CHAPTER II

CLASSIFICATION OF $K3$ SURFACES

May 25 2001

We will present the major results on the classification of $K3$ surfaces: any two $K3$'s are deformation equivalent (Kodaira [??] and Tjurina ??), the global Torelli Theorem holds [Sh,P-S71], corrected by others [Looijenga, Peters]), and the surjectivity of the period map (Kulikov [??], Todorov [??],Looijenga [??]). Here are the precise statements.

Theorem (Kodaira). *Let X, Y be $K3$ surfaces. Then X and Y are directly deformation equivalent, via a family of $K3$ surfaces.*

Global Torelli Theorem (Shafarevich, Piatechki-Shapiro). *Let X, Y be $K3$ surfaces, and suppose there exists an isometry*

$$(H^2(X;\mathbb{Z}),\cup) \xrightarrow{f} (H^2(Y;\mathbb{Z}),\cup)$$

such that the induced map $f_{\mathbf{C}}\colon H^2(X;\mathbf{C}) \to H^2(Y;\mathbf{C})$ is an isomorphism of Hodge structures (a Hodge isometry). Then X is isomorphic to Y. Furthermore if f carries the effective classes of X to the effective classes of Y, then there exists an isomorphism $g\colon Y \xrightarrow{\sim} X$ such that $f = g^*$.

A few comments regarding the second part of the Torelli theorem (the so-called strong version). An effective class is $c_1(D)$, where D is an effective divisor. The effective classes are determined by the Hodge structure plus the knowledge of one Kähler class. It is not true that every Hodge isometry is induced by an isometry (e.g. multiplication by -1), but one can describe exactly those which are not (see (??)). Finally, surjectivity of the period map is the description of all the Hodge structures of $K3$ surfaces, we state this after describing the period space.

The period domain for $K3$ surfaces

Let Λ be the lattice $H^{\oplus 3} \oplus (-E_8)^{\oplus 2}$. For a commutative ring R let $\Lambda_R := \Lambda \otimes_{\mathbb{Z}} R$. The scalar product on Λ extends to a symmetric bilinear unimodular form on Λ_R, with values in R; for $\alpha, \beta \in \Lambda_R$ we denote their product by $\alpha \cdot \beta$ or by (α, β). The *period domain for $K3$ surfaces* is the subset $\Omega \subset \mathbb{P}(\Lambda_{\mathbf{C}})$ defined by

$$\Omega := \{[\omega] \in \mathbb{P}(\Lambda_{\mathbf{C}}) | \quad \omega \cdot \omega = 0, \quad \omega \cdot \bar{\omega} > 0\}.$$

Thus Ω is an open subset in a 20-dimensional smooth quadric. One can define a diffeomorphism

$$\Omega \xrightarrow{\text{diff}} Gr^+(2, \Lambda_{\mathbb{R}}) := \{\text{oriented 2 dim. } V \subset \Lambda_{\mathbb{R}} \text{ with } (\cdot,\cdot)|_V \text{ positive definite}\}$$

by associating to $[\omega] \in \Omega$ the two-dimensional subspace

$$V_\omega := (\mathbf{C}\omega \oplus \mathbf{C}\bar{\omega}) \cap \Lambda_{\mathbb{R}} = \mathbb{R}Re(\omega) \oplus \mathbb{R}Im(\omega),$$

with orientation induced by the standard orientation of $\mathbf{C}\omega$. Since the scalar product on $\Lambda_{\mathbb{R}}$ is of type $(3, 19)$ we see that $Gr^+(2, \Lambda_{\mathbb{R}})$ is connected. Let X be a $K3$ surface. By (1.9) there exists an isometry

$$(H^2(X;\mathbb{Z}), \langle,\rangle) \overset{f}{\cong} \Lambda. \tag{2.1}$$

Such an isometry we call a *marking of* X, and the couple (X, f) a *marked* $K3$. Let $f_{\mathbf{C}}: H^2(X;\mathbf{C}) \to \Lambda_{\mathbf{C}}$ be the isometry obtained extending scalars. A simple computation shows that $f_{\mathbf{C}}(H^{2,0}(X)) \in \Omega$. Thus we have associated to the couple (X, f) a point $P(X, f) \in \Omega$: the "periods"of X. The terminology can be explained as follows: the coordinates of $P(X, f) \in \Omega$ in a given basis are computed by integrating a non-zero holomorphic 2-form φ on the 2-homology classes of a dual basis, and if we do the analogue for elliptic integrals, we are computing the period of a pendulum (??). The period point $P(X, f)$ depends effectively on f: replacing Ω by the orbit space $\mathcal{M} := \Omega/Isom(\Lambda)$ we can define a map

$$\begin{array}{ccc} \{K3\text{'s}\}/\text{isom.} & \overset{\overline{P}}{\to} & \mathcal{M} \\ X & \mapsto & \overline{P(X, f)}. \end{array} \tag{2.2}$$

Thus the Torelli Theorem for $K3$ surfaces states that $\overline{P}$ is injective (plus something more). The last big Theorem we will present is the following.

