. Prove that the geometric representation can be defined in such a way that $V = \mathbb{R}^2$ and B is the standard scalar product.

Exercise 8. Prove that σ_s preserves B for any $s \in S$.

Exercise 9. Let (W, S) be a Coxeter system with graph of type A_{n-1}. Prove that W is isomorphic to the symmetric group S_n for any $n \in \mathbb{Z}_{\geq 1}$. (*Hint: use the geometric representation and identify V with the hyperplane of $\mathbb{R}^n$ where the sum of all coordinates vanishes.*)

Exercise 10. Let $I \subseteq S$, and for any $w \in W_I$ define $\ell_I(w)$ to be the minimal length of an expression of w only involving generators in I.

(1) Use the exchange property to prove that $\ell_I(w) = \ell(w)$ for all $w \in W_I$.
(2) Prove that W_I is a normal subgroup if and only if $st = ts$ for all $s \in I$ and all $t \in S \setminus I$.

Exercise 11. (1) Verify that the definition of the Bruhat order does define an order on W.
(2) Give an alternative definition of the Bruhat order using tw (instead of wt) with $w \in W$ and $t \in T$. Prove that this defines the same order.
(3) Prove that $v \leq w$ if and only if $v^{-1} \leq w^{-1}$, for any $v, w \in W$.

Exercise 12. Draw the Hasse diagram of $I_2(m)$ for any m, and prove that in this case $v < w$ if and only if $\ell(v) < \ell(w)$, for any $v, w \in W$.

Exercise 13. Let $v, w \in W$ with $v < w$, and suppose there is no x such that $v < x < w$. Prove that $\ell(v)$ and $\ell(w)$ do not have the same parity.

Exercise 14. Let W be an affine reflection group on the euclidean space E.

(1) Prove that all W-orbits in E are discrete subsets.
(2) Prove that

$$\bigcup_{H \in \Phi} H$$

is a closed subset of E.

Exercise 15. Let A be a commutative ring with 1 and let $\mathcal{E}$ be the "generic" algebra defined by (W, S) and a choice of coefficients $a_s, b_s \in A$ for all $s \in S$. Prove that

$$T_w \mapsto T_{w^{-1}}$$

extends to a homomorphism $\mathcal{E} \to \mathcal{E}$ of A-modules that is an anti-automorphism of A-algebras.

Exercise 16. Prove that $T_s \mapsto -1$ extends to a homomorphism of $\mathbb{Z}[q, q^{-1}]$-modules $\mathcal{H} \to \mathbb{Z}[q, q^{-1}]$ sending T_w to $(-1)^{\ell(w)}$.

Exercise 17. Compute $R_{x,w}$ for all $x \leq w$ in the case $W = S_3$ (with the usual presentation of type A_2).

Exercise 18. Define a ring homomorphism $\sigma\colon \mathcal{H} \to \mathcal{H}$ similarly as ι, by setting $\sigma(q) = q^{-1}$ and

$$\sigma(T_w) = \varepsilon(w) q^{-\ell(w)} T_w.$$

Prove that this defines an involution $\sigma\colon \mathcal{H} \to \mathcal{H}$ that commutes with ι.

Exercise 19. Verify that C_e, C_s and $C_{st} = C_s C_t$ for all $s, t \in S$ satisfy the theorem about the existence and uniqueness of the elements C_w.

Exercise 20. Assuming $W = S_3$ with the usual presentation, prove that $P_{x,w} = 1$ for all $x \leq w$.

Exercise 21. With any W, prove that $R_{x,w} = q - 1$ and $P_{x,w} = 1$, assuming $x \leq w$ and $\ell(w) = \ell(x) + 1$.

Exercise 22. Using formula (*) of page 64 of the handwritten notes, prove that $P_{x,w}(0) = 1$ for all $x \leq w$, and that $P_{x,w}$ for all $x \leq w$ with $\ell(w) - \ell(x) \leq 2$.