ELLIPTIC EQUATIONS

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Contents

1. Existence with regular data in the linear case

Before stating and proving the existence theorem for linear elliptic equations, we need some tools.

1.1. **Minimization in Banach spaces.** Let E be a Banach space, and let $J : E \to \mathbb{R}$ be a functional.

Definition 1.1. A functional $J : E \to \mathbb{R}$ is said to be *weakly lower semicontinuous* if

$$u_n \rightharpoonup u \quad \Rightarrow \quad J(u) \le \liminf_{n \to +\infty} J(u_n).$$

Definition 1.2. A functional $J : E \to \mathbb{R}$ is said to be *coercive* if

$$\lim_{\|u\|_E \to +\infty} J(u) = +\infty.$$

Example 1.3. If $E = \mathbb{R}$, the function $J(x) = x^2$ is an example of a (weakly) lower semicontinuous and coercive functional. Another example is $J(u) = \|u\|_E$.

Theorem 1.4. *Let E be a reflexive Banach space, and let $J : E \to \mathbb{R}$ be a coercive and weakly lower semicontinuous functional (not identically equal to $+\infty$). Then J has a minimum on E.*

Proof. Let

$$m = \inf_{v \in E} J(v) < +\infty,$$

and let $\{v_n\}$ in E be a minimizing sequence, i.e., v_n is such that

$$\lim_{n \to +\infty} J(v_n) = m.$$

We begin by proving that $\{v_n\}$ is bounded. Indeed, if it were not, there would be a subsequence $\{v_{n_k}\}$ such that

$$\lim_{k \to +\infty} \|v_{n_k}\| = +\infty.$$

Since J is coercive, we will have

$$m = \lim_{n \to +\infty} J(v_n) = \lim_{k \to +\infty} J(v_{n_k}) = +\infty,$$

which is false. Therefore, $\{v_n\}$ is bounded in E and so, being E reflexive, there exists a subsequence $\{v_{n_k}\}$ and an element v of E such that v_{n_k} weakly converges to v as k diverges. Since J is weakly lower semicontinuous, we have

$$m \le J(v) \le \liminf_{k \to +\infty} J(v_{n_k}) = \lim_{n \to +\infty} J(v_n) = m,$$

so that v is a minimum of J. □

1.2. **Hilbert spaces.**

1.2.1. *Linear forms and dual space.* We recall that a Hilbert space H is a vector space where a scalar product $(\cdot|\cdot)$ is defined, which is complete with respect to the distance induced by the scalar product by the formula

$$d(x, y) = \sqrt{(x - y|x - y)}.$$

Examples of Hilbert spaces are $\mathbb{R}$ (with $(x|y) = x\,y$), $\mathbb{R}^N$ (with the "standard" scalar product), ℓ^2, and $L^2(\Omega)$ with

$$(f|g) = \int_\Omega f\,g.$$

Theorem 1.5 (Riesz). *Let H be a separable Hilbert space, and let T be an element of its dual H', i.e., a linear application $T : H \to \mathbb{R}$ such that there exists $C \geq 0$ such that*

$$|\langle T, x\rangle| \leq C\|x\|, \quad \forall x \in H. \tag{1.1}$$

Then there exists a unique y in H such that

$$\langle T, x\rangle = (y|x), \quad \forall x \in H.$$

Proof. Denote by $\{e_h\}$ a complete orthonormal system in H, i.e. a sequence of vectors of H such that $(e_h|e_k) = \delta_{hk}$, and such that, for every x in H, one has

$$x = \sum_{h=1}^{+\infty} (x|e_h) e_h.$$

It is then well known that there exists a bijective isometry $\mathcal{F}$ from H to ℓ^2, defined by $\mathcal{F}(x) = \{(x|e_h)\}$. We claim that $\{\langle T, e_h\rangle\}$ belongs to ℓ^2. Indeed, if

$$y_n = \sum_{h=1}^{n} \langle T, e_h\rangle e_h,$$

we have, by linearity and by (1.1),

$$\sum_{h=1}^{n} (\langle T, e_h\rangle)^2 = \langle T, y_n\rangle \leq C\|y_n\| = C \left(\sum_{h=1}^{n} (\langle T, e_h\rangle)^2 \right)^{\frac{1}{2}},$$

so that

$$\sum_{h=1}^{n} (\langle T, e_h\rangle)^2 \leq C^2,$$

which yields (letting n tend to infinity) that $\{\langle T, e_h\rangle\rangle\}$ belongs to ℓ^2. Therefore, one has, again by linearity and by (1.1),

$$\langle T, x\rangle = \sum_{h=1}^{+\infty} (x|e_h) \langle T, e_h\rangle, \quad \forall x \in H.$$

Let now y be the vector of H defined by

$$y = \sum_{h=1}^{+\infty} \langle T, e_h\rangle e_h.$$

Then, since $\langle T, e_h\rangle = (y|e_h)$, one has

$$\langle T, x\rangle = \sum_{h=1}^{+\infty} (x|e_h)(y|e_h), \quad \forall x \in H,$$

and the right hand side is nothing but the scalar product in ℓ^2 of $\mathcal{F}(x)$ and $\mathcal{F}(y)$. Since $\mathcal{F}$ is an isometry, we then have

$$\langle T, x\rangle = (y|x), \quad \forall x \in H,$$

as desired. Uniqueness follows from the fact that $(y|x) = (z|x)$ for every x in H implies $y = z$ (just take $x = y - z$). □

Corollary 1.6. *The map $T \mapsto y$ is a bijective linear isometry between H' and H.*

Proof. Since $\langle T + S, x\rangle = \langle T, x\rangle + \langle S, x\rangle$, and $\langle \lambda T, x\rangle = \lambda\langle T, x\rangle$, it is clear that the map $T \mapsto y$ is linear. In order to prove that it is an isometry, we have

$$|\langle T, x\rangle| = |(y|x)| \leq \|y\|\|x\|,$$

which implies $\|T\| \leq \|y\|$. Furthermore

$$\|y\|^2 = (y|y) = \langle T, y\rangle \leq \|T\|\|y\|,$$

so that $\|y\| \leq \|T\|$. The map is clearly injective, and it is surjective since the application $x \mapsto (y|x)$ is linear and continuous on H (by Cauchy-Schwartz inequality). □

1.2.2. *Bilinear forms.* An application $a : H \times H \to \mathbb{R}$ such that

$$a(\lambda x + \mu y, z) = \lambda a(x, z) + \mu a(y, z),$$

and

$$a(z, \lambda x + \mu y) = \lambda a(z, x) + \mu a(z, x),$$

for every x and y in H, and for every λ and μ in $\mathbb{R}$, is called *bilinear form*. A bilinear form is said to be *continuous* if there exists $\beta \geq 0$ such that

$$|a(x, y)| \leq \beta\|x\|\|y\|, \quad \forall x, y \in H,$$

and is said to be *coercive* if there exists $\alpha > 0$ such that

$$a(x, x) \geq \alpha\|x\|^2, \quad \forall x \in H.$$

An example of bilinear form on H is the scalar product, which is both continuous (with $\beta = 1$, thanks to the Cauchy-Schwartz inequality), and coercive (with $\alpha = 1$, by definition of the norm in H).

Theorem 1.7. *Let $a : H \times H \to \mathbb{R}$ be a continuous bilinear form. Then there exists a linear and continuous map $A : H \to H$ such that*

$$a(x, y) = (A(x)|y), \quad \forall x, y \in H.$$

Proof. Since a is linear in the second argument and continuous, for every fixed x in H the map $y \mapsto a(x,y)$ is linear and continuous, so that it belongs to H'. By Riesz theorem, there exists a unique vector $A(x)$ in H such that

$$a(x,y) = (A(x)|y), \quad \forall x, y \in H.$$

Since a is linear in the first argument, the map $x \mapsto A(x)$ is linear. Furthermore, by the continuity of a,

$$\|A(x)\|^2 = (A(x)|A(x)) = a(x, A(x)) \leq \beta \|x\| \|A(x)\|,$$

so that $\|A(x)\| \leq \beta \|x\|$, and the map is continuous. □

1.2.3. *Banach-Caccioppoli and Lax-Milgram theorems.*

Theorem 1.8 (Banach-Caccioppoli)**.** *Let (X,d) be a complete metric space, and let $S : X \to X$ be a contraction mapping, i.e., a continuous application such that there exists θ in $[0,1)$ such that*

$$d(S(x), S(y)) \leq \theta\, d(x,y), \quad \forall x, y \in X.$$

Then there exists a unique $\overline{x}$ in X such that $S(\overline{x}) = \overline{x}$.

Proof. Let x_0 in X be fixed, and define $x_1 = S(x_0)$, $x_2 = S(x_1)$, and, in general, $x_n = S(x_{n-1})$. We then have, since S is a contraction mapping,

$$d(x_{n+1}, x_n) = d(S(x_n), S(x_{n-1})) \leq \theta\, d(x_n, x_{n-1}),$$

and iterating we obtain

$$d(x_{n+1}, x_n) \leq \theta^n\, d(x_1, x_0).$$

Therefore, by the triangular inequality,

$$d(x_n, x_m) \leq \sum_{h=m}^{n-1} d(x_{h+1}, x_h) \leq \sum_{h=m}^{n-1} \theta^h\, d(x_1, x_0) = \frac{\theta^m - \theta^n}{1 - \theta}.$$

Since $\{\theta^h\}$ is a Cauchy sequence in $\mathbb{R}$ (being convergent to zero), it then follows that $\{x_n\}$ is a Cauchy sequence in (X,d), which is complete. Therefore, there exists $\overline{x}$ in X such that x_n converges to $\overline{x}$. Since S is continuous, on one hand $S(x_n)$ converges to $S(\overline{x})$, and on the other hand $S(x_n) = x_{n+1}$ converges to $\overline{x}$ so that $\overline{x}$ is a fixed point for S. If there exist $\overline{x}$ and $\overline{y}$ such that $S(\overline{x}) = \overline{x}$ and $S(\overline{y}) = \overline{y}$, then, since S is a contraction mapping,

$$d(\overline{x}, \overline{y}) = d(S(\overline{x}), S(\overline{y})) \leq \theta\, d(\overline{x}, \overline{y}),$$

which implies (since $\theta < 1$) $d(\overline{x}, \overline{y}) = 0$ and so $\overline{x} = \overline{y}$. □

Theorem 1.9 (Lax-Milgram)**.** *Let $a : H \times H \to \mathbb{R}$ be a continuous and coercive bilinear form, and let T be an element of H'. Then there exists a unique $\overline{x}$ in H such that*

$$a(\overline{x}, z) = \langle T, z\rangle, \quad \forall z \in H. \tag{1.2}$$

Proof. Using the Riesz theorem and Theorem 1.7, solving the equation (1.2) is equivalent to find $\overline{x}$ such that

$$a(\overline{x}, z) = (A(\overline{x})|z) = (y|z) = \langle T, z\rangle, \quad \forall z \in H,$$

i.e., to solve the equation $A(\overline{x}) = y$. Given $\lambda > 0$, this equation is equivalent to $\overline{x} = \overline{x} - \lambda A(\overline{x}) + \lambda y$, which is a fixed point problem for the function $S(x) = x - \lambda A(x) + \lambda y$. Since, being A linear, one has

$$S(x_1) - S(x_2) = x_1 - x_2 - \lambda A(x_1) + \lambda A(x_2) = x_1 - x_2 - \lambda A(x_1 - x_2),$$

in order to prove that S is a contraction mapping, it is enough to prove that there exists $\lambda > 0$ such that

$$\|x - \lambda A(x)\| \leq \theta \|x\|,$$

for some $\theta < 1$ and for every x in H. We have

$$\|x - \lambda A(x)\|^2 = \|x\|^2 + \lambda^2 \|A(x)\|^2 - 2\lambda(A(x)|x).$$

Recalling Theorem 1.7 and the definition of A, we have

$$\|A(x)\|^2 \leq \beta^2 \|x\|^2, \quad (A(x)|x) = a(x, x) \geq \alpha \|x\|^2,$$

so that

$$\|x - \lambda A(x)\|^2 \leq (1 + \lambda^2\beta^2 - 2\lambda\alpha)\|x\|^2.$$

If $0 < \lambda < \frac{2\alpha}{\beta^2}$, we have $\theta^2 = 1 + \lambda^2\beta^2 - 2\lambda\alpha < 1$, so that S is a contraction mapping. □

Remark 1.10. Not every function which has a fixed point is a contraction; for example, the identity map (which has infinitely many fixed points), and the function $f : [0, 1] \to [0, 1]$ defined by $f(x) = x^2$ (so that $f(0) = 0$, and $f(1) = 1$), are not contractions. It is therefore useful to have other fixed point theorems, under different assumptions on the map S. This is the case of Schauder's theorem.

