Dirac operators in the affine setting

Paolo Papi
(joint work with Victor G. Kac, Pierluigi Möseneder Frajria)

1. Introduction

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be an infinitesimal symmetric space. The adjoint representation gives a map $\mathfrak{k} \to so(\mathfrak{p})$ and in turn we have a map between the corresponding affinizations $\widehat{\mathfrak{k}} \to \widehat{so(\mathfrak{p})}$. Therefore, given a $\widehat{so(\mathfrak{p})}$-module, it makes sense to ask for its $\widehat{\mathfrak{k}}$-decomposition. Kac and Peterson [6] discovered that this decomposition is finite for level 1 modules. Recall that a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is called *quadratic* if the restriction to $\mathfrak{a}$ of a non-degenerate invariant form on $\mathfrak{g}$ is still non-degenerate. About 20 years ago there was much activity on the problem of classifying the quadratic subalgebras $\mathfrak{a}$ such that level 1 $\widehat{so(\mathfrak{p})}$-modules restrict finitely to $\widehat{\mathfrak{a}}$ and on that of finding actual decompositions. The first goal was achieved, and the above subalgebras might be split into three classes: certain equal rank subalgebras, a list of "exceptional" cases, and the symmetric subalgebras. Decompositions were known for the first two classes and in some instances of the third. Recently, we found a connection with the theory of abelian ideals in Borel subalgebras which allowed us to solve completely the problem (cf. [1]). It turns out that an affine analogue of Kostant's theory of multiplets [9] is the natural framework for a conceptual explanation of our formulas. This has been achieved by letting the Kac-Todorov field [7] play the role of Kostant cubic Dirac operator. We also found an analogue in affine setting of a theorem of Huang and Pandžić [2] which solves a conjecture of Vogan on Dirac cohomology. This result (see Theorem 3.2) allows us to prove a general multiplet theorem (see Theorem 3.1). We plan to investigate the applications of our methods in the context of finite and affine Lie superalgebras. Though this project is still at early stage of development, the construction of the Dirac field can be extended (with careful modifications) to the superalgebra case. We give a concise outline of this construction in Section 2 and in Section 4 we point out some of its consequences, notably a uniform proof of Freudenthal strange formula (4.1) for Lie superalgebras. The main results in the affine setting appear in Section 3.

2. The Dirac field

Let $\mathfrak{g}$ be a basic classical Lie superalgebra and σ an elliptic automorphism of $\mathfrak{g}$ (i.e., diagonalizable with modulus 1 eigenvalues). Let $(\cdot,\cdot)$ be a non-degenerate invariant supersymmetric form and assume that it is σ-invariant. Set $\bar{\mathfrak{g}} = P\mathfrak{g}$, where P is the parity reversing functor. Consider the conformal algebra $R = (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes \bar{\mathfrak{g}}) \oplus \mathbb{C}K \oplus \mathbb{C}K'$ with λ-products

$$[a_\lambda b] = [a,b] + \lambda(a,b)K,\ [a_\lambda \bar{b}] = \overline{[a,b]},\ [\bar{a}_\lambda b] = p(b)\overline{[a,b]},\ [\bar{a}_\lambda \bar{b}] = (b,a)K',$$

K, K' being even central elements. Let $V(R)$ be the corresponding universal vertex algebra, and denote by $V^{k,1}(\mathfrak{g})$ its quotient by the ideal generated by $K - k|0\rangle$ and $K' - |0\rangle$. The relations are the same used in [4] for even variables.

Choose a homogeneous basis $\{x_i\}$ of $\mathfrak{g}$ and let $\{x^i\}$ be its dual basis. We assume that the Casimir operator of $\mathfrak{g}$ acts on $\mathfrak{g}$ as $2gI_{\mathfrak{g}}$. The element

$$G_{\mathfrak{g}} = \sum_i :x^i \overline{x}_i: - \frac{1}{3}\sum_{i,j} :\overline{[x^i, x_j]}\,\overline{x}^j \overline{x}_i: \in V^{k+g,1}(\mathfrak{g})$$

is called the Kac-Todorov operator. To enlighten how $G_{\mathfrak{g}}$ acts on representations, recall from [4] that the vertex algebra $V^{k+g,1}(\mathfrak{g})$ is isomorphic to $V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}})$, where the left factor is the universal affine vertex algebra of level k and the right factor is the universal fermionic vertex algebra. There is a natural notion of (σ-twisted) Spin-Weil module $SW^{\sigma}(\overline{\mathfrak{g}})$ for $F(\overline{\mathfrak{g}})$, hence given a σ-twisted module for $V^k(\mathfrak{g})$ (i.e., a representation M of the twisted affine superalgebra $\widehat{L}(\mathfrak{g}, \sigma)$), we may produce a $\sigma \otimes (-\sigma)$-twisted representation