Theorem(surjectivity of the period map). *Given any $[\omega] \in \Omega$ there exists a marked $K3$ surface (X, f) such that $P(X, f) = [\omega]$.*

Remark. The elements of $Isom(\Lambda)$ act on Ω via holomorphic automorphisms, but the action is not properly discontinuous: there are points with stabilizer of infinite cardinality. In fact if $V_\omega \in Gr^+(2, \Lambda_{\mathbb{R}})$ then $Stab(V_\omega) = Stab(V_\omega^\perp)$ injects into $Isom(V_\omega) \times Isom(V_\omega^\perp)$, and its image may very well be infinite because the restriction of $(\cdot,\cdot)$ to $V_\omega^\perp$ is indefinite (of type $(1, 19)$). More explicitely, an example of $K3$ surface with infinite group of automorphisms is given by $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ a smooth complete intersection of hypersurfaces Y and W, where $Y \in |(1, 1)|$ and $W \in |(2, 2)|$. Assuming that both projections of X to $\mathbb{P}^2$ are finite (this is the case for a generic choice of Y, W), we get that X has two covering involutions which generate a subgroup of $Aut(X)$ isomorphic to $\mathbb{Z}/(2) * \mathbb{Z}/(2)$. (See [Silverman??].) Thus there is no natural structure of complex space on the orbit space $\mathcal{M}$. One gets a quotient which is a complex space (in fact a quasi-projective variety) by fixing a *polarization* (see (??)).

Derivative of the period map and local Torelli

Let $\pi: \mathcal{X} \to B$ be a family of $K3$ surfaces. We are interested in the map

$$\begin{array}{ccc} B & \overset{\overline{P}_\pi}{\longrightarrow} & \mathcal{M} \\ b & \mapsto & \overline{P}(X_b). \end{array}$$

Since $\mathcal{M}$ is not a complex space it is more convenient to lift $\overline{P}_\pi$ to a map from B to Ω. A lift exists when the local system $R^2\pi_*\mathbb{Z}$ is trivial, e.g. if B is simply connected. Assume a trivialization of $R^2\pi_*\mathbb{Z}$ has been chosen, defining for each $b \in B$ an isometry f_b between $H^2(X_b;\mathbb{Z})$ and Λ. Then we have a map

$$\begin{array}{ccc} B & \xrightarrow{P_\pi} & \Omega \\ b & \mapsto & P(X_b, f_b), \end{array}$$

the *period map.* (Of course P_π depends on the trivialization chosen, the notation is slightly imprecise.) A fundamental result of Griffiths [??] (valid for arbitrary families of Kähler manifolds) asserts that the period map is holomorphic, and computes its differential as follows. Let φ be a non-zero holomorphic 2-form on X_b, so that $P_\pi(b) = [\varphi]$. Then we have a canonical isomorphism

$$T_{[\varphi]}\Omega \cong Hom(\varphi, \varphi^\perp/\mathbf{C}\varphi). \tag{2.4}$$

Let

$$T_bB \xrightarrow{\kappa} H^1(X_b;\Theta)$$

be the Kodaira-Spencer map induced by the family $\mathcal{X}$. Griffiths' formula asserts that for $v \in T_bB$

$$\langle dP_\pi(v), \varphi\rangle = \varphi \cup \kappa(v). \tag{2.5}$$

The cup-product is to be understood as cup-product followed by the map

$$H^1(\Omega^2 \otimes \Theta) \to H^1(\Omega^1)$$

obtained contracting Ω^2 with Θ, and furthermore we make the identification

$$H^1(\Omega^1) \cong \varphi^\perp/\mathbf{C}\varphi$$

via the Hodge decomposition. Before stating the local Torelli Theorem we recall that by a Theorem of Kodaira [??] all small deformations of a Kähler manifold are Kähler, hence all small deformations of a $K3$ surface are $K3$ surfaces.