Theorem 1.11 (Schauder)**.** *Let K be a convex, closed, bounded subset of a Banach space, and let $S : K \to K$ be a continuous function such that $\overline{S(K)}$ is compact. Then S has at least a fixed point.*

1.3. **Sobolev spaces.** The Banach spaces where we will look for solutions are space of functions in Lebesgue spaces "with derivatives in Lebesgue spaces" (whatever this means).

Warning: This section is a short summary of the results contained in Chapter IX of the book by H. Brezis (see [2]): we refer to it for further results and proofs.

1.3.1. *Definition of Sobolev spaces.* Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 1$, and let u be a function in $L^1(\Omega)$. We say that u has a *weak* (or *distributional*) *derivative* in the direction x_i if there exists a function v in $L^1(\Omega)$ such that

$$\int_\Omega u\,\frac{\partial \varphi}{\partial x_i} = -\int_\Omega v\,\varphi, \quad \forall \varphi \in C^1_0(\Omega).$$

In this case we define the weak derivative $\frac{\partial u}{\partial x_i}$ as the function v. If u has weak derivatives in every direction, we define its (weak, or distributional) gradient as the vector

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right).$$

If $p \geq 1$, we define the Sobolev space $W^{1,p}(\Omega)$ as

$$W^{1,p}(\Omega) = \left\{u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^N\right\}.$$

The Sobolev space $W^{1,p}(\Omega)$ becomes a Banach space under the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{(L^p(\Omega))^N},$$

and $W^{1,2}(\Omega)$ is a Hilbert space under the scalar product

$$(u|v)_{W^{1,2}(\Omega)} = \int_\Omega u\,v + \int_\Omega \nabla u \cdot \nabla v.$$

For historical reasons the space $W^{1,2}(\Omega)$ is usually denoted by $H^1(\Omega)$: we will use this notation from now on.

Since we will be dealing with elliptic problems with zero boundary conditions, we need to define functions which somehow are "zero" on the boundary of Ω. Since $\partial\Omega$ has zero Lebesgue measure, and functions in $W^{1,p}(\Omega)$ are only defined up to almost everywhere equivalence, there is no "direct" way of defining the boundary value a function u in some Sobolev space. We then give the following definition.

Definition 1.12. We define $W^{1,p}_0(\Omega)$ as the closure of $C^1_0(\Omega)$ in the norm of $W^{1,p}(\Omega)$. If $p = 2$, we will denote $W^{1,2}_0(\Omega)$ by $H^1_0(\Omega)$, which is a Hilbert space.

From now on we will mainly deal with $W^{1,p}_0(\Omega)$.

1.3.2. *Properties of Sobolev spaces.* Since a function in $W_0^{1,p}(\Omega)$ is "zero at the boundary" it is possible to control the norm of u in $L^p(\Omega)$ with the norm of its gradient in the same space. This is known as Poincaré inequality.

Theorem 1.13 (Poincaré inequality)**.** *Let $p \geq 1$; then there exists a constant C, only depending on Ω, N and p, such that*

$$\|u\|_{L^p(\Omega)} \leq C \, \|\nabla u\|_{(L^p(\Omega))^N}, \quad \forall u \in W_0^{1,p}(\Omega). \tag{1.3}$$

Proof. We only give an idea of the proof in dimension 1. Let u belong to $C_0^1((0,1))$. Then

$$u(x) = u(0) + \int_0^x u'(t)\, dt = \int_0^x u'(t)\, dt, \quad \forall x \in (0,1).$$

Thus, by Hölder inequality

$$|u(x)|^p = \left| \int_0^x u'(t)\, dt \right|^p \leq x^{\frac{p}{p'}} \int_0^x |u'(t)|^p \leq \int_0^1 |u'(t)|^p.$$

Integrating this inequality yields the result for $C_0^1((0,1))$ functions. The result for functions in $W_0^{1,p}(\Omega)$ then follows by a density argument. □

As a consequence of Poincaré inequality, we can define on $W_0^{1,p}(\Omega)$ the equivalent norm built after the norm of ∇u in $(L^p(\Omega))^N$. From now on, we define

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{(L^p(\Omega))^N}.$$

Even though functions in $W_0^{1,p}(\Omega)$ should only belong to $L^p(\Omega)$, the assumptions made on the gradient allow to improve the summability of functions belonging to Sobolev spaces. This is what is stated in the following "embedding" theorem.

Theorem 1.14. *Let $1 \leq p < N$, and let $p^* = \frac{Np}{N-p}$ (p^* is called the Sobolev embedding exponent). Then there exists a constant $\mathcal{S}_p$ (depending only on N and p) such that*

$$\|u\|_{L^{p^*}(\Omega)} \leq \mathcal{S}_p \, \|u\|_{W_0^{1,p}(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega). \tag{1.4}$$

Remark 1.15. The fact that p^* is the correct exponent can be easily recovered by a scaling argument. Indeed, if u belongs to $W_0^{1,p}(\mathbb{R}^N)$, then $u(\lambda x)$ belongs to the same space. But then

$$\int_{\mathbb{R}^N} |u(\lambda x)|^q \, dx = \frac{1}{\lambda^N} \int_{\mathbb{R}^N} |u(y)|^q \, dy,$$

and

$$\int_{\mathbb{R}^N} |\nabla u(\lambda x)|^p \, dx = \frac{1}{\lambda^{N-p}} \int_{\mathbb{R}^N} |\nabla u(y)|^p \, dy.$$

Therefore, if (1.4) holds for some constant C (independent on λ) and some exponent q, one should have

$$\frac{N}{q} = \frac{N-p}{p},$$

which implies $q = \frac{Np}{N-p} = p^*$.

By (1.4), the embedding of $W_0^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$ is continuous. We recall that a map $T : X \to Y$ (with X and Y Banach spaces) is said to be **compact** if the closure of $T(B)$ is compact in Y for every bounded set B in X. To obtain compactness of the embedding of $W_0^{1,p}(\Omega)$ in Lebesgue spaces, we cannot consider exponents up to p^*.

Theorem 1.16. *Let $1 \leq p < N$, and let $1 \leq q < p^*$. Then the embedding of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$ is compact.*

Remark 1.17. The fact that the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ is not compact is at the basis for several nonexistence results for equations like $-\Delta u = u^q$ if q is "too large". But this is another story...

An important role will be played by the dual of a Sobolev space. We have the following representation theorem.

Theorem 1.18. *Let $p > 1$, and let T be an element of $(W_0^{1,p}(\Omega))'$. Then there exists F in $(L^{p'}(\Omega))^N$ such that*

$$\langle T, u\rangle = \int_\Omega F \cdot \nabla u, \quad \forall u \in W_0^{1,p}(\Omega).$$

The dual of $W_0^{1,p}(\Omega)$ will be denoted by $W^{-1,p'}(\Omega)$, while the dual of $H_0^1(\Omega)$ is $H^{-1}(\Omega)$.

Remark 1.19. The space $H_0^1(\Omega)$ is a Hilbert space. Therefore, by Theorem 1.5, it is isometrically equivalent to its dual $H^{-1}(\Omega)$. Furthermore, by Poincaré inequality, $H_0^1(\Omega)$ is embedded into $L^2(\Omega)$, which is itself a Hilbert space. Since the embedding is continuous and dense, we also have that the the dual of $L^2(\Omega)$ (which is $L^2(\Omega)$) is embedded into $H^{-1}(\Omega)$. We therefore have

$$H_0^1(\Omega) \subset L^2(\Omega) \equiv (L^2(\Omega))' \subset (H_0^1(\Omega))' = H^{-1}(\Omega).$$

If we identify both $L^2(\Omega)$ and its dual, **and** $H_0^1(\Omega)$ and its dual, we obtain a contradiction (since $H_0^1(\Omega)$ and $L^2(\Omega)$ are different spaces). Therefore, we have to choose which identification to make: which will be that $L^2(\Omega)$ is equivalent to its dual.

Remark 1.20. Since, by Sobolev embedding, $W_0^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, we have by duality that $(L^{p^*}(\Omega))'$ is continuously embedded in $W^{-1,p'}(\Omega)$. If we define

$$p_* = (p^*)' = \frac{Np}{Np - N + p},$$

we then have

$$L^{p_*}(\Omega) \subset W^{-1,p'}(\Omega).$$

If $p = 2$, we have $2_* = \frac{2N}{N+2}$, and the embedding of $L^{2_*}(\Omega)$ into $H^{-1}(\Omega)$.

The final result on Sobolev spaces will be about composition with regular functions.

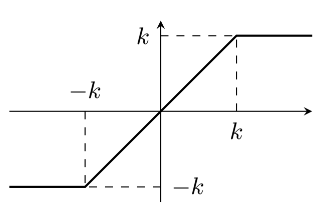
Theorem 1.21 (Stampacchia)**.** *Let $G : \mathbb{R} \to \mathbb{R}$ be a lipschitz continuous functions such that $G(0) = 0$. If u belongs to $W_0^{1,p}(\Omega)$, then $G(u)$ belongs to $W_0^{1,p}(\Omega)$ as well, and*

$$\nabla G(u) = G'(u)\,\nabla u, \quad \textit{almost everywhere in } \Omega. \tag{1.5}$$

Remark 1.22. Recall that a lipschitz continuous function is only almost everywhere differentiable, so that the right-hand side of (1.5) may not be defined. We have however two possible cases: if k is a value such that $G'(k)$ does not exist, either the set $\{u = k\}$ has zero measure (and so, since identity (1.5) only holds almost everywhere, this value does not give any problems), or the set $\{u = k\}$ has positive measure. In this latter case, however, we have both $\nabla u = 0$ and $\nabla G(u) = 0$ almost everywhere, so that (1.5) still holds.

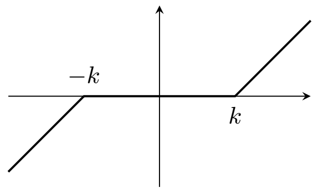
Let $k > 0$; in what follows, we will often use composition of functions in Sobolev spaces with the lipschitz continuous functions

$$T_k(s) = \max(-k, \min(s, k)), \tag{1.6}$$



and

$$G_k(s) = s - T_k(s) = (|s| - k)_+ \text{sgn}(s). \tag{1.7}$$



By Theorem 1.21, we have

$$\nabla T_k(u) = \nabla u\, \chi_{\{|u|\le k\}}, \quad \nabla G_k(u) = \nabla u\, \chi_{\{|u|\ge k\}},$$

almost everywhere in Ω.

1.4. **Weak solutions for elliptic equations.** We have now all the tools needed to deal with elliptic equations.

1.4.1. *Definition of weak solution.* Let $A : \Omega \to \mathbb{R}^{N^2}$ be a matrix-valued measurable function such that there exist $0 < \alpha \le \beta$ such that

$$A(x)\xi \cdot \xi \ge \alpha|\xi|^2, \quad |A(x)| \le \beta, \tag{1.8}$$

for almost every x in Ω, and for every ξ in $\mathbb{R}^N$. We will consider the following uniformly elliptic equation with Dirichlet boundary conditions

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where f is a function defined on Ω which satisfies suitable assumptions. If the matrix A is the identity matrix, problem (1.9) becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e., the Dirichlet problem for the laplacian operator.

1.4.2. *Classical solutions and weak solutions.* Suppose that the matrix A and the functions u and f are sufficiently smooth so that one can "classically" compute $-\mathrm{div}(A(x)\nabla u)$. If φ is a function in $C^1_0(\Omega)$, we can then multiply the equation in (1.9) by φ and integrate on Ω. Since

$$-\mathrm{div}(A(x)\nabla u)\,\varphi = -\mathrm{div}(A(x)\nabla u\,\varphi) + A(x)\nabla u \cdot \nabla\varphi,$$

we get

$$\int_\Omega A(x)\nabla u \cdot \nabla\varphi - \int_\Omega \mathrm{div}(A(x)\nabla u\,\varphi) = \int_\Omega f\,\varphi.$$

By Gauss-Green formula, we have (if ν is the exterior normal to Ω)

$$\int_\Omega \mathrm{div}(A(x)\nabla u\,\varphi) = \int_{\partial\Omega} A(x)\nabla u \cdot \nu\,\varphi = 0,$$

since φ has compact support in Ω. Therefore, if u is a classical solution of (1.9), we have

$$\int_\Omega A(x)\nabla u\cdot\nabla v=\int_\Omega f\,v,\quad \forall v\in C_0^1(\Omega).$$

We now remark that there is no need for A, u, φ and f to be smooth in order for the above identity to be well defined. It is indeed enough that A is a bounded matrix, that u and φ belong to $H_0^1(\Omega)$, and that f is in $L^2(\Omega)$ (or in $L^{2_*}(\Omega)$, thanks to Sobolev embedding, see Remark 1.20).

We therefore give the following definition.

