ADDENDUM TO "ON THE KERNEL OF THE AFFINE DIRAC OPERATOR"

VICTOR G. KAC
PIERLUIGI MÖSENEDER FRAJRIA
PAOLO PAPI

After the paper [3] was published, we realized that a key step in the proof of its main result, Theorem 1.1, could be achieved in a more direct way, which makes the proof independent from the results of [2].

We retain the same notation as in [3]. In [1, Ch. 10, § 5] it is shown that $(\cdot,\cdot)_{|\mathfrak{h}_0\times\mathfrak{h}_0}$ is nondegenerate, thus we can define dually a form $(\cdot,\cdot)$ on $\mathfrak{h}_0^*$. Denote by Δ the set of $\mathfrak{h}_0$-weights of $\mathfrak{g}$ and let $(\mathfrak{h}_0)^*_{\mathbb{R}}$ be the real span of Δ. As proven in [1, Ch. 10, § 5] one can always choose the form $(\cdot,\cdot)$ in such a way that $(\cdot,\cdot)_{|(\mathfrak{h}_0)^*_{\mathbb{R}}\times(\mathfrak{h}_0)^*_{\mathbb{R}}}$ is positive definite. We shall assume that the form $(\cdot,\cdot)$ satisfies this condition. This is not restrictive, and it has been tacitly used in the proof of Theorem 1.1.

Observe that, since μ stabilizes $\mathfrak{h}_0$, it permutes the set Δ of $\mathfrak{h}_0$-weights of $\mathfrak{g}$, hence $\mu((\mathfrak{h}_0)_{\mathbb{R}}) = (\mathfrak{h}_0)_{\mathbb{R}}$. In particular we have the orthogonal decomposition

$$(\mathfrak{h}_0)^*_{\mathbb{R}} = ((\mathfrak{h}_0)^*_{\mathbb{R}} \cap (\mathfrak{h}^\mu)^*) \oplus ((\mathfrak{h}_0)^*_{\mathbb{R}} \cap \mathfrak{h}^*_{\mathfrak{p}}).$$

Recall that a weight $\Lambda \in \widehat{\mathfrak{h}}^*$ is said to be dominant if $(\Lambda,\alpha)\in\mathbb{R}$ for any $\alpha\in\widehat{\Delta}$ and $(\Lambda,\alpha)\geq 0$ for $\alpha\in\widehat{\Delta}^+$. If we write $\Lambda = k\Lambda_0 + \Lambda_{|\mathfrak{h}_0} + (\Lambda_0,\Lambda)\delta$ then Λ dominant implies $k=(\Lambda,\delta)\geq 0$.

It is shown in [1, Ch. 10, § 5] that Δ generates $\mathfrak{h}_0$ over $\mathbb{C}$. This implies that $\lambda\in(\mathfrak{h}_0)^*_{\mathbb{R}}$ if and only if $(\lambda,\alpha)\in\mathbb{R}$ for any $\alpha\in\Delta$. In particular if $\Lambda\in\widehat{\mathfrak{h}}^*$ is such that $(\Lambda,\alpha)\in\mathbb{R}$ for any $\alpha\in\widehat{\Delta}$, then $\Lambda_{|\mathfrak{h}_0}\in(\mathfrak{h}_0)^*_{\mathbb{R}}$. Thus we have an orthogonal decomposition

$$\Lambda_{|\mathfrak{h}_0} = \Lambda_{|\mathfrak{h}_0^\mu} + \Lambda_{|\mathfrak{h}_\mathfrak{p}}$$

with

$$\Lambda_{|\mathfrak{h}_0^\mu} \in (\mathfrak{h}_0)^*_{\mathbb{R}} \cap (\mathfrak{h}_0^\mu)^* \quad \Lambda_{|\mathfrak{h}_\mathfrak{p}} \in (\mathfrak{h}_0)^*_{\mathbb{R}} \cap \mathfrak{h}^*_{\mathfrak{p}}.$$

Recall that $\Lambda\in\widehat{\mathfrak{h}}^*$ is said to be integral if $2\frac{(\Lambda,\alpha)}{(\alpha,\alpha)}\in\mathbb{Z}$ for any simple root α. We set for shortness $\|\lambda\| = \|(\varphi^*_{\mathfrak{a}})^{-1}(\lambda)\|$ whenever $\lambda\in\varphi^*_{\mathfrak{a}}((\widehat{\mathfrak{h}}^\mu)^*)$.

Proposition. *Suppose that $\Lambda\in\widehat{\mathfrak{h}}^*$ is dominant integral. Let ν be a weight of $L(\Lambda)\otimes F^\sigma(\mathfrak{p})$ such that*

$$\|\nu + \widehat{\rho}_{\mathfrak{a}\sigma}\| = \|\Lambda + \widehat{\rho}_\sigma\|.$$

Then there is $w \in \widehat{W}_\sigma$ *such that*

$$w(\Lambda + \widehat{\rho}_\sigma) = (\varphi_{\mathfrak{a}}^*)^{-1}(\nu + \widehat{\rho}_{\mathfrak{a}\sigma}). \tag{1}$$