(2.6) LOCAL TORELLI FOR $K3$ SURFACES (KODAIRA, TYURINA (??). *Let (X, f) be a marked $K3$ surface, and set $p := P(X, f)$. The map of germs*

$$Def(X) \xrightarrow{m} (\Omega, p) \tag{2.7}$$

induced by the universal family parametrized by $Def(X)$ (with marking infuced by f) is an isomorphism.

PROOF. The deformation space of X is smooth. In fact a holomorphic symplectic form on X defines an isomorphism $\Theta \cong \Omega^1$, hence $H^2(X;\Theta) \cong H^2(X;\Omega^1)$, and this cohomology space vanishes by the Hodge decomposition: thus deformations of X are unobstructed. Since $Def(X)$ is smooth it suffices to prove that the differential of m is an isomorphism. The Kodaira-Spencer map for $Def(X)$ identifies the tangent space to $Def(X)$ with $H^1(X;\Theta)$; since the map

$$H^1(X;\Theta) \xrightarrow{\varphi\cup\cdot} H^1(X;\Omega)$$

is an isomorphism, we conclude that dm is an isomorphism.

Periods and divisors of a $K3$

Let X be a $K3$ surface: let $S_X, T_X \subset H^2(X;\mathbb{Z})$ be the sublattices defined by

$$S_X := c_1(Pic(X)), \quad T_X := S_X^{\perp}.$$

Thus S_X is isomorphic to the Neron-Severi group of X; T_X is the *transcendental lattice of* X. By Lefschetz' $(1,1)$-Theorem S_X is completely determined by the periods of X: if φ is a symplectic form on X, then

$$S_X = H^2(X;\mathbb{Z}) \cap \varphi^{\perp}.$$

In fact the left-hand side is $H^{1,1}_{\mathbb{Z}}(X) := H^{1,1}(X) \cap H^2(X;\mathbb{Z})$ by the $(1,1)$-Theorem; on the other hand $H^{1,1}_{\mathbb{Z}}(X)$ is contained in the right-hand side, and viceversa a class in the right-hand side is also orthogonal to $\overline{\varphi}$ because it is real, and hence belongs to $H^{1,1}_{\mathbb{Z}}(X)$. If $S \subset \Lambda$ is a sublattice we let $\Omega_S \subset \Omega$ be given by

$$\Omega_S := \{[\omega] \in \Omega | \ \omega \perp S\}.$$

Thus if (X,f) is a marked $K3$ then

$$P(X,f) \in \Omega_S \text{ if and only if } f^{-1}(S) \subset S_X. \tag{2.8}$$

For $v \in \Lambda$ we set $\Omega_v := \Omega_{\mathbb{Z}v}$.

(2.9) Any two $K3$'s are deformation equivalent

Kodaira proved the stated result [??] by first showing that any $K3$ may be deformed into an elliptic one, and then by proving that elliptic $K3$'s form a connected family. We give a similar proof: we prove that any $K3$ may be deformed in a double cover of $\mathbb{P}^2$ ramified over a smooth sextic, the result follows because double planes of this type clearly from a connected family. We need a density result for Hodge structures. For $k \in \mathbb{Z}$ let $P_{2k} \subset \Lambda$ be the subset of non-zero primitive vectors with norm $2k$, and let

$$\Omega_{2k} := \bigcup_{v \in P_{2k}} \Omega_v.$$

We wish to prove that

$$\Omega_{2k} \text{ is dense in } \Omega. \tag{2.10}$$

For F a field let $Q_F \subset \mathbb{P}(\Lambda_F)$ be the cone of isotropic lines in Λ_F. Let $[\omega] \in \Omega$; then

$$\omega^{\perp} \cap Q_{\mathbb{R}} = [V_{\omega}^{\perp}] \cap Q_{\mathbb{R}}.$$

Since the restriction of $(\cdot,\cdot)$ to $V_{\omega}^{\perp}$ is of type $(1,19)$ there exists

$$[v] \in \omega^{\perp} \cap Q_{\mathbb{R}}. \tag{2.11}$$

Since $Q_{\mathbb{Q}}$ is not empty, it is dense in $Q_{\mathbb{R}}$, hence there exists a sequence $\{[v_n]\}$ converging to $[v]$, where $v_n \in P_0$. We wish to replace the vectors v_n by elements of P_{2k}, where k is arbitrary.