Definition 1.23. Let f be a function in $L^{2_*}(\Omega)$. A function u in $H_0^1(\Omega)$ is a *weak solution* of (1.9) if

$$\int_\Omega A(x)\nabla u\cdot\nabla v=\int_\Omega f\,v,\quad \forall v\in H_0^1(\Omega). \tag{1.10}$$

If u is a weak solution of (1.9), and u is sufficiently smooth in order to perform the same calculations as above “going backwards”, then it can be proved that u is a “classical” solution of (1.9). The study of the assumptions on f and A such that a weak solution is also a classical solution goes beyond the purpose of this text.

1.4.3. *Existence of solutions (using Lax-Milgram).*

Theorem 1.24. *Let f be a function in $L^{2_*}(\Omega)$. Then there exists a unique solution u of (1.9) in the sense of (1.10).*

Proof. We will use Lax-Milgram theorem. Indeed, if we define the bilinear form $a : H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{R}$ by

$$a(u,v)=\int_\Omega A(x)\nabla u\cdot\nabla v,$$

and the linear and continuos (thanks to Sobolev embedding) functional $T : H_0^1(\Omega)\to\mathbb{R}$ by

$$\langle T,v\rangle=\int_\Omega f\,v,$$

solving problem (1.9) in the sense of (1.10) amounts to finding u in $H_0^1(\Omega)$ such that

$$a(u,v)=\langle T,v\rangle,\quad \forall v\in H_0^1(\Omega),$$

which is exactly the result given by Lax-Milgram theorem. In order to apply the theorem, we have to check that a is continuous and coercive (the fact that it is bilinear being evident). We have, by (1.8), and by Hölder inequality,

$$|a(u,v)| \leq \int_\Omega |A(x)||\nabla u||\nabla v| \leq \beta \, \|u\|_{H_0^1(\Omega)} \, \|v\|_{H_0^1(\Omega)} \, ,$$

so that a is continuous. Furthermore, again by (1.8), we have

$$a(u,u) = \int_\Omega A(x)\nabla u \cdot \nabla u \geq \alpha \int_\Omega |\nabla u|^2 = \alpha \, \|u\|^2_{H_0^1(\Omega)} \, ,$$

so that a is also coercive. □

1.4.4. *Existence of solutions (using minimization).* If the matrix A satisfies (1.8) and is symmetrical, existence and uniqueness of solutions for (1.9) can be proved using minimization of a suitable functional.

Theorem 1.25. *Let f be a function in $L^{2_*}(\Omega)$, and let $J : H_0^1(\Omega) \to \mathbb{R}$ be defined by*

$$J(v) = \frac{1}{2}\int_\Omega A(x)\nabla v \cdot \nabla v - \int_\Omega f \, v, \quad \forall v \in H_0^1(\Omega).$$

Then J has a unique minimum u in $H_0^1(\Omega)$, which is the solution of (1.9) in the sense of (1.10).

Proof. We begin by proving that J is coercive and weakly lower semicontinuous on $H_0^1(\Omega)$, so that a minimum will exist by Theorem 1.4. Recalling (1.8) and using Hölder and Sobolev inequalities, we have

$$\begin{aligned} J(v) &\geq \frac{\alpha}{2}\int_\Omega |\nabla v|^2 - \|f\|_{L^{2_*}(\Omega)} \, \|v\|_{L^{2^*}(\Omega)} \\ &\geq \frac{\alpha}{2}\|v\|^2_{H_0^1(\Omega)} - \mathcal{S}_2 \, \|f\|_{L^{2_*}(\Omega)} \, \|v\|_{H_0^1(\Omega)} \, , \end{aligned}$$

and the right hand side diverges as the norm of u in $H_0^1(\Omega)$ diverges, so that J is coercive. Let now $\{v_n\}$ be a sequence of functions which is weakly convergent to some v in $H_0^1(\Omega)$. Since f belongs to $L^{2_*}(\Omega)$, and v_n converges weakly to v in $L^{2^*}(\Omega)$, we have

$$\lim_{n\to+\infty} \int_\Omega f \, v_n = \int_\Omega f \, v,$$

so that the weak lower semicontinuity of J is equivalent to the weak lower semicontinuity of

$$K(v) = \int_\Omega A(x)\nabla v \cdot \nabla v.$$

By (1.8) we have

$$K(v-v_n)=\int_\Omega A(x)\,\nabla(v-v_n)\cdot\nabla(v-v_n)\geq 0,$$

which, together with the symmetry of A, implies

$$2\int_\Omega A(x)\nabla v\cdot\nabla v_n-\int_\Omega A(x)\nabla v\cdot\nabla v\leq\int_\Omega A(x)\nabla v_n\cdot\nabla v_n. \tag{1.11}$$

Since ∇v_n converges weakly to ∇v in $(L^2(\Omega))^N$, and since $A(x)\nabla v$ is fixed in the same space, we have

$$\lim_{n\to+\infty}\int_\Omega A(x)\nabla v\cdot\nabla v_n=\int_\Omega A(x)\nabla v\cdot\nabla v,$$

so that taking the inferior limit in both sides of (1.11) implies

$$K(v)=\int_\Omega A(x)\nabla v\cdot\nabla v\leq\liminf_{n\to+\infty}\int_\Omega A(x)\nabla v_n\cdot\nabla v_n=\liminf_{n\to+\infty}K(v_n),$$

which means that K is weakly lower semicontinuous on $H_0^1(\Omega)$, as desired.

Let now u be a minimum of J on $H_0^1(\Omega)$. We are going to prove that it is unique. Indeed, if u and v are both minima of J, one has

$$J(u)\leq J\Big(\frac{u+v}{2}\Big),\quad J(v)\leq J\Big(\frac{u+v}{2}\Big),$$

that is,

$$J(u)+J(v)\leq 2J\Big(\frac{u+v}{2}\Big),$$

which becomes (after cancelling equal terms and multiplying by 4)

$$2\int_\Omega A(x)\nabla u\cdot\nabla u+2\int_\Omega A(x)\nabla u\cdot\nabla u=\int_\Omega A(x)\nabla(u+v)\cdot\nabla(u+v).$$

Using the fact that A is symmetric, expanding the right hand side, and cancelling equal terms, we arrive at

$$\int_\Omega A(x)\nabla u\cdot\nabla u-2\int_\Omega A(x)\nabla u\cdot\nabla v+\int_\Omega A(x)\nabla v\cdot\nabla v\leq 0,$$

which can be rewritten as

$$\int_\Omega A(x)\nabla(u-v)\cdot\nabla(u-v)\leq 0.$$

Using (1.8) we therefore have

$$\alpha\,\|u-v\|^2_{H_0^1(\Omega)}\leq 0,$$

which implies $u=v$, as desired.

We are now going to prove that the minimum u is a solution of (1.9) in the sense of (1.10). Given v in $H_0^1(\Omega)$ and t in $\mathbb{R}$, we have $J(u) \leq J(u+tv)$, that is

$$\frac{1}{2}\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega f u \leq \frac{1}{2}\int_\Omega A(x)\nabla(u+tv)\cdot\nabla(u+tv) - \int_\Omega f(u+tv).$$

Expanding the right hand side, cancelling equal terms, and using the fact that A is symmetric, we obtain

$$t\int_\Omega A(x)\nabla u\cdot\nabla v + \frac{t^2}{2}\int_\Omega A(x)\nabla v\cdot\nabla v - t\int_\Omega f\,v \geq 0.$$

If $t > 0$, dividing by t and then letting t tend to zero implies

$$\int_\Omega A(x)\nabla u\cdot\nabla v - \int_\Omega f\,v \geq 0,$$

while if $t < 0$, dividing by t and then letting t tend to zero implies the reverse inequality. It then follows that

$$\int_\Omega A(x)\nabla u\cdot\nabla v = \int_\Omega f\,v, \quad \forall v \in H_0^1(\Omega),$$

and so u solves (1.9) (in the sense of (1.10)). In order to prove that such a solution is unique, we are going to prove that if u solves (1.9), then u is a minimum of J. Indeed, choosing $u - v$ as test function in (1.10), we have

$$\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega A(x)\nabla u\cdot\nabla v = \int_\Omega f(u-v).$$

This implies

$$J(u) + \frac{1}{2}\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega A(x)\nabla u\cdot\nabla v = J(v) - \frac{1}{2}\int_\Omega A(x)\nabla v\cdot\nabla v,$$

which implies $J(u) \leq J(v)$ since

$$\frac{1}{2}\int_\Omega A(x)\nabla u\cdot\nabla u - \int_\Omega A(x)\nabla u\cdot\nabla v + \frac{1}{2}\int_\Omega A(x)\nabla v\cdot\nabla v$$

is nonnegative by (1.8) since it is equal to

$$\frac{1}{2}\int_\Omega A(x)\nabla(u-v)\cdot\nabla(u-v).$$

□

1.5. **A nonlinear equation.** Let now $a : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\alpha \le a(x,s) \le \beta\,, \quad \forall x \in \Omega\,,\ \forall s \in \mathbb{R}\,, \tag{1.12}$$

with $0 < \alpha \le \beta$ in $\mathbb{R}$. Given a function f in $L^{2_*}(\Omega)$, we ask ourselves whether there exists a weak solution of the equation

$$\begin{cases} -\mathrm{div}(a(x,u)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

that is, a function u in $H_0^1(\Omega)$ such that

$$\int_\Omega a(x,u)\,\nabla u \cdot \nabla v = \int_\Omega f\,v\,, \quad \forall v \in H_0^1(\Omega)\,.$$

Remark that, since $a(x,u)$ belongs to $L^\infty(\Omega)$ being a bounded, the integral on the left hand side is well defined.

Of course, since the equation is nonlinear, Lax-Milgram theorem cannot be applied, so that we will need to follow another approach.

The first idea is to perform a change of variable if a does not depend on the variable x. Indeed, in this case, we define

$$A(s) = \int_0^s a(t)\,dt\,,$$

and $v = A(u)$. Then $\nabla v = A'(u)\,\nabla u = a(u)\,\nabla u$, so that u solves (1.13) if and only if v is a solution in $H_0^1(\Omega)$ of $-\Delta v = f$. Since this latter equation has a unique solution v, and since A is invertible (being strictly increasing as its derivative $a(s)$ is positive), then $u = A^{-1}(v)$ is the solution of (1.13).

Unfortunately, such a trick does not work if the function a depends also on the variable x. In this case, even if we can define

$$v(x) = \int_0^{u(x)} a(x,t)\,dt\,,$$

we have

$$\nabla v = a(x,u)\,\nabla u + \int_0^{u(x)} \nabla a(x,t)\,dt\,,$$

so that an extra term appears (which requires a to be C^1).

Luckily, Lax-Milgram theorem (or a change of variable) is not our only tool. We can for example consider the following functional:

$$J(v) = \frac{1}{2}\int_\Omega a(x,v)\,|\nabla v|^2 - \int_\Omega f\,v\,, \quad v \in H_0^1(\Omega)\,.$$

Using the assumptions on a, it is easy to see that J is both coercive and weakly lower semicontinuous in $H_0^1(\Omega)$ (see the proof of Theorem

1.25), so that there exists at least a minimum u of J on $H_0^1(\Omega)$ by Weierstrass' theorem.

Since u is a minimum, then $J(u) \leq J(u+tv)$ for every t in $\mathbb{R}$, and for every v in $H_0^1(\Omega)$. Starting from this inequality, we arrive to

$$\begin{aligned} 0 \leq \frac{1}{2}\int_\Omega [a(x,u+tv)-a(x,u)]|\nabla u|^2 \\ +t\int_\Omega a(x,u+tv)\nabla u\cdot\nabla v + \frac{t^2}{2}\int_\Omega a(x,u+tv)|\nabla v|^2 - t\int_\Omega f\,v\,. \end{aligned}$$

Dividing by $t > 0$, and letting t tend to zero, we will obtain (should every passage be correct, which is not)

$$0 \leq \frac{1}{2}\int_\Omega a'(x,u)\,|\nabla u|^2\,v + \int_\Omega a(x,u)\,\nabla u\cdot\nabla v - \int_\Omega f\,v\,,$$

and the reverse inequality dividing by $t < 0$ and letting t tend to zero. In other words, u in $H_0^1(\Omega)$ would be such that

$$\int_\Omega a(x,u)\,\nabla u\cdot\nabla v + \frac{1}{2}\int_\Omega a'(x,u)\,|\nabla u|^2\,v = \int_\Omega f\,v\,, \quad \forall v \in H_0^1(\Omega)\,.$$

This identity, however, has at least two problems: first of all, the function a is only continuous, so that $a'(x,s)$ may not exist. This, however, can be solved: just suppose that a has a continuous derivative (remark that this assumption is not needed to prove that J has a minimum). Furthermore, even if a has a continuous derivative, the term

$$\int_\Omega a'(x,u)\,|\nabla u|^2\,v$$

is not necessarily well defined for every v in $H_0^1(\Omega)$; indeed, the term $|\nabla u|^2$ only belongs to $L^1(\Omega)$, while $H_0^1(\Omega)$ functions are not necessarily bounded (this only happens if the space dimension is one). Therefore, we have to restrict the class of test functions we consider in the inequality $J(u) \leq J(u+tv)$: we have to consider functions v in $H_0^1(\Omega)\cap L^\infty(\Omega)$; once again, this is not enough: we also need that $a'(x,s)$ is bounded (note that the assumption "a bounded" does not imply "a' bounded"...).

