Recap

(W, S) Coxeter system

$$\mathcal{H} = \mathcal{H}(W,S) \quad \text{Hecke algebra}$$

$$= \bigoplus_{x \in W} \mathbb{Z}[q^{\pm 1}] T_x \qquad \{T_x\}_{x\in W} \quad \text{"std basis"}$$

$$M = (m_{xw})_{x,w \in W}$$

$$= \bigoplus_{w \in W} \mathbb{Z}[q^{\pm 1}] C_w \qquad \{C_w\}_{w \in W} \quad \text{"KL-basis"} \ (\text{KL}, 1979)$$

M is the base change matrix (after extending scalars to $\mathbb{Z}[q^{\pm 1/2}]$) and it is morally given by KL-polynomials:

$$m_{xw} = \varepsilon_w q_w^{1/2} \varepsilon_x q_x^{-1} P_{xw}(q^{-1})$$

Recall that C_w is uniquely determined by

a) $\iota(C_w) = C_w$ (self-duality)

b) $C_w = \sum_{x \in W} m_{xw} T_x$,

where m_{xw} is given as above and $P_{xw}(q) \in \mathbb{Z}[q]$

$P_{ww}(q) = 1$, $\deg P_{xw} \leq \frac{1}{2}\left(\ell(w) - \ell(x) - 1\right)$ $(x < w)$

$P_{xw} = 0$ if $x \nleq w$

Recall also that while proving existence Guido showed that

$$P_{xw} = q^{1-c} P_{sxv} + q^{c} P_{x,v} - \sum_{\substack{z \prec v \\ sz < z}} \mu(z,w)\, q_z^{-1/2} q_w^{1/2} P_{x,z} \qquad (*)$$

where:
- $v = sw$
- $z \prec v$ means that $P_{zv} = \mu(z,v)\, q^{\frac{1}{2}(\ell(v)-\ell(z)-1)}$ + smaller degree terms
- $c = \begin{cases} 0 & x < sx \\ 1 & x > sx \end{cases}$

<u>Example computation KL-polys</u>: let W = dihedral group $I_2(m)$

$\rightsquigarrow S = \{s,t\}$

relations: $s^2 = t^2 = (st)^m = 1$

<u>Claim</u> $P_{xw} = 1 \quad \forall\, x \leq w$ in W

Observe that $W = \{e, s, t, st, ts, \ldots, \underbrace{stst\ldots}_{m-1}, \underbrace{tsts\ldots}_{m-1}, \underbrace{\overset{\overset{tsts\ldots}{\shortparallel}}{stst\ldots}}_{m}\}$

$\rightsquigarrow$ all elements but the longest one have only a reduced expression and for any $w \in W$

$$\{x \in W \mid x \leq w\} = \{x \in W \mid \ell(x) < \ell(w)\} \cup \{w\}$$

We'll show our claim by induction on $\ell(w)$

If $\ell(w)=0$, then $w=e$ and claim verified.

Also if $x=w$, $P_{x,w}=1$, so that we can assume $w>x\geq e$.

Since $\ell(w)>0$, $a\in\{s,t\}$ s.t. $sw<w$. WLOG: $\boxed{a=s}$

Denote $v:=sw$

Observe that if $z<v$ then $\ell(v)-\ell(z)<\ell(w)-\ell(x)$

and we can assume by induction that $P_{z,v}=1$, so that

$z\prec v$ iff $0=\frac{1}{2}\left(\ell(v)-\ell(z)-1\right) \Rightarrow \ell(z)=\ell(w)-2=\ell(v)-1.$

So that the summation has in fact at most one term: the unique

$z<w$ s.t. $\ell(z)=\ell(w)-2$ and $sz<z$.

If $sx<x$, then by ⊛ we have

$$P_{x,w}=P_{sx,v}+qP_{x,v}-1\cdot q^{\frac{\ell(w)-\ell(z)}{2}}P_{xz}\underset{\text{by ind.}}{=}1+q-q=1$$

If $\boxed{sx > x}$ we distinguish two cases $\rightarrow x = v$, $\rightarrow x < v$

(Since $x \leq w$, $v = ws$ and $sx > x \Rightarrow x \leq v$)

$\boxed{x = v}$ then in this case $sx = w$ and $w \nleq v$, by ind

$$P_{x,w} = q \underbrace{P_{sx,v}}_{= 0} + P_{xv} - 1 \cdot q^{\frac{\ell(w)-\ell(z)}{2}} \underbrace{P_{xz}}_{= 0} = 0 + 1 + 0 = 1$$

$\rightarrow x = v \nleq z$

$\boxed{x < v}$ then two cases: $\Rightarrow sx < v$

$\hookrightarrow \ell(sx) = \ell(v)$ but $sx \neq v$ (since $s(sx) < sx$, $sv > v$)

$\boxed{sx < v}$ Then $\ell(x) \leq \ell(v) - 2$ and we find the z

$$P_{xw} = q \cdot 1 + 1 - q = 1$$

$\boxed{\ell(sx) = \ell(v)}$ Then $P_{sx,v} = 0$ and $\nexists z$ between x and v s.t. $\ell(z) = \ell(v) - 1$.

$$P_{xw} = q \cdot 0 + 1 = 1.$$

Looking at the dihedral example
+ Guido's exercises ($P_{x w}=1$ $\forall x \leq w$ s.t. $\ell(w)-2 \leq \ell(x) \leq \ell(w)$)

we are tempted to conjecture that KL-polys are always 1.