(2.12) LEMMA. *Let $w \in P_0$ be primitive of norm zero. Given any $k \in \mathbb{Z}$ there exists a sequence $\{w_n\}$, where $w_n \in P_{2k}$, such that $[w_n] \to [w]$.*

PROOF. An easy exercise shows that we can give an orthogonal decomposition

$$\Lambda = H^{\oplus 2} \oplus \Gamma$$

so that $w \in H^{\oplus 2}$. A straighforward computation shows that there exists a sequence $\{w_n\}$ with $w_n \in H^{\oplus 2}$ primitive of norm $2k$, such that $[w_n] \to [w]$.

Applying this lemma to the points $[v_n]$ of the sequence converging to the point $[v]$ of (2.11) we get that there exists a sequence $\{u_n\}$, with $u_n \in P_{2k}$, such that $\{[u_n]\}$ converges to $[v]$. Thus every point of Ω_v is a limit of points in Ω_{u_n}. Since $\varphi \in \Omega_v$ we conclude that

$$\varphi \in \overline{\bigcup_n \Omega_{u_n}}.$$

This proves (2.10). We are ready to prove (2.9). By the local Torelli Theorem, and by (2.10)-(2.8) we see that every $K3$ may be deformed into a $K3$ surface which has a primitive divisor class of self-intersection $2k$, however we choose $k \in \mathbb{Z}$. Applying again Local Torelli and (2.8) we can further deform the $K3$ to another $K3$ surface X which has a divisor class D of self-intersection $2k$ spanning $Pic(X)$, i.e. $Pic(X) = \mathbb{Z}[D]$. Let's assume $k = 1$: then by Riemann-Roch and Serre duality

$$h^0(D) + h^0(-D) \geq 3.$$

Thus either D or $-D$ is effective; without loss of generality we can assume D is effective. By Riemann-Roch we get that $h^0(D) \geq 3$; an easy argument shows that $h^0(D) = 3$, that the linear system $|D|$ has no base-points, and that the corresponding map

$$X \xrightarrow{\varphi_{|D|}} |D|^* \cong \mathbb{P}^2$$

is a regular double cover ramified over a smooth sextic. Thus every $K3$ is a deformation of a double plane ramified over a smooth sextic: since $K3$'s of this type are all deformation equivalent, we conclude that any two $K3$'s can be deformed one into another.

The classical proof of the Torelli theorem for $K3$ surfaces

This is the proof given by Shafarevich and Piatechki-Shapiro [Sh,P-S71] with corrections due to ??. We will give a very brief outline of the proof, since there exist at least two excellent presentations in book format, by Barth, Peters, Van de Ven [BPVV84] and by Beauville [SémPal85]. The key point is to first prove a Torelli theorem for Kummer surfaces. More precisely, one shows that if a $K3$ surface X has the same Hodge structure as a Kummer surface $\widetilde{K}(T)$, then X is also a Kummer surface, say $X = \widetilde{K}(T')$. One also gets that the weight-2 polarized Hodge structures of T and T' are isomorphic. Shafarevich and Piatechki-Shapiro concluded from this that $T \cong T'$, but the Hodge isometry $H^2(T;\mathbb{Z}) \cong H^2(T';\mathbb{Z})$ alone is not sufficient to get that $T \cong T'$: in fact a 2-dimensional complex torus and its dual (i.e. its Picard variety) have the same weight-2 Hodge structure [Shioda78]. However using the extra information given by the matching of nodal curves, one can show that

$T \cong T'$, and hence $X \cong \widetilde{K}(T)$. The next step consists in verifying that the Hodge structures of Kummer surfaces are dense in the period space: this is similar to the density result (2.10) that we proved in order to show that every $K3$ can be deformed into a double plane. One finishes the proof arguing as follows. Let X, Y be $K3$'s such that $H^2(X;\mathbb{Z})$ and $H^2(Y;\mathbb{Z})$ are Hodge-isometric: by local Torelli, density of Kummer Hodge structures, and Torelli for "Kummer Hodge structures", there exist sequences $\{X_n\}$ and $\{Y_n\}$ of Kummer surfaces approaching X and Y respectively, with an isomorphism $f_n: X_n \cong Y_n$ for every n. One concludes by showing that f_n converges to an isomorphism $X \cong Y$: in fact a volume computation (if we are in the projective category we may use projectivity of the Chow variety of cycles of boundede degree) gives an effective cycle Γ on $X \times Y$ as limit of the graphs of the f_n, and then one shows that Γ contains the graph of an isomorphism between X and Y (this is because $H^{2,0}(X) \neq 0 \neq H^{2,0}(Y)$ and X, Y are minimal).