Thus, under all of these assumptions, any minimum u of J is such that

$$\int_\Omega a(x,u)\,\nabla u\cdot\nabla v + \frac{1}{2}\int_\Omega a'(x,u)\,|\nabla u|^2\,v = \int_\Omega f\,v\,,$$

for every v in $H_0^1(\Omega) \cap L^\infty(\Omega)$, so that it is a weak solution of the equation

$$\begin{cases} -\mathrm{div}(a(x,u)\,\nabla u) + \frac{1}{2}\,a'(x,u)\,|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega. \end{cases}$$

But this is not equation (1.13)! Our "functional" idea is wrong — since the derivative of a product is not the product of derivatives...

It is now time for the right way of proving the existence of solutions for (1.13). Let v in $L^2(\Omega)$ be fixed. Then since a is bounded and strictly positive, there exists a unique solution u in $H_0^1(\Omega)$ of

$$\begin{cases} -\mathrm{div}(a(x,v)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{su } \partial\Omega. \end{cases} \tag{1.14}$$

Thus, the function $S : L^2(\Omega) \to H_0^1(\Omega)$ defined by $S(v) = u$ is well defined. Furthermore, since $H_0^1(\Omega)$ is embedded in $L^2(\Omega)$, S is a function from $L^2(\Omega)$ into itself. Thus, a solution of (1.13) is a fixed point for S. We are going to prove the existence of a fixed point using Schauder's theorem 1.11.

To apply Schauder's theorem, we begin by observing that there exists $R > 0$ such that $\|S(v)\|_{L^2(\Omega)} \leq R$ for every v in $L^2(\Omega)$. Indeed, choosing $u = S(v)$ as test function in the weak formulation of (1.14), and using the fact that $a(x,s) \geq \alpha > 0$, we have

$$\alpha \int_\Omega |\nabla u|^2 \leq \int_\Omega a(x,v)\,|\nabla u|^2 = \int_\Omega f\,u \leq \|f\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)}\,. \tag{1.15}$$

Recalling Poincaré inequality, we get

$$\|u\|_{L^2(\Omega)}^2 \leq C\,\|f\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)}\,,$$

which implies the result choosing $R = C\,\|f\|_{L^2(\Omega)}$. Thus, the ball B_R of $L^2(\Omega)$ centered at the origin, and with radius R, is convex, closed and bounded in $L^2(\Omega)$, and S maps B_R into itself. Let us prove that S is continuous. Let $\{v_n\}$ be a sequence which strongly converges to v in $L^2(\Omega)$, and let $u_n = S(v_n)$ be the solutions of (1.14). Using (1.15) we get, recalling that the norm of u_n in $L^2(\Omega)$ is smaller than R,

$$\alpha \int_\Omega |\nabla u_n|^2 \leq R\,\|f\|_{L^2(\Omega)}\,,$$

and so u_n is bounded in $H_0^1(\Omega)$. Thus, up to subsequences, v_n converges to v almost everywhere, and u_n tends to u weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Therefore, we can pass to the limit in the identities

$$\int_\Omega a(x, v_n)\, \nabla u_n \cdot \nabla z = \int_\Omega f\, z\,, \quad \forall z \in H_0^1(\Omega)\,,$$

to prove that u is a solution of (1.14) (with $a(x, v)$), so that $u = S(v)$ (by uniqueness). Since the limit u does not depend on the extracted subsequences, then $u_n = S(v_n)$ converges to $u = S(v)$ in $L^2(\Omega)$, and so S is continuous.

The compactness of $\overline{S(B_R)}$ is easy to prove, since we have proved that $S(B_R)$ is bounded in $H_0^1(\Omega)$: by Rellich-Kondrachov theorem, the closure of $S(B_R)$ is compact in $L^2(\Omega)$, as desired.

Therefore, by Schauder's theorem, there exists at least a solution u of (1.13).

2. Regularity results

Warning to the reader:
from now, unless explicitly stated, $N \geq 3$.

Thanks to the results of the previous section, we have existence of solutions for linear elliptic equations with data f in $L^{2_*}(\Omega)$. The solution u belongs to $H_0^1(\Omega)$ and (thanks to Sobolev embedding) to $L^{2^*}(\Omega)$. One then wonders whether an increase on the regularity of f will yield more regular solutions.

2.1. Examples.

We are going to study a model case, in which the solution of (1.9) can be explictly calculated. This example will give us a hint on what happens in the general case.

Example 2.1. Let $\Omega = B_{\frac{1}{2}}(0)$, let $N \geq 3$, let $\alpha < N$, and define

$$f(x) = \frac{1}{|x|^\alpha\,(-\log(|x|))}.$$

It is well known that f belongs to $L^p(\Omega)$, with $p = \frac{N}{\alpha}$. We are going to study the regularity of the solution u of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

taking advantage of the fact that the solution will be radially symmetric. Recalling the formula for the laplacian in radial coordinates, we have

$$-\frac{1}{\rho^{N-1}}(\rho^{N-1}u'(\rho))' = \frac{1}{\rho^\alpha\,(-\log(\rho))}.$$

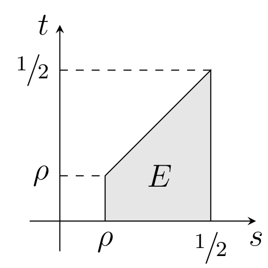
Multiplying by ρ^{N-1} and integrating between 0 and ρ, we obtain

$$\rho^{N-1}\,u'(\rho) = \int_0^\rho \frac{t^{N-1-\alpha}}{\log(t)}\,dt.$$

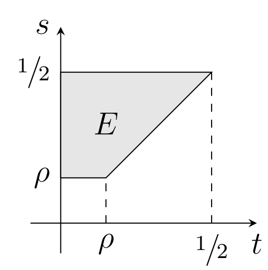
Dividing by ρ^{N-1} and integrating between $\frac{1}{2}$ and ρ we then get (recalling that $u(\frac{1}{2}) = 0$)

$$u(\rho) = -\int_\rho^{\frac{1}{2}} \frac{1}{s^{N-1}} \left(\int_0^s \frac{t^{N-1-\alpha}}{\log(t)}\,dt\right) ds.$$

We are integrating on the set $E = \{(s,t) \in \mathbb{R}^2 : \rho \le s \le \frac{1}{2},\ 0 \le t \le s\}$,



which, after exchanging t with s, becomes $E = \{(t,s) \in \mathbb{R}^2 : 0 \le t \le \frac{1}{2},\ \max(\rho,t) \le s \le \frac{1}{2}\}$,



Exchanging the integration order, we then have

$$\begin{aligned} u(\rho) &= -\int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}}{\log(t)} \left(\int_{\max(\rho,t)}^{\frac{1}{2}} \frac{ds}{s^{N-1}}\right) dt \\ &= \frac{1}{N-2}\int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}}{\log(t)} \left[\left(\frac{1}{2}\right)^{2-N} - (\max(\rho,t))^{2-N}\right] dt \\ &= \frac{2^{N-2}}{N-2}\int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}}{\log(t)} dt - \frac{1}{N-2}\int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}(\max(\rho,t))^{2-N}}{\log(t)}\,dt. \end{aligned}$$

Since $\alpha < N$, the first integral is bounded, so that it is enough to study the behaviour near zero of the function

$$\begin{aligned} v(\rho) &= \int_0^{\frac{1}{2}} \frac{t^{N-1-\alpha}(\max(\rho,t))^{2-N}}{\log(t)}\,dt \\ &= \rho^{2-N} \int_0^{\rho} \frac{t^{N-1-\alpha}}{\log(t)}\,dt + \int_\rho^{\frac{1}{2}} \frac{t^{1-\alpha}}{\log(t)}\,dt \\ &= \rho^{2-N}\, w(\rho) + z(\rho). \end{aligned}$$

It is easy to see (using the de l'Hopital rule), that if $\alpha \neq 2$

$$w(\rho) \approx \frac{\rho^{N-\alpha}}{\log(\rho)}, \quad \text{and} \quad z(\rho) \approx \frac{\rho^{2-\alpha}}{\log(\rho)},$$

as ρ tends to zero, so that, if $\alpha \neq 2$,

$$u(\rho) \approx \frac{\rho^{2-\alpha}}{\log(\rho)},$$

as ρ tends to zero. This implies that u belongs to $L^\infty(\Omega)$ if $\alpha < 2$, while it is in $L^m(\Omega)$, with $m = \frac{N}{\alpha-2}$, if $2 < \alpha < N$. Recalling that f belongs to $L^p(\Omega)$ with $p = \frac{N}{\alpha}$, we therefore have that u belongs to $L^\infty(\Omega)$ if f belongs to $L^p(\Omega)$, and $p > \frac{N}{2}$, while it is in $L^m(\Omega)$, with $m = \frac{Np}{N-2p}$, if f belongs to $L^p(\Omega)$, with $1 < p < \frac{N}{2}$.

If $\alpha = 2$, then

$$w(\rho) \approx \frac{\rho^{N-\alpha}}{\log(\rho)}, \quad \text{and} \quad z(\rho) \approx \log(-\log(\rho)),$$

so that u is in every $L^m(\Omega)$, but not in $L^\infty(\Omega)$, if f belongs to $L^p(\Omega)$ with $p = \frac{N}{2}$. In this case (which we will not study in the following), it can be proved that $\mathrm{e}^{|u|}$ belongs to $L^1(\Omega)$.

Observe that if $\alpha = \frac{N+2}{2}$, so that f belongs to $L^{2_*}(\Omega)$, we get that u belongs to $L^{2^*}(\Omega)$, which is exactly the results we already knew by Sobolev embedding. Also remark that the above example gives informations also if f does not belong to $L^{2_*}(\Omega)$ (i.e., if $\frac{N+2}{2} < \alpha < N$), an assumption under which we do not have any existence results (yet!).

If we want to take $\alpha = N$, we need to change the definition of f. We fix $\beta > 1$ and define

$$f(x) = \frac{1}{|x|^N\,(-\log(|x|))^\beta},$$

which is a function belonging to $L^1(\Omega)$. Performing the same calculations as above, we obtain

$$u(\rho) = \frac{1}{\beta - 1} \int_\rho^{\frac{1}{2}} \frac{dt}{t^{N-1}\,(-\log(t))^{\beta-1}},$$

so that

$$u(\rho) \approx \frac{1}{\rho^{N-2}\,(-\log(\rho))^{\beta-1}},$$

as ρ tends to zero. Observe that in this case f belongs to $L^1(\Omega)$ for every $\beta > 1$, but u belongs to $L^m(\Omega)$, with $m = \frac{N\cdot 1}{N-2\cdot 1} = \frac{N}{N-2}$ if and only if $\beta > 2 - \frac{2}{N}$. If $1 < \beta \le 2 - \frac{2}{N}$, the solution u belongs "only" to $L^m(\Omega)$, for every $m < \frac{N}{N-2}$.

We leave to the interested reader the study of the case $N = 2$.

2.2. **Stampacchia's theorems.** The regularity results we are going to prove now show that the previous example is not just an example. We begin with a real analysis lemma.

Lemma 2.2 (Stampacchia)**.** *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function such that*

$$\psi(h) \le \frac{M\,\psi(k)^\delta}{(h-k)^\gamma}, \quad \forall h > k > 0, \tag{2.16}$$

where $M > 0$, $\delta > 1$ and $\gamma > 0$. Then $\psi(d) = 0$, where

$$d^\gamma = M\,\psi(0)^{\delta-1}\,2^{\frac{\delta\gamma}{\delta-1}}.$$

Proof. Let n in $\mathbb{N}$ and define $d_n = d(1 - 2^{-n})$. We claim that

$$\psi(d_n) \le \psi(0)\,2^{-\frac{n\gamma}{\delta-1}}. \tag{2.17}$$

Indeed, (2.17) is clearly true if $n = 0$; if we suppose that it is true for some n, then, by (2.16),

$$\psi(d_{n+1}) \le \frac{M\,\psi(d_n)^\delta}{(d_{n+1} - d_n)^\gamma} \le M\,\psi(0)^\delta\,2^{-\frac{n\gamma\delta}{\delta-1}}\,2^{(n+1)\gamma}\,d^{-\gamma} = \psi(0)\,2^{-\frac{(n+1)\gamma}{\delta-1}},$$

which is (2.17) written for $n + 1$. Since (2.17) holds for every n, and since ψ is non increasing, we have

$$0 \le \psi(d) \le \liminf_{n\to+\infty} \psi(d_n) \le \lim_{n\to+\infty} \psi(0)^{\delta-1}\,2^{-\frac{n\gamma}{\delta-1}} = 0,$$

as desired. □

The first result (due to Guido Stampacchia, see [6]), deals with bounded solutions for (1.9).