Proof. Observe that $(\varphi_{\mathfrak{a}}^*)^{-1}(\nu + \widehat{\rho}_{\mathfrak{a}\sigma})$ is a $\widehat{\mathfrak{h}}^\mu$-weight of $L(\Lambda) \otimes F^\sigma(\mathfrak{p}) \otimes F^\sigma(\mathfrak{a}) = L(\Lambda) \otimes F^\sigma(\mathfrak{g})$, thus

$$(\varphi_{\mathfrak{a}}^*)^{-1}(\nu + \widehat{\rho}_{\mathfrak{a}\sigma}) = (\lambda + \widehat{\rho}_\sigma - s)_{|\widehat{\mathfrak{h}}^\mu}$$

with λ a weight of $L(\Lambda)$ and $s \in \mathcal{S}$, where

$$\mathcal{S} = \{\xi \in \widehat{\mathfrak{h}}^* \mid \xi = \sum_{\beta \in \widehat{\Delta}^+} n_\beta \beta,\ 0 \le n_\beta \le \dim \widehat{L}(\mathfrak{g}, \sigma)_\beta,\ n_\beta = 0 \ \ a.e.\}.$$

Since $(\lambda + \widehat{\rho}_\sigma - s)(K) = k + g > 0$, we can find $v \in \widehat{W}_\sigma$ such that $v(\lambda + \widehat{\rho}_\sigma - s)$ is dominant. The set of weights of $L(\Lambda)$ is $\widehat{W}_\sigma$-invariant and the same holds for $\widehat{\rho}_\sigma - \mathcal{S}$, hence we can write $v(\lambda + \widehat{\rho}_\sigma - s) = \lambda' + \widehat{\rho}_\sigma - s'$. It follows that $\|\lambda + \widehat{\rho}_\sigma - s\| = \|\lambda' + \widehat{\rho}_\sigma - s'\|$, so we have

$$\|\Lambda + \widehat{\rho}_\sigma\|^2 - \|\lambda + \widehat{\rho}_\sigma - s\|^2 = (\Lambda + \widehat{\rho}_\sigma + \lambda' + \widehat{\rho}_\sigma - s', \Lambda - \lambda' + s').$$

Since $\Lambda - \lambda' + s'$ is a sum of positive roots and $\Lambda + \widehat{\rho}_\sigma$, $\lambda' + \widehat{\rho}_\sigma - s'$ are both dominant, we obtain that

$$\|\Lambda + \widehat{\rho}_\sigma\| \ge \|\lambda + \widehat{\rho}_\sigma - s\|.$$

On the other hand

$$\|\nu + \widehat{\rho}_{\mathfrak{a}\sigma}\| = \|(\lambda + \widehat{\rho}_\sigma - s)_{|\widehat{\mathfrak{h}}^\mu}\| \le \|\lambda + \widehat{\rho}_\sigma - s\| \le \|\Lambda + \widehat{\rho}_\sigma\|$$

so, since $\|\nu + \widehat{\rho}_{\mathfrak{a}\sigma}\| = \|\Lambda + \widehat{\rho}_\sigma\|$, we obtain equalities. Since $\Lambda + \widehat{\rho}_\sigma$ is regular we find that

$$0 = \|\Lambda + \widehat{\rho}_\sigma\|^2 - \|\lambda + \widehat{\rho}_\sigma - s\|^2 = (\Lambda + \widehat{\rho}_\sigma + \lambda' + \widehat{\rho}_\sigma - s', \Lambda - \lambda' + s')$$

implies $\Lambda = \lambda'$ and $s' = 0$, so $\Lambda + \widehat{\rho}_\sigma = v(\lambda + \widehat{\rho}_\sigma - s)$. Moreover $\|(\lambda + \widehat{\rho}_\sigma - s)_{|\widehat{\mathfrak{h}}^\mu}\| = \|\lambda + \widehat{\rho}_\sigma - s\|$ implies $\|(\lambda + \widehat{\rho}_\sigma - s)_{|\mathfrak{h}_\mathfrak{p}}\| = 0$. Since $(\lambda + \widehat{\rho}_\sigma - s)_{|\mathfrak{h}_0} \in (\mathfrak{h}_0)^*_\mathbb{R}$ and the form $(\cdot, \cdot)$ is positive definite on $(\mathfrak{h}_0)^*_\mathbb{R}$ we obtain that $(\lambda + \widehat{\rho}_\sigma - s)_{|\mathfrak{h}_\mathfrak{p}} = 0$ and $(\lambda + \widehat{\rho}_\sigma - s)_{|\widehat{\mathfrak{h}}^\mu} = \lambda + \widehat{\rho}_\sigma - s$. Thus

$$(\varphi_{\mathfrak{a}}^*)^{-1}(\nu + \widehat{\rho}_{\mathfrak{a}\sigma}) = v^{-1}(\Lambda + \widehat{\rho}_\sigma).$$

Taking $w = v^{-1}$ we obtain (2), as wished. □

Using this Proposition, there is no need to invoke the affine version of the Vogan conjecture to start the proof of Theorem 1.1 (after Lemma 5.1).

References

[1] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, 1978.

[2] V. G. Kac, P. Möseneder Frajria and P. Papi, *Multiplets of representations, twisted Dirac operators and Vogan's conjecture in affine setting*, Adv. Math. (2008), **217** n.6, 2485–2562.

[3] V. G. Kac, P. Möseneder Frajria and P. Papi, *On the Kernel of the affine Dirac operator*, Moscow Math. J. (2008) **8** n.4, 759–788.

[4] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

V.K.: Department of Mathematics, Rm 2-178, MIT, 77 Mass. Ave, Cambridge, MA 02139;
`kac@math.mit.edu`
P.MF.: Politecnico di Milano, Polo regionale di Como, Via Valleggio 11, 22100 Como, ITALY;
`pierluigi.moseneder@polimi.it`
P.P.: Dipartimento di Matematica, Sapienza Università di Roma, P.le A. Moro 2, 00185, Roma , ITALY;
`papi@mat.uniroma1.it`