Torelli and surjectivity of the period map via degenerations

Kulikov [??] (with corrections by Persson and Pinkham [??]) proved an important result which puts in standard form the singular varieties one may obtain as degenerations of $K3$ surfaces: surjectivity of the period map (for projective $K3$'s) follows almost immediately, and R. Friedman [??] was able to deduce from Kulikov's theorem also the Torelli Theorem (for projective $K3$'s). We will state and comment Kulikov's result, we will show how to derive surjectivity of the period map, and we will give a brief sketch of Friedman's proof of Torelli. We start with general considerations on moduli of projective $K3$ surfaces.

Moduli of projective $K3$ surfaces. Let k be a strictly positive integer. Let $v \in P_{2k}$ and $\Omega_v \subset \Omega$ be the submanifold of lines perpendicular to v. Given a *polarized* $K3$ *of degree* $2k$, i.e. a couple (S, H), where S is a $K3$ and H a primitive (indivisible) ample divisor of degree $2k$, we can assume the period point of S lies in Ω_v: in fact a standard result on lattices [??] assures us that there exists an isometry $f: H^2(S;\mathbb{Z}) \xrightarrow{\sim} \Lambda$ such that $f(c_1(H)) = v$, and then $P(S, f) \in \Omega_v$. Let $\Gamma_v < Isom(\Lambda)$ be the subgroup fixing v: as is easily checked Γ_v acts properly discontinuously on Ω_v, so that the quotient has a natural structure of analytic space. Hence the isomorphism classes of (possible) Hodge structures of degree $2k$ polarized $K3$'s are parametrized by the analytic space Ω_v/Γ_v. In fact this space has a natural structure of quasi-projective variety, by classical results of Baily-Borel [??], since Ω_v is a bounded symmetric space, and Γ_v is an arithmetic group acting on it. On the other hand the set of isomorphism classes of degree-$2k$ polarized $K3$'s has a natural structure of quasi-projective variety $\mathcal{F}^0_{2k}$ [??]. (An example of a *moduli space.*) One shows that the period map defines a regular map

$$\pi^0_{2k}: \mathcal{F}^0_{2k} \to \Omega_v/\Gamma_v. \tag{2.13}$$

The Torelli Theorem for projective $K3$ surfaces is the statement that for any $k > 0$ the map $\pi_2 k$ is injective. (This implies the "standard" Torelli Theorem for projective $K3$'s.) What about surjectivity ? The map π^0_k is certainly not surjective: let $k = 4$, then $\mathcal{F}^0_4$ parametrizes isomorphism classes of smooth quartic surfaces in $\mathbb{P}^3$ (we must also include double covers of a smooth quadric ramified over a smooth $(4,4)$-curve and $K3$'s with an elliptic pencil $|F|$ with a section Γ, where $H = \Gamma + 3F$.) As shown in Chapter 1 we can construct a family $\mathcal{X}$ of $K3$'s parametrized by a disc

Δ with the property that for $t \neq 0$ the surface X_t is a smooth quartic and X_0 is the blow-up at the node of a quartic with an ordinary double point. The period map takes Δ to Ω_v/Γ_v and the image of Δ^* lies in $\mathrm{Im}(\pi_{2k})$, however the image of 0 is not contained $\mathrm{Im}(\pi_{2k})$. This example shows that π_{2k} is not surjective, and it also indicates that in order to achiev surjectivity we must enlarge $\mathcal{F}^0_{2k}$ so that it includes also isomorphism classes of *quasi-polarized* $K3$*'s*. To be precise: a degree-$2k$ quasi-polarized $K3$ consists of a couple (S, H) where S is a $K3$ surface and H is a primitive nef divisor of self-intersection $2k$. We recall that nef means that $H \cdot C \geq 0$ for all curves $C \subset S$. An example of a degree-4 quasi-polarized $K3$ is given by the blow-up at the node of a quartic with an ordinary double point, where the nef divisor is the pull-back of the hyperplane section of the singular quartic. There is a quasi-projective moduli space $\mathcal{F}_{2k}$ for quasi-polarized $K3$'s of degree $2k$ and the period map π^0_{2k} of (2.13) extends to a reguar map

$$\pi_{2k}\colon \mathcal{F}_{2k} \to \Omega_v/\Gamma_v. \tag{2.14}$$

The following is the main result on periods of projective $K3$'s.

THEOREM(PIATECHKI-SHAPIRO, SHAFAREVICH, KULIKOV). *The map π_{2k} is an isomorphism.*

Our aim is to show that Kulikov's Theorem on degenerations of $K3$ implies properness of π_{2k} (this implies surjectivity by local Torelli), and to sketch R. Friedman's proof of injectivity of π_{2k} based on Kulikov's Theorem.