Theorem 2.3 (Stampacchia)**.** *Let f belong to $L^p(\Omega)$, with $p > \frac{N}{2}$. Then the solution u of (1.9) belongs to $L^\infty(\Omega)$, and there exists a constant C, only depending on N, Ω, p and α, such that*

$$\|u\|_{L^\infty(\Omega)} \leq C \, \|f\|_{L^p(\Omega)} \,. \tag{2.18}$$

Proof. Let $k > 0$ and choose $v = G_k(u)$ as test function in (1.9) ($G_k(s)$ has been defined in (1.7)). Defining $A_k = \{x \in \Omega : |u(x)| \geq k\}$ one then has, since $\nabla v = \nabla u \, \chi_{A_k}$ by Theorem 1.21, and using (1.8)

$$\alpha \int_{A_k} |\nabla G_k(u)|^2 \leq \int_\Omega A(x)\nabla u \cdot \nabla u \, \chi_{A_k} = \int_\Omega f \, G_k(u) = \int_{A_k} f \, G_k(u).$$

Using Sobolev inequality (in the left hand side), and Hölder inequality (in the right hand side), one has

$$\frac{\alpha}{\mathcal{S}_2^2} \left(\int_{A_k} |G_k(u)|^{2^*} \right)^{\frac{2}{2^*}} \leq \left(\int_{A_k} |f|^{2_*} \right)^{\frac{1}{2_*}} \left(\int_{A_k} |G_k(u)|^{2^*} \right)^{\frac{1}{2^*}} .$$

Simplifying equal terms, we thus have

$$\int_{A_k} |G_k(u)|^{2^*} \leq \left(\frac{\mathcal{S}_2^2}{\alpha} \right)^{2^*} \left(\int_{A_k} |f|^{2_*} \right)^{\frac{2^*}{2_*}} .$$

Recalling that f belongs to $L^p(\Omega)$, and that $p > 2_*$ since $p > \frac{N}{2}$, we have (again by Hölder inequality)

$$\int_{A_k} |G_k(u)|^{2^*} \leq \left(\frac{\mathcal{S}_2^2 \, \|f\|_{L^p(\Omega)}}{\alpha} \right)^{2^*} m(A_k)^{\frac{2^*}{2_*} - \frac{2^*}{p}} .$$

We now take $h > k$, so that $A_h \subseteq A_k$, and $G_k(u) \geq h - k$ on A_h. Thus,

$$(h-k)^{2^*} \, m(A_h) \leq \left(\frac{\mathcal{S}_2^2 \, \|f\|_{L^p(\Omega)}}{\alpha} \right)^{2^*} m(A_k)^{\frac{2^*}{2_*} - \frac{2^*}{p}} ,$$

which implies

$$m(A_h) \leq \left(\frac{\mathcal{S}_2^2 \, \|f\|_{L^p(\Omega)}}{\alpha} \right)^{2^*} \frac{m(A_k)^{\frac{2^*}{2_*} - \frac{2^*}{p}}}{(h-k)^{2^*}} .$$

We define now $\psi(k) = m(A_k)$, so that

$$\psi(h) \leq \frac{M \, \psi(k)^\delta}{(h-k)^\gamma} ,$$

where

$$M = \left(\frac{\mathcal{S}_2^2 \, \|f\|_{L^p(\Omega)}}{\alpha} \right)^{2^*} , \quad \delta = \frac{2^*}{2_*} - \frac{2^*}{p} , \quad \gamma = 2^* .$$

The assumption $p > \frac{N}{2}$ implies $\delta > 1$, so that applying Lemma 2.2, we have that $\psi(d) = 0$, where

$$d^{2^*} = C(\Omega, N, p)\, M.$$

Since $m(A_d) = 0$, we have $|u| \leq d$ almost everywhere, which implies

$$\|u\|_{L^\infty(\Omega)} \leq d = C(N, \Omega, p, \alpha)\ \|f\|_{L^p(\Omega)}\,,$$

as desired. □

Remark 2.4. Observe that, in order to prove the previous theorem, we did not use two of the properties of the equation: that the matrix A is bounded from above (we only used its ellipticity) and, above all, the fact that the equation was *linear*: in other words, the proof above also holds for every uniformly elliptic operator (for example, for the equation studied in §5 of the previous section).

The second results deals with the case of unbounded solutions.

Theorem 2.5 (Stampacchia)**.** *Let f belong to $L^p(\Omega)$, with $2_* \leq p < \frac{N}{2}$. Then the solution u of (1.9) belongs to $L^m(\Omega)$, with $m = p^{**} = \frac{Np}{N-2p}$, and there exists a constant C, only depending on N, Ω, p and α, such that*

$$\|u\|_{L^{p^{**}}(\Omega)} \leq C\ \|f\|_{L^p(\Omega)}\,. \tag{2.19}$$

Proof. We begin by observing that if $p = 2_*$, then $p^{**} = 2^*$, so that the result is true in this limit case by the Sobolev embedding. Therefore, we only have to deal with the case $p > 2_*$.

The original proof of Stampacchia used a linear interpolation theorem; i.e., it is typical of a linear framework. We are going to give another proof, following [1], which makes use of a technique that can be applied also in a nonlinear context.

Let $k > 0$ be fixed, let $\gamma > 1$ and choose $v = |T_k(u)|^{2\gamma-2}\, T_k(u)$ as test function in (1.9) ($T_k(s)$ has been defined in (1.6)). We obtain, by Theorem 1.21,

$$(2\gamma - 1)\int_\Omega A(x)\nabla u \cdot \nabla T_k(u)\, |T_k(u)|^{2\gamma-2} = \int_\Omega f\, |T_k(u)|^{2\gamma-2}\, T_k(u).$$

Using (1.8), and observing that $\nabla u = \nabla T_k(u)$ where $\nabla T_k(u) \neq 0$, we then have

$$\alpha\,(2\gamma - 1)\int_\Omega |\nabla T_k(u)|^2\, |T_k(u)|^{2\gamma-2} \leq \int_\Omega |f|\, |T_k(u)|^{2\gamma-1}.$$

Since, again by Theorem 1.21, $|\nabla T_k(u)|^2\,|T_k(u)|^{2\gamma-2} = \frac{1}{\gamma^2}\,|\nabla |T_k(u)|^\gamma|^2$, we have

$$\frac{\alpha\,(2\gamma-1)}{\gamma^2}\int_\Omega |\nabla |T_k(u)|^\gamma|^2 \le \int_\Omega |f|\,|T_k(u)|^{2\gamma-1}.$$

Using Sobolev inequality (in the left hand side), and Hölder inequality (in the right hand one), we obtain

$$\frac{\alpha\,(2\gamma-1)}{\mathcal{S}_2^2\gamma^2}\left(\int_\Omega |T_k(u)|^{\gamma 2^*}\right)^{\frac{2}{2^*}} \le \|f\|_{L^p(\Omega)}\left(\int_\Omega |T_k(u)|^{(2\gamma-1)p'}\right)^{\frac{1}{p'}}.$$

We now choose γ so that $\gamma 2^* = (2\gamma-1)p'$, that is $\gamma = \frac{p^{**}}{2^*}$ (as it is easily seen). With this choice, $\gamma > 1$ if and only if $p > 2_*$ (which is true). Since $p < \frac{N}{2}$, we also have $\frac{2}{2^*} > \frac{1}{p'}$, and so

$$\left(\int_\Omega |T_k(u)|^{p^{**}}\right)^{\frac{2}{2^*}-\frac{1}{p'}} \le C(N,\Omega,p,\alpha)\,\|f\|_{L^p(\Omega)}\,.$$

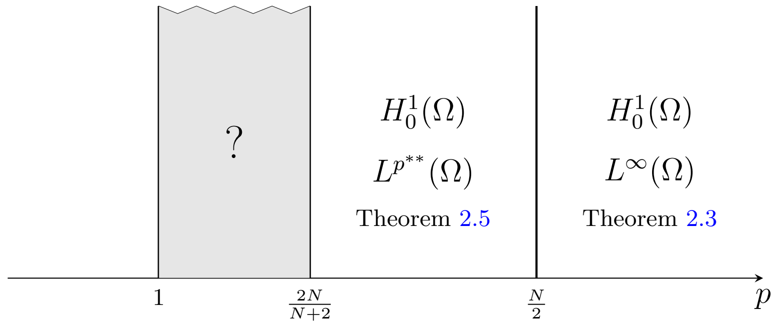
Observing that $\frac{2}{2^*} - \frac{1}{p'} = \frac{1}{p^{**}}$, we have therefore proved that

$$\|T_k(u)\|_{L^{p^{**}}(\Omega)} \le C(N,\Omega,p,\alpha)\,\|f\|_{L^p(\Omega)}\,,\quad \forall k > 0.$$

Letting k tend to infinity, and using Fatou lemma (or the monotone convergence theorem), we obtain the result. □

Remark 2.6. The results of theorems 2.3 and 2.5 are somehow "natural" if we make a mistake... Indeed, let u be the solution of $-\Delta u = f$, with f in $L^p(\Omega)$. Then, if we read the equation, we have that u has two derivatives in $L^p(\Omega)$, so that it belongs to $W_0^{2,p}(\Omega)$. By Sobolev embedding, u then belongs to $W_0^{1,p^*}(\Omega)$ and, again by Sobolev embedding, to $L^{p^{**}}(\Omega)$ (or to $L^\infty(\Omega)$ if $p > \frac{N}{2}$). The "mistake" here is to deduce from the fact that the sum of (some) derivatives of u belongs to $L^p(\Omega)$, the fact that all derivatives are in the same space. Surprisingly, it turns out that, in the case of the laplacian, the fact that $-\Delta u$ belongs to $L^p(\Omega)$ actually implies that u is in $W_0^{2,p}(\Omega)$ (this is the so-called Calderun-Zygmund theory), so that the "mistake" is not an actual one...

Summarizing the results of this section, we have the following picture.



We will deal with the "?" part in the forthcoming section (actually, in all the forthcoming sections).

3. Existence via duality for measure data

We are now going to deal with existence results for data which do not belong to $L^{2_*}(\Omega)$ (i.e., they are not in $H^{-1}(\Omega)$), so that neither Lax-Milgram theorem nor minimization techniques can be applied. Before going on, we need some definitions.

3.1. **Measures.** We recall that a *nonnegative measure* on Ω is a set function $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ defined on the σ-algebra $\mathcal{B}(\Omega)$ of Borel sets of Ω (i.e., the smallest σ-algebra containing the open sets) such that $\mu(\emptyset) = 0$ and such that

$$\mu\Big(\bigcup_{n=1}^{+\infty} E_n\Big) = \sum_{n=1}^{+\infty} \mu(E_n),$$

for every sequence $\{E_n\}$ of *disjoint* sets in $\mathcal{B}(\Omega)$. This latter property is called σ-additivity. A σ-additive measure μ is also σ-subadditive, i.e., one has

$$\mu\Big(\bigcup_{n=1}^{+\infty} E_n\Big) \leq \sum_{n=1}^{+\infty} \mu(E_n),$$

for every sequence $\{E_n\}$ of sets in $\mathcal{B}(\Omega)$. A nonnegative measure μ is also *monotone*, i.e., one has that

$$A \subseteq B \quad \text{implies} \quad \mu(A) \leq \mu(B).$$

A measure μ is said to be *regular* if for every E in $\mathcal{B}(\Omega)$ and for every $\varepsilon > 0$ there exist an open set A_ε, and a closed set C_ε, such that

$$C_\varepsilon \subseteq E \subseteq A_\varepsilon, \quad \mu(A_\varepsilon \setminus C_\varepsilon) < \varepsilon.$$

A measure μ is said to be *bounded* if $\mu(\Omega) < +\infty$. The set of nonnegative, regular, bounded measures on Ω will be denoted by $\mathcal{M}^+(\Omega)$. We define the set of *bounded Radon measures* on Ω as

$$\mathcal{M}(\Omega) = \{\mu_1 - \mu_2,\ \mu_i \in \mathcal{M}^+(\Omega)\}.$$

Given a measure μ in $\mathcal{M}(\Omega)$, there exists a unique pair (μ^+, μ^-) in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\Omega)$ such that

$$\mu = \mu^+ - \mu^-,$$

and such there exist E^+ and E^- in $\mathcal{B}(\Omega)$, *disjoint sets*, such that

$$\mu^\pm(E) = \mu(E \cap E^\pm), \quad \forall E \in \mathcal{B}(\Omega).$$

The measures μ^+ and μ^- are the *positive* and *negative* parts of the measure μ. Given a measure μ in $\mathcal{M}(\Omega)$, the measure $|\mu| = \mu^+ + \mu^-$ is said to be the *total variation* of the measure μ. If we define

$$\|\mu\|_{\mathcal{M}(\Omega)} = |\mu|(\Omega),$$

the vector space $\mathcal{M}(\Omega)$ becomes a Banach space, which turns out to be the dual of $C^0(\overline{\Omega})$.