$$X(M) = M \otimes SW^{-\sigma}(\overline{\mathfrak{g}})$$

of $V^{k+g,1}(\mathfrak{g})$. It turns out that $(\sigma \otimes (-\sigma))(G_{\mathfrak{g}}) = -G_{\mathfrak{g}}$, so that $Y^{X(M)}(G_{\mathfrak{g}}, z) = \sum_{n\in\mathbb{Z}} G_n^X z^{-n-\frac{3}{2}}$. Given a quadratic σ-stable subsuperalgebra $\mathfrak{a} \subset \mathfrak{g}$, we have an embedding $V^{k+1,g}(\mathfrak{a}) \subset V^{k+1,g}(\mathfrak{g})$, so that we may consider the field $G_{\mathfrak{g}} - G_{\mathfrak{a}}$, which turns out to act on $M \otimes SW^{-\sigma}(\overline{\mathfrak{p}})$ where $\mathfrak{p} = \mathfrak{a}^{\perp}$. We introduce the Kac-Todorov operator as

$$D_{\mathfrak{g},\mathfrak{a}} = (G_{\mathfrak{g}} - G_{\mathfrak{a}})_0^{M\otimes SW^{-\sigma}(\overline{\mathfrak{p}})}.$$

3. Main Theorems

Throughout this Section, $\mathfrak{g}$ is a Lie algebra, σ an elliptic automorphism of $\mathfrak{g}$ preserving the form and $\mathfrak{a}$ a quadratic subalgebra.

Write $\mathfrak{g} = \oplus_{\overline{j}\in\mathbb{R}/\mathbb{Z}}\mathfrak{g}^{\overline{j}}$, $\mathfrak{a} = \oplus_{\overline{j}\in\mathbb{R}/\mathbb{Z}}\mathfrak{a}^{\overline{j}}$, $\mathfrak{a}^{\overline{j}} = \mathfrak{a} \cap \mathfrak{g}^{\overline{j}}$.

Assumption. *We assume that there exists an elliptic automorphism of $\mathfrak{g}$ preserving the form, commuting with σ, and such that a Cartan subalgebra $\mathfrak{t}$ of the joint fixed points of σ and μ is a Cartan subalgebra of $\mathfrak{a}^{\overline{0}}$.*

Denote by $\mathfrak{h}_0$ the Cartan subalgebra $Cent_{\mathfrak{g}^{\overline{0}}}(\mathfrak{t})$ of $\mathfrak{g}^{\overline{0}}$ and decompose it as $\mathfrak{h}_0 = \mathfrak{t} \oplus \mathfrak{h}_{\mathfrak{p}}$. Let $\widehat{W}_{\sigma}$ be the Weyl group of $\widehat{L}(\mathfrak{g}, \sigma)$ and $\widehat{\mathfrak{h}} = \mathfrak{h}_0 \oplus \mathbb{C}K \oplus \mathbb{C}d$ the Cartan subalgebra. Set $\mathfrak{t}^{aff} = \mathfrak{t} \oplus \mathbb{C}K \oplus \mathbb{C}d$. We prove that the subgroup $\widehat{W}(\mu) = \{w \in \widehat{W}_{\sigma} \mid w\mu = \mu w\}$ is isomorphic to the group generated by the reflections s_{β} in the vectors $\beta = \alpha_{|\mathfrak{t}^{aff}}$ (where we stipulate that $s_{\beta} = Id$ if $\alpha_{|\mathfrak{t}^{aff}}$ is isotropic). We prove that the latter group is a Coxeter group which contains the Weyl group of $\widehat{L}(\mathfrak{a}, \sigma)$ as a reflection subgroup. Let $\widehat{W}'$ be the corresponding set of minimal right coset representatives. Let $\widehat{\rho}_{\sigma}$, $\widehat{\rho}_{\mathfrak{a},\sigma}$ be ρ-vectors for $\widehat{L}(\mathfrak{g}, \sigma)$, $\widehat{L}(\mathfrak{a}, \sigma)$ respectively.