Kulikov's Theorem on degenerations of $K3$ surfaces. Let $f\colon \mathcal{X} \to \Delta$ be a family of compact complex analytic spaces with parameter space the unit disc Δ, i.e. f is a proper surjective map. Thus $X_t := f^{-1}(t)$ is a compact analytic space and we think of the collection $\{X_t\}_{t\in\Delta}$ as a family of compact analytic spaces. If X_t is smooth for $t \neq 0$ we say X_0 is a *degeneration of the smooth varieties X_t ($t \neq 0$)*. The base-change of order n of $\mathcal{X}$ is the fiber product

$$\begin{array}{ccc} \mathcal{Y} = \Delta \times_\Delta \mathcal{X} & \longrightarrow & \mathcal{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta & \xrightarrow{u_n} & \Delta \end{array}$$

where $u_n(t) := t^n$. Let $f\colon \mathcal{X} \to \Delta$ be a degeneration. By Hironaka's Theorem on resolution of singularities and Mumford's theory of toroidal embeddings there exists a base change $\mathcal{Y}$ of $\mathcal{X}$ and a resolution $\mathcal{Z}$ of $\mathcal{Y}$ such that the "new"degeneration $h\colon \mathcal{Z} \to \Delta$ is semistable. This results says that for certain purposes we can assume every degeneration is semistable. Now let $f\colon \mathcal{X} \to \Delta$ be a semistable degeneration of $K3$'s. An easy argument shows that since $K_{X_t} \sim 0$ for all $t \neq 0$ the canonical bundle of $\mathcal{X}^*$ is trivial and hence

$$K_{\mathcal{X}} \sim \sum_i n_i V_i, \tag{2.15}$$

where the V_i's are the components of X_0.

(2.16) Theorem(Kulikov [Kulikov77]). *Let $\mathcal{X} \to \Delta$ be a semistable degeneration of $K3$'s, and assume that all irreducible components of X_0 are projective. There exists a semistable degeneration $\mathcal{Y} \to \Delta$ with $\mathcal{Y}$ birational to $\mathcal{X}$, with $\mathcal{Y}^* \to \Delta^*$ isomorphic to $\mathcal{X}^* \to \Delta*$, such that $K_{\mathcal{Y}}$ is trivial.*

A degeneration of $K3$'s is a *Kulikov degenerations* if its canonical bundle is trivial. These degenerations can be described explicitely.

(2.17) Theorem(Kulikov [Kulikov77]). *Let $\mathcal{Y} \to \Delta$ be a Kulikov degeneration of $K3$'s: then it belongs to one of the following three types, distinguished by the nilpotency index of the logarithm of the monodromy action on cohomology (denoted by N).*

I Y_0 *is smooth (hence a $K3$): $N = 0$.*

II Y_0 *is a chain of surfaces, the end surfaces are rational, the "middle"surfaces are elliptically (birationally) ruled, the double curves of the end surfaces are sections of the anticanonical bundle, an elliptically ruled surface meets each nearby surface in a section of the ruling: $N \neq 0$ and $N^2 = 0$.*

III *The dual graph of Y_0 is a triangulation of the sphere, each component of Y_0 is rational and the sum of its double curves is a cycle of rational curves giving a section of the anticanoniacl bundle: $N^2 \neq 0$ and $N^3 = 0$.*

(2.18) Example. We consider pencils of quartic surfaces in $\mathbb{P}^3$, of the form

$$\{S_\infty + tS_0\}_{t\in\Delta},$$

where S_0 is a singular quartic and S_∞ is a smooth quartic. Proceeding as in (??) we produce from this (if necessary we introduce a base change) a degeneration of $K3$'s whose smooth fibers are isomorphic to the smooth surfaces of the pencil and whose central fiber is the union of components birational to the components of S_0 (maybe with some extra components). If the singularities of S_0 are ordinary nodes then after a base change of order 2 we get a Type I degeneration. If S_0 is the union of two quadrics then we get a Type II degeneration with no elliptically ruled components. If S_0 is the union of 4 planes we get a Type III degeneration whose dual graph is the triangulation of the sphere given by a thetrahedron.