A bounded Radon measure μ is said to be *concentrated* on a Borel set E if $\mu(B) = \mu(B \cap E)$ for every Borel set B. In this case, we will write $\mu \llcorner E$. For example, we have $\mu^\pm = \mu \llcorner E^\pm$, with $E^\pm$ as above.

Given two Radon measures μ and ν, we say that μ is *absolutely continuous* with respect to ν if $\nu(E) = 0$ implies $\mu(E) = 0$. In this case we will write $\mu << \nu$. Two Radon measures μ and ν are said to be *orthogonal* if there exists a set E such that $\mu(E) = 0$, and $\nu = \nu \llcorner E$. In this case, we will write $\mu \perp \nu$. For example, given a Radon measure μ, we have $\mu^+ \perp \mu^-$.

Theorem 3.1. *Let ν be a nonnegative Radon measure. Given a Radon measure μ, there exists a unique pair (μ_0, μ_1) of Radon measures such that*

$$\mu = \mu_0 + \mu_1, \quad \mu_0 << \nu, \quad \mu_1 \perp \nu.$$

Proof. Suppose that μ is nonnegative, and define

$$\mathcal{A} = \{\mu(E) : E \in \mathcal{B}(\Omega),\ \nu(E) = 0\}.$$

Let $\alpha = \sup \mathcal{A}$, and let E_n be a maximizing sequence, i.e., a sequence of Borel sets such that

$$\lim_{n\to+\infty} \mu(E_n) = \alpha, \quad \nu(E_n) = 0.$$

If we define E as the union of the E_n, clearly $\nu(E) = 0$ (since ν is σ-subadditive), and $\mu(E) = \alpha$ (since $\mu(E) \geq \mu(E_n)$ for every n). Define now

$$\mu_1 = \mu \llcorner E, \quad \mu_0 = \mu - \mu_1.$$

Clearly, $\mu_1 \perp \nu$ (since $\nu(E) = 0$, and since μ_1 is concentrated on E by definition). On the other hand, if $\nu(B) = 0$, then $\mu_0(B) = 0$; and indeed, if it were $\mu_0(B) > 0$ for some $B \neq E$, then

$$0 < \mu_0(B) = \mu(B) - \mu(B \cap E) = \mu(B \setminus E),$$

so that $B \cup E$ will be such that $\nu(B \cup E) = 0$, and

$$\mu(B \cup E) = \mu(E) + \mu(B \setminus E) = \alpha + \mu(B \setminus E) > \alpha,$$

thus contradicting the definition of α.

As for uniqueness, if $\mu = \mu_0 + \mu_1 = \mu_0' + \mu_1'$, then $\mu_0 - \mu_0' = \mu_1' - \mu_1$. If $\nu(B) = 0$, we will have $(\mu_1 - \mu_1')(B) = 0$. Since $\mu_1 - \mu_1'$ is also orthogonal with respect to ν, this implies that $(\mu_1 - \mu_1')(E) = 0$ for every Borel set E, so that $\mu_1 = \mu_1'$, hence $\mu_0 = \mu_0'$.

If the measure μ has a sign, it is enough to apply the result to μ^+ and μ^-. □

Examples of bounded Radon measures are the Lebesgue measure $\mathcal{L}^N$ concentrated on a bounded set of $\mathbb{R}^N$, or the measure defined by

$$\delta_{x_0}(E) = \begin{cases} 1 & \text{if } x_0 \in E, \\ 0 & \text{if } x_0 \notin E, \end{cases}$$

which is called the *Dirac's delta* concentrated at x_0. We clearly have $\delta_{x_0} \perp \mathcal{L}^N$. Another example of Radon measure is the measure defined by

$$\mu(E) = \int_E f(x)\,dx,$$

with f a function in $L^1(\Omega)$. In this case $\mu << \mathcal{L}^N$, and

$$\mu^{\pm}(E) = \int_E f^{\pm}(x)\,dx, \quad |\mu|(E) = \int_E |f(x)|\,dx.$$

Therefore, $L^1(\Omega) \subset \mathcal{M}(\Omega)$. For sequences of measures, we have two notions of convergence: the weak*:

$$\int_\Omega \varphi\,d\mu_n \to \int_\Omega \varphi\,d\mu, \quad \forall \varphi \in C_0^0(\Omega),$$

and the *narrow convergence*:

$$\int_\Omega \varphi\,d\mu_n \to \int_\Omega \varphi\,d\mu, \quad \forall \varphi \in C_{\mathrm{b}}^0(\Omega).$$

For positive measures, narrow convergence is equivalent to weak* convergence and convergence of the "masses" (i.e., $\mu_n(\Omega)$ converges to $\mu(\Omega)$). If x_n is a sequence in Ω which converges to a point x_0 on $\partial\Omega$, then δ_{x_n} converges to zero for the weak* convergence (since the measure δ_{x_0} is indeed the zero measure in Ω), but not for the narrow convergence.

Measures can be approximated (in either convergence) by sequences of bounded functions.

Before dealing with existence results for elliptic equations with measure data, we will begin with a particular case.

3.2. **Duality solutions for L^1 data.** Let f and g be two functions in $L^\infty(\Omega)$, and let u and v be the solutions of

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\mathrm{div}(A^*(x)\,\nabla v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

where A^* is the transposed matrix of A (note that A^* satisfies (1.8) with the same constants as A). Since both u and v belong to $H_0^1(\Omega)$, u can be chosen as test function in the formulation of weak solution for v, and *vice versa*. One obtains

$$\int_\Omega u\,g = \int_\Omega A^*(x)\nabla v \cdot \nabla u = \int_\Omega A(x)\nabla u \cdot \nabla v = \int_\Omega f\,v.$$

In other words, one has

$$\int_\Omega u\,g = \int_\Omega f\,v,$$

for every f and g in $L^\infty(\Omega)$, where u and v solve the corresponding problems with data f and g respectively. Clearly, both u and v belong to $L^\infty(\Omega)$ by Theorem 2.3, but we remark that the two integrals are well-defined also if f only belongs to $L^1(\Omega)$, and u only belongs to $L^1(\Omega)$ (always maintaining the assumption that g — and so v — is a bounded function). This fact inspired to Guido Stampacchia the following definition of solution for (1.9) if the datum is in $L^1(\Omega)$.

Definition 3.2. Let f belong to $L^1(\Omega)$. A function u in $L^1(\Omega)$ is a *duality solution* of (1.8) with datum f if one has

$$\int_\Omega u\,g = \int_\Omega f\,v,$$

for every g in $L^\infty(\Omega)$, where v is the solution of

$$\begin{cases} -\mathrm{div}(A^*(x)\,\nabla v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 3.3 (Stampacchia)**.** *Let f belong to $L^1(\Omega)$. Then there exists a unique duality solution of (1.8) with datum f. Furthermore, u belongs to $L^q(\Omega)$, for every $q < \frac{N}{N-2}$.*

Proof. Let $p > \frac{N}{2}$ and define the linear functional $T : L^p(\Omega) \to \mathbb{R}$ by

$$\langle T, g\rangle = \int_\Omega f\, v.$$

By Theorem 2.3, the functional is well-defined; furthermore, since (2.18) holds, there exists $C > 0$ such that

$$|\langle T, g\rangle| \leq \int_\Omega |f|\,|v| \leq \|f\|_{L^1(\Omega)}\,\|v\|_{L^\infty(\Omega)} \leq C\,\|f\|_{L^1(\Omega)}\,\|g\|_{L^p(\Omega)}\,,$$

so that T is continuous on $L^p(\Omega)$. By Riesz representation Theorem for L^p spaces, there exists a unique function u_p in $L^{p'}(\Omega)$ such that

$$\langle T, g\rangle = \int_\Omega u_p\, g, \quad \forall g \in L^p(\Omega).$$

Since $L^\infty(\Omega) \subset L^p(\Omega)$, we have

$$\int_\Omega u_p\, g = \langle T, g\rangle = \int_\Omega f\, v, \quad \forall g \in L^\infty(\Omega),$$

so that u_p is a duality solution of (1.9), as desired. We claim that u_p does not depend on p; indeed, if for example $p > q > \frac{N}{2}$, we have

$$\int_\Omega u_p\, g = \int_\Omega f\, v = \int_\Omega u_q\, g, \quad \forall g \in L^\infty(\Omega),$$

so that $u_p = u_q$ in $L^1(\Omega)$ (and so they are almost everywhere the same function). Therefore, there exists a unique function u which is a duality solution of (1.9), and it belongs to $L^{p'}(\Omega)$ for every $p > \frac{N}{2}$; i.e., u belongs to $L^q(\Omega)$ for every $q < \frac{N}{N-2}$, as desired. □

Remark that the fact that u belongs to $L^q(\Omega)$ for every $q < \frac{N}{N-2}$ is consistent with the results of the last part of Example 2.1 (the case $\alpha = N$).

3.3. **Duality solutions for measure data.** The case of $L^1(\Omega)$ data is only a particular one, since $L^1(\Omega)$ is a subset of $\mathcal{M}(\Omega)$. However, recalling that $\mathcal{M}(\Omega)$ is the dual of $C^0(\overline{\Omega})$, the proof of Theorem 3.3 could be performed in exactly the same way if one knew that the solution of (1.9) were not only bounded, but also *continuous* on Ω if the datum is in $L^p(\Omega)$ with $p > \frac{N}{2}$. This is exactly the case if the boundary of Ω is sufficiently regular.

Theorem 3.4 (De Giorgi)**.** *Let Ω be of class C^1, and let f be in $L^p(\Omega)$, with $p > \frac{N}{2}$. Then the solution u of (1.9) with datum f belongs to $C^0(\overline{\Omega})$, and there exists a constant C_p such that*

$$\|u\|_{C^0(\overline{\Omega})} \leq C_p\,\|f\|_{L^p(\Omega)}\,.$$

Thanks to the previous result, we thus have the following existence result.

Theorem 3.5. *Let μ be a measure in $\mathcal{M}(\Omega)$. Then there exists a unique duality solution of (1.8) with datum μ, i.e., a function u in $L^1(\Omega)$ such that*

$$\int_\Omega u\, g = \int_\Omega v\, d\mu, \quad \forall g \in L^\infty(\Omega),$$

where v is the solution of (1.9) with datum g and matrix A^*. Furthermore, u belongs to $L^q(\Omega)$, for every $q < \frac{N}{N-2}$.

3.4. **Regularity of duality solutions.** If the datum f belongs to $L^p(\Omega)$, with $1 < p < 2_*$, then the duality solution of (1.9) is more regular.

Theorem 3.6. *Let f belong to $L^p(\Omega)$, $1 < p < 2_*$. Then the duality solution u of (1.8) belongs to $L^{p^{**}}(\Omega)$, $p^{**} = \frac{Np}{N-2p}$.*

Proof. Let $q = \frac{Np}{Np-N+2p}$, and define $T : L^q(\Omega) \to \mathbb{R}$ as in the proof of Theorem 3.3. We then have

$$|\langle T, g\rangle| \leq \int_\Omega |f|\,|v| \leq \|f\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)}.$$

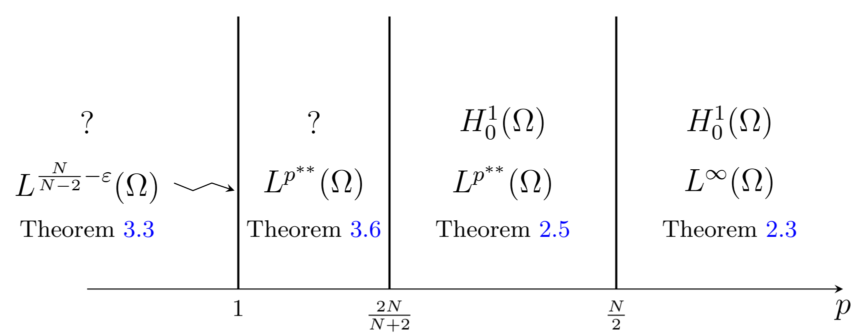
By Theorem 2.5, the norm of v in $L^r(\Omega)$ is controlled by a constant times the norm of g in $L^s(\Omega)$, with $r = s^{**}$. Taking $r = p'$, this gives $s = q$; hence,

$$|\langle T, g\rangle| \leq C \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)},$$

so that the function u which represents T belongs to $L^{q'}(\Omega)$; since we have $q' = \frac{Np}{N-2p}$, the result is proved. □

Once again, the fact that u belongs to $L^{p^{**}}(\Omega)$ is consistent with the results of Example 2.1 (the case $\frac{N+2}{2} < \alpha < N$).