This is false and thanks to the failure of our conjecture

we have interesting stuff in representation theory

and geometry!

In fact, if you're patient enough you can compute

Exercise $W = S_4$ $\quad S = \{\underset{s_1}{(12)}, \underset{s_2}{(23)}, \underset{s_3}{(34)}\}$. Then $P_{s_2, s_2 s_1 s_3 s_2} = 1+q$

In fact, we have this surprising result

Theorem (Polo 1999, Caselli 2004)
geometry, alg. comb.

Let $P \in \mathbb{Z}_{\geq 0}[q]$ be such that $P(0)=1$.

Then $\exists$ $N \geq 2$, $x, y \in S_N$ s.t. $P = P_{x,y}$

$\rightsquigarrow$ Every poly with non-negative coefficients and constant term $=1$ is a KL-poly!

Relevance of KL-polys

Since their birth KL-polys turned out to be relevant in several areas.

In particular, already in [KL79] connections to representation theory (of Weyl groups, of ss cplx Lie alg's) to geometry (of flag varieties) to algebraic combinatorics were (sometimes conjecturally) discussed.

REP. THEORY:
- Repn's of Hecke alg's can be constructed by using KL-polys
- If $\mathfrak{g}$ is a ss cplx Lie alg, W its Weyl gp, S simple reflections ($\leftrightarrow$ simple roots) then KL-polys "control" multiplicities of simple modules in Jordan-Hölder filtration of Verma [Conjectured in 1979, was proven shortly after by geometric methods by Beilinson-Bernstein & Brylinski-Kashiwara. Algebraic proof is consequence of a more recent (~2014) paper by Elias-Williamson]

From [KL79] "Our polynomials P_{yw} appear to be very closely related with the structure of singularities of Schubert varieties. More precisely P_{yw} can be regarded as a measure for the failure of local Poincaré duality on the Schubert variety $\mathcal{B}_w$ in a nbh of a point in $\mathcal{B}_y$"

- <u>GEOMETRY</u> : KL-polynomials are Poincaré polynomials for graded rings coming from geometry. More precisely, they compute the local intersection cohomology of Schubert varieties of a flag variety G/B, where G is a reductive group whose Weyl group is W (and B Borel)

- <u>COMBINATORICS</u> : many interesting combinatorial features!! I'll state here two conjectures:

① $P_{xw} \in \mathbb{Z}_{\geq 0}[q] \ \forall x, w \in W$ (for any W)

[stated already by KL in 1979, was known to hold for Weyl groups given the geometric interpretation. Proven for any Coxeter gp by Elias-Williamson]

② $P_{x,w}$ only depends on Bruhat interval $[x,w]$

[STILL OPEN! F. Brenti & collaborators worked a lot on it. Next Friday: talk at ARTS on related stuff]

Graphically:

"principal block of cat. $\mathcal{O}$"

category of semisimple perverse sheaves on G/B

$$\begin{tikzcd}
\mathrm{rep}_{\mathbb{C}}(\mathfrak{g}) \supset \mathcal{O}_0 \arrow[dr, dashed, "①"'] & & \mathrm{Perv}(G/B) \subset D^b(G/B) \arrow[dl, dashed, "②"] \\
& \mathcal{H}(W,S) &
\end{tikzcd}$$

"decategorification"

(morally: taking Groth grps get Hecke alg)

combinatorial world

① Grothendieck group of $\mathcal{O}_0$ has two bases: $\{M_x\}_{x\in W}$ "Verma mods"

$\{L_w\}_{w\in W}$ "simple mods"

[where $M_x = M(x\cdot 0)$, $x\cdot 0 = x(\rho)-\rho$, $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$]

KL-intuition: $M_x \dashrightarrow T_x$

$L_w \dashrightarrow C_w$

$\rightsquigarrow [M_x : L_w] = P_{xw}(1)$

In fact, we can refine this giving to any coeff of P_{xw} a representation theoretical meaning.

② We'll need to know a little more to be able to even state what's happening.

Vague idea of old pf of KL-conj. on mult. of simples in Vermas:

$$\text{rep. of } \mathfrak{g} \underset{\text{"localization"}}{\rightsquigarrow} \mathcal{D}_{G/B}\text{-mod} \underset{\text{"Riemann Hilbert corresp."}}{\rightsquigarrow} \mathrm{Perv}(G/B) \rightsquigarrow \mathcal{H}$$

"sheaves of diff ops on G/B"
$\sim$ alg. diff. equ's on G/B

$\mathbb{C}$-coeff's

$\uparrow$ passing through l-adic perverse sheaves ☹

Soergel 1990: New proof using "modules over coinvariants"

$\rightsquigarrow$ Functor $\mathbb{V} : \mathcal{O}_0 \longrightarrow \text{mod-}C \qquad C = \mathrm{End}(P_{w_0})$, $P_{w_0} \in \mathcal{O}_0$

Thm (Soergel) $\mathbb{V}(P_x) \overset{\text{Ⓐ}}{=} IH^\bullet\left(\overline{B^\vee x B^\vee}/B^\vee\right)$

<u>Good features:</u> • $\mathbb{V}(P_x)$ has "elementary" definition

• KL-conj. can be reformulated algebraically

$$\begin{tikzcd} \mathcal{O}_0 \arrow[dr, dashed] & \mathrm{Perv}(G/B) \arrow[d, dashed] & \text{mod-}C \arrow[dl, dashed] \\ & \mathcal{H} & \end{tikzcd}$$