(2.19) Exercise. Let $F_\infty, F_0 \in \mathbf{C}[x_0, x_1, x_2]_6$, and $C_\infty := (F_\infty)$, $C_0 := (F_0)$ be the associated divisors. We assume C_∞ is a reduced and smooth. Let $C_t := (F_0 + tF_\infty)$, and $W_t \to \mathbb{P}^2$ the double cover branched over C_t. The surfaces W_t (for $t \in \mathbb{A}^1$) fit together to form the fibers of a family $f\colon \mathcal{W} \to \mathbb{A}^1$ with generic member a $K3$ surface: in fact $\mathcal{W}$ is the double cover of $\mathbb{P}^2 \times \mathbb{A}^1$ branched over $\bigcup_{t\in\mathbb{A}^1} C_t$. If we restrict the family to a sufficiently small disc $\Delta \subset \mathbb{A}^1$ centered around 0, we get a degeneration of $K3$ surfaces $f\colon \mathcal{W} \to \Delta$ (yes, we abuse notation). Compute a Kulikov model of $\mathcal{W}$ when

(1) $C_0 = 2E$, E a smooth cubic,
(2) $C_0 = 2L + Q$, L a line, Q a quartic intersecting transversely L,
(3) $C_0 = 2L_1 + 2L_2 + 2L_3$, where L_1, L_2, L_3 are lines not passing through the same point.

We will not give the proof of Kulikov's Theorems: we will limit ourselves to a few comments. The idea of the proof is simple, to understand it let's reason backwards:

assume that $\mathcal{Y} \to \Delta$ is a Kulikov degeneration and that $\mathcal{X}$ is obtained starting from $\mathcal{Y}$ by a series of blow-ups centered in the fiber Y_0. Then in the expression (2.15) we may assume $n_i \geq 0$, and the coefficient of the "last exceptional divisor" is maximal. Thus in order to prove Kulikov's Theorem (2.16) we try to invert this process (notice that although the expression (2.15) is determined only up to addition of multiples of the trivial divisor $X_0 = \sum_i V_i$, the components whose coefficient is maximal are uniquely determined). Assume $K_{\mathcal{X}}$ is not trivial: then the coefficients appearing in (2.15) are not all equal, and since X_0 is connected there exists a component with maximal coefficient meeting a component whose coefficient is stricly smaller than the maximum. Renumbering if necessary we can suppose the component with maximal coefficient is V_1, and that V_2 is a component meeting V_1 with non maximal coefficient. Adding $0 \sim -n_1 X_0$ to Formula (2.15) we get

$$K_{\mathcal{X}} \sim -\sum_{i>1} m_i V_i,$$

where $m_i \geq 0$ and $m_2 > 0$. Let $D_{1i} := V_1 \cap V_i$; by adjunction and the relation $V_1 \sim -\sum_{i>1} V_i$,

$$K_{V_1} \sim -\sum_{i>1} (m_i + 1) D_{1i}.$$

Thus the Kodaira dimension of V_1 is $-\infty$, hence V_1 is either $\mathbb{P}^2$ or rationally ruled. If $V_1 \cong \mathbb{P}^2$ and $m_2 = 2$, or if V_1 is minimally ruled and D_{12} is a section of the ruling we can blow down V_1 and we get a new semistable degeneration $\mathcal{X}_1 \to \Delta$, whose central fiber has one component less. In reality things are not that simple. Applying an old construction of Hironaka we can see what might be in store for us. Start with a Kulikov degeneration $\mathcal{Y}$, and let $C_1, C_2 \subset Y_0$ be smooth curves intersecting transversely in two points p, q. Let $\rho\colon \mathcal{W} \to (\mathcal{Y} \setminus p)$ be the map obtained by first blowing up $C_1 \setminus p$ and then the proper transform of $C_2 \setminus p$. Let $\theta\colon \mathcal{Z} \to (\mathcal{Y} \setminus q)$ be the map obtained by reversing the order of the blow-ups: first blow up $C_2 \setminus q$ and then the proper transform of $C_1 \setminus q$. The manifolds $\rho^{-1}(\mathcal{Y} \setminus \{p, q\})$ and $\theta^{-1}(\mathcal{Y} \setminus \{p, q\})$ are isomorphic: gluing them together we get a semistable degeneration $\mathcal{Y} \to \Delta$. The components of the central fiber whose coefficients are maximal (in Formula (2.15)) are the two exceptional divisors, say V_1 and V_2. Both these components are non-minimally ruled surfaces, hence they can not be blown down "globally". In fact we see that in order to recover $\mathcal{Y}$ from $\mathcal{X}$ we must contract $\mathcal{X}$ along each of the divisors $(V_1 \setminus F_1)$, $(V_2 \setminus F_2)$ where F_i is the reducible fiber of the ruling of V_i. The process of blowing down pieces of $\mathcal{X}$ and gluing them back is formalized by Persson, Pinkham [PersPink81].