The picture at the end of Section 2 can now be improved as follows.



4. Existence via approximation for measure data

The result of Theorem 3.5 is somewhat unsatisfactory: even though it proves that there exists a unique solution by duality of (1.9) if the datum belongs to $\mathcal{M}(\Omega)$, it only states that the solution belongs to some Lebesgue space, and does not say anything about the gradient of such a solution. In order to prove gradient estimates on the duality solution we have to proceed in a different way.

Theorem 4.1. *Let μ belong to $\mathcal{M}(\Omega)$. Then the unique duality solution of (1.8) with datum f belongs to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$.*

Proof. Let f_n be a sequence of $L^\infty(\Omega)$ functions which converges to μ in $\mathcal{M}(\Omega)$, with the property that $\|f_n\|_{L^1(\Omega)} \leq \|\mu\|_{\mathcal{M}(\Omega)}$, and let u_n be the unique solution in $H_0^1(\Omega)$ of

$$\begin{cases} -\operatorname{div}(A(x)\,\nabla u_n) = f_n & \text{in } \Omega, \\ \quad u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $k > 0$ and choose $v = T_k(u_n)$ as test function of the weak formulation for u_n. We obtain, recalling that $\nabla u_n = \nabla T_k(u_n)$ where $\nabla T_k(u_n) \neq 0$, and using (1.8),

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq \int_\Omega A(x)\nabla u_n \cdot \nabla T_k(u_n) = \int_\Omega f_n T_k(u_n) \leq k\, \|\mu\|_{\mathcal{M}(\Omega)},$$

where in the last passage we have used that $|T_k(u_n)| \leq k$. Using Sobolev embedding in the left hand side, we have

$$\frac{\alpha}{\mathcal{S}_2^2} \left(\int_\Omega |T_k(u_n)|^{2^*} \right)^{\frac{2}{2^*}} \leq k\, \|\mu\|_{\mathcal{M}(\Omega)}.$$

Observing that $|T_k(u_n)| = k$ on the set $A_{n,k} = \{x \in \Omega : |u_n(x)| \geq k\}$, we have

$$\frac{\alpha}{\mathcal{S}_2^2} k^2 \left(m(A_{n,k})\right)^{\frac{2}{2^*}} \leq k\, \|\mu\|_{\mathcal{M}(\Omega)},$$

which implies

$$m(A_{n,k}) \leq C \Big(\frac{\|\mu\|_{\mathcal{M}(\Omega)}}{k} \Big)^{\frac{N}{N-2}},$$

with C depending only on N and α. Now we fix $\lambda > 0$, and we have

$$\{|\nabla u_n| \geq \lambda\} = \{|\nabla u_n| \geq \lambda, |u_n| < k\} \cup \{|\nabla u_n| \geq \lambda, |u_n| \geq k\},$$

so that

$$\{|\nabla u_n| \geq \lambda\} \subset \{|\nabla u_n| \geq \lambda, |u_n| < k\} \cup A_{n,k}.$$

Since

$$m(\{|\nabla u_n| \geq \lambda, |u_n| < k\}) \leq \frac{1}{\lambda^2} \int_\Omega |\nabla T_k(u_n)|^2 \leq \frac{k\ \|\mu\|_{\mathcal{M}(\Omega)}}{\lambda^2},$$

we have

$$m(\{|\nabla u_n| \geq \lambda\}) \leq \frac{k\ \|\mu\|_{\mathcal{M}(\Omega)}}{\lambda^2} + C\Big(\frac{\|\mu\|_{\mathcal{M}(\Omega)}}{k}\Big)^{\frac{N}{N-2}},$$

for every $k > 0$. If we choose $k = \lambda^{\frac{N-2}{N-1}}\ \|\mu\|_{\mathcal{M}(\Omega)}^{\frac{1}{N-1}}$, the above inequality becomes

$$m(\{|\nabla u_n| \geq \lambda\}) \leq C\Big(\frac{\|\mu\|_{\mathcal{M}(\Omega)}}{\lambda}\Big)^{\frac{N}{N-1}}.$$

Let $q < \frac{N}{N-1}$ be fixed, and let $t > 0$. Then

$$\begin{aligned}\int_\Omega |\nabla u_n|^q &= \int_{\{|\nabla u_n| < t\}} |\nabla u_n|^q + \int_{\{|\nabla u_n| \geq t\}} |\nabla u_n|^q \\ &\leq t^q\, m(\Omega) + (q-1) \int_t^{+\infty} \lambda^{q-1}\, m(\{|\nabla u_n| \geq \lambda\})\, d\lambda \\ &\leq t^q\, m(\Omega) + C(q-1)\, \|f\|_{L^1(\Omega)}^{\frac{N}{N-1}} \int_t^{+\infty} \lambda^{q-1-\frac{N}{N-1}}\, d\lambda \\ &= t^q\, m(\Omega) + \frac{C(q-1)}{\frac{N}{N-1} - q} \frac{\|\mu\|_{\mathcal{M}(\Omega)}^{\frac{N}{N-1}}}{t^{\frac{N}{N-1}-q}}.\end{aligned}$$

Choosing $t = \|\mu\|_{\mathcal{M}(\Omega)}$, we obtain

$$\int_\Omega |\nabla u_n|^q \leq C_q\ \|\mu\|_{\mathcal{M}(\Omega)}^q\,, \tag{4.20}$$

so that u_n is bounded in $W_0^{1,q}(\Omega)$, with $q < \frac{N}{N-1}$. Note that C_q diverges as q tends to $\frac{N}{N-1}$. Therefore, up to subsequences, u_n converges to some function u_q weakly in $W_0^{1,q}(\Omega)$ and strongly in $L^1(\Omega)$. Since u_n, being a weak solution, is such that

$$\int_\Omega u_n\, g = \int_\Omega f_n\, v, \quad \forall g \in L^\infty(\Omega),\ \forall n \in \mathbb{N},$$

we can pass to the limit as n tends to infinity to have

$$\int_\Omega u_q\, g = \int_\Omega v\, d\mu, \quad \forall g \in L^\infty(\Omega),$$

so that u_q (which belongs to $W_0^{1,q}(\Omega)$ for *some* $q < \frac{N}{N-1}$) is **the** duality solution of (1.9) with datum μ. This fact is true for *every* $q < \frac{N}{N-1}$, so that u_q does not depend on q. It then follows that the duality solution u of (1.9) belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. □

Remark 4.2. If $\mu = f$ is a function in $L^1(\Omega)$, and f_n converges to f strongly in $L^1(\Omega)$, we have that f_n is a Cauchy sequence in $L^1(\Omega)$. Thus, if we repeat the proof of the previous theorem working with $u_n - u_m$, using the linearity of the operator, and "keeping track" of $f_n - f_m$, we find that (4.20) becomes

$$\int_\Omega |\nabla u_n - u_m|^q \leq C_q \, \|f_n - f_m\|^q_{L^1(\Omega)} ,$$

for every $q < \frac{N}{N-1}$. Since $\{f_n\}$ is a Cauchy sequence in $L^1(\Omega)$, it then follows that u_n is a Cauchy sequence in $W^{1,q}_0(\Omega)$, for every $q < \frac{N}{N-1}$. This implies that u_n *strongly* converges to the solution u in $W^{1,q}_0(\Omega)$, for every $q < \frac{N}{N-1}$, so that (up to subsequences) ∇u_n converges to ∇u almost everywhere in Ω.

Remark 4.3. If $\mu = f$ is a function in $L^1(\Omega)$, and we repeat the proof of the previous theorem working with $u_n - v_n$, where v_n is the solution of (1.9) with a datum g_n which converges to f in $L^1(\Omega)$, we find as before that

$$\int_\Omega |\nabla (u_n - v_n)|^q \leq C \, \|f_n - g_n\|^q_{L^1(\Omega)} , \tag{4.21}$$

for every $q < \frac{N}{N-1}$. Since $\{f_n - g_n\}$ tends to zero in $L^1(\Omega)$, it then follows that $u_n - v_n$ tends to zero in $W^{1,q}_0(\Omega)$, for every $q < \frac{N}{N-1}$. In other words, the solution u found by approximation does not depend on the sequence we choose to approximate the datum f. We already knew this fact (since every approximating sequence converges to the duality solution which is unique), but this different proof may be useful if, for example, the differential operator is not linear, but allows to prove (4.21) in some way, so that the concept of duality solution is not available.

If the datum f is "more regular", one expects solutions with an increased regularity. We already know, from Theorem 3.6, that the summability of u increases with the summability of f, but what happens to the gradient? Recall that if the datum f is "regular" (i.e., if it belongs to $L^{2_*}(\Omega)$), the summability of u increases with that of f, but the gradient of u always belongs to $(L^2(\Omega))^N$. Surprisingly, this is not the case for "bad" solutions, as the following theorem shows.

Theorem 4.4. *Let f be a function in $L^m(\Omega)$, $1 < m < 2_*$. Then the duality solution of (1.9) belongs to $W^{1,m^*}_0(\Omega)$, $m^* = \frac{Nm}{N-m}$.*

Proof. Let $f_n = T_n(f)$, and let u_n be the unique solution of

$$\begin{cases} -\mathrm{div}(A(x)\,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Since we already know that u_n will converge to the duality solution of (1.9), it is clear that in order to prove the result it will be enough to prove an *a priori* estimate on u_n in $W_0^{1,m^*}(\Omega)$. In order to do that, we fix $h > 0$ and choose $\varphi_h(u_n) = T_1(G_h(u_n))$ as test function in the weak formulation for u_n. If we define $B_h = \{x \in \Omega : h \le |u_n| \le h+1\}$, and $A_h = \{x \in \Omega : |u_n| \ge h\}$ (for the sake of simplicity, we omit the dependence on n on the sets), we obtain, recalling (1.8),

$$\alpha \int_{B_h} |\nabla u_n|^2 \le \int_\Omega A(x)\nabla u_n \cdot \nabla \varphi_h(u_n) = \int_\Omega f_n\, \varphi_h(u_n) \le \int_{A_k} |f|.$$

Let now $0 < \lambda < 1$; we can then write

$$\begin{aligned}
\int_\Omega \frac{|\nabla u_n|^2}{(1+|u|)^\lambda} &= \sum_{h=0}^{+\infty} \int_{B_h} \frac{|\nabla u_n|^2}{(1+|u_n|)^\lambda} \le \sum_{h=0}^{+\infty} \frac{1}{(1+h)^\lambda} \int_{B_h} |\nabla u_n|^2 \\
&\le \sum_{h=0}^{+\infty} \frac{1}{\alpha(1+h)^\lambda} \int_{A_h} |f| = \sum_{h=0}^{+\infty} \frac{1}{\alpha(1+h)^\lambda} \sum_{k=h}^{+\infty} \int_{B_k} |f| \\
&= \sum_{k=0}^{+\infty} \int_{B_k} |f| \sum_{h=0}^{k} \frac{1}{\alpha(1+h)^\lambda} \\
&\le C \sum_{k=0}^{+\infty} \int_{B_k} |f|\,(1+k)^{1-\lambda} \le C \int_\Omega |f|(1+|u_n|)^{1-\lambda} \\
&\le C\, \|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^{(1-\lambda)m'} \right)^{\frac{1}{m'}}.
\end{aligned}$$

Let now $q > 1$ be fixed. Then, by Sobolev and Hölder inequality,

$$\begin{aligned}
\frac{1}{\mathcal{S}_q^q} \left(\int_\Omega |u_n|^{q^*} \right)^{\frac{q}{q^*}} &\le \int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{\lambda \frac{q}{2}}} (1+|u_n|)^{\lambda \frac{q}{2}} \\
&\le \left(\int_\Omega \frac{|\nabla u_n|^2}{(1+|u|)^\lambda} \right)^{\frac{q}{2}} \left(\int_\Omega (1+|u_n|)^{\frac{\lambda q}{2-q}} \right)^{1-\frac{q}{2}} \\
&\le C\, \|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^{(1-\lambda)m'} \right)^{\frac{q}{2m'}} \\
&\quad \times \left(\int_\Omega (1+|u_n|)^{\frac{\lambda q}{2-q}} \right)^{1-\frac{q}{2}}.
\end{aligned}$$

We now choose λ and q in such a way that

$$(1-\lambda)m' = q^* = \frac{\lambda q}{2-q}.$$

This implies

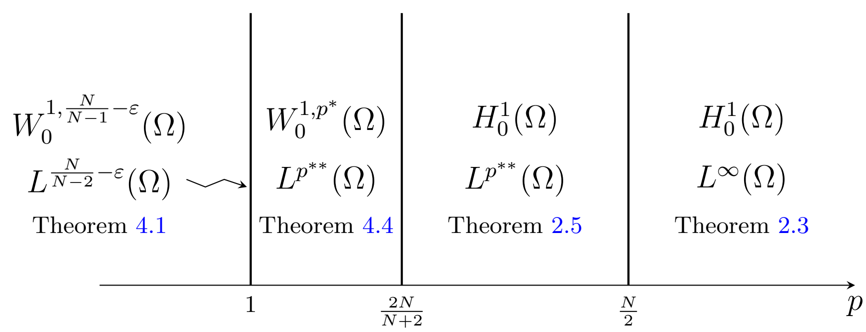
$$\lambda = \frac{N(2-q)}{N-q}, \quad q = m^* = \frac{Nm}{N-m}.$$

It is easy to see that $1 < m < 2_*$ implies $0 < \lambda < 1$, as desired. We thus have

$$\left(\int_\Omega |u_n|^{q^*}\right)^{\frac{q}{q^*}} \le C \int_\Omega |\nabla u_n|^q \le C\, \|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^{q^*}\right)^{1-\frac{q}{2m}}.$$

Since $\frac{q}{q^*} > 1 - \frac{q}{2m}$ is true (being equivalent to $m < \frac{N}{2}$), we obtain from the first and third term that u_n is bounded in $L^{q^*}(\Omega)$ (which is again $L^{m^{**}}(\Omega)$, see Theorem 2.5) by a constant depending (among other quantities) on the norm of f in $L^m(\Omega)$. Once u_n is bounded, the boundedness of $|\nabla u_n|$ in $L^q(\Omega)$ (with $q = m^*$) then follows comparing the second and the third term. □

We can now draw the complete picture.