Theorem 3.1. [5, Theorem 1.1] *In the above setup, assume furthermore that $(\Lambda + \widehat{\rho}_{\sigma})_{|\mathfrak{h}_{\mathfrak{p}}} = 0$. Then the following decomposition into a direct sum of irreducible $\widehat{L}(\mathfrak{a}, \sigma)$-modules holds:*

$$Ker\,(D) = 2^{\lfloor \frac{\mathrm{rank}(\mathfrak{g}^{\overline{0}})) - \mathrm{rank}(\mathfrak{a}^{\overline{0}}) + 1}{2} \rfloor} \sum_{w\in\widehat{W}'} V(w(\Lambda + \widehat{\rho}_{\sigma}) - \widehat{\rho}_{\mathfrak{a}\,\sigma}).$$

By taking $\Lambda = 0$ and considering a symmetric subalgebra we recover via a multiplet approach the results obtained in previous papers for both the equal and non-equal rank cases. The proof proceeds along the lines of the finite-dimensional case, up to the fact that Parthasarathy's Dirac inequality is replaced by the following theorem, which can be viewed as an affine analogue of the "Vogan conjecture".

Theorem 3.2. [4, Theorem 8.1] *Let $\mathfrak{g}$ be a semisimple Lie algebra, σ an elliptic automorphism of $\mathfrak{g}$ and $\mathfrak{a}$ a reductive quadratic subalgebra. Assume that the centralizer in $\mathfrak{g}^{\overline{0}}$ of Cartan subalgebra of $\mathfrak{a}$ is a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}^{\overline{0}}$. Fix $\Lambda \in \widehat{\mathfrak{h}}^*$ such that $\Lambda + \widehat{\rho}_\sigma$ is in the Tits cone of $\widehat{L}(\mathfrak{g}, \sigma)$ and let M be a highest weight module for $\widehat{L}(\mathfrak{g}, \sigma)$ with highest weight Λ. Let f be a holomorphic $\widehat{W}_\sigma$-invariant function on the Tits cone. Suppose that a twisted highest weight $\widehat{L}(\mathfrak{a}, \sigma)$-module of highest weight μ occurs in the Dirac cohomology of M. Then $f(\Lambda + \widehat{\rho}_\sigma) = f(\mu + \widehat{\rho}_{\mathfrak{a},\sigma})$.*

4. Perspectives on the Lie superalgebra case

One of the key properties of the classical Dirac operator is the existence of a nice formula for its square. The replacement of the latter formula in our case is a nice expression for $[G_{\mathfrak{g}\,\lambda} G_{\mathfrak{g}}]$. Let now $\mathfrak{g}$ be a basic classical superalgebra, $\sigma = I_{\mathfrak{g}}$ and $M = L(\Lambda)$ be a highest module w.r.t. some positive system. If v is an highest weight vector in M, we compute that $G_0^X(v \otimes 1) = v \otimes (\overline{h}_{\overline{\Lambda}+\rho}) \cdot 1$ (here $\overline{\Lambda} = \Lambda_{|\mathfrak{h}_0}$ and h_μ is defined by $\mu(h) = (h, h_\mu)$ for $\mu \in \mathfrak{h}_0^*$). By the above nice expression, $v \otimes 1$ is an eigenvector for $(G_0^X)^2$, so taking Λ such that $\overline{\Lambda} = -\rho$, we get the Freudenthal "strange" formula

$$(\rho, \rho) = \frac{g}{12} \mathrm{sdim} \mathfrak{g}. \tag{4.1}$$

For other (non-uniform) proofs of (4.1) see [8]. We also have a twisted version of this formula, which is an analogue of the "very strange formula". By applying the Zhu functor π_{Zhu} to our Dirac operator $D_{\mathfrak{g},\mathfrak{a}}$, we obtain a "finite-dimensional" Dirac operator in superalgebra setting which we are going to study in more detail. We have verified that $\pi_{Zhu}(D_{\mathfrak{g},\mathfrak{g}_0})$ is the Dirac operator defined in [3].

References

[1] P. Cellini, V. G. Kac, P. Möseneder Frajria and P. Papi, Adv. Math. **207** (2006), 156–204.
[2] J.-S. Huang and P. Pandžić, J. Amer. Math. Soc. **15** (2002), no. 1, 185–202.
[3] J.-S. Huang and P. Pandžić, Transform. Groups **10** (2005), 201–209.
[4] V. G. Kac, P. Möseneder Frajria and P. Papi, Adv. Math. (2008), **217** n.6, 2485–2562.
[5] ———, Moscow. Math. J. (2008), **8** n.4, 759–788.
[6] V. G. Kac and D. Peterson, Proc. Natl. Acad. Sci. USA **78** (1981), 3308–3312.
[7] V. G. Kac and I. T. Todorov, Comm. Math. Phys. **102** (1985), no. 2, 337–347.
[8] V. G. Kac, M. Wakimoto, Progress in Math., 123, Boston: Birkhäuser, 1994, pp. 415–456.
[9] B. Kostant, Duke Math. J. **100** (1999), no. 3, 447–501.

Dip. di Matematica, Sapienza Università di Roma, P.le A. Moro 2, 00185, Roma , ITALY;
papi@mat.uniroma1.it