Surjectivity of the period map via Kulikov's Theorem. Let (S, H) be a degree-$2k$ polarized $K3$ surface. Then $3H$ is very ample [??], and by Riemann-Roch $\dim |3H| = 9k + 2$. Thus we can realize S as a degree-$18k$ surface in $\mathbb{P}^{9k+2}$, where the divisibility of $\mathcal{O}_S(1)$ is equal to 3: these surfaces are parametrized by an open subset U_{2k} of a suitable Hilbert scheme $Hilb_{2k}$. Composing the quotient map $U_{2k} \to \mathcal{F}^0_{2k}$ with the period map π^0_{2k} of (2.13) we get a regular map

$$\widetilde{\pi}^0_{2k}\colon U_{2k} \to \Omega_v / \Gamma_v,$$

whose image is Zariski-open by the local Torelli Theorem. We must show that if $p \notin \mathrm{Im}(\widetilde{\pi}^0_{2k})$ then p is the period point of a quasi-polarized $K3$ surface of degree $2k$.

Since $\mathrm{Im}(\widetilde{\pi}^0_{2k})$ is Zariski-dense in Ω_v/Γ_v there exists a smooth curve $T \subset \Omega_v/\Gamma_v$ containing p, with $(T \setminus \{p\}) \subset \mathrm{Im}(\widetilde{\pi}^0_{2k})$. There exists a finite cover $m\colon B \to T$, with B smooth, and a family of surfaces $f\colon \mathcal{X} \to B$, with the following properties:

(1) each X_b is a surface in $\mathbb{P}^{9k+2}$ parametrized by a point of $Hilb_{2k}$,
(2) if $m(b) \in (T \setminus \{p\})$ then $(X_b, \mathcal{O}_{X_b}(1/3))$ is a polarized $K3$ of degree $2k$, and $m(b)$ is the period point of X_b.

Let $0 \in B$ be a point such that $m(0) = p$, and let $\Delta \subset B$ be a small disc centered at 0, so that X_b is smooth for all $b \neq 0$: thus restricting $\mathcal{X} \to B$ over Δ we get a degeneration of $K3$ surfaces. Taking a suitable base-change we can assume that the new degeneration $\mathcal{Z} \to \Delta$ is semistable, and that the logarithm of monodromy is 0. Notice also that all the components of the central fiber are projective surfaces. Thus by Theorem (2.16) there exists a Kulikov model $\mathcal{Y} \to \Delta$, with $\mathcal{Y}^* \cong \mathcal{Z}^*$, and by (2.17) the central fiber is a $K3$ surface. On $\mathcal{Z}$ we have a line bundle $\mathcal{L}$ whose restriction to Z_b for $b \neq 0$ is the degree-$2k$ polarization. Since $\mathcal{X}$ is isomorphic to $\mathcal{Z}$ in codimension 1, there corresponds to $\mathcal{L}$ a line-bundle $\mathcal{L}'$ on $\mathcal{X}$. If $\mathcal{L}'_0$ is nef we are done, because then $(X_0, \mathcal{L}'_0)$ is a degree-$2k$ quasi-polarized $K3$ whose period point is p. Now $\mathcal{L}'_0$ is not necessarily nef, however one can perform a finite number of birational modifications (so-called *Type 0 modifications* []) starting from $\mathcal{X}$, so that at the end we get a new family of $K3$'s with a line bundle $\mathcal{L}''$ restricting to a quasi-polarization on every fiber.

References

[BPVV84] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, 1984.

[Kulikov77] V. S. Kulikov, *Degenerations of $K3$ surfaces and Enriques surfaces*, Math. USSR Izvestija **11** (1977), 957-989.

[PersPink81] U. Persson, H. Pinkham, *Degeneration of surfaces with trivial canonical bundle*, Ann. of Math. **113** (1981), 45-66.

[SémPal85] *Géometrie des surfaces $K3$: modules et périodes*, Astérisque 126, Société mathématique de France, 1985.

[Sh,P-S71] I. Piatechki-Shapiro, I.R. Shafarevich, *A Torelli theorem for algebraic surfaces of type $K3$*, Math. USSR Izvestija **5** (1971), 547-588.

[Shioda78] T. Shioda, *The period map of abelian surfaces*, J. Fac. Sci. Univ. Tokyo **25** (1978), 47-59.