5. Nonuniqueness for distributional solutions

If the datum μ is a measure, we have proved in Theorem 4.1 that the sequence u_n of approximating solutions is bounded in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Therefore, and up to subsequences, u_n weakly converges to the solution u in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Choosing a $C_0^1(\Omega)$ test function φ in the formulation (1.10) for u_n, we obtain

$$\int_\Omega A(x)\nabla u_n \cdot \nabla\varphi = \int_\Omega f_n\, \varphi,$$

which, passing to the limit, yields

$$\int_\Omega A(x)\nabla u \cdot \nabla\varphi = \int_\Omega \varphi\, d\mu \quad \forall \varphi \in C_0^1(\Omega),$$

so that u is a solution *in the sense of distributions* of (1.9). Since the definition of solution in the sense of distributions can always be given (even when the notion of duality solution is unavailable due for example to the operator being nonlinear), one may wonder whether there is a way of proving uniqueness of distributional solutions (not passing through duality solutions).

The following example is due to J. Serrin (see [5]). Let $\varepsilon > 0$ and $A^\varepsilon(x)$ be the symmetric matrix defined by

$$a_{ij}^\varepsilon(x) = \delta_{ij} + (a_\varepsilon - 1)\frac{x_i\, x_j}{|x|^2}.$$

If $a_\varepsilon = \frac{N-1}{\varepsilon(N-2+\varepsilon)}$, then the function

$$w^\varepsilon(x) = x_1\, |x|^{1-N-\varepsilon}$$

is a solution in the sense of distributions of

$$-\operatorname{div}(A^\varepsilon(x)\,\nabla w^\varepsilon) = 0, \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{5.22}$$

Indeed, if we rewrite $w(x) = x_1|x|^\alpha$ and

$$a_{ij}(x) = \delta_{ij} + \beta\frac{x_i\, x_j}{|x|^2},$$

simple (but tedious) calculations imply

$$w_{x_1}(x) = |x|^\alpha + \alpha x_1^2|x|^{\alpha-2}, \quad w_{x_i}(x) = \alpha x_1 x_i |x|^{\alpha-2},$$

so that

$$\sum_{i=1}^N a_{ij}(x)\, w_{x_i}(x) = \delta_{1j}|x|^\alpha + (\alpha\beta + \alpha + \beta)x_1 x_j |x|^{\alpha-2}.$$

Therefore,

$$(A(x)\,\nabla w)_{x_1} = \alpha x_1|x|^{\alpha-2} + (\alpha\beta + \alpha + \beta)[2x_1|x|^{\alpha-2} + (\alpha-2)x_1^3|x|^{\alpha-4}],$$

and

$$(A(x)\,\nabla w)_{x_j} = (\alpha\beta + \alpha + \beta)[x_1|x|^{\alpha-2} + (\alpha-2)x_1 x_j^2|x|^{\alpha-4}],$$

so that

$$\operatorname{div}(A(x)\nabla w) = x_1|x|^{\alpha-2}[\alpha + (N-1+\alpha)(\alpha\beta + \alpha + \beta)].$$

Given $0 < \varepsilon < 1$, if we choose $\alpha = 1 - N - \varepsilon$, and $\beta = \frac{N-1}{\varepsilon(N-2+\varepsilon)} + 1$, we have

$$\alpha + (N-1+\alpha)(\alpha\beta + \alpha + \beta) = 0,$$

so that w is a solution of (5.22) if $x \neq 0$. To prove that w^ε is a solution in the sense of distributions in the whole $\mathbb{R}^N$, let φ be a function in $C^1_0(\Omega)$, and observe that since $|A^\varepsilon(x)\nabla w^\varepsilon|$ belongs to $L^1(\Omega)$, we have

$$\int_{\mathbb{R}^N} A^\varepsilon(x)\nabla w^\varepsilon \cdot \nabla\varphi = \lim_{r\to 0^+} \int_{\mathbb{R}^N\setminus B_r(0)} A^\varepsilon(x)\nabla w^\varepsilon \cdot \nabla\varphi.$$

Using Gauss-Green formula, and recalling that w^ε is a solution of the equation outside the origin, we have

$$\int_{\mathbb{R}^N} A^\varepsilon(x)\nabla w^\varepsilon \cdot \nabla\varphi = -\lim_{r\to 0^+} \int_{\partial B_r(0)} \varphi\, A^\varepsilon(x)\nabla w^\varepsilon \cdot \nu \, d\sigma,$$

where ν is the exterior normal to $B_r(0)$, i.e., $\nu = \frac{x}{r}$. By a direct computation,

$$A^\varepsilon(x)\nabla w^\varepsilon \cdot \frac{x}{r} = Q x_1 |r|^{\alpha-1},$$

with $Q = 1 + \alpha\beta + \alpha + \beta = -\frac{N-1}{\varepsilon}$. Therefore, recalling the value of α, and rescaling to the unit sphere,

$$-\int_{\partial B_r(0)} \varphi\, A^\varepsilon(x)\nabla w^\varepsilon \cdot \nu \, d\sigma = \frac{N-1}{\varepsilon}\frac{1}{r^\varepsilon} \int_{\partial B_1(0)} \varphi(ry) x_1 \, d\sigma.$$

Using again the Gauss-Green formula, we have

$$\int_{\partial B_1(0)} \varphi(ry) x_1 \, d\sigma = r \int_{B_1(0)} e_1 \cdot \nabla\varphi(rx)\, dx,$$

where $e_1 = (1, 0, \ldots, 0)$. Therefore, since $0 < \varepsilon < 1$, we have

$$\lim_{r\to 0^+} \int_{\partial B_r(0)} \varphi\, A^\varepsilon(x)\nabla w^\varepsilon \cdot \nu \, d\sigma = \lim_{r\to 0^+} r^{1-\varepsilon} \int_{B_1(0)} e_1 \cdot \nabla\varphi(rx)\, dx = 0,$$

so that w^ε is a solution in the sense of distributions of $-\mathrm{div}(A^\varepsilon \nabla w^\varepsilon) = 0$ in the whole $\mathbb{R}^N$.

Let now $\Omega = B_1(0)$ be the unit ball, and let v_ε be the unique solution of

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x)\, \nabla v^\varepsilon) = \mathrm{div}(A^\varepsilon(x)\, \nabla x_1) & \text{in } \Omega, \\ v^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

which exists since $\mathrm{div}(A^\varepsilon(x)\, \nabla x_1)$ is a regular function belonging to $H^{-1}(\Omega)$ (as can be easily seen). Therefore, the function $z^\varepsilon = v^\varepsilon + x_1$ is the unique solution in $H^1(\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x)\, \nabla z^\varepsilon) = 0 & \text{in } \Omega, \\ z^\varepsilon = x_1 & \text{on } \partial\Omega, \end{cases}$$

so that the function $u^\varepsilon = w^\varepsilon - z^\varepsilon$ is a solution in the sense of distributions of

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x)\,\nabla u^\varepsilon) = 0 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

which is not identically zero since z^ε belongs to $H^1(\Omega)$, while w^ε belongs to $W^{1,q}_0(\Omega)$ for every $q < q_\varepsilon = \frac{N}{N-1+\varepsilon}$. Hence, the problem

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x)\,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has infinitely many solutions in the sense of distributions, which can be written as $u = \overline{u} + t\,u^\varepsilon$, t in $\mathbb{R}$, where $\overline{u}$ is the duality solution.

One may observe that the solution found by approximation belongs to $W^{1,q}_0(\Omega)$ for *every* $q < \frac{N}{N-1}$, while the solution of the above example belongs to $W^{1,q}_0(\Omega)$ for *some* $q < \frac{N}{N-1}$, and that we are not allowed to take $\varepsilon = 0$ since in this case a_ε diverges. Thus one may hope that there is still uniqueness of the solution obtained by approximation. However, it is possible to modify Serrin's example in dimension $N \geq 3$ (see [4]) to find a nonzero solution in the sense of distributions for

$$\begin{cases} -\mathrm{div}(B^\varepsilon(x)\,\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which belongs to $W^{1,q}_0(\Omega)$, for every $q < \frac{N}{N-1}$. Here

$$B^\varepsilon(x) = \begin{pmatrix} 1 + (a_\varepsilon - 1)\frac{x_1^2}{x_1^2+x_2^2} & (a_\varepsilon - 1)\frac{x_1\,x_2}{x_1^2+x_2^2} & 0 \\ (a_\varepsilon - 1)\frac{x_1\,x_2}{x_1^2+x_2^2} & 1 + (a_\varepsilon - 1)\frac{x_2^2}{x_1^2+x_2^2} & 0 \\ 0 & 0 & I \end{pmatrix},$$

where I is the identity matrix in $\mathbb{R}^{N-2}$, and a_ε is as above, with ε fixed so that $w^\varepsilon(x) = x_1\,(\sqrt{x_1^2 + x_2^2})^{\varepsilon-1}$ belongs to $W^{1,q}(\mathbb{R}^2)$ for every $q < 2$.

On the other hand, in dimension $N = 2$ there is a unique solution in the sense of distributions belonging to $W^{1,q}_0(\Omega)$, for every $q < 2$. The proof of this fact uses Meyers' regularity theorem for linear equations with regular data.

Theorem 5.1 (Meyers)**.** *Let A be a matrix which satisfies (1.8). Then there exists $p > 2$ (p depends on the ratio $\frac{\alpha}{\beta}$ and becomes larger as $\frac{\alpha}{\beta}$ tends to 1) such that if u is a solution of (1.9) with datum f belonging to $L^\infty(\Omega)$, then u belongs to $W^{1,p}_0(\Omega)$.*

Theorem 5.2. *Let $N = 2$. Then there exists a unique solution in the sense of distributions of (1.9) such that u belongs to $W_0^{1,q}(\Omega)$, for every $q < 2$.*

Proof. Since the equation is linear, it is enough to prove that if u is such that

$$\int_\Omega A(x)\,\nabla u \cdot \nabla \varphi = 0, \quad \forall \varphi \in C_0^1(\Omega),$$

then $u = 0$. Since u belongs to $W_0^{1,q}(\Omega)$, for every $q < 2$, it is enough to prove that

$$\int_\Omega A(x)\,\nabla u \cdot \nabla \varphi = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega),$$

for some $p > 2$, implies $u = 0$. Let B be a subset of Ω, and let v_B be the solution of

$$\begin{cases} -\mathrm{div}(A^*(x)\,\nabla v_B) = \chi_B & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By Meyers' theorem, v_B belongs to $W_0^{1,p}(\Omega)$, for some $p > 2$. Hence

$$\int_\Omega A(x)\,\nabla u \cdot \nabla v_B = 0,$$

while, choosing u as test function in the weak formulation for v_B (which can be done using a density argument and the regularity of ∇v_B), we have

$$\int_\Omega A^*(x)\,\nabla v_B \cdot \nabla u = \int_B u.$$

Therefore,

$$\int_B u = 0, \quad \forall B \subseteq \Omega,$$

and this implies $u \equiv 0$. □

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